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The Explicit Solution of Constrained LP-Based Receding Horizon Control

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Abstract

For discrete-time linear time-invariant systems with constraints on inputs and states, we develop an algorithm to determine explicitly, as a function of the initial state, the solution to optimal control problems that can be formulated using a linear program. In particular, we focus our attention on a receding horizon control scheme where the performance criterion is based on a mixed $1/\infty$-norm (i.e., 1-norm with respect to time and ∞-norm with respect to space). We show that the optimal control profile is a piecewise linear and continuous function of the initial state. Thus, when the optimal control problem is solved at each time step according to a moving horizon scheme, the on-line computation of the resultant MPC controller is reduced to a simple linear function evaluation, instead of the typical expensive linear program required up to now. The technique proposed has both theoretical and practical advantages. From a theoretical point of view, the explicit solution gives insight on the action of the controller in different regions of the state space, and highlights conditions of degeneracy. From a practical point of view, the proposed technique is attractive for a wide range of applications where the simplicity of the on-line computational complexity is a crucial requirement.

Keywords: Model predictive control, constraints, piecewise linear control, multi-parametric programming, linear programming.

1 Introduction

As we extend the class of system descriptions beyond the class of linear systems, *linear systems with constraints* are probably the most important class in practice and the most studied. It is well accepted that for these systems, in general, stability and good performance can only be achieved by a nonlinear control law. The most popular approaches for designing non-linear controllers for linear systems with constraints fall into two categories: anti-windup and model predictive control.

Anti-windup schemes assume that a well functioning linear controller is available for small excursions from the nominal operating point. This controller is augmented by an anti-windup scheme to take care of situations when constraints are violated. Kothare *et al.* [28] reviewed numerous different anti-windup schemes and showed that they differ only in their choice of two static matrix parameters. The least conservative stability test for these schemes can be formulated in terms of a Linear Matrix Inequality (LMI) [29]. The systematic and automatic *synthesis* of anti-windup schemes which guarantee closed loop stability and achieve some kind of optimal performance, has remained largely elusive though some promising steps were achieved recently [31]. Despite these drawbacks anti-windup schemes are widely used in practice because in most SISO situations they are simple to design and work adequately.

Model Predictive Control (MPC) has become the accepted standard for complex constrained multivariable control problems in the process industries. Here at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. At the next time-step the computation is repeated starting from the new state and over a shifted horizon, leading to a moving horizon policy. The solution relies on a linear dynamic model, respects all input and output constraints, and optimizes a linear or quadratic performance index. Thus, as much as the performance index together with various constraints can be used to express true performance objectives, the performance of MPC is excellent. Over the last decade a solid theoretical foundation for MPC has emerged so that in real-life large scale MIMO applications controllers with non-conservative stability guarantees can be designed routinely and with ease. The big drawback of MPC is the relatively formidable on-line computational effort which limits its applicability to slow and/or small problems.

In this paper we show how to move off-line all the computations necessary for the implementation of MPC, while preserving all its other characteristics. This should largely increase MPC's range of applicability to problems where anti-windup schemes and other ad hoc techniques dominated up to now.

From a different point of view we show in effect how to solve the equivalent of the Hamilton-Jacobi-Bellman equation for discrete-time linear constrained systems. Rather than gridding the state space in some ad hoc fashion we discover the inherent underlying controller structure and provide its most efficient parameterization.

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The described approach is close in spirit to the techniques proposed earlier for the explicit solution to MPC problems based on a *quadratic* performance index [6], and for hybrid systems [4]. Besides the computational advantages mentioned above, these techniques provide insight into the basic structure of the MPC controller. The results in this paper elucidate MPC schemes based on linear programming (LP), which have been investigated by several authors [36, 22, 33].
The paper is organized as follows. The basics of MPC based on the minimization of a mixed $1/\infty$-norm are reviewed first to derive the linear program which needs to be solved to determine the optimal control action. The conditions for stability of such an MPC scheme are investigated. We note that the linear program depends on the current state which appears linearly in the constraints, and treat the LP as a *multi-parametric linear program* (mp-LP) [21, 20]. Recasting the MPC problem as an mp-LP allows one to solve it explicitly [13], and to analyze its properties, in particular to show that it is a piecewise affine function of the state vector. The paper concludes with an example of a double integrator, which illustrates the different features of the method, and highlights the structural properties of the $1/\infty$-norm receding horizon controller.

2 Model Predictive Control

Consider the problem of regulating to the origin the discrete-time linear time-invariant system

$$\begin{cases} x(t+1) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t) \end{cases} \qquad (1)$$

while fulfilling the constraints

$$y_{\min} \le y(t) \le y_{\max},\ u_{\min} \le u(t) \le u_{\max}{}^1 \qquad (2)$$

at all time instants $t \ge 0$. In (1)-(2), $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input, and output vector respectively, $y_{\min} \le y_{\max}$ ($u_{\min} \le u_{\max}$) are $p(m)$-dimensional vectors[2], and the pair (A, B) is stabilizable.
Model Predictive Control (MPC) solves such a constrained regulation problem in the following way. Assume that a full measurement of the state $x(t)$ is available at the current time t. Then, the optimization problem

$$\begin{array}{ll} \min\limits_{U\triangleq\{u_t,\dots,u_{t+N_u-1}\}} & \left\{ J(U,x(t)) = \|Px_{t+N_y|t}\|_\infty + \sum\limits_{k=0}^{N_y-1} \|Qx_{t+k|t}\|_\infty + \|Ru_{t+k}\|_\infty \right\} \\ \text{subj. to} & y_{\min} \le y_{t+k|t} \le y_{\max},\ k=1,\dots,N_c \\ & u_{\min} \le u_{t+k} \le u_{\max},\ k=0,1,\dots,N_c \\ & x_{t|t} = x(t) \\ & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k},\ k \ge 0 \\ & u_{t+k} = 0,\ N_u \le k \le N_y - 1 \end{array} \qquad (3)$$

is solved at each time t, where $x_{t+k|t}$ denotes the predicted state vector at time $t+k$, obtained by applying the input sequence $u_t, \dots, u_{t+k-1}$ to model (1) starting from the state $x(t)$, and $\|Vx\|_\infty \triangleq max_{j=1,\dots,m}\,(V^j x)$, and $V^i = i$-th row of $V \in \mathbb{R}^{m\times n}$.
In (3), we assume that Q, $R \in \mathbb{R}^{n\times n}$ are non-singular matrices, $P \in \mathbb{R}^{m\times n}$ is full column rank matrix, and $N_y \ge N_c \ge N_u$.
Let $U^*(t) = \{u_t^*, \dots, u_{t+N_u-1}^*\}$ be the optimal solution of (3). Then at time t

$$u(t) = u_t^* \qquad (4)$$

is applied as input to system (1). The optimization (3) is repeated at time $t+1$, based on the new state $x(t+1)$, yielding a *moving* or *receding horizon* control strategy. The two main issues regarding this policy are the feasibility of the optimization problem (3) and stability of the resulting closed-loop system. When $N_c < \infty$ there is no guarantee that the optimization problem (3) will remain feasible at all future time steps t, as "blind alleys" might be entered by the system. On the other hand, setting $N_c = \infty$ leads to an optimization problem with an infinite number of constraints, which is impossible to handle. In the next section the stability issue is addressed.

2.1 Stability Through the Terminal Weight

In general, stability is a complex function of the various tuning parameters N_u, N_y, N_c, P, Q, and R. For applications it is most useful to impose some conditions on N_y, N_c and P so that stability is guaranteed for all nonsingular Q and R and leave Q and R as free parameters to tune the performance. Sometimes the optimization problem (3) is augmented with a so called "stability constraint" (see [5] for a survey of different constraints proposed in the literature). This additional constraint imposed over the prediction horizon explicitly forces the state vector either to shrink in some norm or to reach an invariant set at the end of the prediction horizon.
Problem (3) is slightly different from the standard MPC formulation, as ∞-norms are used instead of 2-norms. Therefore, the standard stability results cannot be directly applied. One possibility is to choose $P = 0$ and add the end-point constraint $x_{t+N_y|t} = 0$ to (3). Provided that the problem is feasible at time $t = 0$, the end-point constraint implies persistence of solutions (i.e., feasibility at each time step) and stability, as shown in [25], although the constraint has a negative effect on performance, especially for small N_y. Another possibility is to relax the end-point constraint by adopting a

[1]Constraints (2) can be equivalently rewritten in the more general form $Dy(t) + Eu(t) \le f$.

[2]More general, we can allow only some components of the inputs or outputs to be constrained (e.g. $u^i_{\min} = -\infty$). In (2), constraints relating to unconstrained input and output components are simply removed.

dual-mode approach [30], namely, by defining an invariant set around the origin, and constrain the terminal state $x_{t+N_y|t}$ to lie in that set.

In this paper, rather than constraining the final state, we weight $x_{t+N_y|t}$ as in (3). Assuming that the constraint horizon N_c is long enough so that the shifted optimal input sequence $\{u^*_{t+1}, \dots, u^*_{t+N_u-1}, 0\}$ is feasible at the next time step $t+1$ [34, 36], the following theorem shows that, by appropriately choosing the terminal weight P, the control law (3) stabilizes system (1) asymptotically

Theorem 1 *Let A be a stable matrix, and let the origin be an equilibrium for system (1). If there exists a full column rank matrix P such that*

$$-\|Px\|_\infty + \|PAx\|_\infty + \|Qx\|_\infty \leq 0 \qquad (5)$$

is satisfied for all $x \in \mathbb{R}^n$, then the MPC law (3)-(4) stabilizes system (1), in that

$$\begin{aligned} \lim_{t\to\infty} x(t) &= 0 \\ \lim_{t\to\infty} u(t) &= 0 \end{aligned}$$

while fulfilling the input and output constraints $u_{\min} \leq u(t) \leq u_{\max}$, $y_{\min} \leq y(t) \leq y_{\max}$.

Proof: The proof follows from standard Lyapunov arguments. Let $V(t)$ be the minimum of the optimization problem (3) obtained for the minimizer $U^*(t) = \{u^*_t, u^*_{t+1}, \dots, u^*_{t+N_u-1}\}$, and consider the sequence $U_{\text{shift}} \triangleq \{u^*_{t+1}, \dots, u^*_{t+N_u-1}, 0\}$. As U_{shift} is feasible at time $t+1$ by assumption,

$$\begin{aligned} &V(t+1) - V(t) \leq -\|Qx(t)\|_\infty - \|Ru(t)\|_\infty \\ &- \|Px^*_{t+N_y|t}\|_\infty + \|Px^*_{t+N_y+1|t}\|_\infty + \|Qx^*_{t+N_y|t}\|_\infty \end{aligned} \qquad (6)$$

As the condition (5) is satisfied for $x = x^*_{t+N_y|t}$, $V(t)$ is a decreasing sequence. Since $V(t)$ is lower-bounded by 0, there exists $V_\infty = \lim_{t\to\infty} V(t)$, which implies $V(t+1) - V(t) \to 0$. Therefore, each term of the sum

$$\|Qx(t)\|_\infty + \|Ru(t)\|_\infty \qquad (7)$$

converges to zero as well, which proves the theorem as Q and R are nonsingular. □

The question now arises if matrices P and Q satisfying (5) exist, and how to find them. Let us focus on a simpler problem by removing the decreasing factor $\|Qx\|_\infty$ from condition (5)

$$-\|\tilde{P}x\|_\infty + \|\tilde{P}Ax\|_\infty \leq 0 \qquad (8)$$

The existence and the construction of a matrix $\tilde{P}$ that satisfies condition (8), has been addressed in different forms by several authors [7, 8, 32, 26, 27, 24, 23, 9, 10, 11, 12]. There are two equivalent ways of tackling this problem: Finding a Lyapunov function for system (1)

$$\Psi(x) = \|\tilde{P}x\|_\infty \qquad (9)$$

with $P \in \mathbb{R}^{m\times n}$ and ∞-norm instead of the usual 2-norm [32], or equivalently computing a symmetrical positively invariant polyhedral set [12] for system (1). Differently from the 2-norm case, the condition that the matrix A has all the eigenvalues in the open disk $\|\lambda_i(A)\| < 1$ is not sufficient for the existence of a Lyapunov function (9) with $m = n$ [7]. The following theorem, proved in [26, 32], states necessary and sufficient condition for the existence of a the Lyapunov function (9):

Theorem 2 *Function (9) is a Lyapunov function of system (1) if and only if there exist a matrix $H \in \mathbb{C}^{m\times m}$ such that*

$$\begin{aligned} \tilde{P}A - H\tilde{P} &= 0 \qquad (10) \\ \|H\|_\infty &< 1 \qquad (11) \end{aligned}$$

In [26] the authors proposed an efficient way to compute a Lyapunov function (9) by constructing matrices $\tilde{P}$ and H satisfying conditions (10)–(11). The resulting matrix $\tilde{P}$ is square provided the following assumption is satisfied:

Assumption 1 *The matrix A in (1) has distinct eigenvalues $\lambda_i = \mu_i + j\sigma_i$ situated in the open square $|\mu_i| + |\sigma_i| < 1$*

In [32] the author shows how to construct matrices $\tilde{P}$ and H in (10)–(11) with the only assumption that A is stable. However this approach has the drawback, that the number m of rows the matrix $\tilde{P}$ depends on the position of the eigenvalues of the matrix A, more precisely m may go to infinity as $|\lambda_i| \to 1$.

By using results in [26, 32], the construction of a matrix P satisfying condition (5) can be performed by computing matrices $\tilde{P}$ and H satisfying conditions (10)–(11) as in [26] or [32], and then computing P by exploiting the result of the following theorem.

Theorem 3 *Let $\varepsilon = 1 - \|H\|_\infty$, $\rho = \|Q\tilde{P}^{-1}\|_\infty$, the square matrix:*

$$P = \frac{\rho}{\varepsilon}\tilde{P} \qquad (12)$$

satisfies condition (5).

Proof: Note that matrix P satisfies

$$PA = HP \qquad (13)$$

By substituting (13) into (5) we obtain

$$\begin{aligned} &-\|Px\|_\infty + \|PAx\|_\infty + \|Qx\|_\infty = \\ &= -\|Px\|_\infty + \|HPx\|_\infty + \|Qx\|_\infty \leq \\ &\leq (\|H\|_\infty - 1\|)\|Px\|_\infty + \|Qx\|_\infty \leq \\ &\leq (\|H\|_\infty - 1\|)\|Px\|_\infty + \|Q\tilde{P}^{-1}\|_\infty\|\tilde{P}x\|_\infty \end{aligned} \qquad (14)$$

which proves the theorem. □

Remark 1 In case matrix A does not satisfy Assumption 1, system (1) can be pre-stabilized by a linear controller such that Assumption 1 is satisfied, without taking care of the constraints. Then, the output vector can be augmented by including the original (now state-dependent) inputs, and saturation constraints can be mapped into additional output constraints in (3).

In [33] the authors use a different approach to construct P provided the matrix A is stable and not defective. It can be proven that the resultant P has $m = 2^{n-n_0-1} + n2^{n_0-1}$ number of rows where n_0 is the algebraic multiplicity of the zero eigenvalues of matrix A.

Remark 2 If P is given in advance rather than computed as in Theorem 3, condition (5) can be tested numerically, either by enumeration (3^{2n} LPs) or, more conveniently, through a mixed-integer linear program ($(5n+1)$ continuous variables + $4n$ integer variables).

3 Piecewise Linear Solution of Constrained MPC with $1/\infty$-Norm

The MPC formulation (3) can be rewritten as a linear program by using the following standard approach. The sum of components of any vector $\{\varepsilon_1^x, \ldots, \varepsilon_{N_y}^x, \varepsilon_1^u, \ldots, \varepsilon_{N_u}^u\}$ that satisfies

$$\begin{array}{l} -\mathbf{1_n}\varepsilon_k^x \le Qx_{t+k|t} \;\; k = 1,2,\ldots,N_y \\ -\mathbf{1_n}\varepsilon_k^x \le -Qx_{t+k|t} \;\; k = 1,2,\ldots,N_y \\ -\mathbf{1_m}\varepsilon_{k+1}^u \le Ru_{t+k} \;\; k = 0,1,\ldots,N_u-1 \\ -\mathbf{1_m}\varepsilon_{k+1}^u \le -Ru_{t+k} \;\; k = 0,1,\ldots,N_u-1 \end{array} \tag{15}$$

represents an upper bound on $J^*(U, x(t))$, where $\mathbf{1_k}$ is a column vector of length k of ones,

$$x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \tag{16}$$

and the inequalities (15) should be intended as componentwise.

Similarly to what was shown in [14], it is easy to prove that the vector $z \triangleq \{\varepsilon_1^x, \ldots, \varepsilon_{N_y}^x, \varepsilon_1^u, \ldots, \varepsilon_{N_u}^u, u_t, \ldots, u_{t+N_u-1}\}$ that satisfies equations (15) and simultaneously minimizes $J(z) = \varepsilon_1^x + \ldots + \varepsilon_{N_u}^x + \varepsilon_1^u + \ldots + \varepsilon_{N_u}^u$ also solves the original problem, i.e. the same optimum $J^*(v_0^{T-1}, x(t))$ is achieved. Therefore, problem (3) can be reformulated as the following LP problem

$$\begin{array}{ll} \min\limits_z & J(z) \\ \text{subj. to} & -\mathbf{1_n}\varepsilon_k^x \le \pm\left[QA^k x(t) + Q\sum_{j=0}^{k-1} A^j B u_{k-1-j}\right], \\ & k = 1,\ldots,N_y \\ & -\mathbf{1_m}\varepsilon_{k+1}^u \le \pm Ru(k), \;\; k = 0,1,\ldots,N_u-1 \\ & y_{\min} \le CA^k x(t) + C\sum_{j=0}^{k-1} A^j B u_{k-1-j} \le y_{\max}, \\ & k = 1,\ldots,N_c \\ & u_{\min} \le u_{t+k} \le u_{\max}, \;\; k = 0,1,\ldots,N_u-1 \\ & x_{t|t} = x(t) \\ & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k}, \;\; k \ge 0 \\ & u_{t+k} = 0, \;\; N_u \le k \le N_y \end{array} \tag{17}$$

Problem (17) can be rewritten in the more compact form

$$\begin{array}{ll} \min\limits_z & J(z, x(t)) = f^T z \\ \text{subj. to} & Gz \le S + Fx(t) \end{array} \tag{18}$$

Problem (18) depends on the current state $x(t)$, implementation of MPC requires the on-line solution of an LP at each time step. Although efficient LP solvers based on simplex methods or interior point methods are available, computing the input $u(t)$ demands significant on-line computation effort. Rather than solving the LP on line, we follow the ideas of [6, 3], and propose an approach where all computation is moved off line. The idea is based on the observation that in (3) the state $x(t) \in \mathbb{R}^n$ can be considered as a vector of parameters. In other words, the state feedback control law is defined implicitly as the solution of the optimization problem (3) as a function of the parameter $x(t)$. Our goal is to make this dependence explicit. In fact, by treating $x(t)$ as a vector of parameters, the LP becomes a *multiparametric* LP (mp-LP).

As we will describe in the next section, we use the algorithm developed in [15] for solving the mp-LP formulated above. Once the multi-parametric problem (17) has been solved off line, i.e. the solution $z_t^* = f(x(t))$ of (18) has been found, the model predictive controller (3) is available explicitly, as the optimal input $u(t)$ consists simply of m components of z_t^*

$$u(t) = [0 \;\ldots 0 \; I \; 0 \;\ldots\; 0] f(x(t)). \tag{19}$$

3.1 Multi-Parametric Linear Programs

Problem (18) is known in the literature as multiparametric linear program. The operations research community has addressed parameter variations in mathematical programs at two levels: *sensitivity analysis*, which characterizes the change of the solution with respect to small perturbations of the parameters, and *parametric programming*, where the characterization of the solution for a full range of parameter values is sought. More precisely, programs which depend only on one scalar parameter are referred to as *parametric programs*, while problems depending on a vector of parameters are referred to as *multi-parametric programs*.

The first method for solving multi-parametric linear programs was formulated by Gal and Nedoma [21], and later only a few authors have dealt with multi-parametric linear [20, 19, 35], nonlinear [18], quadratic [17, 6], and mixed-integer [15, 16, 1] program solvers.

Parametric programming systematically subdivides the space of parameters into characteristic regions, which depict the feasibility and corresponding performance as a function of the parameters. In [13] we proposed a new algorithm which, rather than visiting different bases of the associated LP tableau [21], is based on the direct exploration of the parameter space [17, 6]. Therefore, the approach is different from the usual methods based on the simplex tableau. Our definition of optimality intervals, also called "critical regions", is directly related to the one in [2, 19]. The resulting algorithm for solving multi-parametric linear programs has computational advantages, namely the simplicity of its implementation in a recursive form, and the possibility to look for parametric solutions within a given polyhedral region of the parameter-space without solving the problem globally.

In the following we recall some known properties of the optimal value function $J^*(x(t))$ and of the optimizer $z^*(x(t))$, see [13].

Theorem 4 *Consider the multi-parametric linear program (18) and suppose that for each $x(t)$ the solution of the linear program (18), if it exist, is unique. Then the set of feasible states $X_f \subseteq X$ is convex, the optimizer $z(x) : X_f \mapsto \mathbb{R}^s$ is continuous and piecewise affine, and the optimal solution $V(x) : X_f \mapsto \mathbb{R}$ is continuous, convex and piecewise linear.*

Remark 3 If the problem (18) has multiple solutions for some $x(t)$, then it is dual degenerate in some region of the state-space. In this case, one can always choose a partition of such a region and a particular optimizer so that the function $z(x)$ is continuous.
Because of (19), we can establish the analytical properties of the controller (3), (4) through the following corollary of Theorem 4

Corollary 1 *The control law $u(t) = f(x(t))$, $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, defined by the optimization problem (3) and (4) is continuous and piecewise affine.*

4 Examples

Consider the double integrator

$$y(t) = \frac{1}{s^2}u(t), \tag{20}$$

and its equivalent discrete-time state-space representation

$$\begin{cases} x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} x(t) \end{cases} \tag{21}$$

obtained by sampling (20) with $T = 1$.
We want to regulate the system to the origin while minimizing the performance measure

$$\sum_{t=0}^{1} \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{t+k|t} \right\|_\infty + |0.8u_{t+k}| \tag{22}$$

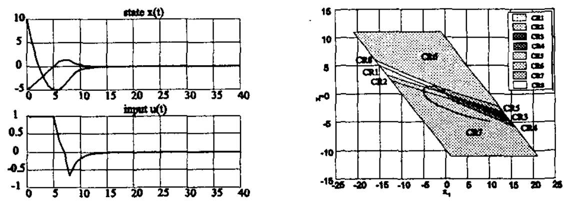
subject to the input constraints

$$-1 \leq u_{t+k} \leq 1, \ k = 0, 1 \tag{23}$$

and the state constraints

$$-10 \leq x_{t+k|t} \leq 10, \ k = 1, 2 \tag{24}$$

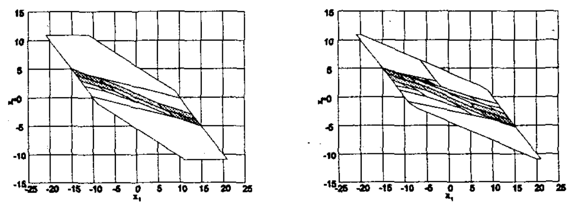
This task is addressed by using the MPC algorithm (3) where $N_y = 2$, $N_u = 2$, $Q = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 0.8$.
The solution of the mp-LP problem was computed in 13.57 s by using a Pentium III-300Mhz and the corresponding polyhedral partition of the state-space is depicted in Fig. 1(b). Note that region #6 and #7 correspond to the saturated controller.
The same example was solved increasing number of degrees of freedom N_u. The corresponding partitions are reported in Fig. 2. Note that white regions correspond to the saturated controller $u(t) = -1$ in the upper part and $u(t) = -1$ in the lower part. The off-line computation times and number of regions are reported in Table 1. Note that by increasing the number of free control moves N_u, the control law appears to change only far away from the origin, the larger N_u the more in the periphery.



(a) Closed-loop MPC

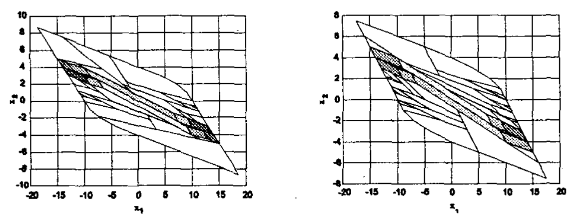
(b) Polyhedral partition of the state-space and closed-loop MPC trajectories

Figure 1: Double integrator example



(a) $N_u = 3$

(b) $N_u = 4$



(c) $N_u = 5$

(d) $N_u = 6$

Figure 2: Partition of the state space for the MPC controller

5 Conclusions

In this paper we formulated a model predictive controller for linear systems subject to input and state constraints based on a $1/\infty$-norm performance objective, and we gave conditions on the weighting matrices for its stability. We provided the explicit solution of such an MPC scheme, and shown that it is a piecewise linear function of the state vector. Further extensions of the basic set-up include trajectory following, suppression of measured disturbances, time-varying constraints, and

Free moves N_u	Computation time (s)	N. of regions N_r
2	13.57	8
3	28.50	16
4	48.17	28
5	92.61	37
6	147.53	44

Table 1: Off-line computation times and number of regions for the double integrator example

the output feedback problem. The mp-LP approach of this paper can also be extended to other norms as 1/1, $\infty/1$, ∞/∞ norms (where x/y stands for x-norm with respect to time and y-norm with respect to space).

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