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The Chromatic Index of Ring Graphs

By *Lilian Shaffer*

Abstract. The goal of graph edge coloring is to color a graph G with as few colors as possible such that each edge receives a color and that adjacent edges, that is, different edges incident to a common vertex, receive different colors. The chromatic index, denoted $\chi'(G)$, is the minimum number of colors required for such a coloring to be possible. There are two important lower bounds for $\chi'(G)$ on every graph: maximum degree, denoted $\Delta(G)$, and density, denoted $\omega(G)$. Combining these two lower bounds, we know that every graph's chromatic index must be at least $\Delta(G)$ or $\omega(G)$, whichever is greater.

In this paper, we prove that the chromatic index of every ring graph is exactly equal to this lower bound.

1 Introduction

In graph theory, a *graph* G is a set of vertices, denoted $V(G)$, and a corresponding set of edges, denoted $E(G)$. Each of these edges in $E(G)$ connects some vertex in $V(G)$ to some vertex in $V(G)$. Such a graph is *undirected* if all edges in $E(G)$ edges are bidirectional. While this paper is concerned with *multigraphs*, graphs which allow vertices to be connected by multiple edges, we simply say "graph" for brevity. Coloring graphs by various parameters has been a long-standing problem in graph theory. An *edge coloring* of a graph G is an assignment of colors to the edges of G such that no two edges adjacent to a common vertex receive the same color. The *chromatic index*, denoted $\chi'(G)$, is the least number of colors required for an edge coloring of G.

When we talk of a *ring graph* G, we mean a finite undirected multigraph constructed from a single cycle. This is done by replacing each edge with some number of edges—possibly no edges, possibly one edge, and possibly multiple edges. This means that there will be no loops, that is, there will be no edges connecting a vertex to itself.

The *degree* of a vertex v, denoted $d(v)$, is the number of edges incident to v. The *maximum degree* of a graph G, denoted $\Delta(G)$, is the largest degree of any vertex in G. The *maximum multiplicity* of G, denoted $\mu(G)$, is the greatest number of edges connecting any two vertices. The *density* of G, denoted $\omega(G)$ and defined $\max_{H\subseteq G, |V(H)|\geq 3}\{\frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)|\rfloor}\}$,

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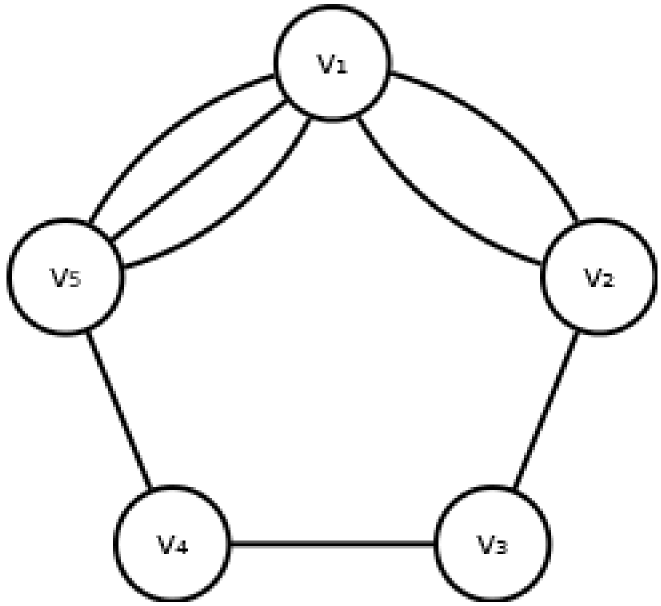

Figure 1: A ring graph with 5 vertices.

provides a conceptual way of describing how dense a graph is in terms of the sizes of edges.

It is known that both $\Delta(G)$ and $\lceil\omega(G)\rceil$ are lower bounds for the chromatic index of G. That is, $\chi'(G) \geq max\{\Delta(G), \lceil\omega(G)\rceil\}$. A graph G is said to be *exact* if $\chi'(G) = max\{\Delta(G), \lceil\omega(G)\rceil\}$.

A graph is *bipartite* if its vertices can be divided into two disjoint sets such that no edges connect any vertex to another vertex in its set. In 1916, Dénes König [1] proved that, for every bipartite graph G, $\chi'(G) = \Delta(G)$, making every bipartite graph exact. Because every ring graph with an even number of vertices is bipartite, every ring graph with an even number of vertices is exact.

Over time, people developed upper bounds for $\chi'(G)$. In 1949, Claude Shannon [2] proved that $\chi'(G) \leq \lfloor\frac{3}{2}\Delta(G)\rfloor$, and later, in 1964, V. G. Vizing [4] proved that $\chi'(G) \leq \Delta(G) + \mu(G)$. These upper bounds work for all graphs, and are thus very useful, but in specific cases, we can do better.

We prove in this paper that every ring graph with an odd number of vertices is exact.

Theorem 1.1. *Let* G *be a ring graph on* $n = 2k+1$ *vertices. Then* $\chi'(G) = max\{\Delta(G), \lceil\omega(G)\rceil\}$.

We will first define our notation and terminology, and we will introduce a few theorems and lemmas we will be using. Then we will prove the main theorem using a proof by cases.

2 Preliminaries

For notation, we define $V(G) = \{v_1, v_2, ..., v_n\}$ to be the set of all vertices in G and E(G) to be the set of all edges in G. Additionally, we will define that $k_i = |E(v_i, v_{i+1})|$, and we will say that $v_{n+1} = v_1$. As is standard, we will use $\Delta(G)$, $d(v_i)$, $\omega(G)$, and $\chi'(G)$ to represent the maximum degree of G, the degree of vertex v_i, the density of G, and the chromatic index of G, respectively. We know that, for every graph G, $\chi'(G) \geq max\{\Delta(G), \lceil\omega(G)\rceil\}$ (see Stiebitz [3]).

Throughout this paper, we use König's Line Coloring Theorem, proved by König [1] in 1916.

Theorem 2.1. *The chromatic index of any bipartite graph is equal to its maximum vertex degree.*

Throughout this proof, we will use several different but equivalent representations of the density of G. By definition, we know that

$$\lceil\omega(G)\rceil = \max_{X\subseteq V(G),|X|\geq 2}\left\lceil\frac{|E(G[X])|}{\lfloor\frac{1}{2}|X|\rfloor}\right\rceil$$

when $|V(G)| \geq 2$. If $|V(G)| \leq 1$, $\omega(G)$ is defined to be 0. For ring graphs, we will show that we can take $X = V(G)$.

Lemma 2.2. *Let* G *be a ring graph where* $\omega(G) \geq \Delta(G)$. *Then* $\lceil\omega(G)\rceil = \left\lceil\frac{|E(G)|}{\lfloor\frac{1}{2}|V(G)|\rfloor}\right\rceil$.

Proof. Recall that $\chi'(G) \geq max\{\Delta(G), \lceil\omega(G)\rceil\}$.

If $X = V(G)$, then $G[X] = G[V(G)] = G$.

If $X \subsetneq V(G)$, then G[X] is a bipartite graph. By Theorem 2.1,

$$\chi'(G) = \Delta(G[X]) \leq \Delta(G) \leq \lceil\omega(G)\rceil$$

So

$$\lceil\omega(G)\rceil = \left\lceil\frac{|E(G)|}{\lfloor\frac{1}{2}|V(G)|\rfloor}\right\rceil = \max_{X\subseteq V(G),|X|\geq 2}\left\lceil\frac{|E(G[X])|}{\lfloor\frac{1}{2}|X|\rfloor}\right\rceil$$

□

When |V(G)| is odd, like in our investigation, this is equivalent to

$$\lceil\omega(G)\rceil = \left\lceil\frac{2|E(G)|}{|V(G)|-1}\right\rceil$$

We know that $|E(G)| = \sum_{i=1}^{n} k_i$ and $|V(G)| = n = 2k+1$, so we can write

$$\lceil\omega(G)\rceil = \left\lceil\frac{2\sum_{i=1}^{n} k_i}{2k+1-1}\right\rceil = \left\lceil\frac{\sum_{i=1}^{n} k_i}{k}\right\rceil$$

3 Proof of Theorem 1.1

We assume that $|V(G)|$ is odd. Otherwise, G is a bipartite graph, and so $\chi'(G) = \Delta(G)$.
Case 1: $k_i = 0$ for some $i = 1, 2, ..., n$.

We assume without loss of generality that $k_n = 0$. For every ring graph with an odd number of vertices, no even-numbered vertices are connected, and v_n and v_1 are the only odd-numbered vertices that may be connected. Since $k_n = 0$, we separate the vertices of G into two discrete sets, the even-numbered vertices and the odd-numbered vertices, showing that G is bipartite. By Theorem 2.1, G is exact.
Case 2: There exists a vertex v_i such that $d(v_i) < \Delta(G)$, and $k_i > 0$ for all $i = 1, 2, ..., n$.

Assume $\chi'(F) = max\{\Delta(F), \lceil\omega(F)\rceil\}$ for every ring graph F where $|V(F)| = |V(G)|$ and $|E(F)| < |E(G)|$.

As in our preliminaries, we assume that $n = 2k+1$ and $v_{n+1} = v_1$. Without a loss of generality, we let $v_i = v_1$.

Take a near-perfect matching $M = \{v_2v_3, v_4v_5, ..., v_{2k}v_{2k+1}\}$. $H = G - M$ is also a ring graph. Because there is a single edge removed from each vertex other than v_1, it must remove a single edge from whichever vertex has the greatest degree.

$$\Delta(H) = \Delta(G) - 1$$

Note that H has the same number of vertices as G, and it has k fewer edges, so we can calculate the density of H.

$$\lceil\omega(H)\rceil = \left\lceil \frac{\sum_{i=1}^{n} k_i - k}{k} \right\rceil$$

By algebra.

$$\lceil\omega(H)\rceil = \left\lceil \frac{\sum_{i=1}^{n} k_i}{k} - 1 \right\rceil$$

$$\lceil\omega(H)\rceil = \left\lceil \frac{\sum_{i=1}^{n} k_i}{k} \right\rceil - 1$$

The first term is clearly $\omega(G)$.

$$\lceil\omega(H)\rceil = \lceil\omega(G)\rceil - 1$$

We can combine these terms for $\Delta(H)$ and $\omega(H)$.

$$max\{\Delta(H), \lceil\omega(H)\rceil\} = max\{\Delta(G) - 1, \lceil\omega(G)\rceil - 1\}$$

$$max\{\Delta(H), \lceil\omega(H)\rceil\} = max\{\Delta(G), \lceil\omega(G)\rceil\} - 1$$

Using induction, $\chi'(H) = max\{\Delta(H), \lceil\omega(H)\rceil\} = max\{\Delta(G), \lceil\omega(G)\rceil\} - 1$. Add one more color for M, the near-perfect matching, to get a coloring for G.

$$max\{\Delta(G), \lceil\omega(G)\rceil\} \leq \chi'(G) \leq \chi'(H) + 1 = max\{\Delta(G), \lceil\omega(G)\rceil\}$$

Thus, $\chi'(G) = max\{\Delta(G), \lceil\omega(G)\rceil\}$.

Case 3: G is r-regular, and $k_i > 0$ for all $i = 1, 2, ..., n$.

Once again assume $\chi'(F) = max\{\Delta(F), \lceil\omega(F)\rceil\}$ for every ring graph F where $|V(F)| = |V(G)|$ and $|E(F)| < |E(G)|$.

Because G is r-regular, for every vertex v_i, $d(v_i) = r$. So, $k_1 + k_2 = k_2 + k_3 = k_3 + k_4 = ... = k_{n-1} + k_n = k_n + k_1$. From this, we can see that $k_1 = k_3 = ... = k_n$ (every odd index is equivalent) and $k_2 = k_4 = ... = k_{n-1}$ (every even index is equivalent). However, we can also see that $k_{n-1} = k_1$, showing that every index, even and odd, is equivalent. This means that every $k_i = \frac{r}{2}$, $\Delta(G) = r$, and $\lceil\omega(G)\rceil = \left\lceil \frac{\frac{r}{2}(2k+1)}{\lfloor\frac{1}{2}(2k+1)\rfloor} \right\rceil = \left\lceil r\frac{2k+1}{2k} \right\rceil \geq \Delta(G) + 1$.

Let M be a near-perfect matching of G. $H = G - M$ is also a ring graph, where $|E(H)| = |E(G)| - k$, $\Delta(H) = \Delta(G)$, and $\lceil\omega(H)\rceil = \left\lceil \frac{\frac{r}{2}(2k+1)-k}{\lfloor\frac{1}{2}(2k+1)\rfloor} \right\rceil = \lceil\omega(G)\rceil - 1$.

We know that $\lceil\omega(H)\rceil = \lceil\omega(G)\rceil - 1 \geq (\Delta(G) + 1) - 1 = \Delta(G) = \Delta(H)$.

Because $|E(H)| < |E(G)|$, we know $\chi'(H) = max\{\Delta(H), \lceil\omega(H)\rceil\} = \lceil\omega(H)\rceil$.

Recall that $\lceil\omega(H)\rceil = \lceil\omega(G)\rceil - 1$.

Because we can color G by adding a single color to H for the near-perfect matching, we know that $\chi'(G) \leq \chi'(H) + 1 = \lceil\omega(G)\rceil$.

Thus, $\chi'(G) = \lceil\omega(G)\rceil$ and G is exact.

So, in all cases, G is exact. □

4 Conclusion

Now that we have proven the chromatic index of ring graphs on odd vertices, and because we have known the chromatic index of ring graphs on even vertices, we have the ability to quickly calculate the chromatic index of any ring graph. It should be investigated if this result can be used to help refine any edge coloring algorithms for more complex graphs.

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