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Hamiltonian-Driven Adaptive Dynamic Programming With Efficient Experience Replay

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***Abstract*— This article presents a novel efficient experience-replay-based adaptive dynamic programming (ADP) for the optimal control problem of a class of nonlinear dynamical systems within the Hamiltonian-driven framework. The quasi-Hamiltonian is presented for the policy evaluation problem with an admissible policy. With the quasi-Hamiltonian, a novel composite critic learning mechanism is developed to combine the instantaneous data with the historical data. In addition, the pseudo-Hamiltonian is defined to deal with the performance optimization problem. Based on the pseudo-Hamiltonian, the conventional Hamilton–Jacobi–Bellman (HJB) equation can be represented in a filtered form, which can be implemented online. Theoretical analysis is investigated in terms of the convergence of the adaptive critic design and the stability of the closed-loop systems, where parameter convergence can be achieved under a weakened excitation condition. Simulation studies are investigated to verify the efficacy of the presented design scheme.**

***Index Terms*— Hamilton–Jacobi–Bellman (HJB) equation, Hamiltonian-driven adaptive dynamic programming (ADP), pseudo-Hamiltonian, quasi-Hamiltonian, relaxed excitation condition.**

I. Introduction

RECENT development in control theory and machine learning has promoted reinforcement learning (RL) and adaptive dynamic programming (ADP) for performance optimization of the decision-making problem with long-term cumulative reward. A classical optimal control theory laid the solid theoretical foundation for the dynamical optimization problem, where Pontryagin's maximum principle (PMP) and Bellman's dynamic programming (DP) are the centerpieces of the optimal control theory [1]. Based on PMP, the optimal control policy over finite horizon depends on solving a two-point boundary value problem (TPBVP) [2]. However, solving the TPBVP for general nonlinear dynamical systems remains to be an open problem. On the other hand, the DP suffers from the "curse of dimensionality" [3]. In contrast, RL and ADP are intelligent methods to assist the agent make intelligent decisions to optimize the cumulative reward based on online collected data [4], [5], [6], [7], [8], [9]. Successful application of RL and ADP to control applications can be found in recent literature [10], [11], [12], [13], [14].

A. Related Work

In ADP and RL-based adaptive optimal controller design, the fundamental considerations are twofold. First, the system during the learning process has to be stable, and the closed-loop signals should be bounded to guarantee that the online learning scheme is feasible. As a typical iterative ADP algorithm, in policy iteration, the initial iterative policy should be admissible to ensure that the closed-loop stability in each iteration is stable [15], [16], [17]. However, the initial admissible policy is difficult to be obtained for complex systems with unknown dynamics. Second, the learning convergence of ADP and RL is desired to guarantee that the optimal control policy can be obtained [18], [19], [20]. For both synchronous and iterative ADP algorithms [21], [22], [23], [24], the persistent of excitation (PE) condition on the collected data is critical to the learning convergence to the optimum [25]. However, the PE condition is difficult to be satisfied because it requires the signal to have sufficient rich information over the infinite horizon [26]. In addition, the PE condition requires the signal to be frequency-rich, which might lead to undesired oscillation in the system evolution. Therefore, it is desired to obtain a convergent and stable ADP online learning algorithm without the stringent PE condition.

Experience replay is an efficient method in RL, which repeatedly utilizes the collected historical data for intelligent decision-making [27]. It has been successfully combined with neural networks for autonomous agents to directly learn from the experience based on sequential actions in

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the environment [28], [29]. Recently, the experience-replay technique has also been employed an adaptive control theory to obviate the stringent PE condition, which has been successfully applied to engineering problems, including adaptive cruise control [30], wastewater treatment process [31], and wind farm control [32]. In concurrent learning for model reference adaptive control [33], the collected data are selected and stored based on a rank condition on the data matrix, which is less restrictive than the PE condition and easy to be verified during the online process. However, the concurrent learning technique requires state derivative to be measurable [34], which might not be feasible in applications. In composite learning, the time-interval integral with filtered signals is designed to construct a novel residual to avoid the measurement of time derivation of plant states [35], where the concept of interval excitation (IE) condition is presented to be weaker than the PE condition and can ensure the parameter estimation convergence [36]. The IE condition is further applied to the adaptive optimal control for linear systems in [37] to provide the adaptive solution with convergence to the solution to the algebraic Riccati equation. However, the optimal control problem of nonlinear system requires solving the Hamilton–Jacobi–Bellman (HJB) equation, which is a nonlinear partial differential equation and difficult to be solved due to the inherent nonlinearity. This article aims to develop an efficient ADP algorithm to solve the HJB equation for nonlinear systems without requiring the PE condition.

The Hamiltonian-driven framework is developed in [38]. Three subproblems for optimal control are categorized as policy evaluation problems for a fixed admissible policy, the performance comparison problem with different admissible policies, and the performance improvement for a given admissible policy. It is shown that the Hamiltonian plays a critical role in the performance optimization problem, and the classical policy iteration algorithm can be viewed as a successive minimization of the iterative Hamiltonian [38]. The Hamiltonian-driven framework is later applied to the performance optimization with intermittent feedback, where the effect of intermittent feedback on the communication bandwidth and the control performance of the iterative ADP algorithms is investigated [39]. In addition, the effect of the residual resulting from the function approximator on the convergence of iterative ADP algorithm is investigated in [40], where a sufficient condition to ensure the closed-loop stability in each iteration and the convergence to the optimum is developed. Recent extensions of the Hamiltonian-driven ADP have been made to solve the differential games [41] and multiobjective optimization [42]. In this article, the Hamiltonian-driven ADP is extended with value function approximation to provide novel defined quasi-Hamiltonian and pseudo-Hamiltonian, which can efficiently combine the instantaneous data with the historical data.

1) Contributions: The contributions of this article are threefold. First, to solve the policy evaluation problem for nonlinear systems with a given admissible policy, the quasi-Hamiltonian is defined to utilize the historical data efficiently. On this basis, an online filtered signal is designed to yield the filtered Bellman equation, which can be solved with a relaxed excitation. Second, for the policy optimization problem for nonlinear systems, the pseudo-Hamiltonian is presented to parameterize the HJB equation, which is named the filtered HJB equation, and efficiently takes the instantaneous data and the online collected data into consideration. Finally, to solve the filtered Bellman equation and filtered HJB equation, a relaxed excitation on the filtered signals is considered to ensure the online learning convergence without the requirement of the stringent PE condition.

2) Structure: The remainder of this article is organized as follows. The optimal control problem of nonlinear dynamical in continuous time with its background knowledge is presented in Section II. In Section III, the quasi-Hamiltonian is presented for efficient data utilization for policy evaluation, where the filtered Bellman equation is developed. A composite critic learning algorithm is presented to solve the filtered Bellman equation with a relaxed excitation condition. Moreover, the filtered HJB equation is derived in Section IV, where the pseudo-Hamiltonian is defined to provide the quadratic parameterization for the filtered HJB equation. On this basis, novel actor–critic learning is developed to solve the filtered HJB equation without the requirement of the PE condition. The simulation study with the presented efficient Hamiltonian-driven ADP algorithm is investigated in Section V. The concluding remarks are made in Section VI.

B. Preliminaries

The following definitions are required for subsequent discussions.

Definition 1 (PE [43]): A vector signal $y(t) \in \mathbb{R}^p$ is PE in $\mathbb{R}^p$ with an excitation level $\beta_1 > 0$, provided that for all $t \in \mathbb{R}^+$, there exist constants $T_{\text{PE}} \in \mathbb{R}^+$ and $\beta_2 > \beta_1$ such that

$$\beta_1 I_{p\times p} \leq \int_t^{t+T_{\text{PE}}} y(\tau)y^{\text{T}}(\tau)\mathrm{d}\tau \leq \beta_2 I_{p\times p}.$$

Definition 2 (Relaxed Excitation Condition): A vector signal $y(t) \in \mathbb{R}^p$ is exciting over interval $[t_a, t_b]$ with an excitation level $\beta_1 > 0$, provided that there exist constants $\beta_1 \in \mathbb{R}^+$ and $\beta_2 \in \mathbb{R}^+$ such that

$$\beta_1 I_{p\times p} \leq \int_{t_a}^{t_b} y(\tau)y^{\text{T}}(\tau)\mathrm{d}\tau \leq \beta_2 I_{p\times p}$$

where $\beta_2 > \beta_1 > 0$.

II. Problem Statement

We consider the following continuous-time nonlinear dynamical system:

$$\dot{x} = f(x) + g(x)u, x(t_0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the system state, $u \in \mathbb{R}^m$ denotes the control input, and the initial condition x_0 is given. In addition, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and $g(\cdot) : \mathbb{R}^{n\times m} \to \mathbb{R}^n$ are the system dynamics. On a compact set Ω_x, the functions $f(x)$ and $g(x)$ are locally Lipschitz functions and satisfy $\|f(\cdot)\| \leq \eta_f \|\cdot\|$ and $\|g(\cdot)\| \leq \eta_g$ with positive constants η_f and η_g. Moreover, it is assumed that $f(0) = 0$, which implies that the origin is an equilibrium of the system.

For the optimal control problem, the performance function under consideration takes the following form:

$$V_u(x_0) = \int_{t_0}^{\infty} r(x(\tau), u(\tau)) d\tau \tag{2}$$

with the reward function $r(x, u) = Q(x) + u^{\mathrm{T}} R u$, where $Q(x)$ is a positive semidefinite function and R is a symmetric positive definite matrix.

Definition 3 (Admissible Policy [44]): For the nonlinear dynamical system (1), a feedback policy $u : \mathbb{R}^n \to \mathbb{R}^m$ is said to be admissible on the compact $\Omega \subset \mathbb{R}^n$, denoted as $u \in \mathbb{R}^n$, provided that the following conditions hold.

1) $u(\cdot)$ is continuous on Ω.
2) $u(0) = 0$.
3) The closed-loop system is stable.
4) The performance $V(x_0)$ is finite.

Define the Hamiltonian as

$$H\left(u, x, \frac{\partial V_u(x)}{\partial x}\right) = r(x, u) + \left[\frac{\partial V_u(x)}{\partial x}\right]^{\mathrm{T}} [f(x) + g(x)u]. \tag{3}$$

The value function $V_u(x)$ can be obtained by solving the Bellman equation

$$0 = H\left(u, x, \frac{\partial V_u(x)}{\partial x}\right). \tag{4}$$

Define the optimal value function and optimal control policy as

$$\begin{aligned} V_*(x) &= \min_{u(\cdot)} \int_{t_0}^{\infty} r(x(\tau), u(\tau)) d\tau \\ u_*(x) &= \arg\min_{u(\cdot)} \int_{t_0}^{\infty} r(x(\tau), u(\tau)) d\tau. \end{aligned} \tag{5}$$

According to the optimal control theory [44], the optimal value function and the optimal control policy satisfy the HJB equation

$$\begin{aligned} 0 &= \min_{u(\cdot)} H\left(u, x, \frac{\partial V_*(x)}{\partial x}\right) \\ &= H\left(u_*, x, \frac{\partial V_*(x)}{\partial x}\right). \end{aligned} \tag{6}$$

Applying the stationary condition to the Hamiltoinan, one can obtain the optimal policy with the optimal value gradient as

$$u_*(x) = -\frac{1}{2} R^{-1} g^{\mathrm{T}}(x) \frac{\partial V_*(x)}{\partial x}. \tag{7}$$

Inserting $u_*(x)$ back into (6), the HJB equation can be equivalently expressed as

$$\begin{aligned} 0 = Q(x) &+ \left[\frac{\partial V_*(x)}{\partial x}\right]^{\mathrm{T}} f(x) \\ &- \frac{1}{4}\left[\frac{\partial V_*(x)}{\partial x}\right]^{\mathrm{T}} g(x) R^{-1} g^{\mathrm{T}}(x) \frac{\partial V_*(x)}{\partial x}. \end{aligned} \tag{8}$$

Remark 1: For linear system with the dynamics $\dot{x} = Ax + Bu$ and the reward function $r(x, u) = x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u$, according to the linear quadratic optimal control theory, the value function with a stabilizing feedback policy $u = Kx$ takes the quadratic form $V_u(x) = x^{\mathrm{T}} P_u x$. Accordingly, the Hamiltonian for linear quadratic optimal control is defined as

$$\begin{aligned} H\left(u, x, \frac{\partial V_u(x)}{\partial x}\right) &= x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u + x^{\mathrm{T}} A^{\mathrm{T}} P_u x + u^{\mathrm{T}} B^{\mathrm{T}} P_u x \\ &\quad + x^{\mathrm{T}} P_u A x + x^{\mathrm{T}} P_u B u. \end{aligned} \tag{9}$$

The optimal value function $V_*(x) = x^{\mathrm{T}} P x$ can be obtained by solving the algebraic Riccati equation

$$A^{\mathrm{T}} P + P A - P B R^{-1} B^{\mathrm{T}} P + Q = 0. \tag{10}$$

Applying the stationary condition, the optimal control can be calculated as $u_*(x) = -R^{-1} B^{\mathrm{T}} P x$. □

As investigated by the Hamiltonian-driven ADP framework [38], the fundamental issues in the optimal control problem are the policy evaluation and policy optimization. For the policy evaluation problem, an admissible policy is given and the design object is to learn the corresponding value function satisfying the Bellman equation. For the policy optimization problem, in addition to the adaptive critic network which approximates the optimal value function, the online actor is added to learn the optimal control policy simultaneously. In the following, the traditional Hamiltonian is extended as quasi-Hamiltonian and pseudo-Hamiltonian for policy evaluation and policy optimization problems with efficient experience replay.

III. Hamiltonian-Driven Efficient Policy Evaluation

A. Instantaneous Bellman Equation

Using the function approximators, such as neural network, the value function and value gradient for the fixed policy $u(x)$ can be denoted as the critic network, i.e.,

$$\begin{aligned} V_u(x) &= W_u^{\mathrm{T}} \phi(x) + \sigma_u(x) \\ \frac{\partial V_u(x)}{\partial x} &= \left[\frac{\partial \phi(x)}{\partial x}\right]^{\mathrm{T}} W_u + \frac{\partial \sigma_u(x)}{\partial x} \end{aligned} \tag{11}$$

where $W_u \in \mathbb{R}^{\ell}$ is the critic network weight, $\phi(x) \in \mathbb{R}^{\ell}$ and $((\partial \phi(x))/\partial x) \in \mathbb{R}^{\ell \times n}$ are the basis function and basis gradient, respectively, and $\sigma_u(x) \in \mathbb{R}$ and $((\partial \sigma_u(x))/\partial x) \in \mathbb{R}^{\ell}$ are the value function approximation residual and its gradient, respectively. Based on the universal approximation theorem, the ideal critic weight is defined as

$$W_u = \arg\min_{W} \left\| V_u(x) - W^{\mathrm{T}} \phi(x) \right\|$$

and the ideal value function approximation residual

$$\sigma_u(x) = \min_{W} \left\| V_u(x) - W^{\mathrm{T}} \phi(x) \right\|$$

is bounded in the sense that there exists a positive constant $\bar{\sigma}_u$ such that $\sup_{x \in \Omega} \|\sigma_u(x)\| \le \bar{\sigma}_u$.

Using the value function parameterization (11), the Bellman equation (4) can be rewritten using the critic network as

$$\begin{aligned} 0 = \left[\frac{\partial \sigma_u(x)}{\partial x}\right]^{\mathrm{T}} &[f(x) + g(x)u] \\ &+ W_u^{\mathrm{T}} \frac{\partial \phi(x)}{\partial x} [f(x) + g(x)u] + r(x, u). \end{aligned} \tag{12}$$

Definition 4: Consider the dynamical system (1) with the reward function $r(x, u)$ and the filtered signals in (15). The quasi-Hamiltonian is defined as

$$H_q(W, \varphi, r) = W^{\mathrm{T}}\varphi + r \tag{13}$$

for all $(W, \varphi, r) \in \mathbb{R}^{\ell} \times \mathbb{R}^{\ell} \times \mathbb{R}$.

Consider the following notations:

$$\delta_u(x(t)) = -\left[\frac{\partial \sigma_u(x(t))}{\partial x(t)}\right]^{\mathrm{T}}[f(x(t)) + g(x(t))u(t)]$$
$$\varphi_u(x(t), u(t)) = \frac{\partial \phi(x(t))}{\partial x(t)}[f(x(t)) + g(x(t))u(t)].$$

For notations simplicity, in the following, we denote $\varphi_u(x(t), u(t))$, $r(x(t), u(t))$, and $\delta_u(x(t))$ as $\varphi_u(t)$, $r_u(t)$, and $\delta_u(t)$, respectively. Then, the Bellman equation (12) implies that

$$\begin{aligned} H_q(W_u, \varphi_u(t), r_u(t)) &= W_u^{\mathrm{T}}\varphi_u(t) + r_u(t) \\ &= \delta_u(t). \end{aligned} \tag{14}$$

In (14), the signals $\varphi_u(t)$ and $r_u(t)$ depend on current system state $x(t)$ and control input $u(t)$. Therefore, (14) is referred to as the instantaneous Bellman equation.

B. Filtered Bellman Equation

To begin with, define the following filtered signals:

$$\begin{aligned} \Delta_u(t) &= \int_0^t e^{-\kappa(t-\tau)}\delta_u(x(\tau))d\tau \\ \Phi_u(t) &= \int_0^t e^{-\kappa(t-\tau)}\varphi_u(x(\tau), u(\tau))d\tau \\ Y_u(t) &= \int_0^t e^{-\kappa(t-\tau)}r(x(\tau), u(\tau))d\tau \end{aligned} \tag{15}$$

which can be equivalently obtained using the online filters as follows:

$$\begin{aligned} \dot{\Delta}_u(t) &= -\kappa\Delta_u(t) + \delta_u(x(t)), \quad \Delta_u(0) = 0 \\ \dot{\Phi}_u(t) &= -\kappa\Phi_u(t) + \varphi_u(x(t), u(t)), \quad \Phi_u(0) = 0 \\ \dot{Y}_u(t) &= -\kappa Y_u(t) + r(x(t), u(t)), \quad Y_u(0) = 0. \end{aligned} \tag{16}$$

From the signals definition in (15) and the instantaneous Bellman equation (14), one has

$$\begin{aligned} \Delta_u(t) &= \int_0^t e^{-\kappa(t-\tau)}\left[W_u^{\mathrm{T}}\varphi_u(\tau) + r_u(\tau)\right]d\tau \\ &= W_u^{\mathrm{T}}\int_0^t e^{-\kappa(t-\tau)}\varphi_u(\tau)d\tau + \int_0^t e^{-\kappa(t-\tau)}r_u(\tau)d\tau \\ &= W_u^{\mathrm{T}}\Phi_u(t) + Y_u(t). \end{aligned} \tag{17}$$

In (17), the terms $\Phi(t)$ and $Y_u(t)$ depend on the information of state and control input on the interval $[0, t]$. In addition, for admissible policy u, the boundedness of $\varphi_u(t)$ and $r_u(t)$ further implies the boundedness of $\Phi(t)$ and $Y_u(t)$. In this article, (17) is referred to as the filtered Bellman equation.

With the filtered signals in (17), the quasi-Hamiltonian is defined as

$$H_q(W_u, \Phi_u(t), Y_u(t)) = W_u^{\mathrm{T}}\Phi_u(t) + Y_u(t).$$

Accordingly, the filtered Bellman equation can be expressed as $\Delta_u(t) = H_q(W_u, \Phi_u(t), Y_u(t))$.

Remark 2: The quasi-Hamiltonian can be viewed as a finite-dimensional parameterization of the Hamiltonian $H(u, x, ((\partial V_u(x))/\partial x))$. The advantage of such parameterization can be summarized as follows.

1) The Hamiltonian $H(u, x, ((\partial V_u(x))/\partial x))$ depends on instantaneous data $\{x(t), u(t)\}$, as shown in (3). As investigated in [38], the Hamiltonian (3) can serve as the temporal difference learning error for continuous-time systems.
2) The quasi-Hamiltonian is defined based on the filtered signals $\{\Phi_u(t), Y_u(t)\}$, which are filters of historic data during the interval $[0, t]$. In this article, the quasi-Hamiltonian is viewed as the novel temporal difference error for critic learning.

Therefore, the quasi-Hamiltonian stands for more data collection and is used in the following design.

C. Efficient Critic Learning

For the policy evaluation with a given admissible policy, the critic learning aims to adapt the critic output

$$V_u(x) = \hat{W}_u^{\mathrm{T}}\phi(x) \tag{18}$$

to minimize the critic learning objective

$$J_u\big(\hat{W}_u(t)\big) = \frac{1}{2}\|e_u(t)\|^2 \tag{19}$$

where $e_u(t)$ is the instantaneous critic learning error defined as

$$e_u(t) = \hat{W}_u^{\mathrm{T}}(t)\varphi_u(t) + r_u(t). \tag{20}$$

One can observe that $e_u(t) \to \delta_u(t)$ as $\hat{W}_u(t) \to W_u$ according to the instantaneous Bellman equation (14). The conventional critic learning is designed based on the gradient descent, i.e.,

$$\begin{aligned} \dot{\hat{W}}_u(t) &= -\gamma_u\frac{\partial J_u\big(\hat{W}_u(t)\big)}{\partial \hat{W}_u(t)} \\ &= -\gamma_u\varphi_u(t)\big[\hat{W}_u^{\mathrm{T}}(t)\varphi_u(t) + r_u(t)\big]. \end{aligned} \tag{21}$$

The convergence of the critic learning depends on the PE condition on the signal $\varphi_u(t)$ [21], which is stringent for adaptive systems.

In the following, we consider the efficient critic learning based on the filtered Bellman equation (17). First, we define the composite critic learning error as

$$\varepsilon_u(\tau, t) = \hat{W}_u^{\mathrm{T}}(t)\Phi_u(\tau) + Y_u(\tau) \tag{22}$$

which combines the current critic weight estimation $\hat{W}_u(t)$ and historical information $\{\Phi_u(\tau), Y_u(\tau)\}$. In contrast to the instantaneous learning objective (19), the efficient critic learning aims to minimize the composite learning objective

$$J_u(t) = \frac{1}{2}\int_0^t \frac{\|\varepsilon_u(\tau, t)\|^2}{\left[1 + \Phi_u^{\mathrm{T}}(\tau)\Phi_u(\tau)\right]^2}d\tau. \tag{23}$$

Note that $\hat{W}_u(t)$ is independent of $\{\Phi_u(\tau), Y_u(\tau)\}$. Applying the chain rule, one can obtain the composite learning objective gradient as

$$\begin{aligned}\frac{\partial J_u(t)}{\partial \hat{W}_u(t)} &= \int_0^t \frac{\varepsilon_u(\tau,t)\Phi_u(t)}{\left[1+\Phi_u^{\mathrm{T}}(\tau)\Phi_u(\tau)\right]^2}d\tau\\ &= \int_0^t \frac{\Phi_u(\tau)\Phi_u^{\mathrm{T}}(\tau)}{\left[1+\Phi_u^{\mathrm{T}}(\tau)\Phi_u(\tau)\right]^2}d\tau \cdot \hat{W}_u(t)\\ &\quad+\int_0^t \frac{\Phi_u(\tau)Y_u(\tau)}{\left[1+\Phi_u^{\mathrm{T}}(\tau)\Phi_u(\tau)\right]^2}d\tau. \end{aligned}\tag{24}$$

Denote

$$E_u(t) = \int_0^t \frac{\Phi_u(\tau)\Phi_u^{\mathrm{T}}(\tau)}{\left[1+\Phi_u^{\mathrm{T}}(\tau)\Phi_u(\tau)\right]^2}d\tau \tag{25}$$

$$F_u(t) = \int_0^t \frac{\Phi_u(\tau)Y_u(\tau)}{\left[1+\Phi_u^{\mathrm{T}}(\tau)\Phi_u(\tau)\right]^2}d\tau \tag{26}$$

which can be obtained using the filter design

$$\begin{aligned}\dot{E}_u(t) &= \frac{\Phi_u(t)\Phi_u^{\mathrm{T}}(t)}{\left[1+\Phi_u^{\mathrm{T}}(t)\Phi_u(t)\right]^2}, \quad E_u(0)=0\\ \dot{F}_u(t) &= \frac{\Phi_u(t)Y_u(t)}{\left[1+\Phi_u^{\mathrm{T}}(t)\Phi_u(t)\right]^2}, \quad F_u(0)=0.\end{aligned}$$

Then, the composite learning objective gradient can be rewritten as

$$\frac{\partial J_u(t)}{\partial \hat{W}_u(t)} = E_u(t)\cdot\hat{W}_u(t) + F_u(t). \tag{27}$$

The critic learning is designed as

$$\begin{aligned}\dot{\hat{W}}_u(t) &= -\gamma_u \frac{\partial J_u(t)}{\partial \hat{W}_u(t)}\\ &= -\gamma_u\left[E_u(t)\cdot\hat{W}_u(t) + F_u(t)\right]. \end{aligned}\tag{28}$$

Theorem 1: Denote $\bar{\Phi}_u(t) = ((\Phi_u(t))/(1+\Phi_u^{\mathrm{T}}(t)\Phi_u(t)))$ and suppose that there exists T_u and β_u such that $\bar{\Phi}_u(t)$ satisfies the relaxed excitation condition. Then, for an admissible policy u, the following holds.

1) All the closed-loop signals L_∞-stable on $[0, T_u)$.
2) The critic weight learning error $\tilde{W}_u(t)$ converges to a small neighborhood of the origin exponentially on $[T_u, +\infty)$.

Proof: The first proposition is a standard result in adaptive control and can be referred to existing literature [45].

Next, we consider the second proposition. For the critic learning (28), we consider the Lyapunov candidate $L_u(\tilde{W}_u(t)) = (1/2)\tilde{W}_u^{\mathrm{T}}(t)\gamma_u^{-1}\tilde{W}_u(t)$. Differentiating $L_u(\tilde{W}_u(t))$ yields

$$\begin{aligned}\dot{L}_u\left(\tilde{W}_u(t)\right) &= -\tilde{W}_u^{\mathrm{T}}(t)\gamma_u^{-1}\dot{\hat{W}}_u(t)\\ &= \tilde{W}_u^{\mathrm{T}}(t)\left[E_u(t)\cdot\hat{W}_u(t)+F_u(t)\right]. \end{aligned}\tag{29}$$

From (17), one has

$$E_u(t)\cdot W_u(t) + F_u(t) = G_u(t) \tag{30}$$

where

$$G_u(t) = \int_0^t \frac{\Phi_u(\tau)\Delta_u(\tau)}{\left[1+\Phi_u^{\mathrm{T}}(\tau)\Phi_u(\tau)\right]^2}d\tau. \tag{31}$$

Then,

$$\begin{aligned}&E_u(t)\cdot\hat{W}_u(t)+F_u(t)\\ &\quad= E_u(t)\cdot\hat{W}_u(t)+F_u(t)\\ &\qquad+G_u(t)-E_u(t)\cdot W_u(t)-F_u(t)\\ &\quad= G_u(t)-E_u(t)\cdot\tilde{W}_u(t). \end{aligned}\tag{32}$$

Inserting (32) into (29) yields

$$\begin{aligned}&\dot{L}_u\left(\tilde{W}_u(t)\right)\\ &\quad\le \frac{1}{2}\gamma_G\left\|\tilde{W}_u(t)\right\| - \lambda_{\min}(E_u(t))\left\|\tilde{W}_u(t)\right\|^2\\ &\quad= -(1-\beta_u)\lambda_{\min}(E_u(t))\left\|\tilde{W}_u(t)\right\|^2 + \beta_u\lambda_{\min}(E_u(t))\\ &\qquad\times\left[\frac{\gamma_G}{2\beta_u\lambda_{\min}(E_u(t))}\left\|\tilde{W}_u(t)\right\| - \left\|\tilde{W}_u(t)\right\|^2\right]. \end{aligned}\tag{33}$$

Based on Young's inequality $2ab - a^2 \le b^2$, one has

$$\begin{aligned}&\frac{\gamma_G}{2\beta_u\lambda_{\min}(E_u(t))}\left\|\tilde{W}_u(t)\right\| - \left\|\tilde{W}_u(t)\right\|^2\\ &\qquad\qquad\le \frac{\gamma_G^2}{4\beta_u^2\lambda_{\min}^2(E_u(t))}. \end{aligned}\tag{34}$$

Finally, one has

$$\begin{aligned}\dot{L}_u\left(\tilde{W}_u(t)\right) &\le -cL_u\left(\tilde{W}_u(t)\right) + d\\ c &= 2\gamma_u(1-\beta_u)\lambda_{\min}(E_u(t))\\ d &= \frac{\gamma_G^2}{4\beta_u^2\lambda_{\min}^2(E_u(t))} \end{aligned}\tag{35}$$

which implies that the critic weight learning error $\tilde{W}_u(t)$ converges to a small neighborhood of the origin exponentially according to the Lyapunov stability extension theorem [46]. This completes the proof. ■

Remark 3: As shown in (28), the critic learning depends on the signals $E_u(t)$ and $F_u(t)$ to tune the critic weight $\hat{W}_u(t)$. This novel critic learning is a combination of both instantaneous data and historic data. First, as shown in (25), the signals $E_u(t)$ and $F_u(t)$ are defined based on the filtered signals $Y_u(t)$ and $\Phi_u(t)$. Second, based on (16), $Y_u(t)$ and $\Phi_u(t)$ are filtered signals of $\varphi_u(x(t), u(t))$ and $r(x(t), u(t))$ during the interval $[0, t]$. Therefore, the presented critic learning (28) also takes the historic data into account.

To this end, the Hamiltonian-driven ADP with efficient replay for the policy evaluation problem with a given admissible policy is shown in Fig. 1, where the quasi-Hamiltonian plays an important role in the critic learning.

IV. Hamiltonian-Driven Efficient Policy Optimization

A. Instantaneous HJB Equation

Using the neural network, the value function and value gradient for the fixed policy $u(x)$ can be denoted as the critic network, i.e.,

$$\begin{aligned}V_*(x) &= W_*^{\mathrm{T}}\phi(x) + \sigma_*(x)\\ \frac{\partial V_*(x)}{\partial x} &= \left[\frac{\partial\phi(x)}{\partial x}\right]^{\mathrm{T}}W_* + \frac{\partial\sigma_*(x)}{\partial x} \end{aligned}\tag{36}$$

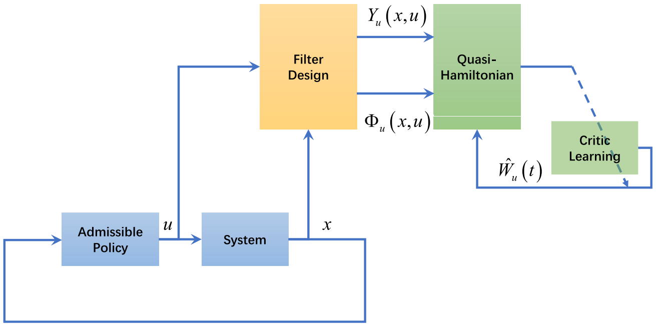

Fig. 1. Hamiltonian-driven ADP with efficient replay for critic learning.

where $W_* \in \mathbb{R}^\ell$ is the critic network weight, $\phi(x)$ and $((\partial\phi(x))/\partial x)$ are the basis function and basis gradient, respectively, and $\sigma_*(x)$ and $((\partial\sigma_*(x))/\partial x)$ are the value function approximation residual and its gradient, respectively. Based on the universal approximation theorem, the ideal critic weight is defined as

$$W_* = \arg\min_W \left\|V_*(x) - W^{\mathrm{T}}\phi(x)\right\|$$

and the ideal value function approximation residual

$$\sigma_*(x) = \min_W \left\|V_*(x) - W^{\mathrm{T}}\phi(x)\right\|$$

is bounded in the sense that there exists a positive constant $\bar{\sigma}_*$ such that $\sup_{x\in\Omega}\|\sigma_*(x)\| \le \bar{\sigma}_*$.

With the value function approximation, the HJB equation can be expressed as

$$\begin{aligned}0 = {} & Q(x) + W_*^{\mathrm{T}}\mu(x) - \frac{1}{4}W_*^{\mathrm{T}}\Theta(x)W_* \\ & - \frac{1}{4}\left[\frac{\partial\sigma_*(x)}{\partial x}\right]^{\mathrm{T}} g(x)R^{-1}g^{\mathrm{T}}(x)\frac{\partial\sigma_*(x)}{\partial x} \\ & - \frac{1}{2}\left[\frac{\partial\sigma_*(x)}{\partial x}\right]^{\mathrm{T}} g(x)R^{-1}g^{\mathrm{T}}(x)\left[\frac{\partial\phi(x)}{\partial x}\right]^{\mathrm{T}} W_* \\ & + \left[\frac{\partial\sigma_*(x)}{\partial x}\right]^{\mathrm{T}} f(x)\end{aligned} \tag{37}$$

with the following notations:

$$\begin{aligned}\mu(x) &= \frac{\partial\phi(x)}{\partial x}f(x) \\ \Theta(x) &= \frac{\partial\phi(x)}{\partial x}g(x)R^{-1}g^{\mathrm{T}}(x)\left[\frac{\partial\phi(x)}{\partial x}\right]^{\mathrm{T}}.\end{aligned} \tag{38}$$

Definition 5: Consider the dynamical system (1) with the signals $\{\mu(x), \Theta(x)\}$ in (38). The pseudo-Hamiltonian is defined as

$$H_p(W, Q, \mu, \Theta) = Q + W^{\mathrm{T}}\mu - \frac{1}{4}W^{\mathrm{T}}\Theta W \tag{39}$$

for all $\{W, Q, \mu, \Theta\} \in \mathbb{R}^\ell \times \mathbb{R} \times \mathbb{R}^\ell \times \mathbb{R}^{\ell\times\ell}$.

Then, the HJB equation (37) can be expressed using the pseudo-Hamiltonian as

$$\begin{aligned}& H_q(W_*, Q(x), \mu(x), \Theta(x)) \\ & \quad = Q(x) + W_*^{\mathrm{T}}\mu(x) - \frac{1}{4}W_*^{\mathrm{T}}\Theta(x)W_* \\ & \quad = \delta_*(x)\end{aligned} \tag{40}$$

with

$$\begin{aligned}\delta_*(x) = {} & \frac{1}{4}\left[\frac{\partial\sigma_*(x)}{\partial x}\right]^{\mathrm{T}} g(x)R^{-1}g^{\mathrm{T}}(x)\frac{\partial\sigma_*(x)}{\partial x} \\ & + \frac{1}{2}\left[\frac{\partial\sigma_*(x)}{\partial x}\right]^{\mathrm{T}} g(x)R^{-1}g^{\mathrm{T}}(x)\left[\frac{\partial\phi(x)}{\partial x}\right]^{\mathrm{T}} W_* \\ & - \left[\frac{\partial\sigma_*(x)}{\partial x}\right]^{\mathrm{T}} f(x).\end{aligned} \tag{41}$$

Note that in (41), the signals $Q(x)$, $\mu(x)$, and $\Theta(x)$ depend on the current state $x(t)$. Therefore, (41) is referred to as the instantaneous HJB equation.

Based on the value function approximation (36), denote the approximate optimal control as

$$\bar{u}_*(x) = -\frac{1}{2}R^{-1}g^{\mathrm{T}}(x)\left[\frac{\partial\phi(x)}{\partial x}\right]^{\mathrm{T}} W_* \tag{42}$$

which satisfies

$$\begin{aligned}\delta_*(x) &= Q(x) + W_*^{\mathrm{T}}\mu(x) - \frac{1}{4}W_*^{\mathrm{T}}\Theta(x)W_* \\ &= W_*^{\mathrm{T}}\varphi_*(x, \bar{u}_*) + r(x, \bar{u}_*) \\ &= H_q(W_*, \varphi_*(x, \bar{u}_*), r(x, \bar{u}_*)) \\ &= H_p(W_*, Q(x), \mu(x), \Theta(x))\end{aligned} \tag{43}$$

where $\varphi_*(x, \bar{u}_*)$ is defined as

$$\begin{aligned}\varphi_*(x, \bar{u}_*) &= \frac{\partial\phi(x)}{\partial x}[f(x) + g(x)\bar{u}_*] \\ &= \mu(x) - \frac{1}{2}\Theta(x)W_*.\end{aligned} \tag{44}$$

B. Filtered HJB Equation

Define the following filtered signals:

$$\begin{aligned}Q_f(t) &= \int_0^t e^{-\kappa(t-\tau)}Q(x(\tau))d\tau \\ \Theta_f(t) &= \int_0^t e^{-\kappa(t-\tau)}\Theta(x(\tau))d\tau \\ \mu_f(t) &= \int_0^t e^{-\kappa(t-\tau)}\mu(x(\tau))d\tau.\end{aligned} \tag{45}$$

The pseudo-Hamiltonian with the above filtered signals is

$$\begin{aligned}& H_q\big(W_*, Q_f, \mu_f, \Theta_f\big) \\ & \quad = Q_f(t) + W_*^{\mathrm{T}}\mu_f(t) - \frac{1}{4}W_*^{\mathrm{T}}\Theta_f(t)W_*.\end{aligned} \tag{46}$$

Based on the instantaneous HJB equation (40), one can obtain the following filtered HJB equation as:

$$\begin{aligned}\delta_f(t) &= \int_0^t e^{-\kappa(t-\tau)}\delta_*(x(\tau))d\tau \\ &= H_q\big(W_*, Q_f, \mu_f, \Theta_f\big).\end{aligned} \tag{47}$$

C. Efficient Actor-Critic Learning

Note that in the filtered HJB equation (47), the terms $\{Q_f(t), \mu_f(t), \Theta_f(t)\}$ contain both the instantaneous and historical information of the system. In the following, we design the efficient actor–critic learning to solve the filtered HJB equation (47).

The adaptive critic network is designed as

$$\hat{V}_c(x(t)) = \hat{W}_c^{\mathrm{T}}(t)\phi(x(t)). \tag{48}$$

The actor network is designed as

$$u_a(x(t)) = -\frac{1}{2}R^{-1}g^{\mathrm{T}}(x(t))\left[\frac{\partial\phi(x(t))}{\partial x(t)}\right]^{\mathrm{T}}\hat{W}_a(t). \tag{49}$$

The critic learning is defined as

$$\begin{aligned} e_c(t) &= \hat{W}_c^{\mathrm{T}}(t)\frac{\partial\phi(x(t))}{\partial x(t)}[f(x(t)) + g(x(t))u_a(t)] \\ &\quad + r(x(t), u_a(t)) \\ &= \hat{W}_c^{\mathrm{T}}(t)\mu(x(t)) - \frac{1}{2}\hat{W}_c^{\mathrm{T}}(t)\Theta(x(t))\hat{W}_a(t) \\ &\quad + Q(x(t)) + \frac{1}{4}\hat{W}_a^{\mathrm{T}}(t)\Theta(x(t))\hat{W}_a(t) \end{aligned} \tag{50}$$

which completely depends on the instantaneous system data. In contrast, based on the pseudo-Hamiltonian (39), we present the composite critic learning error as

$$\begin{aligned} \varepsilon_c(\tau, t) &= H_q\big(\hat{W}_c(t), Q_f(\tau), \mu_f(\tau), \Theta_f(\tau)\big) \\ &= \hat{W}_c^{\mathrm{T}}(t)\mu_f(\tau) - \frac{1}{2}\hat{W}_c^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t) \\ &\quad + Q_f(\tau) + \frac{1}{4}\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t). \end{aligned} \tag{51}$$

Denote

$$\begin{aligned} r_a(\tau, t) &= Q_f(\tau) + \frac{1}{4}\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t) \\ \varphi_a(\tau, t) &= \mu_f(\tau) - \frac{1}{2}\Theta_f(\tau)\hat{W}_a(t). \end{aligned} \tag{52}$$

Then,

$$\varepsilon_c(\tau, t) = \hat{W}_c^{\mathrm{T}}(t)\varphi_a(\tau, t) + r_a(\tau, t). \tag{53}$$

The critic learning objective is considered as

$$\bar{J}_c\big(\hat{W}_c(t)\big) = \frac{J_c\big(\hat{W}_c(t)\big)}{B_c(t)} \tag{54}$$

with

$$\begin{aligned} J_c\big(\hat{W}_c(t)\big) &= \frac{1}{2}\int_0^t \|\varepsilon_c(\tau, t)\|^2 d\tau \\ B_c(t) &= 1 + \int_0^t \big[\varphi_a^{\mathrm{T}}(\tau, t)\varphi_a(\tau, t)\big]^2 d\tau. \end{aligned} \tag{55}$$

Using the gradient descent rule, the critic weight update is designed as

$$\begin{aligned} \dot{\hat{W}}_c(t) &= -\gamma_c\frac{\partial\bar{J}_c\big(\hat{W}_c(t)\big)}{\partial\hat{W}_c(t)} \\ &= -\gamma_c\frac{1}{B_c(t)}\frac{\partial J_c\big(\hat{W}_c(t)\big)}{\partial\hat{W}_c(t)}. \end{aligned} \tag{56}$$

The details about calculation of $((\partial J_c(\hat{W}_c(t)))/(\partial\hat{W}_c(t)))$ are provided in Appendix A.

For the actor network (49), we design

$$\dot{\hat{W}}_a(t) = \gamma_a\left\{-K_{\mathrm{aa}}\hat{W}_a(t) + \frac{K_{\mathrm{ca}}F_c(t)}{B_c(t)} + \frac{F_a(t)}{4[B_c(t)]^2}\right\} \tag{57}$$

with

$$\begin{aligned} F_a(t) &= \int_0^t \Theta_f(\tau)\hat{W}_a(t)\varphi_a^{\mathrm{T}}(t, \tau)\hat{W}_c(t)d\tau \\ F_c(t) &= \int_0^t \varphi_a^{\mathrm{T}}(t, \tau)\hat{W}_c(t)d\tau. \end{aligned} \tag{58}$$

The details about the calculation of $F_a(t)$ and $F_c(t)$ are provided in Appendix B.

Theorem 2: Suppose that there exists T_a and β_a such that

$$\int_0^{T_a} \varphi_a(\tau, t)\varphi_a^{\mathrm{T}}(\tau, t)d\tau \succ \beta_a \cdot I_{N\times N}. \tag{59}$$

Then, with the actor–critic learning in (56) and (57), the system state $x(t)$, the critic weight learning error $\tilde{W}_c(t)$, and the actor weight learning error $\tilde{W}_a(t)$ are uniformly ultimately bounded.

Proof: First, the boundedness of closed-loop signals during the interval $[0, T_s]$ can be obtained based on a classical adaptive learning analysis and interested readers can be referred to [21] and [45].

In the following, we investigate the convergence of closed-loop signals on the interval $[T_s, +\infty]$. Consider the Lyapunov candidate

$$\begin{aligned} L\big(x, \tilde{W}_c, \tilde{W}_a\big) &= V_*(x) + L_c\big(\tilde{W}_c\big) + L_a\big(\tilde{W}_a\big) \\ L_c\big(\tilde{W}_c\big) &= \frac{1}{2}\tilde{W}_c^{\mathrm{T}}\gamma_c^{-1}\tilde{W}_c, \quad L_a\big(\tilde{W}_a\big) = \frac{1}{2}\tilde{W}_a^{\mathrm{T}}\gamma_a^{-1}\tilde{W}_a. \end{aligned} \tag{60}$$

With the online actor (49), the system dynamics is

$$\dot{x} = f(x) - \frac{1}{2}g(x)R^{-1}g^{\mathrm{T}}(x(t))\left[\frac{\partial\phi(x(t))}{\partial x(t)}\right]^{\mathrm{T}}\hat{W}_a(t). \tag{61}$$

Considering the value function approximation (36) and applying the chain rule, one has

$$\begin{aligned} \dot{V}_*(x) &= \left[\frac{\partial V_*(x(t))}{\partial x(t)}\right]\dot{x} \\ &= W_*^{\mathrm{T}}\frac{\partial\phi(x(t))}{\partial x(t)}f(x) - \frac{1}{2}W_*^{\mathrm{T}}\frac{\partial\phi(x(t))}{\partial x(t)}g(x)R^{-1} \\ &\quad \times g^{\mathrm{T}}(x(t))\left[\frac{\partial\phi(x(t))}{\partial x(t)}\right]^{\mathrm{T}}\hat{W}_a(t) + \delta_a \\ &= W_*^{\mathrm{T}}\mu(x) - \frac{1}{2}W_*^{\mathrm{T}}\Theta(x)\hat{W}_a(t) + \delta_a \end{aligned} \tag{62}$$

with

$$\delta_a = \left[\frac{\partial\phi(x)}{\partial x}\right]^{\mathrm{T}}\left[f(x) - \frac{1}{2}g(x)R^{-1}g^{\mathrm{T}}(x)\left[\frac{\partial\phi(x)}{\partial x}\right]^{\mathrm{T}}\hat{W}_a(t)\right]$$

satisfying

$$\|\delta_a\| \le L_{d\phi}L_f\|x\| + \frac{1}{2}\lambda_{\min}(R)L_g^2L_{d\phi}^2\big(\big\|\tilde{W}_a(t)\big\| + \|W_*\|\big).$$

Adding and subtracting $(1/2)W_*^{\mathrm{T}}\Theta(x)W_*$ to (62) yields

$$\begin{aligned}\dot{V}_*(x) &= W_*^{\mathrm{T}}\varphi_*(x,\bar{u}_*) + \frac{1}{2}W_*^{\mathrm{T}}\Theta(x)\tilde{W}_a(t) + \delta_a \\ &= \delta_*(x) - Q(x) - \frac{1}{4}W_*^{\mathrm{T}}\Theta(x)W_* \\ &\quad + \frac{1}{2}W_*^{\mathrm{T}}\Theta(x)\tilde{W}_a(t) + \delta_a. \qquad (63)\end{aligned}$$

As shown in (56) and (77), the critic learning can be rewritten as

$$\dot{\hat{W}}_c(t) = -\gamma_c\frac{1}{[B_c(t)]^2}\int_0^t \varphi_a(t,\tau)\varepsilon_c(t,\tau)d\tau. \qquad (64)$$

From (53) and (47), the critic learning error satisfies

$$\begin{aligned}\varepsilon_c(\tau,t) &= Q_f(\tau) + \frac{1}{4}\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t) \\ &\quad + \hat{W}_c^{\mathrm{T}}(t)\varphi_a(\tau,t) \\ &= Q_f(\tau) + \frac{1}{4}\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t) \\ &\quad + \hat{W}_c^{\mathrm{T}}(t)\varphi_a(\tau,t) + \delta_f(\tau) - W_*^{\mathrm{T}}\varphi_*(\tau) \\ &\quad - Q_f(\tau) - \frac{1}{4}W_*^{\mathrm{T}}\Theta_f(\tau)W_* \\ &= \frac{1}{4}\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t) - \frac{1}{4}W_*^{\mathrm{T}}\Theta_f(\tau)W_* \\ &\quad + \hat{W}_c^{\mathrm{T}}(t)\varphi_a(\tau,t) - W_*^{\mathrm{T}}\varphi_*(\tau) + \delta_f(\tau) \\ &= \frac{1}{4}\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t) - \frac{1}{4}W_*^{\mathrm{T}}\Theta_f(\tau)W_* \\ &\quad - \frac{1}{2}\hat{W}_c^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t) + \frac{1}{2}W_*^{\mathrm{T}}\Theta_f(\tau)W_* \\ &\quad - \tilde{W}_c^{\mathrm{T}}(t)\mu_f(\tau) + \delta_f(\tau) \\ &= \frac{1}{2}\tilde{W}_c^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t) + \frac{1}{4}\tilde{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\tilde{W}_a(t) \\ &\quad - \tilde{W}_c^{\mathrm{T}}(t)\mu_f(\tau) + \delta_f(\tau) \\ &= \delta_f(\tau) - \tilde{W}_c^{\mathrm{T}}(t)\varphi_a(\tau,t) + \frac{1}{4}\tilde{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\tilde{W}_a(t).\end{aligned}$$

Inserting the above equation into the critic learning (64) yields

$$\begin{aligned}\dot{\hat{W}}_c(t) &= \gamma_c\frac{\int_0^t \varphi_a(t,\tau)\varphi_a^{\mathrm{T}}(\tau,t)d\tau}{[B_c(t)]^2}\tilde{W}_c(t) \\ &\quad - \gamma_c\frac{\int_0^t \varphi_a(t,\tau)\tilde{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\tilde{W}_a(t)d\tau}{4[B_c(t)]^2} \\ &\quad - \gamma_c\frac{\int_0^t \varphi_a(t,\tau)\delta_f(\tau)d\tau}{[B_c(t)]^2}. \qquad (65)\end{aligned}$$

Then,

$$\begin{aligned}\dot{L}_c\big(\tilde{W}_c\big) &= -\tilde{W}_c^{\mathrm{T}}(t)\gamma_c^{-1}\dot{\hat{W}}_c(t) \\ &= -\tilde{W}_c^{\mathrm{T}}(t)\frac{\int_0^t \varphi_a(t,\tau)\varphi_a^{\mathrm{T}}(\tau,t)d\tau}{[B_c(t)]^2}\tilde{W}_c(t) \\ &\quad + \tilde{W}_c^{\mathrm{T}}(t)\frac{\int_0^t \varphi_a(t,\tau)\delta_f(\tau)d\tau}{[B_c(t)]^2} + \frac{\Lambda(t,\tau)}{4[B_c(t)]^2} \qquad (66)\end{aligned}$$

where

$$\Lambda(t,\tau) = \int_0^t \tilde{W}_c^{\mathrm{T}}(t)\varphi_a(t,\tau)\tilde{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\tilde{W}_a(t)d\tau. \qquad (67)$$

Consider the fact that

$$W_* = \hat{W}_c(t) + \tilde{W}_c(t) = \hat{W}_a(t) + \tilde{W}_a(t). \qquad (68)$$

Then, the term $\Lambda(t,\tau)$ can be further written as

$$\begin{aligned}\Lambda(t,\tau) &= -\int_0^t \tilde{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)W_*\varphi_a^{\mathrm{T}}(t,\tau)W_*d\tau \\ &\quad + \int_0^t \tilde{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)W_*\varphi_a^{\mathrm{T}}(t,\tau)\tilde{W}_c(t)d\tau \\ &\quad + \int_0^t \tilde{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\tilde{W}_c(t)\varphi_a^{\mathrm{T}}(t,\tau)W_*d\tau \\ &\quad + \tilde{W}_a^{\mathrm{T}}(t)F_a(t). \qquad (69)\end{aligned}$$

In addition, for the actor learning (57), one has

$$\begin{aligned}\dot{L}_a\big(\tilde{W}_a\big) &= -\tilde{W}_a^{\mathrm{T}}(t)\gamma_a^{-1}\dot{\hat{W}}_a(t) \\ &= \tilde{W}_a^{\mathrm{T}}(t)K_{\mathrm{aa}}\hat{W}_a(t) - \tilde{W}_a^{\mathrm{T}}(t)\frac{F_a(t)}{4[B_c(t)]^2} \\ &\quad - \tilde{W}_a^{\mathrm{T}}(t)K_{\mathrm{ca}}\frac{\int_0^t \varphi_a^{\mathrm{T}}(t,\tau)\hat{W}_c(t)d\tau}{B_c(t)}. \qquad (70)\end{aligned}$$

Recall the fact in (68), and then,

$$\begin{aligned}\dot{L}_a\big(\tilde{W}_a\big) &= \tilde{W}_a^{\mathrm{T}}(t)K_{\mathrm{aa}}\hat{W}_a(t) - \tilde{W}_a^{\mathrm{T}}(t)\frac{F_a(t)}{4[B_c(t)]^2} \\ &\quad - \tilde{W}_a^{\mathrm{T}}(t)K_{\mathrm{ca}}\frac{\int_0^t \varphi_a^{\mathrm{T}}(t,\tau)d\tau}{B_c(t)}W_* \\ &\quad + \tilde{W}_a^{\mathrm{T}}(t)K_{\mathrm{ca}}\frac{\int_0^t \varphi_a^{\mathrm{T}}(t,\tau)d\tau}{B_c(t)}\tilde{W}_c(t). \qquad (71)\end{aligned}$$

Collecting the results in (63), (66), and (71), one has

$$\begin{aligned}\dot{L}(Z) &\le -Z^{\mathrm{T}}\begin{bmatrix} P_{\mathrm{xx}} & 0 & 0 \\ 0 & P_{\mathrm{cc}} & P_{\mathrm{ca}} \\ 0 & P_{\mathrm{ac}} & P_{\mathrm{aa}} \end{bmatrix}Z + Z^{\mathrm{T}}d + c \\ &= -Z^{\mathrm{T}}P_Z Z + Z^{\mathrm{T}}d + c \qquad (72)\end{aligned}$$

with

$$\begin{aligned}P_{\mathrm{xx}} &= qI_{n\times n}, \quad P_{\mathrm{aa}} = K_{\mathrm{aa}} \\ P_{\mathrm{cc}} &= \frac{\int_0^t \varphi_a(t,\tau)\varphi_a^{\mathrm{T}}(\tau,t)d\tau}{[B_c(t)]^2} \\ P_{\mathrm{ac}} &= P_{\mathrm{ca}}^{\mathrm{T}} = -\frac{K_{\mathrm{ca}}\int_0^t \varphi_a^{\mathrm{T}}(t,\tau)d\tau}{2B_c(t)} \\ &\quad - \frac{\int_0^t \Theta_f(\tau)W_*\varphi_a^{\mathrm{T}}(t,\tau)d\tau}{8[B_c(t)]^2} \\ &\quad - \frac{\int_0^t \Theta_f(\tau)\varphi_a^{\mathrm{T}}(t,\tau)W_*d\tau}{8[B_c(t)]^2} \\ d_x &= L_{d\phi}L_f, \quad d_c = \frac{\int_0^t \varphi_a(t,\tau)\delta_f(\tau)d\tau}{[B_c(t)]^2} \\ d_a &= -\frac{\int_0^t \Theta_f(\tau)W_*\varphi_a^{\mathrm{T}}(t,\tau)W_*d\tau}{4[B_c(t)]^2} + K_{\mathrm{aa}}W_* \\ &\quad - \frac{K_{\mathrm{ca}}\int_0^t \varphi_a^{\mathrm{T}}(t,\tau)d\tau}{B_c(t)}W_* + \frac{1}{2}\Theta(x)W_* \\ &\quad + \frac{1}{2}\lambda_{\min}(R)L_g^2L_{d\phi}^2 \\ c &= \delta_*(x) - \frac{1}{4}W_*^{\mathrm{T}}\Theta(x)W_* \\ &\quad + \frac{1}{2}\lambda_{\min}(R)L_g^2L_{d\phi}^2\|W_*\|. \qquad (73)\end{aligned}$$

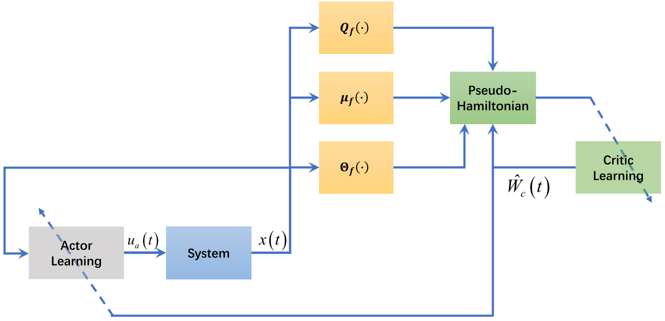

Fig. 2. Hamiltonian-driven ADP with efficient replay for actor–critic learning.

Note that $P_{\rm xx}$ and $P_{\rm aa}$ are positive definite matrices. From condition (59), the matrix $P_{\rm cc}$ is also positive definite. In addition, the Shur complement of $P_{\rm cc}$ is positive definite, i.e., $P_{\rm cc} - P_{\rm ca}P_{\rm aa}^{-1}P_{\rm ac} > 0$, provided that $K_{\rm ca} \ll K_{\rm aa}$. Then, the matrix P_Z can be guaranteed to be positive definite. In addition, with the normalization term $B_c(t)$, the terms d is bounded. Based on the Lyapunov extension theorem [46], the augmented state Z is uniformly ultimately bounded. This completes the proof. ■

To this end, the Hamiltonian-driven ADP with efficient replay for the policy optimization problem is shown in Fig. 2, where the pseudo-Hamiltonian lays the foundation for the actor–critic learning.

Remark 4: In Sections III and IV, we extend the traditional Hamiltonian to quasi-Hamiltonian and pseudo-Hamiltonian, respectively. With the quasi-Hamiltonian, the Bellman equation is parameterized as a linear equation of the unknown critic weight W_u. For the HJB equation, the pseudo-Hamiltonian is presented to produce the filtered HJB equation, which is quadratic in the optimal weight W_*, which is compared to the linear parameterized model [33], [35], [36], [37]. In addition, the presented learning does not require probing noise to guarantee the convergence to the optimum, which can obviate the unnecessary oscillation in the closed-loop signals. □

Remark 5: In existing ADP methods [47], [48], the experience-replay technique is based on concurrent learning [34], of which the historic data are collected and updated based on a matrix rank condition. However, the matrix rank condition has to be checked throughout the implementation to guarantee that the collected data are sufficiently excited, which might increase the computational complexity. In contrast, the experience-replay technique in this article is based on the online filters design (15) and (45), which can be efficiently obtained and has lower computational complexity.

V. Simulation Study

In this section, we investigate the simulation example of optimal stabilization problem with nonlinear dynamical system to demonstrate the efficacy of the presented efficient Hamiltonian-driven ADP. To be specific, the van der pol's oscillator is considered with the dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - \frac{1}{2}(1 - x_1^2)x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1 \end{bmatrix} u \tag{74}$$

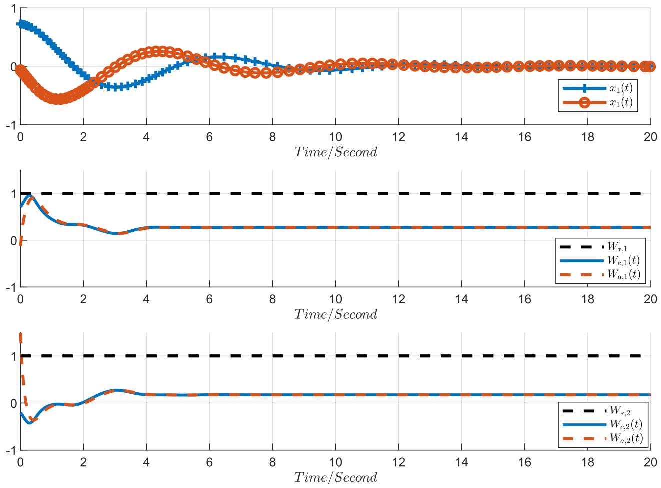

Fig. 3. State evolution and online actor–critic learning for policy optimization problem using conventional approach [21] without probing noise. The optimal critic weight is denoted as $W_* = [W_{*,1} \ W_{*,2}]^{\rm T} = [1 \ 1]^{\rm T}$, as illustrated by the black dashed lines. The adaptive critic learning is denoted as $W_c(t) = [W_{c,1}(t) \ W_{c,2}(t)]^{\rm T}$, as illustrated by the blue solid lines. The online actor learning is denoted as $W_a(t) = [W_{a,1}(t) \ W_{a,2}(t)]^{\rm T}$ and illustrated by the red solid lines.

where $x = [\, x_1 \ \ x_2 \,] \in \mathbb{R}^2$ is the system state and $u \in \mathbb{R}$ is the control input. According to [49], with the performance $V_u(x_0) = \int_{t_0}^{\infty} r(x(\tau), u(\tau))d\tau$ and the reward function $r(x, u) = x_1^2 + u^2$, the optimal value function satisfying the HJB equation

$$0 = x_1^2 + \left[\frac{\partial V_*(x)}{\partial x}\right]^{\rm T} \left[x_2 {-} x_1 - \frac{1}{2}(1 - x_1^2)x_2 \right] - \frac{1}{4}\left[\frac{\partial V_*(x)}{\partial x}\right]^{\rm T} \begin{bmatrix} 0 \\ x_1 \end{bmatrix} \begin{bmatrix} 0 & x_1 \end{bmatrix} \frac{\partial V_*(x)}{\partial x} \tag{75}$$

can be determined as $V_*(x) = x_1^2 + x_2^2$. On this basis, the optimal control policy can be further obtained as $u_* = -x_1 x_2$. In addition, the optimal value function can be parameterized as $V_*(x) = W_*^{\rm T}\phi(x)$ using the basis function $\phi(x) = [\, x_1^2 \ \ x_2^2 \,]^{\rm T}$ and optimal weight $W_* = [\, 1 \ \ 1 \,]^{\rm T}$. Note that with the above value function parameterization, the value function approximation residual $\sigma_*(x)$ is identically equal to zero.

The critic learning rate γ_c should be bigger than the actor learning rate γ_a to guarantee the closed-loop stability and actor–critic learning convergence, as investigated in Theorems 1 and 2. In addition, the filter parameter κ should be small enough to guarantee that the relaxed excitation is satisfied.

Case 1 (Conventional Actor-Critic Learning Without Probing Noise): With the conventional actor–critic learning [21], the evolution of system state and actor–critic weights is shown in Fig. 3. One can observe that in the absence of PE condition, the convergence of the actor–critic learning to the optimal weight cannot be guaranteed.

Case 2 (Conventional Actor-Critic Learning With Probing Noise): To satisfy the PE condition, we add the probing noise $n_u(t)$ into the actor network, i.e.,

$$n_u(t) = e^{-\frac{t}{10}}[\sin(t) + 2\sin(2t)]. \tag{76}$$

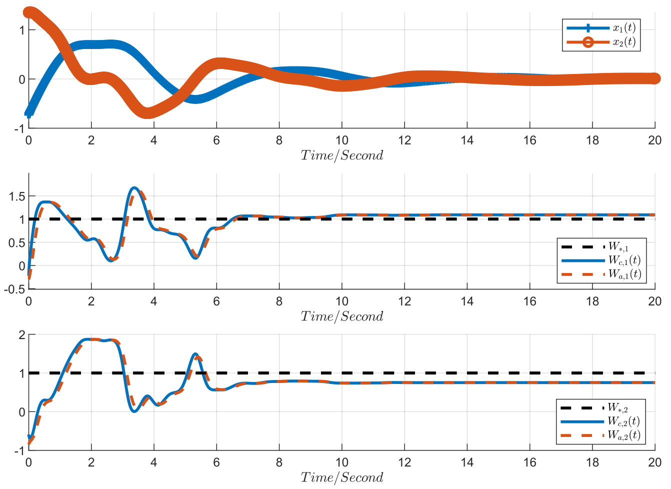

Fig. 4. State evolution and online actor–critic learning for policy optimization problem using conventional approach [21] with probing noise. The optimal critic weight is denoted as $W_* = [W_{*,1}\ W_{*,2}]^{\mathrm{T}} = [1\ 1]^{\mathrm{T}}$, as illustrated by the black dashed lines. The adaptive critic learning is denoted as $W_c(t) = [W_{c,1}(t)\ W_{c,2}(t)]^{\mathrm{T}}$, as illustrated by the blue solid lines. The online actor learning is denoted as $W_a(t) = [W_{a,1}(t)\ W_{a,2}(t)]^{\mathrm{T}}$ and illustrated by the red solid lines.

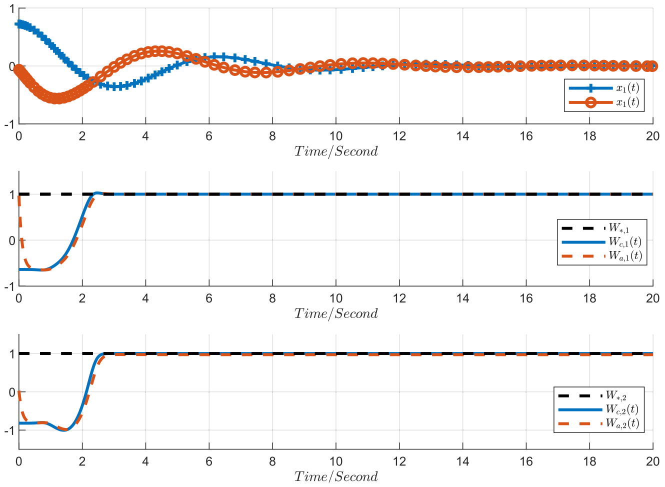

Fig. 5. State evolution and online actor–critic learning for policy optimization problem using presented efficient Hamiltonian-driven ADP. The optimal critic weight is denoted as $W_* = [W_{*,1}\ W_{*,2}]^{\mathrm{T}} = [1\ 1]^{\mathrm{T}}$, as illustrated by the black dashed lines. The adaptive critic learning is denoted as $W_c(t) = [W_{c,1}(t)\ W_{c,2}(t)]^{\mathrm{T}}$, as illustrated by the blue solid lines. The online actor learning is denoted as $W_a(t) = [W_{a,1}(t)\ W_{a,2}(t)]^{\mathrm{T}}$ and illustrated by the red solid lines.

Accordingly, the system response is shown in Fig. 4. One can observe that the satisfaction of PE condition contributes to the actor–critic learning convergence. However, the probing noise also brings undesired oscillation in the evolution of both system state and online actor–critic learning. In addition, the probing noise also results in a bias in the actor–critic learning and the actor–critic weights converges to a neighborhood set around the origin.

Case 3 (Hamiltonian-Driven ADP With Experience-Replay): To relax the stringent PE condition while ensuring the actor–critic learning convergence, we apply the presented Hamiltonian-driven ADP with efficient experience-replay technique to the system, where the results are shown in Fig. 5. Finally, one can observe that the actor–critic weights converge exponentially to the optimal critic weights within 1.5 s. In addition, compared to the case using probing noise, one can observe that there is no undesired oscillation in the closed-loop signals.

VI. Conclusion

In this article, we extend the classical Hamiltonian based on the value function approximation to develop efficient data-based ADP learning. The quasi-Hamiltonian and the pseudo-Hamiltonian are proposed to cope with the policy evaluation and policy optimization problems, respectively. With the quasi-Hamiltonian and pseudo-Hamiltonian, the historical and instantaneous data are simultaneously utilized through additional filters design. Compared with the stringent PE condition, the presented efficient Hamiltonian-driven ADP ensures the learning convergence with a relaxed excitation condition. In addition, the probing noise is also obviated to cancel the undesired oscillation in the closed-loop responses. Simulation examples are investigated to verify the presented design in the end. Future works aim to extend the presented design to distributed optimization and control problems.

Appendix A
Calculation of $((\partial J_c(\hat{W}_c(t)))/(\partial \hat{W}_c(t)))$

Applying the chain rule to the critic learning object (54) and considering (52), one has

$$\begin{aligned}\frac{\partial J_c\big(\hat{W}_c(t)\big)}{\partial \hat{W}_c(t)} &= \int_0^t \varphi_a(t,\tau)\varepsilon_c(t,\tau)d\tau\\ &= \int_0^t \varphi_a(t,\tau)\big[\varphi_a^{\mathrm{T}}(t,\tau)\hat{W}_c(t) + r_a(t,\tau)\big]d\tau.\end{aligned}$$

Define

$$\begin{aligned}M_c(t) &= \int_0^t \varphi_a(t,\tau)\varphi_a^{\mathrm{T}}(t,\tau)d\tau\\ N_c(t) &= \int_0^t \varphi_a(t,\tau)r_a(t,\tau)d\tau. \qquad (77)\end{aligned}$$

Then, the objective gradient can be further expressed as

$$\frac{\partial J_c\big(\hat{W}_c(t)\big)}{\partial \hat{W}_c(t)} = M_c(t)\cdot \hat{W}_c(t) + N_c(t). \qquad (78)$$

From (52), one has

$$\begin{aligned}M_c(t) = &\frac{1}{4}\int_0^t \big[\Theta_f(\tau)\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\big]d\tau\\ &+ \int_0^t \big[\mu_f(\tau)\mu_f^{\mathrm{T}}(\tau)\big]d\tau\\ &- \frac{1}{2}\int_0^t \big[\mu_f(\tau)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\big]d\tau\\ &- \frac{1}{2}\int_0^t \big[\Theta_f(\tau)\hat{W}_a(t)\mu_f^{\mathrm{T}}(\tau)\big]d\tau. \qquad (79)\end{aligned}$$

Using the Kronecker property, one has

$$\begin{aligned}
&\int_0^t \big[\Theta_f(\tau)\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\big]d\tau \\
&\quad= \mathrm{vec}_{M,M}^{-1}\big[\psi_{\Theta\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\big)\big] \\
&\int_0^t \big[\mu_f(\tau)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\big]d\tau \\
&\quad= \int_0^t \mathrm{vec}_{M,M}^{-1}\big[\psi_{\mu\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a^{\mathrm{T}}(t)\big)\big]d\tau \\
&\int_0^t \big[\Theta_f(\tau)\hat{W}_a(t)\mu_f^{\mathrm{T}}(\tau)\big]d\tau \\
&\quad= \mathrm{vec}_{M,M}^{-1}\big[\psi_{\Theta\mu}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\big)\big]
\end{aligned} \tag{80}$$

where

$$\begin{aligned}
\psi_{\Theta\Theta}(t) &= \int_0^t \big[\Theta_f(\tau)\otimes\Theta_f(\tau)\big]d\tau \\
\psi_{\mu\Theta}(t) &= \int_0^t \big[\Theta_f(\tau)\otimes\mu_f(\tau)\big]d\tau \\
\psi_{\Theta\mu}(t) &= \int_0^t \big[\mu_f(\tau)\otimes\Theta_f(\tau)\big]d\tau.
\end{aligned} \tag{81}$$

Denote $\psi_{\mu\mu}(t) = \int_0^t \mu_f(\tau)\mu_f^{\mathrm{T}}(\tau)d\tau$. Then,

$$\begin{aligned}
M_c(t) = {}& \frac{1}{4}\mathrm{vec}_{M,M}^{-1}\big[\psi_{\Theta\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\big)\big] \\
&- \frac{1}{2}\int_0^t \mathrm{vec}_{M,M}^{-1}\big[\psi_{\mu\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a^{\mathrm{T}}(t)\big)\big]d\tau \\
&- \frac{1}{2}\mathrm{vec}_{M,M}^{-1}\big[\psi_{\Theta\mu}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\big)\big] \\
&+ \psi_{\mu\mu}(t).
\end{aligned} \tag{82}$$

Second, inserting (52) into (77) yields

$$\begin{aligned}
N_c(t) = {}& \int_0^t \big[\mu_f(\tau)Q_f(\tau)\big]d\tau \\
&+\frac{1}{4}\int_0^t \big[\mu_f(\tau)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t)\big]d\tau \\
&-\frac{1}{2}\int_0^t \big[\Theta_f(\tau)\hat{W}_a(t)Q_f(\tau)\big]d\tau \\
&-\frac{1}{8}\int_0^t \big[\Theta_f(\tau)\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t)\big]d\tau.
\end{aligned}$$

Using the Kronecker property, one has

$$\begin{aligned}
&\int_0^t \big[\mu_f(\tau)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t)\big]d\tau \\
&\quad= \mathrm{vec}_{M,M}^{-1}\big[\psi_{\mu\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a^{\mathrm{T}}(t)\big)\big]\cdot\hat{W}_a(t) \\
&\int_0^t \big[\Theta_f(\tau)\hat{W}_a(t)Q_f(\tau)\big]d\tau \\
&\quad= \mathrm{vec}_{M,1}^{-1}\big[\psi_{\Theta Q}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\big)\big] \\
&\int_0^t \big[\Theta_f(\tau)\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_a(t)\big]d\tau \\
&\quad= \mathrm{vec}_{M,1}^{-1}\big[\psi_{\Theta\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\big)\big]\cdot\hat{W}_a(t)
\end{aligned}$$

where $\psi_{\Theta Q}(t) = \int_0^t [\Theta_f(\tau)Q_f(\tau)]d\tau$. Denote $\psi_{\mu Q}(t) = \int_0^t [\mu_f(\tau)Q_f(\tau)]d\tau$. Then, $N_c(t)$ can be calculated as

$$\begin{aligned}
N_c(t) = {}& \psi_{\mu Q}(t) - \frac{1}{2}\mathrm{vec}_{M,1}^{-1}\big[\psi_{\Theta Q}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\big)\big] \\
&+ \frac{1}{4}\mathrm{vec}_{M,M}^{-1}\big[\psi_{\mu\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a^{\mathrm{T}}(t)\big)\big]\cdot\hat{W}_a(t) \\
&- \frac{1}{8}\mathrm{vec}_{M,1}^{-1}\big[\psi_{\Theta\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\big)\big]\cdot\hat{W}_a(t).
\end{aligned} \tag{83}$$

Appendix B
Calculation of $F_a(t)$ and $F_c(t)$

Based on the definition of $\varphi_a(t,\tau)$ in (52), the term $F_a(t)$ can be rewritten as

$$\begin{aligned}
F_a(t) = {}& \bigg[\int_0^t \Theta_f(\tau)\hat{W}_a(t)\mu_f^{\mathrm{T}}(\tau)d\tau\bigg]\cdot\hat{W}_c(t) \\
&- \frac{1}{2}\bigg[\int_0^t \Theta_f(\tau)\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)d\tau\bigg]\cdot\hat{W}_c(t).
\end{aligned}$$

Recalling the facts in (80), one has

$$\begin{aligned}
F_a(t) = {}& \mathrm{vec}_{M,M}^{-1}\big[\psi_{\Theta\mu}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\big)\big]\cdot\hat{W}_c(t) \\
&- \frac{1}{2}\mathrm{vec}_{M,M}^{-1}\big[\psi_{\Theta\Theta}(t)\cdot \mathrm{vec}\big(\hat{W}_a(t)\hat{W}_a^{\mathrm{T}}(t)\big)\big]\cdot\hat{W}_c(t).
\end{aligned}$$

Similarity, inserting the definition of $\varphi_a(t,\tau)$ (52) into F_c in (58) yields

$$\begin{aligned}
F_c(t) &= \int_0^t \bigg[\mu_f^{\mathrm{T}}(\tau) - \frac{1}{2}\hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\bigg]\hat{W}_c(t)d\tau \\
&= \int_0^t \mu_f^{\mathrm{T}}(\tau)\hat{W}_c(t)d\tau - \frac{1}{2}\int_0^t \hat{W}_a^{\mathrm{T}}(t)\Theta_f(\tau)\hat{W}_c(t)d\tau \\
&= \bigg[\int_0^t \mu_f^{\mathrm{T}}(\tau)d\tau\bigg]\hat{W}_c(t) - \frac{1}{2}\hat{W}_a^{\mathrm{T}}(t)\bigg[\int_0^t \Theta_f(\tau)d\tau\bigg] \\
&\quad \times \hat{W}_c(t).
\end{aligned}$$

Denote the signals $\Theta_{\mathrm{ff}}(t) = \int_0^t \Theta_f(\tau)d\tau$ and $\mu_{\mathrm{ff}}(t) = \int_0^t \mu_f(\tau)d\tau$.

To this end, $F_c(t)$ can be online calculated as

$$F_c(t) = \mu_{\mathrm{ff}}^{\mathrm{T}}(t)\hat{W}_c(t) - \frac{1}{2}\hat{W}_a^{\mathrm{T}}(t)\Theta_{\mathrm{ff}}(t)\hat{W}_c(t).$$

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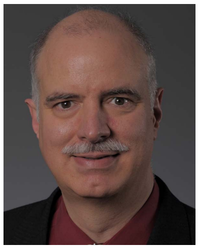

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