Mathematics and Statistics Faculty Research & Creative Works

Mathematics and Statistics

01 Jan 2022

Fundamental Structure of General Stochastic Dynamical Systems: High-Dimension Case

Haoyu Wang

Xiaoliang Gan

Wenqing Hu
Missouri University of Science and Technology, huwen@mst.edu

Ping Ao

Research Article

Fundamental Structure of General Stochastic Dynamical Systems: High-Dimension Case

Haoyu Wang [ID],[1] Xiaoliang Gan [ID],[2] Wenqing Hu [ID],[3] and Ping Ao [ID][4]

[1]*Department of Mathematics, Shanghai University, Shanghai 200444, China*
[2]*School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China*
[3]*Department of Mathematics, Missouri University of Science and Technology, Rolla, MO 65401, USA*
[4]*Department of Physics, Shanghai University, Shanghai 200444, China*

Correspondence should be addressed to Haoyu Wang; wanghaoyu042@163.com

No one has proved that mathematically general stochastic dynamical systems have a special structure. Thus, we introduce a structure of a general stochastic dynamical system. According to scientific understanding, we assert that its deterministic part can be decomposed into three significant parts: the gradient of the potential function, friction matrix and Lorenz matrix. Our previous work proved this structure for the low-dimension case. In this paper, we prove this structure for the high-dimension case. Hence, this structure of general stochastic dynamical systems is fundamental.

1. Introduction

Stochastic differential equations are widely used to describe random phenomena in complex systems in physics, biology, and chemistry. For such a stochastic dynamical system, researchers usually build an appropriate mathematical model based on basic scientific laws and analyze or simulate it to gain insights about its complex phenomena. However, these models are proposed to solve specific scientific problems [1–3]. Until now, the general theory for stochastic differential equations has been limited.

To gain a deeper understanding of the dynamic behaviors of general stochastic dynamical systems requires the exploration of their intrinsic mechanisms. In 2005, Ao [4] proposed Ao decomposition, which demonstrates that the deterministic part of general stochastic dynamical systems can be decomposed into three significant parts: the friction force, gradient of the potential function, and Lorenz force. This inspired much work [5, 6]. We discuss the scientific significance of these three terms.

1.1. Potential Function. From the biological point of view, the potential function can be explained by evolution theory. As we know, the fundamental nature of biology is determined by evolution. To explain adaptation and speciation, Darwin [7] formulated the theory of evolution based on natural selection. Accordingly, Fisher [8] proposed the fundamental theorem of natural selection, indicating that the increase rate of mean fitness by natural selection is equal to its genetic variance in fitness. In 1932, Wright [9] proposed the fitness landscape concept, by which evolutionary adaptation may be seen as a hill-climbing process on the mean fitness landscape until a local mean fitness peak is reached. In 1940, Waddington [10] proposed the developmental landscape, which is equivalent to the fitness landscape. Wright's fitness landscape and Fisher's fundamental theorem of natural selection have been widely used to interpret adaptation as mean fitness maximization (see Figure 1).

This phenomenon is often illustrated on mathematical landscapes as balls rolling downhill. A ball experiencing
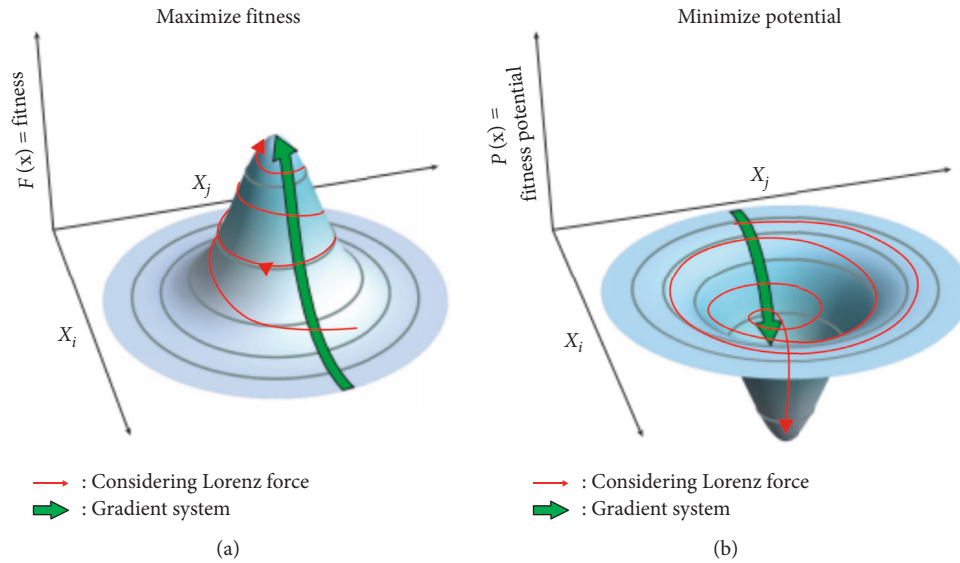
FIGURE 1: Evolution is described as populations moving on landscapes. This is represented as either (a) a maxima on a landscape of fitness; or (b) a minima on a landscape of fitness potential. They are different ways to visualize the same process.

gravity tends to a minimum of the gravitational potential energy $p(x)$ as a function of its spatial position x. The force on the ball is given by the local slope, $f = -(dp(x)/dx)$.

The potential landscape is also called a potential function or energy function, which has been applied in fields such as physics, biology, and chemistry [11, 12]. It can compare the relative stability of different attractors [13], account for the transition rates between neighboring steady states induced by noise [14], and provide an intuitive picture that reveals the essential mechanism underlying the complex system [15]. In physics, the potential function is closely related to the non-equilibrium thermodynamic framework [16]; in chemistry, it provides useful explanations for protein folding [17, 18]; in biology, it has been used to explore basic problems in evolution such as the robustness, adaptability, and efficiency of real biological networks [19]. Until now, the general existence of the potential function remains unsolved. Researchers such as Prigogine et al. [20–22] have insisted that the potential function does not exist in non-equilibrium systems because they have not found it.

1.2. The Friction Matrix. The friction matrix (the frictional force) represents dissipation. In this case, the energy of a dynamical system decreases, so the potential function decreases and the corresponding fitness increases. A system that has only friction is a gradient system (see green thick arrow in Figure 1).

1.3. The Lorenz Matrix. Interestingly, Wright's fitness landscape theory [9] cannot explain the Red Queen hypothesis proposed by Valen [23], which illustrates that the biotic interactions between species provide a driving force resulting in endless evolution for some species even if the physical environment is unchanged. This is because he neglects the Lorenz matrix (Lorenz force). If considering the Lorenz force, the population flow on a landscape is not

directly down the gradient of the potential function. It also swirls (see red thin arrow in Figure 1).

The above scientific understanding indicates that these three components exist in general stochastic dynamical systems, but this decomposition lacks rigorous mathematical proof. This paper is the first to prove that the deterministic part of general high-dimension stochastic dynamical systems can be decomposed into three components: the diffusion force, gradient of the potential function, and curl flux. Our previous work proved this structure for the low-dimension case (when the dimension $n = 1, 2$). On this basis, we prove this structure for the high-dimension case (when $n \geq 4$, and $(n(n-1)/2)$ is an even number, i.e., $n = 4, 5, 8, 9, \ldots, 4i, 4i+1, \ldots$, when $i = 1, 2, 3, \ldots$). Apart from theoretical significance, our result has important guiding significance for applications in both mathematics and subjects such as biology and physics. The potential function provides intuitive and global landscapes. Real dynamical systems are complex and usually have more than one steady state, so the potential function has a wide range of applications in real dynamical systems. For example, Hu and Xu [24] studied the phenomenon of multi-stable chaotic attractors existing in generalized synchronization for a driving and response system named Rössler system. Angeli and Sontag [25] studied the emergence of multi-stability and hysteresis in those monotone input/output systems that arise, under positive feedback, starting from monotone systems with well-defined steady-state responses. Liu and You [26] studied multi-stability, existence of almost periodic solutions of a class of recurrent neural networks with bounded activation functions and all criteria they proposed can be easily extended to fit many concrete forms of neural networks such as Hopfield neural networks, or cellular neural networks, etc.. The potential function has provided a general and unified perspective for researchers to investigate different types of dynamical systems.

The rest of this paper is organized as follows. Section 2 introduces Ao decomposition for general stochastic differential equations and proposes our problem: proving the equivalence of the Langevin equation and the equation after Ao decomposition. In Section 3, we reduce this problem to proving the existence of solutions for first-order partial differential equations, and we accomplish this proof in Section 4.

2. A-Type Decomposition for General Stochastic Differential Equations

The Langevin equation in physics, which has the form of a general stochastic differential equation, is usually a more accurate description of physical processes than the purely deterministic one [27–30]. Here, we use the physicists' notation for the noise, and we can write this equation in the form

$$\dot{q} = f(q, t) + \zeta(q, t). \tag{1}$$

We discuss this equation in n-dimensional real Euclidean space. The state variable $q = (q_1(t), q_2(t), \ldots, q_n(t))$ is a function of time t, and the component functions q_i ($i = 1, 2, \ldots, n$) of the state variable $q = (q_1, q_2, \ldots, q_n)$ are independent. We assume that $f(q, t)$ is an infinitely differentiable smooth function. The noise $\zeta(q, t)$ is a function of t and the state variable q, and is almost nowhere differentiable. We consider the case that $\zeta(q, t)$ is n-dimensional white Gaussian noise with mean

$$\langle \zeta(q, t) \rangle = 0, \tag{2}$$

and covariance

$$\langle \zeta(q, t) \zeta^\tau(q, t') \rangle = 2D(q) \delta(t - t'). \tag{3}$$

The superscript τ denotes the transpose of a matrix (vector), $\delta(t - t')$ is the Dirac delta function, $\langle \cdot \rangle$ indicates the average over the noise distribution, and the diffusion matrix $D(q)$ is symmetric and positive semi-definite.

Noticing a new formulation of equation (1) [6, 15, 31], we propose Ao decomposition, under which equation (1) can be formally decomposed into

$$[S(q(t)) + A(q(t))]\dot{q} = -\nabla\varphi(q(t)) + \xi(q, t), \tag{4}$$

where $S(q(t))$ is a symmetric semi-definite matrix (which we call the "friction matrix"), $A(q(t))$ is an antisymmetric matrix (the "Lorenz matrix"), $\varphi(q)$ is a real and single-valued function of $q_1, q_2, \ldots, q_n$, and $\xi(q, t)$ is n-dimensional white Gaussian noise with mean

$$\langle \xi(q, t) \rangle = 0, \tag{5}$$

and covariance

$$\langle \xi(q, t) \xi^\tau(q, t') \rangle = 2S(q) \delta(t - t'). \tag{6}$$

It should be noted that equations (3) and (6) are manifestations of the fluctuation-dissipation theorem, where $D(q)$ and $S(q)$ reflect dissipation, and the covariance structures $\langle \zeta(q, t) \zeta^\tau(q, t') \rangle$ and $\langle \xi(q, t) \xi^\tau(q, t') \rangle$ reflect fluctuation [32].

Our main problem is to prove the equivalence of equations (1) and (4). We can also prove that $\varphi(q(t))$ in equation (4) is a potential function.

3. Reduction of Problem into Partial Differential Equations (PDEs)

To show that equation (1) is equivalent to equation (4), we first show that equation (4) implies equation (1). To this end, we assume that the function matrix $[S(q) + A(q)]$ is invertible and the components q_i ($i = 1, 2, \ldots, n$) of the state variable $q = (q_1, q_2, \ldots, q_n)^\tau$ are independent. If they are not independent, the dimension can be reduced to $n\prime, n\prime < n$, until they are independent. Therefore, the equations of this system are linearly independent. Equation (4) can be straightforwardly transformed to

$$\dot{q} = -[S(q(t)) + A(q(t))]^{-1}\nabla\varphi(q(t)) + \zeta(q, t), \tag{7}$$

where $\zeta(q, t)$ is noise that takes the form $\zeta(q, t) = [S(q) + A(q)]^{-1}\xi(q, t)$. To match equation (1), we can then set $f(q) = -[S(q) + A(q)]^{-1}\nabla\varphi(q)$. Notice that with the explicit representation of $\zeta(q, t)$ in terms of $S(q), A(q)$, and $\xi(q, t)$, as well as equation (6), we can calculate

$$
\begin{aligned}
\langle \zeta(q, t) \rangle &= \langle [S(q) + A(q)]^{-1}\xi(q, t) \rangle \\
&= [S(q) + A(q)]^{-1}\langle \xi(q, t) \rangle = 0, \\
\langle \zeta(q, t) \zeta^\tau(q, t') \rangle &= \langle [S(q) + A(q)]^{-1}\xi(q, t)\xi^\tau(q, t')[S(q) + A(q)]^{-\tau} \rangle \\
&= [S(q) + A(q)]^{-1}\langle \xi(q, t)\xi^\tau(q, t') \rangle [S(q) + A(q)]^{-\tau} \\
&= 2[S(q) + A(q)]^{-1}S(q)[S(q) + A(q)]^{-\tau}\delta(t - t').
\end{aligned}
\tag{8}
$$

Comparing the above two calculations with equations (2) and (6), we see that we have

$$D(q) = [S(q) + A(q)]^{-1}S(q)[S(q) + A(q)]^{-\tau}, \tag{9}$$

which gives an explicit representation of $D(q)$.

Next, we consider the problem of whether equation (1) implies equation (4). In fact, transforming from equations (1) to (4) requires much more effort. In this case, we need to

obtain $S(q)$, $A(q)$, and $\varphi(q)$ from the general dynamic equation (1). We propose heuristic inference. While not a rigorous mathematical proof, it can lead to a reformulation of the problem into PDEs. The main idea of this heuristic inference is that equations (1) and (4) can describe the same dynamical behaviors in $\mathbb{R}^n$. Hence we may replace $\dot{q}$ in equation (4) by the right side of equation (1) to obtain

$$[S(q(t)) + A(q(t))][f(q(t)) + \zeta(q(t),t)] = -\nabla\varphi(q(t)) + \xi(q(t),t). \tag{10}$$

Regarding t as a parameter in $q(t)$, the above equation can be written as

$$[S(q) + A(q)][f(q) + \zeta(q,t)] = -\nabla\varphi(q) + \xi(q,t), \tag{11}$$

which has a deterministic part that is differentiable up to an arbitrary order, and a random part that is nondifferentiable everywhere. From the point of view of physics, the two kinds of noises $\zeta(q,t)$ and $\xi(q,t)$ have the same source. Inspired by this, we may assume that we can establish a classification,

$$[S(q) + A(q)]f(q) = -\nabla\varphi(q), \tag{12}$$

$$[S(q) + A(q)]\zeta(q,t) = \xi(q,t). \tag{13}$$

This subjective decomposition is the key to understanding Ao decomposition, which results in the consistency of stable points between a stochastic dynamical system and the corresponding dynamical system. Therefore, the generalized Lyapunov function of the stochastic dynamical system is equivalent to the Lyapunov function of the corresponding dynamical system. It must be noted that the A-type integral derived from Ao decomposition is a new integral that is different from the Itô and Stratonovich integrals. In one dimension, the A-type integral is simplified to the α-type [33], where $\alpha = 1$ (Itô corresponds to $\alpha = 0$, and Stratonovich to $\alpha = 0.5$). In the high-dimensional case, the A-type integral is not usually the α-type [34].

Combining equations (3) and (6) with (13), we obtain

$$\begin{aligned}\langle \xi(q,t)\xi^\tau(q,t')\rangle &= \langle [S(q) + A(q)]\zeta(q,t)\zeta^\tau(q,t')[S(q) + A(q)]^\tau\rangle \\ &= 2[S(q) + A(q)]D(q)[S(q) + A(q)]^\tau\delta(t-t') \\ &= 2[S(q) + A(q)]D(q)[S(q) - A(q)]\delta(t-t') = 2S(q)\delta(t-t'),\end{aligned} \tag{14}$$

which implies

$$[S(q) + A(q)]f(q) = -\nabla\varphi(q), \tag{15}$$

$$[S(q) + A(q)]D(q)[S(q) - A(q)] = S(q). \tag{16}$$

From the physical point of view, equation (16) is a generalized Einstein relation in greater than one dimension. From equation (16), we have

$$\begin{aligned}D(q) &= [S(q) + A(q)]^{-1} \cdot \frac{1}{2}\{[S(q) + A(q)] + [S(q) - A(q)]\} \cdot [S(q) - A(q)]^{-1} \\ &= \frac{1}{2}\{[S(q) - A(q)]^{-1} + [S(q) + A(q)]^{-1}\} \\ &= \frac{1}{2}\{[S(q) + A(q)]^{-\tau} + [S(q) + A(q)]^{-1}\},\end{aligned} \tag{17}$$

where the symmetric part of $[S(q) + A(q)]^{-1}$ is $(1/2)[(S(q) + A(q))^{-1} + (S(q) + A(q))^{-\tau}]$. This is the diffusion matrix $D(q)$ defined in equation (3). Hence we can rewrite the identity

$$[S(q) + A(q)][S(q) + A(q)]^{-1} = I, \tag{18}$$

as

$$[S(q) + A(q)][D(q) + Q(q)] = I, \tag{19}$$

where $Q(q)$ is an anti-symmetric unknown matrix function and I is the identity matrix. Substituting equation (19) in equation (15), we obtain

$$[D(q) + Q(q)]^{-1}f(q) = -\nabla\varphi(q). \tag{20}$$

From equation (15), it is easy to obtain that if $\dot{q} = f(q^*) = 0$, $\varphi(q^*) = 0$. Moreover, by equation (15) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(q) = \dot{q}^{\tau}\nabla\varphi(q) = -\dot{q}^{\tau}[S(q) + A(q)]\dot{q} = -\dot{q}^{\tau}S(q)\dot{q} \leq 0. \tag{21}$$

Thus $\varphi(q)$ satisfies $\dot{\varphi}(q) \leq 0$ for all $q \in \mathbb{R}^n$, and it is proven that $\varphi(q)$ is a potential function.

Assuming equation (1) holds true, equation (3) is given, and thus $D(q)$ is known. We see that to obtain equation (4), we just have to show that there exists an anti-symmetric matrix $Q(q)$ and potential function $\varphi(q)$ that satisfy equation (20). Assuming basic integrability conditions on $f(q), D(q)$, and $Q(q)$, by the classical Helmholtz decomposition we obtain that it suffices to show that the curl part of the vector field $[D(q) + Q(q)]^{-1}f(q)$ vanishes, i.e.,

$$\nabla \times \left\{ [D(q) + Q(q)]^{-1}f(q) \right\} = 0, \tag{22}$$

where equation (22) is a family of $(n(n-1)/2)$ first–order quasilinear partial differential equations for the coefficients of $Q(q)$ in equation (20).

We notice that according to the above heuristic inference, equation (22) is a *sufficient* condition for equations $(1) \Rightarrow (4)$. In fact, if equation (22) holds true, then by Helmholtz decomposition, there exists a function $\varphi = \varphi(q)$ such that equation (20) holds true, with the anti-symmetric matrix $Q(q)$ from (22). Moreover, with $D(q)$ from equation (3) and $Q(q)$ from equation (22), we can construct the matrix $S(q) + A(q) = [D(q) + Q(q)]^{-1}$, where $S(q)$ is symmetric and $A(q)$ is anti-symmetric, and $S(q)$ and $A(q)$ satisfy equations (15) and (16). Thus we can construct the noise $\xi(q, t)$ from equation (13), which, together with equation (12), implies that we can construct equation (4) from equation (1).

Our problem has now been reduced to proving the existence of solution $Q(q)$ to equation (22), which is a first-order PDE. The rest of the paper is dedicated to the investigation of this first-order PDE system in dimension $n \geq 4$.

4. Existence of Solutions to First-Order Quasilinear Partial Differential Equations in Dimension $n \geq 4$

Obviously, $q = (q_1, q_2, \ldots, q_n)$, $n \geq 4$. Assume that

$$Q(q) = \begin{pmatrix} 0 & Q_{12}(q) & \cdots & Q_{1n}(q) \\ -Q_{12}(q) & 0 & \cdots & Q_{2n}(q) \\ \vdots & \vdots & \ddots & \vdots \\ -Q_{1n}(q) & -Q_{2n}(q) & \cdots & 0 \end{pmatrix}. \tag{23}$$

From equation (3), we can assume that

$$D(q) = \begin{pmatrix} d_{11}(q) & d_{12}(q) & \cdots & d_{1n}(q) \\ d_{12}(q) & d_{22}(q) & \cdots & d_{2n}(q) \\ \vdots & \vdots & \ddots & \vdots \\ d_{1n}(q) & d_{2n}(q) & \cdots & d_{nn}(q) \end{pmatrix}. \tag{24}$$

Let vector $Q = (Q_{12}, Q_{13}, \ldots, Q_{1n}, Q_{23}, \ldots, Q_{2n}, \ldots, Q_{n-1,n})$.

By equation (22), we can assume $(D + Q)^{-1} = R$,

$$R(q) = \begin{pmatrix} R_{11}(Q) & R_{12}(Q) & \cdots & R_{1n}(Q) \\ R_{21}(Q) & R_{22}(Q) & \cdots & R_{2n}(Q) \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1}(Q) & R_{n2}(Q) & \cdots & R_{nn}(Q) \end{pmatrix}. \tag{25}$$

Therefore, according to the matrix-valued cross-product rule, equation (22) can be transformed to

$$\begin{cases} \dfrac{\partial}{\partial q_1}(f_1 R_{21} + f_2 R_{22} + \cdots + f_n R_{2n}) - \dfrac{\partial}{\partial q_2}(f_1 R_{11} + f_2 R_{12} + \cdots + f_n R_{1n}) = 0, \\ \vdots \\ \dfrac{\partial}{\partial q_{n-1}}(f_1 R_{n1} + f_2 R_{n2} + \cdots + f_n R_{nn}) - \dfrac{\partial}{\partial q_n}\left(f_1 R_{n-1,1} + \cdots + f_n R_{n-1,n}\right) = 0. \end{cases} \tag{26}$$

According to the composite function derivation rule, we obtain

$$\begin{cases} f_1^{(1)} R_{21} + f_1 R_{21}^{(1)} + \cdots - f_1^{(2)} R_{11} - f_1 R_{11}^{(2)} - \cdots - f_n^{(2)} R_{1n} - f_n R_{1n}^{(2)} = 0, \\ \vdots \\ f_1^{(n-1)} R_{n1} + f_1 R_{n1}^{(n-1)} + \cdots - f_1^{(n)} R_{n-1,1} - f_1 R_{n-1,1}^{(n)} - \cdots - f_n R_{n-1,n}^{(n)} = 0, \end{cases} \tag{27}$$

where superscripts $(1), (2), \ldots, (n)$ denote the partial derivatives corresponding to $q_1, q_2, \ldots, q_n$, respectively. This is obviously a first-order n-dimensional quasilinear system of partial differential equations consisting of $(n(n-1)/2)$ equations and $(n(n-1)/2)$ unknown functions. Our goal is to prove the existence of solutions for PDEs (27).

We first consider the matrix form of PDEs (27),

$$R_x(x, y) + A(x, y)R_y(x, y) + B(x, y)R(x, y) + C(x, y) = 0. \tag{28}$$

The independent variables are $x = (q_1, \ldots, q_i)$ and $y = (q_i, \ldots, q_n)$, where $1 < i < n$. If A has no real eigenvalues at any point in a region, PDEs (28) are elliptic in this region, and they obey the rule that the equations do not explicitly contain time. Because A is a real matrix, this may occur only when $(n(n-1)/2)$ is an even number. Obviously, when the multiplicities of eigenvalues of matrix A at each point (x, y) are constant in the whole region, then the order of every sub-block of the Jordan standard form of A is constant in the whole region, such that there exists a nonsingular matrix $T(x, y)$ satisfying

$$J = TAT^{-1} = \operatorname{diag}\{J_0, \overline{J}, \ldots, J_\varepsilon, \overline{J}_\varepsilon\}, \tag{29}$$

where

$$J_0 = \operatorname{diag}\{\lambda_1(x, y), \ldots, \lambda_p(x, y)\},$$

$$J_i = \begin{pmatrix} \lambda_{p+j}(x, y) & \cdots & \cdots & 0 \\ 1 & \lambda_{p+j}(x, y) & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & \lambda_{p+j}(x, y) \end{pmatrix}. \tag{30}$$

Then we assume the following:

(i) T and A belong to function space $C_\alpha^1(G)$;

(ii) The order of every Jordan sub-block of matrix $J = TAT^{-1}$ is constant in the whole region G;

(iii) The eigenvalue $\lambda_j(x, y) \in C_\alpha(\overline{G})$.

Let $v = TR$. Equation (28) can be transformed to

$$v_x + Jv_y + \text{zero} - \text{order term} = 0. \tag{31}$$

Then we study the solution of system

$$v_x + Jv_y = 0. \tag{32}$$

This system can be decomposed into a sub-system with the form

$$\begin{cases} \dfrac{\partial v_{j_1}}{\partial x} + J_j \dfrac{\partial v_{j_1}}{\partial y} = 0, \\ \dfrac{\partial v_{j_2}}{\partial x} + \overline{J}_j \dfrac{\partial v_{j_2}}{\partial y} = 0, \end{cases} \tag{33}$$

where $j = 0, 1, \ldots, s$, and v_{j_1}, v_{j_2} are real vector functions whose dimensions equal the order of matrix J_i. Let $\omega_j = v_{j_1} + iv_{j_2}$, where i is the imaginary unit. Then equation (33) can be written in the form

$$\frac{\partial \omega_j}{\partial x} + J_j \frac{\partial \omega_j}{\partial y} = 0, \quad j = 0, 1, \ldots, s. \tag{34}$$

Using operator notation,

$$\begin{cases} \dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right), \\ \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right). \end{cases} \tag{35}$$

When $j = 0$, the sub-system can be decomposed into p equations like

$$(1 - i\lambda)\omega_{\overline{z}} + (1 + i\lambda)\omega_z = 0, \tag{36}$$

where the value of λ is selected from $\lambda_1, \ldots, \lambda_p$. This case has been solved [35].

When $j \neq 0$, the sub-system can be transformed to

$$\left(1 - i\lambda_{p+j}\right)\omega_{j\overline{z}} + \left(1 + i\lambda_{p+j}\right)\omega_{jz} + ie\left(\omega_{jz} - \omega_{j\overline{z}}\right) = 0, \quad j = 1, 2, \ldots, s, \tag{37}$$

where

$$e = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{38}$$

Therefore, we consider the system

$$(1 - i\lambda)\omega_{\overline{z}} + (1 + i\lambda)\omega_z + ie(\omega_z - \omega_{\overline{z}}) = 0, \tag{39}$$

where e has the form of equation (38). We assume that the pair of eigenvalues $(\lambda, \overline{\lambda})$ is r-fold and the corresponding linear independent eigenvector is only one, so the complex

vector ω in PDEs (39) is r-dimensional. Obviously, without loss of generality, we can assume λ in PDEs (39) satisfies $\operatorname{Im}\lambda > 0$, and PDEs (39) are uniformly elliptic,

$$\operatorname{Im}\lambda(z) \geq \varepsilon_0 > 0, \quad z \in G. \tag{40}$$

We divide both sides of PDEs (39) by $1 - i\lambda$ to obtain

$$\left(I - \frac{i}{1 - i\lambda}e\right)\omega_{\overline{z}} + \left(\frac{1 + i\lambda}{1 - i\lambda}I + \frac{i}{1 - i\lambda}e\right)\omega_z = 0, \tag{41}$$

where I is a unit matrix of order r. Because the linear transformation $((1 + i\xi)/(1 - i\xi))$ maps the half-plane $\operatorname{Im}\xi > \varepsilon_0$ to disk $|\xi| \leq \rho < 1$, we obtain the function

$$\tilde{q}_0(z) = \frac{1 + i\lambda}{1 - i\lambda}, \tag{42}$$

which satisfies $|\tilde{q}_0(z)| \le \rho < 1$, where ρ is some positive constant.

Definition 1. The $(n(n-1)/2) \times (n(n-1)/2)$ matrix $A = (a_{jk})$ is called quasi-diagonal if

$$a_{jk} = \begin{cases} 0, & \text{when } j < k, \\ a_{j+m,k+m}, & \text{when } 1 \le k+m \le j+m \le n. \end{cases} \tag{43}$$

A quasi-diagonal matrix is a lower triangular matrix, we set

   (i) a_0 represents an element of the main diagonal;
   (ii) $a_j, 1 \le j \le n-1$, represent elements of the jth diagonal under the main diagonal.

Because the coefficient matrix of $\omega_{\bar{z}}$ and ω_z in PDEs (41) is quasi-diagonal, PDEs (41) can be written as

$$\omega_{\bar{z}} + \tilde{Q}(z)\omega_z = 0, \tag{44}$$

where $\tilde{Q}(z)$ is quasi-diagonal, and the element $\tilde{q}_0$ in main diagonal of $\tilde{Q}(z)$. The first equation of PDEs (41) is

$$\omega_{1\bar{z}} + \tilde{Q}(z)\omega_{1z} = 0, \tag{45}$$

which is a Beltrami equation. Because $\lambda \in C_\alpha(\overline{G})$, $\tilde{q}_0 \in C_\alpha(\overline{G})$, we can extend $\tilde{q}_0$ such that it belongs to $C_\alpha(\theta)$ and maintain 0 outside a large enough circle. With $|\tilde{q}_0| \le \rho < 1$, we obtain the solution $\xi(z) \in C_\alpha^1(\theta)$ of Beltrami equation (45) [36].

Under the coordinate transformation $\xi = \xi(z)$, PDEs (44) change to standard form,

$$\omega_{\bar{\xi}} + \left[\overline{\xi}_{\bar{z}}\left(I - \tilde{q}_0\tilde{Q}\right)\right]^{-1}\left[\xi_z\left(-\tilde{q}_0 I + \tilde{Q}\right)\right]\omega_\xi = 0. \tag{46}$$

If we let Q represent the coefficient matrix of ω_ξ, and still use z as the independent variable, we have

$$\omega_{\bar{z}} + Q\omega_z = 0. \tag{47}$$

A. Douglis derived the quasi-diagonal form of Q by introducing the algebra of hypercomplex numbers [36, 37].

Definition 2 (see [36]). $a = \sum_{k=0}^{r-1} a_k e^k$ is called a hypercomplex number, where e is defined by equation (38), a_k is a complex number, and a_0 is the complex number part of a, and $\sum_{k=1}^{r-1} a_k e^k$ is the nilpotent part of a. Note that $|a| = \sum_{k=0}^{r-1} |a_k|$, where a_k is the kth component of a. A hypercomplex function is a map from the plane into this algebra, and it has the form

$$\omega(x, y) = \sum_{k=0}^{r-1} \omega_k(x, y)e^k, \tag{48}$$

where each ω_k is complex-valued.

Using Definition 2, we can write PDEs (47) as

$$\omega_{0\bar{z}} + \sum_{k=1}^{r-1} e^k\left(\omega_{\bar{z}}^k + \sum_{j=0}^{k-1} q_{k-j}\omega_{jz}\right) = 0. \tag{49}$$

Let

$$\omega = \sum_{k=0}^{r-1} \omega_k e_k,$$
$$q = \sum_{k=1}^{r-1} q_k e_k. \tag{50}$$

Note the differential operator,

$$D \equiv \frac{\partial}{\partial\bar{z}} + q_m\frac{\partial}{\partial z}, \tag{51}$$

where q_m is the mth column vector of q. Using the nilpotency of e, we have

$$D\omega = \sum_{k=0}^{r-1} e_k\omega_{k\bar{z}} + \sum_{k=1}^{r-1}\sum_{j=1}^{r-1} e^{j+k}q_k\omega_{jz} = \sum_{k=0}^{r-1} e^k\omega_{k\bar{z}}$$
$$+ \sum_{k=1}^{r-1} e_k\left(\sum_{j=0}^{k-1} q_{k-j}\omega_{jz}\right). \tag{52}$$

Therefore, PDEs (49) can be written as

$$D\omega = 0. \tag{53}$$

We define a generating solution.

Definition 3. A hypercomplex function space that has bounded continuous derivatives up to order k defined in set N is represented by $B^k(N)$, and a hypercomplex function space whose k-order derivative with index α is Holder continuous in $B^k(N)$ is represented by $B_\alpha^k(N)$. A module of hypercomplex function space $B_\alpha^0(N)$ is represented by $|\cdot, N|_{0,\alpha}$,

$$|\omega, E|_{0,\alpha} = \sup_{z \in N} |\omega(z)| + \sup_{z_1 \neq z_2, z_1, z_2 \in N} \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^\alpha}. \tag{54}$$

A hypercomplex function $t(z)$ is called a generating solution of operator D if:

   (1) $t(Z)$ has the form $t(z) = z + \sum_{k=1}^{r-1} e^k t_k(z) = z + T(z)$;
   (2) $T(z) \in B^1(\theta)$;
   (3) $Dt(z) = 0, z \in \theta$.

We prove the existence of a generating solution to PDEs (32). We assume that

$$q_k \in C_\alpha^0(G), \quad k = 1, 2, \ldots, r-1, \ 0 < a < 1, \tag{55}$$

and $q_k(z)$ can be extended to θ such that they all belong to $C_\alpha^0(\theta)$ and are equal to zero outside a large enough circle. Let

$$t_0(z) = z,$$

$$t_k(z) = -\sum_{j=0}^{k-1} J_G\left(q_{k-j}\frac{\partial}{\partial z}t_j(z)\right), \quad k = 1, 2, \ldots, r-1, \quad (56)$$

where J_G is an integral operator,

$$(J_G f)(z) = -\frac{1}{\pi}\int\int_G \frac{f(\xi)}{\xi - z}d\xi d\eta. \quad (57)$$

By the property of operator J_G and the assumption on $q_k(z)$, we obtain

$$t(z) = \sum_{k=0}^{r-1} e^k t_k(z) \in B_\alpha^1(\theta). \quad (58)$$

By equation (52), we obtain

$$Dt(z) = \frac{\partial}{\partial\overline{z}}t + q\frac{\partial}{\partial z}t = \sum_{k=0}^{r-1}e^k\frac{\partial}{\partial\overline{z}}t_k + \sum_{k=1}^{r-1}e^k\sum_{j=0}^{k-1}q_{k-j}\frac{\partial}{\partial z}t_j$$

$$= \sum_{k=1}^{r-1}e^k\left(-\sum_{j=0}^{k-1}q_{k-j}\frac{\partial}{\partial z}t_j(z)\right) \quad (59)$$

$$+ \sum_{k=1}^{r-1}e^k\sum_{j=0}^{k-1}q_{k-j}\frac{\partial}{\partial z}t_j = 0.$$

Therefore, $t(z)$ is the generating solution.

$t(z)$ has the property of a positive constant M such that

$$\left|\frac{1}{t(\zeta) - t(z)}\right| \le \frac{M}{|\zeta - z|}, \quad \zeta \ne z, \quad (60)$$

where $(1/(t(\zeta) - t(z)))$ is another sign of $(t(\zeta) - t(z))^{-1}$. The inverse exists because the complex part $\zeta - z$ of $t(\zeta) - t(z)$ is not zero.

Next, we consider two corresponding boundary value problems of original nonhomogeneous PDEs (28), the nonlinear Riemann boundary value problem and nonlinear Riemann-Hilbert boundary value problem. PDEs (26) can be written in the following form in the sense of Douglis algebra:

$$D\omega = F(z, \omega), \quad (61)$$

where differential operator

$$D = \frac{\partial}{\partial\overline{z}} + q(z)\frac{\partial}{\partial z}, \quad (62)$$

where $q(z)$ is a known nilpotent hypercomplex function, and

$$F(z, \omega) = \sum_{k=0}^{r-1} F_k(z, \overline{z}, \omega_0, \ldots, \omega_{r-1}, \overline{\omega}_0, \ldots, \overline{\omega}_{r-1})e^k, \quad (63)$$

where F_k is a known complex value function of all its variables.

Assume that $I(z, \omega) = I(z, \overline{z}, \omega, \overline{\omega})$ is a hypercomplex function of independent variable z and hypercomplex variable ω. Define the Gateaux first-order differential of I about $\omega, \overline{\omega}$,

$$\left(\delta_\omega^h\right)I(z, \omega) = \lim_{\alpha\to 0}\frac{1}{\alpha}[I(z, \overline{z}, \omega + \partial h, \overline{\omega}) - I(z, \overline{z}, \omega, \overline{\omega})],$$

$$\left(\delta_{\overline{\omega}}^{\overline{h}}\right)I(z, \omega) = \lim_{\alpha\to 0}\frac{1}{\alpha}[I(z, \overline{z}, \omega, \overline{\omega} + \alpha\overline{h}) - I(z, \overline{z}, \omega, \overline{\omega})]. \quad (64)$$

We can similarly define the second-order Gateaux differential $(\delta_\omega^h)^2 I, \delta_\omega^h \delta_{\overline{\omega}}^{\overline{h}}I, (\delta_{\overline{\omega}}^{\overline{h}})^2 I$, and so on. We utilize Γ to represent a simple smooth closed contour in the complex plane Γ, whose positive direction is counterclockwise. It divides θ into bounded interior region G^+ and external unbounded region G^-. Assume that

$$M(\tau, \omega^{(1)}, \omega^2) = \sum_{k=0}^{r-1}M_k(\tau, \omega_0^{(1)}, \ldots, \omega_{r-1}^{(1)}, \omega_0^{(2)}, \ldots, \omega_{r-1}^{(2)})e^k, \quad (65)$$

is a known hypercomplex function on variable $\tau \in \Gamma$, with hypercomplex elements

$$\omega^{(1)} = \sum_{k=0}^{r-1}\omega_k^{(1)}e^k,$$

$$\omega^{(2)} = \sum_{k=0}^{r-1}\omega_k^{(2)}e^k. \quad (66)$$

Assume that $g(\tau)$ is a hypercomplex function that satisfies the Holder condition on Γ, whose complex part $g_0(\tau)$ is not zero forever on Γ. We also introduce the integer notation

$$n = \text{Ind}g(\tau) = \frac{1}{2\pi i}\int_\Gamma d\text{In}g_0(\tau). \quad (67)$$

Now we can introduce the corresponding nonlinear Riemann boundary value problem.

Definition 4 (Nonlinear Riemann boundary value problem). Assume that G^+ is a bounded and simply connected region in plane θ, whose boundary Γ is a smooth closed curve, and the positive direction of Γ causes G^+ to be located to the left. Note the complement of $G^+ + \Gamma$ as G^-, where the origin of coordinates is located in G^+. In the whole plane θ, we seek the normal block solution $\omega(z)$ to PDEs (61), such that a nonlinear boundary value in Γ that satisfies

$$\omega^+(\tau) = g(\tau)\omega^-(\tau) + M(\tau, \omega^+(\tau), \omega^-(\tau)), \quad \tau \in \Gamma, \quad (68)$$

has definite order $m - n$ at infinity, where m is an integer.

Then we consider the linear Riemann boundary value problem,

$$\begin{cases} D\omega + A\omega + B\overline{\omega} = C(z), & z \in \theta\backslash\Gamma, \\ \omega^+(\tau) = \omega^-(\tau) + \gamma(\tau), & \tau \in \Gamma, \\ \omega^-(\infty) = 0. \end{cases} \quad (69)$$

Assume that $A(z), B(z), C(z) \in L^{p,2}(\theta), p > 2$, are known hypercomplex functions. Note

$$\alpha = \frac{p-2}{p}, \tag{70}$$

and known hypercomplex function $\gamma(\tau) \in B_\alpha^0(\Gamma)$ on Γ. Then we represent the generating solution of differential operator D by $t(z)$. Before stating our main theorem and proof, we introduce four lemmas. The proofs of these lemmas can refer to Appendix.

Lemma 1. *The Cauchy-type integral*

$$\varphi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(\zeta)}{t(\zeta) - t(z)} dt(\zeta), \tag{71}$$

is a block hyperanalytic function that is equal to 0 at infinity,

$$\begin{aligned} \varphi^+(z) &\in B_\alpha^0(G^+ + \Gamma), \\ \varphi^-(z) &\in B_\alpha^0(G^- + \Gamma), \end{aligned} \tag{72}$$

and the estimate establishes

$$\begin{aligned} \left|\varphi^+, G^+ + \Gamma\right|_{0,\alpha} &\leq N^{(1)}|\gamma, \Gamma|_{0,\alpha}, \\ \left|\varphi^-, G^- + \Gamma\right|_{0,\alpha} &\leq N^{(1)}|\gamma, \Gamma|_{0,\alpha}, \end{aligned} \tag{73}$$

where $N^{(1)}$ is a positive constant related to $\alpha, q(z)$, and Γ. In addition, the boundary value condition satisfies on Γ,

$$\varphi^+(\tau) - \varphi^-(\tau) = \gamma(\tau), \quad \tau \in \Gamma. \tag{74}$$

Lemma 2. *Assume hypercomplex functions $A(z), B(z), f(z) \in L^{p,2}(\theta)$, $p > 2$. The integral operator $\widetilde{K}$ is defined by*

$$(\widetilde{K}f)(z) = \frac{1}{2\pi i} \iint_\theta \frac{t_\xi(\zeta)}{\theta i + eb(\zeta)} \frac{f(\zeta)}{t(\zeta) - t(z)} d\xi d\eta, \tag{75}$$

where f is a hypercomplex function in the whole plane θ. Hypercomplex functional $(\widetilde{K}f)(z)$ satisfies:

(i) $|(\widetilde{K}f)(z)| \leq M|f, \theta|_{p,2}$, $z \in \theta$;

(ii) $|(\widetilde{K}f)(z_1) - (\widetilde{K}f)(z_2)| \leq M|f, \theta|_{p,2}|z_1 - z_2|^\alpha$, $z_1, z_2 \in \theta$;

(iii) *For any real number $R > 1$, when $|z| \geq R$,*

$$|(\widetilde{K}f)(z)| \leq M|f, \theta|_{p,2}|z|^{-\alpha}; \tag{76}$$

(iv) *Hypercomplex functional $\omega(z) = (\widetilde{K}f)(z)$ satisfies the following system in the Sobolev sense:*

$$D\omega + A\omega + B\overline{\omega} = f(z). \tag{77}$$

In (i)–(iii), M is a positive constant only relative to q, A, B, p, and R. The positive number in (ii) and (iii) is

$$\alpha = \frac{p-2}{p}. \tag{78}$$

According to Lemma 2, operator $(\widetilde{K}f)(z)$ is zero at infinity and continuous in the whole plane θ. Then we can establish the expression and estimate of the solution of boundary value problem (69).

Lemma 3. *Boundary value problem (69) has a unique solution,*

$$\omega(z) = \varphi(z) + \widetilde{K}(-A\varphi - B\overline{\varphi} + C), \tag{79}$$

where

$$\varphi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(\zeta)}{t(\zeta) - t(z)} dt(\zeta), \tag{80}$$

and $\widetilde{K}$ is an integral operator defined by Lemma 2.

Next, we introduce two estimates of solution $\omega(z)$.

Lemma 4. *For solution $\omega(z)$ of boundary value problem (69), the following estimates hold true:*

$$\begin{aligned} \left|\omega^+, G^+ + \Gamma\right|_{0,\alpha} &\leq N\left(|\gamma, \Gamma|_{0,\alpha} + |C, \theta|_{p,2}\right), \\ \left|\omega^-, G^- + \Gamma\right|_{0,\alpha} &\leq N\left(|\gamma, \Gamma|_{0,\alpha} + |C, \theta|_{p,2}\right), \end{aligned} \tag{81}$$

where N is a positive constant only relative to $p, q(z), A(z), B(z)$, and Γ.

The same as above, if hypercomplex function $\omega(z) \in B_\alpha^0(G^+ + \Gamma) \cap B_\alpha^0(G^- + \Gamma)$, define

$$|\omega, \theta|_{0,\alpha} = |\omega^+, G^+ + \Gamma|_{0,\alpha} + |\omega^-, G^- + \Gamma|_{0,\alpha}. \tag{82}$$

For solution $\omega(z)$ of (69), from estimates (81), we can deduce that

$$|\omega, \theta|_{0,\alpha} \leq 2N\left(|\gamma, \Gamma|_{0,\alpha} + |C, \theta|_{p,2}\right). \tag{83}$$

Now, we go back to the research on seeking solutions for nonlinear Riemann boundary value problems (61) and (68). The corresponding hypercomplex function is represented by $X(z)$, which means the determined hyperanalytic function defined in the whole plane θ, and it satisfies boundary value condition $X^+(\tau) = g(\tau)X^-(\tau)$ on Γ, and has $-n$ order at infinity. Because it has complex number parts that are not zero everywhere, it has inverse $(1/X(z))$.

Making the substitution

$$\omega(z) = X(z)\widetilde{\omega}(z), \tag{84}$$

The new hypercomplex function $\widetilde{\omega}(z)$ satisfies

$$\begin{cases} D\widetilde{\omega} = \dfrac{1}{X(z)} F(z, X(z)\widetilde{\omega}(z)), & z \in \theta \backslash \Gamma, \\[2mm] \widetilde{\omega}^+(\tau) = \widetilde{\omega}^-(\tau) + \dfrac{1}{X^+(\tau)} M(\tau, X^+(\tau)\widetilde{\omega}^+(\tau)), \\[2mm] X^-(\tau)\widetilde{\omega}^-(\tau), \quad \tau \in \Gamma, \\[2mm] |\widetilde{\omega}(z)| = O(|z|^m), \quad |z| \longrightarrow \infty. \end{cases} \tag{85}$$

By the properties of $X(z)$,

$$\begin{aligned} &\frac{1}{X(z)} F(z, X(z)\widetilde{\omega}(z)), \\ &\frac{1}{X^+(\tau)} M(\tau, X^+(\tau)\widetilde{\omega}^+(\tau), X^-(\tau)\widetilde{\omega}^-(\tau)), \end{aligned} \tag{86}$$

have the same behavior as $F(z, \omega)$, $M(\tau, \omega^+(\tau))$, and $\omega^-(\tau))$. If $m \leq -1$, when $|z| \longrightarrow \infty$, $\widetilde{\omega}(z) \longrightarrow 0$. If $m \geq 0$, there must exist a hyperpolynomial whose order is within m,

$$P(z) = \sum_{j=0}^{m} a_j [t(z)]^{m-j}, \tag{87}$$

where all a_j are hypercomplex constants, such that under the transformation

$$\omega(z) = X(z)[\widetilde{\omega}(z) + P(z)]. \tag{88}$$

A new hypercomplex function $\widetilde{\omega}(z)$ satisfies a similar system and boundary value condition, and when $|z| \longrightarrow \infty$, $\widetilde{\omega}(z) \longrightarrow 0$. That only needs adding some degenerating condition on hypercomplex function $F(z, \omega)$ and its first order and second order Gateaux differential at infinity. We omit the specific condition.

Finally, we only need to consider the solution of a nonlinear boundary value problem,

$$\begin{cases} D\omega = F(z, \omega), & z \in \theta \backslash \Gamma, \\ \omega^+(\tau) = \omega^-(\tau) + M(\tau, \omega^+(\tau), \omega^-(\tau)), & \tau \in \Gamma, \\ \omega^-(\infty) = 0. \end{cases} \tag{89}$$

We assume the following:

(i) Hypercomplex function $F(z, 0) \in L^{p,2}(\theta)$, $p > 2$. For every fixed $z \in \theta$, the first- and second-order Gateaux differential of $F(z, \omega)$ related to $\omega, \overline{\omega}$ exist and are continuous. For all hypercomplex elements $\omega, \delta_\omega^h F, \delta_{\overline{\omega}}^{\overline{h}} F, (\delta_\omega^h)^2 F, \delta_\omega^h \delta_{\overline{\omega}}^{\overline{h}} F, (\delta_{\overline{\omega}}^{\overline{h}})^2 F$, their coefficients about $h, \overline{h}, h^2, h\overline{h}, \overline{h}^2$ belong to $L^{p,2}(\theta)$, $p > 2$.

(ii) Note that $\alpha = ((p-2)/p)$. Assume that for any hypercomplex $\omega^{(1)}(\tau), \omega^{(2)}(\tau) \in B_\alpha^0(\Gamma)$, hypercomplex function $M(\tau, \omega^{(1)}(\tau), \omega^{(2)}(\tau))$ as a function of τ belongs to $B_\alpha^0(\Gamma)$. There exists a positive constant Q, such that for any hypercomplex function

$$\omega^{(1)}(\tau), \omega^{(2)}(\tau), \widetilde{\omega}^{(1)}(\tau), \widetilde{\omega}^{(2)}(\tau) \in B_\alpha^0(\Gamma), \tag{90}$$

we have

$$\begin{aligned} &\left| M(\tau, \omega^{(1)}(\tau), \omega^{(2)}(\tau)) - M(\tau, \widetilde{\omega}^{(1)}(\tau), \widetilde{\omega}^{(2)}(\tau)), \Gamma \right|_{0,\alpha} \\ &\leq Q \left[\left| \omega^{(1)}(\tau) - \widetilde{\omega}^{(1)}(\tau), \Gamma \right|_{0,\alpha} + \left| \omega^{(2)}(\tau) - \widetilde{\omega}^{(2)}(\tau), \Gamma \right|_{0,\alpha} \right]. \end{aligned} \tag{91}$$

Theorem 1. *Under the above assumptions, if positive constant Q in inequality (91) satisfies*

$$2NQ < 1, \tag{92}$$

where N is the positive constant appearing on the right side of estimate equations (81) or (83), the solution of nonlinear boundary value problem (89), which can be written as $\omega(z) \in B_\alpha^0(G^+ + \Gamma) \cap B_\alpha^0(G^- + \Gamma)$, must exist, and it can be constructed by a successive approximation and continuity method.

Proof. We introduce the parameter $\lambda, 0 \leq \lambda \leq 1$, and consider the boundary value problem with parameter λ,

$$\begin{cases} D\omega = \lambda F(z, \omega), & z \in \theta \backslash \Gamma, \\ \omega^+(\tau) = \omega^-(\tau) + \lambda M(\tau, \omega^+(\tau), \omega^-(\tau)), & \tau \in \Gamma, \\ \omega^-(\infty) = 0. \end{cases} \tag{93}$$

When $\lambda = 1$, we discuss boundary value problem (89). When $\lambda = 0$, boundary value problem (93) has unique solution $\omega(z) = 0$. Assume that there exists a solution $\omega(z, \lambda_0) \in B_\alpha^0(G^+ + \Gamma) \cap B_\alpha^0(G^- + \Gamma)$ of (93) for value $\lambda_0, 0 \leq \lambda_0 < 1$. We want to prove that there exists a certain positive constant ε, independent of λ_0, such that for all $\lambda, \lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon$, (93) has solution $\omega(z, \lambda) \in B_\alpha^0(G^+ + \Gamma) \cap B_\alpha^0(G^- + \Gamma)$. Therefore, we can deduce that boundary value problem (93) has a solution when $\lambda = 1$, starting from $\lambda = 0$ and by finite steps, i.e., we prove the existence of a solution to PDEs (93). Note that hypercomplex function $\omega^{(0)}(z) = \omega(z; \lambda_0)$.

We demonstrate that when condition (92) is established, $|\omega^{(0)}, \theta|_{0,\alpha}$ is bounded. Actually, because $\omega^{(0)}$ satisfies

$$\begin{aligned} D\omega^0 &= \lambda_0 F(z, \omega^{(0)}) = \lambda_0 \left(F(z, \omega^{(0)}) - F(z, 0) + F(z, 0) \right) \\ &= \lambda_0 \left(\int_0^1 (\delta_\omega^h) F(z, s\omega^{(0)}) ds + \int_0^1 (\delta_{\overline{\omega}}^{\overline{h}}) F(z, s\omega^{(0)}) ds \right) \\ &\quad + \lambda_0 F(z, 0), \end{aligned} \tag{94}$$

where $h = \omega^{(0)}, \overline{h} = \overline{\omega^{(c)}}$, we use estimate (83) to obtain

$$\begin{aligned} \left| \omega^{(0)}, \theta \right|_{0,\alpha} &\leq 2N \left[\left| M(\tau, \omega^{(0)+}(\tau), \omega^{(0)-}(\tau)), \Gamma \right|_{0,\alpha} + |F(z, 0), \theta|_{p,2} \right] \\ &\leq 2NQ \left| \omega^{(0)}, \theta \right|_{0,\alpha} + 2N \left(|M(\tau, 0, 0), \Gamma|_{0,\alpha} + |F(z, 0), \theta|_{p,2} \right). \end{aligned} \tag{95}$$

By condition (92), we can obtain

$$\left| \omega^{(0)}, \theta \right|_{0,\alpha} \le \frac{2N}{1 - 2NQ} \left(\left| M(\tau, 0, 0), \Gamma \right|_{0,\alpha} + \left| F(z, 0), \theta \right|_{p,2} \right). \tag{96}$$

Now, we regard $\omega^{(0)}$ as a zeroth order approximation, and successively determine the sequence of hypercomplex function $\omega^{(l)}(z)$ according to the form

$$\begin{cases} D\omega^{(l)} = \lambda \left[\left(\delta_\omega^h \right) F\left(z, \omega^{(l-1)} \right) + \left(\delta_{\overline{\omega}}^{\overline{h}} \right) F\left(z, \omega^{(l-1)} \right) \right] + \lambda F\left(z, \omega^{(l-1)} \right), & z \in \theta \backslash \Gamma, \\ \omega^{(l)+}(\tau) = \omega^{(l)-}(\tau) + \lambda M\left(\tau, \omega^{(l-1)+}(\tau), \omega^{(l+1)-}(\tau) \right), & \tau \in \Gamma, \\ \omega^{(l)-}(\infty) = 0, \end{cases} \tag{97}$$

where $h = \omega^{(l)} - \omega^{(l-1)}, \overline{h} = \overline{\omega^{(l)} - \omega^{(l-1)}}, l \ge 1$. Since (97) is a linear boundary value problem about $\omega^{(l)}$, if $\omega^{(l-1)}(z) \in B_\alpha^0(G^+ + \Gamma) \cap B_\alpha^0(G^- + \Gamma)$, then by Lemmas 3 and 4, (97) has a solution $\omega^{(l)}(z)$ that belongs to $B_\alpha^0(G^+ + \Gamma) \cap B_\alpha^0(G^- + \Gamma)$. We utilize estimate (83) to easily prove that if $\left| \omega^{(l-1)}, \theta \right|_{0,\alpha}$ is bounded, $\left| \omega^{(l)}, \theta \right|_{0,\alpha}$ is also bounded. Therefore, hypercomplex function sequence $\omega^{(l)}(z)$ is uniformly bounded, relying on module $\left| \cdot, \theta \right|_{0,\alpha}$.

Now we prove the convergence of sequence $\omega^{(l-1)}(z)$. We consider difference

$$\eta^{(l-1)} = \omega^{(l)} - \omega^{(l-1)}, \quad l \ge 1, \tag{98}$$

satisfying, when $l = 1$,

$$\begin{cases} D\eta^{(0)} = (\lambda - \lambda_0) F\left(z, \omega^{(0)} \right) + \lambda \left[\left(\delta_\omega^h \right) F\left(z, \omega^{(0)} \right) + \left(\delta_{\overline{\omega}}^{\overline{h}} \right) F\left(z, \omega^{(0)} \right) \right], & z \in \theta \backslash \Gamma, \\ \eta^{(0)+}(\tau) = \eta^{(0)-}(\tau) + (\lambda - \lambda_0) M\left(\tau, \omega^{(0)+}(\tau), \omega^{(0)-}(\tau) \right), & \tau \in \Gamma, \\ \eta^{(0)-}(\infty) = 0, \end{cases} \tag{99}$$

where $h = \eta^{(0)}, \overline{h} = \overline{\eta^{(0)}}$. When $l > 1$,

$$\begin{cases} D\eta^{(l-1)} = \lambda \left[\left(\delta_\omega^h \right) F\left(z, \omega^{(l-1)} \right) + \left(\delta_{\overline{\omega}}^{\overline{h}} \right) F\left(z, \omega^{(l-1)} \right) \right] + \lambda \left[\delta_{\omega,\omega}^{h_1, h_1} f(z) + 2\delta_{\omega,\overline{\omega}}^{h_1, \overline{h_1}} f(z) + \delta_{\overline{\omega},\overline{\omega}}^{\overline{h_1}, \overline{h_1}} f(z) \right], & z \in \theta \backslash \Gamma, \\ \eta^{(l-1)+}(\tau) = \eta^{(l-1)-}(\tau) + \lambda \left[M\left(\tau, \omega^{(l-1)+}(\tau), \omega^{(l-1)-}(\tau) \right) - M\left(\tau, \omega^{(l-2)+}(\tau), \omega^{(l-2)-}(\tau) \right) \right], & \tau \in \Gamma, \\ \eta^{(l-1)-}(\infty) = 0, \end{cases} \tag{100}$$

where $h = \eta^{(l-1)}, \overline{h} = \overline{\eta^{(l-1)}}, h_1 = \eta^{(l-2)}, \overline{h_1} = \overline{\eta^{(l-2)}}$, but

$$\delta_{\omega,\overline{\omega}}^{h_1, \overline{h_1}} f(z) = \int_0^1 (1 - s)\left(\delta_\omega^{h_1} \right)^2 F\left(z, s\omega^{(l-1)}(z) + (1 - s)\omega^{(l-2)}(z) \right) ds. \tag{101}$$

Similarly, we can define $\delta_{\omega,\overline{\omega}}^{h_1, \overline{h_1}} f(z), \delta_{\overline{\omega},\overline{\omega}}^{\overline{h_1}, \overline{h_1}} f(z)$. Now we can utilize (83) and assumption (i) on $F(z, \omega)$ to estimate $\left| \eta^{(l-1)}, \theta \right|_{0,\alpha} (l \ge 1)$:

$$\begin{aligned} \left| \eta^{(0)}, \theta \right|_{0,\alpha} &\le 2(\lambda - \lambda_0) N \left(\left| F\left(z, \omega^{(0)} \right), \theta \right|_{p,2} + \left| M\left(\tau, \omega^{(0)+}(\tau), \omega^{(0)-}(\tau) \right), \Gamma \right|_{0,\alpha} \right) \\ &\le 2(\lambda - \lambda_0) N \left(Q\left| \omega^{(0)}, \theta \right|_{0,\alpha} + \left| M(\tau, 0, 0), \Gamma \right|_{0,\alpha} + N^{(4)}\left| \omega^{(0)}, \theta \right|_{0,\alpha} + \left| F(z, 0), \theta \right|_{p,2} \right) \\ &= (\lambda - \lambda_0) N^{(5)}, \end{aligned} \tag{102}$$

where $N^{(4)}, N^{(5)}$ are positive constants independent of λ, λ_0. Then, from the property of the hypercomplex function $|\eta^{(l-2)}\overline{\eta^{(l-2)}}| \le |\eta^{(l-2)}|^2$, we have the recurrent inequality

$$\left|\eta^{(l-1)}, \theta\right|_{0,\alpha} \le 2N\lambda|Q|\eta^{(l-2)}, \theta\big|_{0,\alpha} + N^{(6)}\big|\eta^{(l-2)}, \theta\big|_{0,\alpha}^2, \quad (103)$$

where $N^{(6)}$ is a positive constant independent of λ, λ_0. We can write (103) as

$$\left|\eta^{(l-1)}, \theta\right|_{0,\alpha} \le \lambda\left(2NQ + 2NN^{(6)}\big|\eta^{(l-2)}, \theta\big|_{0,\alpha}\right)\big|\eta^{(l-2)}, \theta\big|_{0,\alpha}, \quad l > 1. \tag{104}$$

Obviously, as the hypercomplex function sequence $\omega^{(l)}(z)$ converges according to module $|\cdot, \theta|_{0,\alpha}$, we only need

$$\lambda\left(2NQ + 2NN^{(6)}\big|\eta^{(0)}, \theta\big|_{0,\alpha}\right) < 1. \tag{105}$$

If inequality (105) holds, then we only need

$$2NQ + 2NN^{(6)}\big|\eta^{(0)}, \theta\big|_{0,\alpha} < 1, \tag{106}$$

or by estimate (102),

$$2NQ + 2NN^{(6)}(\lambda - \lambda_0)N^{(5)} < 1, \tag{107}$$

i.e.,

$$\lambda - \lambda_0 < \frac{1 - 2NQ}{2NN^{(5)}N^{(6)}}. \tag{108}$$

When inequality (108) holds, hypercomplex function sequence $\omega^{(l)}(z)$ has the limit $\omega(z; \lambda)$, and obviously, $\omega(z; \lambda) \in B_\alpha^0(G^+ + \Gamma) \cap B_\alpha^0(G^- + \Gamma)$. Now we prove the limit function $\omega(z; \lambda)$ obtained in this way is the solution of boundary value problem (93). Actually, it is apparent that $\omega(z; \lambda)$ satisfies the boundary value condition on Γ of (93). Because module $|\omega^{(l)}, \theta|_{0,\alpha}$ is uniformly bounded, $D\omega^{(l)}$ uniformly converges to $D\omega$ in any compact subset on $\theta\backslash\Gamma$, ω satisfies equation (93), and we utilize Lemma 3, assumption (ii) on $M(\tau, \omega^+(\tau), \omega^-(\tau))$, and the uniform boundedness of $|\omega^{(l)}, \theta|_{0,\alpha}$ to prove that when $|z| \longrightarrow \infty$, the relation

$$\left|\omega^{(l)}(z)\right| = O(|z|^{-\alpha}), \tag{109}$$

is uniformly established. Therefore, limit function $\omega(z; \lambda)$ also has the property $\omega^-(\infty, \lambda) = 0$. Here we find

$$\varepsilon = \frac{1 - 2NQ}{2NN^{(5)}N^{(6)}} > 0, \tag{110}$$

which is independent of λ_0, such that for all λ: $\lambda_0 \le \lambda \le \lambda_0 + \varepsilon$, the solution of boundary value problem (93) exists.   □

Definition 5 (Nonlinear Riemann-Hilbert boundary value problem). To seek a continuous solution $\omega(z)$ to PDEs (61) in the region $\overline{G^+}$, such that the nonlinear boundary value condition

$$\text{Re}\,\omega = \psi(z, \omega), \quad z \in \Gamma,$$
$$\frac{1}{2\pi}\int_\Gamma \text{Im}\,\omega(\zeta)|d\phi(\zeta)| = \chi(\omega), \tag{111}$$

is satisfied in boundary Γ, where $\phi(\zeta)$ is a conformal mapping from region G^+ to the unit disk, $\psi(z, \omega)$ is a known real hypercomplex function of variable $z \in \Gamma$ and hypercomplex element ω, and $\chi(\omega)$ is a known real hyperfunctional of hypercomplex function ω.

We assume:

(i) Nilpotent function $q(z) \in B_\nu^0(G^+ + \Gamma)$, $\quad 0 < \nu < 1$;

(ii) Hypercomplex function $F(z, \omega)$ and its first- and second-order Gateaux differential $\delta_\omega^h F, \delta_{\overline{\omega}}^h F, (\delta_\omega^h)^2 F, \delta_\omega^h \delta_{\overline{\omega}}^h F, (\delta_{\overline{\omega}}^h)^2 F$ about $\omega, \overline{\omega}$ are continuous for all variable elements, and bounded according to module $|\cdot, G^+ \cup \Gamma|_0$, this boundary is noted as $\widetilde{M}$ and is a positive constant;

(iii) Real-valued hypercomplex function $\psi(z, \omega(z))$, as function of $z \in \Gamma$, belongs to $B_\nu^1(\Gamma)$ for any hypercomplex element $\omega(z) \in B_\nu^1(\Gamma)$, and for any hypercomplex function $\omega^{(1)}(z), \omega^{(2)}(z)$ of $B_\nu^1(\Gamma)$, we have the equation

$$\left|\psi\big(z, \omega^{(1)}(z)\big) - \psi\big(z, \omega^{(2)}(z)\big), \Gamma\right|_{0,\nu} \le L\big|\omega^{(1)}(z) - \omega^{(2)}(z), \Gamma\big|_{0,\nu}, \tag{112}$$

where L is a positive constant independent of $\omega^{(1)}(z), \omega^{(2)}(z)$. Assume that hypercomplex function $\psi(z, 0)$ is bounded according to module $|\cdot; \Gamma|_{0,\nu}$;

(iv) Real-valued hyperfunctional $\chi(\omega) \in B_\nu^1(\Gamma)$, and any hyperfunction $\omega^{(1)}(z), \omega^{(2)}(z) \in B_\nu^1(\Gamma)$ has

$$\left|\chi\big(\omega^{(1)}(z)\big) - \chi\big(\omega^{(2)}(z)\big)\right| \le N\big|\omega^{(1)}(z) - \omega^{(2)}(z), \Gamma\big|_{0,\nu}, \tag{113}$$

where N is a positive constant independent of $\omega^{(1)}(z), \omega^{(2)}(z)$. Assume that nonnegative number $|\chi(0)|$ is finite.

Theorem 2. *When the above assumptions (i)–(iv) on nilpotent function $q(z)$, hypercomplex function $F(z, \omega)$, real-valued hypercomplex function $\psi(z, \omega)$, and real-valued hyperfunctional $\chi(\omega)$ satisfy, when the positive constant L and N are small enough in inequalities (112) and (113), solution of nonlinear Riemann–Hilbert boundary value problem (61), (111)*

$$\omega(z) \in B_\nu^1(G^+ + \Gamma), \tag{114}$$

exists.

Before the proof, we establish an estimate equation.

Lemma 5. *Assume that hypercomplex function $A(z), B(z), \tilde{g}(z) \in B_\nu^0(G + \Gamma)$, and*

$$\left|A, G^+ + \Gamma\right|_{0,\nu} \le M^{(0)},$$
$$\left|B, G^+ + \Gamma\right|_{0,\nu} \le M^{(0)}, \tag{115}$$

where $M^{(0)}$ is a positive constant. For the solution $\omega \in B_\nu^1(G^+ + \Gamma)$ of linear boundary value problem

$$\begin{cases} D\omega = A\omega + B\overline{\omega} + \widetilde{g}, & \text{in } G^+, \\[2mm] \mathrm{Re}\,\omega|_\Gamma = \widetilde{\psi}(z), & z \in \Gamma, \\[2mm] \dfrac{1}{2\pi}\displaystyle\int_\Gamma \mathrm{Im}\,\omega(\zeta)|d\phi(\zeta)| = c, \end{cases} \tag{116}$$

there exist positive constants $M^{(1)}$, $M^{(2)}$, $M^{(3)}$ which are only relevant to region G^+, positive integer r, positive constant $M^{(0)}$, and ν, such that

$$\left|\omega, G^+ + \Gamma\right|_{0,\nu} \le M^{(1)}\left|\widetilde{\psi}, \Gamma\right|_{0,\nu} + M^{(2)}|c| + M^{(3)}\left|\widetilde{g}, G^+ + \Gamma\right|_{0,\nu}, \tag{117}$$

holds, where $\widetilde{\psi}(z) \in B_\nu^0(\Gamma)$ is a real-valued hypercomplex function and c is a real hypercomplex number.

Now we are in position to give proof of Theorem 2.

Proof. We consider a family of nonlinear Riemann-Hilbert boundary value problems with real parameter λ, $0 \le \lambda \le 1$:

$$\begin{cases} D\omega = \lambda F(z, \omega), & z \in G^+, \\[2mm] \mathrm{Re}\,\omega|_\Gamma = \lambda\psi(z, \omega), & z \in \Gamma, \\[2mm] \dfrac{1}{2\pi}\displaystyle\int_\Gamma \mathrm{Im}\,\omega(\zeta)|d\phi(\zeta)| = \lambda\chi(\omega). \end{cases} \tag{118}$$

We want to prove that for every λ: $0 \le \lambda \le 1$, the solution $\omega(z; \lambda)$ of boundary value problem (118) exists, and belongs to $B_\nu^1(G^+ + \Gamma)$; then $\omega(z; 1)$ is the solution of the above Riemann-Hilbert boundary value problems (61) and (111).

Obviously, when $\lambda = 0$, boundary value problem (118) has only the zero solution $\omega(z; 0) = 0$. If it has solution $\omega(z; \lambda) \in B_\nu^1(G^+ + \Gamma)$ for $\lambda = \lambda_1$, $0 \le \lambda_1 < 1$, we prove there must exist a positive constant $\varepsilon < 1$, independent of λ_1, such that (118) has solution $\omega(z; \lambda_2) \in B_\nu^1(G^+ + \Gamma)$ for λ_2: $\lambda_1 < \lambda_2 < \lambda_1 + \varepsilon \le 1$; then we can start from the existence of a solution to (118) for $\lambda = 0$, through finite steps, to obtain $\omega(z; 1)$.

Now we assume that $\omega(z; \lambda_1) \in B_\nu^1(G^+ + \Gamma)$ is the solution of boundary value problem (118) when $\lambda = \lambda_1$, $0 \le \lambda_1 < 1$. Note that $\omega^{(0)}(z) = \omega(z; \lambda_1)$, $z \in G^+ + \Gamma$, and by the following, a sequence of linear boundary value problems ($j \ge 1$):

$$\begin{cases} D\omega^{(j)} = \lambda\left[\left(\delta_\omega^h\right)F\left(z, \omega^{(j-1)}\right) + \left(\delta_{\overline{\omega}}^{\overline{h}}\right)F\left(z, \omega^{(j-1)}\right) + F\left(z, \omega^{(j-1)}\right)\right], & z \in G^+, \\[2mm] \mathrm{Re}\,\omega^{(j)}|_\Gamma = \lambda\psi\left(z, \omega^{(j-1)}\right), & z \in \Gamma, \\[2mm] \dfrac{1}{2\pi}\displaystyle\int_\Gamma \mathrm{Im}\,\omega^{(j)}(\zeta)|d\phi(\zeta)| = \lambda\chi\left(\omega^{(j-1)}\right), & 0 < \lambda < 1. \end{cases} \tag{119}$$

We can determine $\omega^{(j)}(z; \lambda)$, one by one, where $h = \omega^{(j)} - \omega^{(j+1)}$, $\overline{h} = \overline{\omega^{(j)}} - \overline{\omega^{(j-1)}}$. Similar to the proof of the linear Riemann boundary problem, the solution of linear boundary problem (119) exists [38]. If $\omega^{(j-1)}(z; \lambda) \in B_\nu^1(G^+ + \Gamma)$, $\omega^{(j)}(z; \lambda) \in B_\nu^1(G^+ + \Gamma)$. Assuming $\omega^{(0)}(z) = \omega(z; \lambda) \in B_\nu^1(G^+ + \Gamma)$, the hypercomplex

function sequences $\omega^{(j)}(z; \lambda_1)$ successively determined by (119) all belong to $B_\nu^1(G^+ + \Gamma)$. Then we prove the convergence of hypercomplex function sequence $\omega^{(j)}(z; \lambda)$ according to module $|\cdot, G^+ + \Gamma|_{0,\nu}$. So, we consider difference $\omega^{(j)} - \omega^{(j+1)}$, which satisfies

$$\begin{cases} D\left(\omega^{(j)} - \omega^{(j-1)}\right) = \lambda\left[\left(\delta_\omega^h\right)F\left(z, \omega^{(j-1)}\right) + \left(\delta_{\overline{\omega}}^{\overline{h}}\right)F\left(z, \omega^{(j-1)}\right)\right] + \lambda\left[\delta_{\omega,\omega}^{(h_1, h_1)}f(z) + 2\delta_{\omega,\overline{\omega}}^{(h_1, \overline{h_1})}f(z) + \delta_{\overline{\omega},\overline{\omega}}^{(\overline{h_1}, \overline{h_1})}f(z)\right], & z \in G^+, \\[2mm] \mathrm{Re}\left(\omega^{(j)} - \omega^{(j-1)}\right)\big|_\Gamma = \lambda\left[\psi\left(z, \omega^{(j-1)}\right) - \psi\left(z, \omega^{(j-2)}\right)\right], & z \in \Gamma, \\[2mm] \dfrac{1}{2\pi}\displaystyle\int_\Gamma \mathrm{Im}\left(\omega^{(j)} - \omega^{(j-1)}\right)(\zeta; \lambda)|d\phi(\zeta)| = \lambda\left[\chi\left(\omega^{(j-1)}\right) - \chi\left(\omega^{(j-2)}\right)\right], & j > 1, \end{cases} \tag{120}$$

where $\delta_{\omega,\omega}^{(h_1,h_1)}$ has the same meaning as before. For boundary value problem (120), we use Lemma 5 to obtain

$$
\begin{aligned}
\left|\omega^{(j)} - \omega^{(j-1)}, G^+ + \Gamma\right|_{0,\nu} \leq{} & M^{(1)}\left|\lambda\left(\psi\left(z,\omega^{(j-1)}\right) - \psi\left(z,\omega^{(j-2)}\right)\right), \Gamma\right|_{0,\nu} \\
& + M^{(2)}\left|\lambda\left(\chi\left(\omega^{(j-1)}\right) - \chi\left(\omega^{(j-2)}\right)\right)\right| \\
& + M^{(3)}4\lambda\widetilde{M}\left|\omega^{(j-1)} - \omega^{(j-2)}, G^+ + \Gamma\right|_{0,\nu}^2.
\end{aligned}
\tag{121}
$$

We use inequality (112), (113) in assumption condition (iii), (iv), and (121), we obtain recursive estimate equation

$$
\begin{aligned}
\left|\omega^{(j)} - \omega^{(j-1)}, G^+ + \Gamma\right|_{0,\nu} \leq{} & M^{(1)}L\lambda\left|\omega^{(j-1)} - \omega^{(j-2)}, \Gamma\right|_{0,\nu} \\
& + M^{(2)}N\lambda\left|\omega^{(j-1)} - \omega^{(j-2)}, \Gamma\right|_{0,\nu} + 4\lambda M^{(3)}\widetilde{M}\left|\omega^{(j-1)} - \omega^{(j-2)}, G^+ + \Gamma\right|_{0,\nu}^2 \\
\leq{} & \lambda\left[M^{(1)}L + M^{(2)}N + 4M^{(3)}\widetilde{M}\left|\omega^{(j-1)} - \omega^{(j-2)}, G^+ + \Gamma\right|_{0,\nu}\right] \cdot \left|\omega^{(j-1)} - \omega^{(j-2)}, G^+ + \Gamma\right|_{0,\nu} \\
={} & \lambda\left(H^{(1)} + H^{(2)}\left|\omega^{(j-1)} - \omega^{(j-2)}, G^+ + \Gamma\right|_{0,\nu}\right) \times \left|\omega^{(j-1)} - \omega^{(j-2)}, G^+ + \Gamma\right|_{0,\nu},
\end{aligned}
\tag{122}
$$

where

$$
\begin{aligned}
H^{(1)} &= M^{(1)}L + M^{(2)}N, \\
H^{(2)} &= 4M^{(3)}\widetilde{M},
\end{aligned}
\tag{123}
$$

are positive constants independent of j. From inequality (122), if

$$
\lambda\left(H^{(1)} + H^{(2)}\left|\omega^{(1)} - \omega^{(0)}, G^+ + \Gamma\right|_{0,\nu}\right) < 1,
\tag{124}
$$

$\omega^{(j)}(z;\lambda)$ must converge. Assume that the limit function of $\omega^{(j)}(z;\lambda)$ is $\omega(z;\lambda)$ when $j \longrightarrow \infty$. From the structure of $\omega^{(j)}(z;\lambda)$, we can prove $\omega(z;\lambda)$ is a solution of boundary value problem (118), and $\omega(z;\lambda) \in B_\nu^1(G^+ + \Gamma)$. To establish inequality (124), it must hold true that

$$
H^{(1)} + H^{(2)}\left|\omega^{(1)} - \omega^{(0)}, G^+ + \Gamma\right|_{0,\nu} < 1.
\tag{125}
$$

Now we estimate module $|\cdot, G^+ + \Gamma|_{0,\nu}$ of the internal term $\omega^{(1)} - \omega^{(0)}$. By definition,

$$
\begin{cases}
D\left(\omega^{(1)} - \omega^{(0)}\right) = \lambda\left(\delta_\omega^h\right)F\left(z,\omega^{(0)}\right) + \lambda\left(\delta_{\overline{\omega}}^{\overline{h}}\right)F\left(z,\omega^{(0)}\right) + (\lambda - \lambda_1)F\left(z,\omega^{(0)}\right), & z \in G^+, \\[2mm]
\operatorname{Re}\left(\omega^{(1)} - \omega^{(0)}\right)\big|_\Gamma = (\lambda - \lambda_1)\psi\left(z,\omega^{(0)}\right), & z \in \Gamma, \\[2mm]
\dfrac{1}{2\pi}\displaystyle\int_\Gamma \operatorname{Im}\left(\omega^{(1)} - \omega^{(0)}\right)|d\phi(\zeta)| = (\lambda - \lambda_1)\chi\left(\omega^{(0)}\right),
\end{cases}
\tag{126}
$$

and we estimate equation (117) to obtain

$$
\begin{aligned}
\left|\omega^{(1)} - \omega^{(0)}, G^+ + \Gamma\right|_{0,\nu} \leq{} & M^{(1)}(\lambda - \lambda_1)\left|\psi\left(z,\omega^{(0)}\right), \Gamma\right|_{0,\nu} \\
& + M^{(2)}(\lambda - \lambda_1)\left|\chi\left(\omega^{(0)}\right)\right| + (\lambda - \lambda_1)M^{(3)}\left|F\left(z,\omega^{(0)}\right), G^+ + \Gamma\right|_{0,\nu} \\
\leq{} & (\lambda - \lambda_1)M^{(1)}\left[L\left|\omega^{(0)}, \Gamma\right|_{0,\nu} + |\psi(z,0), \Gamma|_{0,\nu}\right] \\
& + (\lambda - \lambda_1)M^{(2)}\left[N\left|\omega^{(0)}, \Gamma\right|_{0,\nu} + |\chi(0)|\right] \\
& + (\lambda - \lambda_1)M^{(3)}\left|F\left(z,\omega^{(0)}\right), G^+ + \Gamma\right|_{0,\nu} \leq H^{(3)}(\lambda - \lambda_1),
\end{aligned}
\tag{127}
$$

where $H^{(3)}$ is a positive constant independent of λ, λ_1. To make inequality (125) hold true, we only need

$$H^{(1)} + H^{(2)}H^{(3)}(\lambda - \lambda_1) < 1. \tag{128}$$

We assume that positive constants L, N in inequalities (112), (113) are small enough that

$$H^{(1)} = LM^{(1)} + NM^{(2)} < 1. \tag{129}$$

From (128), we obtain

$$\lambda - \lambda_1 < \frac{1 - H^{(1)}}{H^{(2)}H^{(3)}}. \tag{130}$$

In a word, we obtain from inequality (130) that we can choose any positive number ε that is less than $((1 - H^{(1)})/H^{(2)}H^{(3)})$ and independent of λ_1, such that for arbitrary λ,

$$\lambda_1 < \lambda \le \lambda_1 + \varepsilon \le 1. \tag{131}$$

The solution $\omega(z; \lambda) \in B_y^1(G^+ + \Gamma)$ of nonlinear boundary value problem (118) exists by the successive approximation method. Then we start from $\lambda = 0$ and, after a finite step, we can obtain

$$\omega(z; 1). \tag{132}$$

Therefore, we have proved the existence of a solution to the initial nonlinear Riemann-Hilbert boundary value problem (61), (111) [39, 40]. □

5. Conclusion

We revealed a fundamental structure of general stochastic dynamical systems proposed by Ao et al.. We demonstrated a scientific understanding of three essential components: the potential function, friction matrix $S(q)$, and Lorenz matrix $A(q)$. Our goal was to prove the equivalence between general stochastic differential equations and equations after A-type decomposition, and then we could assert that the above elements are fundamental components of general stochastic dynamical systems. This problem can be transformed to proof of the existence of solutions for first-order quasilinear partial differential equations. Then we mathematically proved the existence of solutions for these equations. Specifically, when dimension n satisfies $n \ge 4$, and $(n(n-1)/2)$ is an even number $(n = 4, 5, 8, 9, \ldots, 4i, 4i+1, \ldots,$ when $i = 1, 2, 3, \ldots)$, the existence of the generated solutions of the homogeneous equations corresponding to these equations was proved by introducing the hypercomplex algebra proposed by Douglis. Then, by successive approximation and continuous methods, we proved the existence of solutions of the Riemann boundary value problem and Riemann-Hilbert boundary value problem corresponding to first-order quasilinear partial differential equations. Therefore, we proved this fundamental structure of general stochastic dynamical systems for the high-dimensional case.

Appendix

We begin with some definitions and properties of hypercomplex functions, to enable readers to better understand hypercomplex calculation. As mentioned before, hypercomplex numbers and functions are defined in Definition 2, and the hypercomplex function space and corresponding generated solutions are defined in Definition 3.

We briefly discuss norms of hypercomplex numbers in our algebra. For a, as given by Definition 2, Douglis [36] defined the norm

$$|a| = \sum_{k=0}^{r-1} |a_k|. \tag{A.1}$$

The following holds for any hypercomplex numbers a and d:

$$\begin{aligned} |ad| &\le |a||d|, \\ |a + d| &\le |a| + |d|. \end{aligned} \tag{A.2}$$

Furthermore, writing a as $a = a_0 + E$, where E is the nilpotent part of a, the inverse of a is

$$\frac{1}{c} = \frac{1}{a_0}\left[1 - \frac{E}{a_0} + \frac{E^2}{a_0^2} - \cdots + (-1)^{r-1}\frac{E^{r-1}}{a_0^{r-1}}\right], \tag{A.3}$$

where $a_0 \ne 0$. Therefore, we also have the inequality

$$\left|\frac{1}{c}\right| \le \frac{1}{|c_0|}\sum_{k=0}^{r-1}\left(\frac{|E|}{|c_0|}\right)^k. \tag{A.4}$$

Moreover, we have $1 = |c \cdot (1/c)| \le |c||1/c|$, and thus

$$\frac{1}{|c|} \le \left|\frac{1}{c}\right|. \tag{A.5}$$

Second, we list some inequalities concerning the generated solution. We denote generic constants by $M(\cdot)$. We have

$$\left|t_x(z)\right|, \left|t_y(z)\right| \le M(e, f),$$

$$\left|\frac{1}{i + eb}\right| \le M(f),$$

$$\left|\frac{1}{t(\zeta) - t(z)}\right| \le \frac{M(e, f)}{|\zeta - z|}, \quad z \ne \zeta, \tag{A.6}$$

$$\left|\frac{t_x(z)}{i + eb(z)}\right| \le M(e, f),$$

$$\left|\frac{t_x(z)}{i + eb(z)} \cdot \frac{1}{t(\zeta) - t(z)}\right| \le \frac{M(e, f)}{|\zeta - z|}, \quad z \ne \zeta.$$

In each case above, $M(e, f)$ is a constant depending on the bounds on f and the derivatives of t.

Third, we state some properties of hypercomplex functions.

Definition A.1. A domain ϑ is regular if it is bounded and its boundary Γ consists of a finite number of simple closed curves with piecewise-continuous tangents.

Theorem A.1 (Green's identity). *If ϑ is a regular domain (bounded, with boundary Γ consisting of a finite number of simple closed curves with piecewise continuous tangents), and ω and ν are hypercomplex functions in $C^1(\overline{\vartheta})$, then*

$$\iint_{\vartheta} \frac{t_x}{i + eb} [\omega(D\nu) + (D\omega)\nu] \mathrm{d}x\mathrm{d}y = -\int_{\Gamma} \omega\nu\mathrm{d}t(z). \quad (A.7)$$

Theorem A.2 (Plemelj-Privalov theorem [41]). *If Γ is a circle, then the singular integral operator (also called a Hilbert transform) given by the Cauchy principal value integral,*

$$f(t) \longrightarrow \frac{1}{\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - t} \mathrm{d}\zeta, \quad t \in \Gamma, \quad (A.8)$$

behaves invariantly with respect to the class of Hölder continuous functions, denoted by $C^{0,\alpha}(\Gamma)$ for $0 < \alpha < 1$.

Furthermore, Sokhotski proposed the Sokhotski-Plemelj formula [42], and Plemelj proved it [41].

Theorem A.3 (Sokhotski-Plemelj formula). *Let L represent an arbitrary smooth curve, and $F(z)$ a Cauchy-type integral of a real function f,*

$$F(z) = \frac{1}{2\pi i} \int_L \frac{f(t)}{t - z} \mathrm{d}t. \quad (A.9)$$

Let $t_0 = \xi_0 + i\eta_0$ represent an arbitrary fixed point on L excluding the endpoint when $f(t_0) \neq 0$. If $f(t)$ satisfies the following condition on neighborhood $\varepsilon > 0$ of every point on L: for two arbitrary point t_1 and t_2 on these neighborhoods, there exists a constant A satisfies

$$|f(t_1) - f(t_2)| \leq A|t_1 - t_2|^{\lambda}, \quad 0 < \lambda \leq 1, \quad (A.10)$$

limit of $f(z)$ exists. When z approaches point t_0 from the left side of L, this limit is named $F^+(t_0)$, and conversely, when z approaches point t_0 from the right side of L, this limit is named $F^-(t_0)$. They satisfy

$$F^+(t_0) = \frac{1}{2} f(t_0) + \frac{1}{2\pi i} \int_L' \frac{f(t)}{t - t_0} \mathrm{d}t,$$

$$F^-(t_0) = -\frac{1}{2} f(t_0) + \frac{1}{2\pi i} \int_L' \frac{f(t)}{t - t_0} \mathrm{d}t, \quad (A.11)$$

where the symbol $'$ over the integral denotes the Cauchy principal value integral.

Then we give proofs of Lemmas 1–4.

Proof of Lemma 1

Proof. By the Sokhotski-Plemelj formula (see Theorem A.3), the Cauchy-type integral $\varphi(z)$ is a block hyperanalytic

function and satisfies boundary value condition (74). Next, we discuss the property by which boundary value

$$\varphi^+(\tau), \varphi^-(\tau) \in B_{\alpha}^0(\Gamma). \quad (A.12)$$

This is a natural generalization of the Plemelj-Privalov theorem (Theorem A.2 in Appendix) for a usual analytic function. Finally, we start to prove estimate equation (73). Assume that τ_0 is an arbitrary point on Γ, $z \in G^+$, and consider hypercomplex function

$$\xi(z) = \frac{\varphi(z) - \varphi^+(\tau_0)}{(t(z) - t(\tau_0))^{\mu}}, \quad 0 \leq \mu < \alpha, z \in G^+. \quad (A.13)$$

On any single-valued branch in G^+, by the Plemelj-Privalov theorem and the properties of $t(z)$, it is easy to know that the boundary value of this function,

$$\xi^+(\tau) = \frac{\varphi^+(\tau) - \varphi^+(\tau_0)}{(t(\tau) - t(\tau_0))^{\mu}}, \quad \tau \in \Gamma, \quad (A.14)$$

satisfies the Hölder condition on Γ. Similar to an analytic function, we can prove in the whole region G^+,

$$\xi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\xi^+(\zeta)}{t(\zeta) - t(z)} \mathrm{d}t(\zeta). \quad (A.15)$$

Therefore, hypercomplex function $\xi(z)$ is continuous on $G^+ + \Gamma$. The following estimate

$$\left|\frac{\varphi(z) - \varphi^+(\tau_0)}{(t(z) - t(\tau_0))^{\mu}}\right| = |\xi(z)| \leq \max_{\tau \in \Gamma} |\xi^+(\tau)| \leq N^{(4)}, \quad (A.16)$$

holds, in which $N^{(4)}$ is a positive constant. This function is independent of the position of point τ_0 on Γ and number $\mu < \alpha$. Therefore, we obtain

$$|\varphi(z) - \varphi^+(\tau_0)| \leq N^{(5)}|z - \tau_0|^{\alpha}, \quad (A.17)$$

where $N^{(5)}$ is a positive constant, which depends linearly on the Hölder coefficients of the hypercomplex function $\gamma(\tau)$. For point z in region G^-, we can similarly prove the above inequalities. Therefore, we obtain estimate equation (73).

Proof of Lemma 2   □

Proof. We use $X^{(1)}(z, \zeta), X^{(2)}(z, \zeta)$ to represent the basic set of solutions for equations $D\omega = 0$,

$$\Omega^{(1)}(z, \zeta) = X^{(1)}(z, \zeta) + iX^{(2)}(z, \zeta),$$
$$\Omega^{(2)}(z, \zeta) = X^{(1)}(z, \zeta) - iX^{(2)}(z, \zeta). \quad (A.18)$$

Utilizing the properties of hypercomplex function $X^{(j)}(z, \zeta)(j = 1, 2)$:

(1) In $\theta - \zeta$, complex variable z satisfies $DX^j(z, \zeta) + A(z)X^j(z, \zeta) + B(z)\overline{X^j(z, \zeta)} = 0$, $j = 1, 2$;

(2) $X^{(1)}(z, \zeta) = (e^{\omega^{(1)}(z) - \omega^{(1)}(\zeta)}/(2(t(\zeta) - t(z))))$, $X^{(2)}(z, \zeta) = (e^{\omega^{(2)}(z) - \omega^{(2)}(\zeta)}/(2i(t(\zeta) - t(z))))$, where

$$\omega^{(j)}(z) \in B_{\alpha}^0(\theta), j = 1, 2, \alpha = \frac{p - 2}{p}, \quad (A.19)$$

and when $|z| \longrightarrow \infty$, $\omega^{(j)}(z) = O(|z|^{-\alpha})$.

Thus we have

$$\left|\Omega^{(j)}(z,\zeta)\right| \leq \frac{M^{(1)}}{|\zeta - z|}, \quad z \neq \zeta, \ z, \zeta \in \theta (j = 1, 2), \qquad \text{(A.20)}$$

where $M^{(1)}$ is a positive constant related to p, q, A, B. Note that $\omega(z) = (\widetilde{K} f)(z) = \omega^{(1)}(z) + \omega^{(2)}(z)$, where

$$\omega^{(1)}(z) = -\frac{1}{\pi} \int\int_{|\zeta|\leq 1} \left[t_\zeta(\zeta) \omega^{(1)}(z,\zeta) f(\zeta) + \overline{t_\zeta(\zeta)} \Omega^{(2)}(z,\zeta) \overline{f(\zeta)} \right] d\xi d\eta,$$

$$\omega^{(2)}(z) = -\frac{1}{\pi} \int\int_{|\zeta|\geq 1} \left[t_\zeta(\zeta) \omega^{(1)}(z,\zeta) f(\zeta) + \overline{t_\zeta(\zeta)} \Omega^{(2)}(z,\zeta) \overline{f(\zeta)} \right] d\xi d\eta. \qquad \text{(A.21)}$$

Then, by (99), and because $|t_\zeta(z)|$ is bounded, we have

$$\left|\omega^{(1)}(z)\right| \leq M^{(2)} I^{(1)}(z), \quad z \in \theta,$$

$$\left|\omega^{(2)}(z)\right| \leq M^{(2)} I^{(2)}(z), \quad z \in \theta, \qquad \text{(A.22)}$$

where $M^{(2)}$ is a positive constant, and

$$I^{(1)}(z) = \int\int_{|\zeta|\leq 1} \frac{|f(\zeta)|}{|\zeta - z|} d\xi d\eta,$$

$$I^{(2)}(z) = \int\int_{|\zeta|\geq 1} \frac{|f(\zeta)|}{|\zeta - z|} d\xi d\eta. \qquad \text{(A.23)}$$

Note that $(1/p) + (1/q) = 1$, because $p > 2$, $0 < q < 2$,

$$I^{(1)}(z) \leq |f, |z| \leq 1|_p \left(\int\int_{|\zeta|\leq 1} |\zeta - z|^{-q} d\xi d\eta \right)^{1/q} \leq \left(\frac{2\pi}{\alpha q} \right)^{1/q} 2^\alpha |f, |z| \leq 1|_p, \quad \alpha = \frac{p-2}{p},$$

$$I^{(2)}(z) = \int\int_{|\zeta|\leq 1} \frac{|f(1/\zeta)|}{|(1/\zeta) - z||\zeta|^4} d\xi d\eta \leq \left||z|^{-2} f\left(\frac{1}{z}\right), |z| \leq 1 \right|_p \left(\int\int_{|\zeta|\leq 1} |\zeta|^{-q} |1 - \zeta z|^{-q} d\xi d\eta \right)^{1/q}. \qquad \text{(A.24)}$$

If $|z| \leq (1/3)$, because $|\zeta| \leq 1$,

$$|1 - z\zeta| \geq 1 - |z\zeta| \geq 1 - |z| \geq \frac{2}{3}. \qquad \text{(A.25)}$$

Therefore,

$$\int\int_{|\zeta|\leq 1} |\zeta|^{-q} |1 - \zeta z|^{-q} d\xi d\eta \leq \left(\frac{3}{2} \right)^q \int\int_{|\zeta|\leq 1} |\zeta|^{-q} d\xi d\eta \leq M^{(3)}, \qquad \text{(A.26)}$$

where $M^{(3)}$ is a positive constant only related to p. If $|z| \geq (1/3)$,

$$\iint_{|\zeta|\leq 1}|\zeta|^{-q}|1-\zeta z|^{-q}\mathrm{d}\xi\mathrm{d}\eta$$

$$=|z|^{-q}\iint_{|\zeta|\leq 1}|\zeta|^{-q}\left|\zeta-\frac{1}{z}\right|^{-q}\mathrm{d}\xi\mathrm{d}\eta \qquad (A.27)$$

$$\leq|z|^{-q}M^{(4)}\left|\frac{1}{z}\right|^{2-2q}\leq 3^{2-q}M^{(4)},$$

where $M^{(4)}$ is a positive constant only related to p. Then, by Hadamard estimation, we obtain

$$I^{(2)}(z)\leq M^{(5)}\left||z|^{-2}f\left(\frac{1}{z}\right),|z|\leq 1\right|_{p}, \qquad (A.28)$$

where M is a positive constant only related to p, q, A, B.

Next, assume that G is a regular domain, and ψ is a hyperanalytic function outside $\overline{G}$ that holds continuous on boundary ∂G and equals 0 at infinity, so

$$\frac{1}{2\pi i}\int_{\partial G}\frac{\psi(\zeta)}{t(\zeta)-t(z)}\mathrm{d}t(\zeta)=\begin{cases}0, & z\in G,\\[2mm]-\psi(z), & z\notin\overline{G}.\end{cases} \qquad (A.29)$$

Furthermore, assume that $A, B\in L^{p,2}(\theta)$, $2<p<\infty$, and $A\equiv B\equiv 0$ outside $\overline{G}$. If $\Omega^{(1)}$ and $\Omega^{(2)}$ are basic kernels concerning A and B, then

$$\frac{1}{2\pi i}\int_{\partial G}\Omega^{(1)}(z,\zeta)\psi(\zeta)\mathrm{d}t(\zeta)-\Omega^{(2)}(z,\zeta)\overline{\psi(\zeta)\mathrm{d}t(\zeta)}=\begin{cases}0, & z\in G,\\[2mm]-\psi(z), & z\notin\overline{G}.\end{cases} \qquad (A.30)$$

Thus we have

$$\Omega^{(1)}(z,\zeta)=\frac{\Omega_*^{(1)}(z,\zeta)}{2[t(\zeta)-t(z)]},$$
$$\qquad (A.31)$$
$$\Omega^{(2)}(z,\zeta)=\frac{\Omega_*^{(2)}(z,\zeta)}{2[t(\zeta)-t(z)]},$$

where $\Omega_*^{(j)}(z,\zeta)\,(j=1,2)$ satisfies

$$\left|\Omega_*^{(j)}(z,\zeta)\right|\leq M^{(6)}, \quad z,\zeta\in\theta,$$
$$\qquad (A.32)$$
$$\left|\Omega_*^{(j)}(z_1,\zeta)-\Omega_*^{(j)}(z_2,\zeta)\right|\leq M^{(6)}|z_1-z_2|^{\alpha}, \quad z_1,z_2,\zeta\in\theta,$$

where $M^{(6)}$ is a positive constant only related to p, q, A, B,

$$\alpha=\frac{p-2}{p}. \qquad (A.33)$$

Therefore,

$$\Omega^{(1)}(z_1,\zeta)-\Omega^{(1)}(z_2,\zeta)=\frac{\Omega_*^{(1)}(z_1,\zeta)}{2[t(\zeta)-t(z_1)]}-\frac{\Omega_*^{(1)}(z_2,\zeta)}{2[t(\zeta)-t(z_2)]}$$

$$=\frac{1}{2[t(\zeta)-t(z_1)][t(\zeta)-t(z_2)]}\left\{[t(\zeta)-t(z_2)]\Omega_*^{(1)}(z_1,\zeta)-[t(\zeta)-t(z_2)]\Omega_*^{(1)}(z_2,\zeta)\right. \qquad (A.34)$$

$$\left.+[t(\zeta)-t(z_2)]\Omega_*^{(1)}(z_2,\zeta)-[t(\zeta)-t(z_1)]\Omega_*^{(1)}(z_2,\zeta)\right\}.$$

Utilizing property (60) of generating solution $t(z)$ and inequality (A.32), we can obtain the estimate

$$\left|\Omega^{(2)}\left(z_1, \zeta\right) - \Omega^{(2)}\left(z_2, \zeta\right)\right| \le M^{(7)}\left|\zeta - z_1\right|^{-1}\left|z_1 - z_2\right|^{\alpha} + \left|\zeta - z_1\right|^{-1}\left|\zeta - z_2\right|^{-1}\left|z_1 - z_2\right|, \tag{A.35}$$

where $M^{(7)}$ is a positive constant only related to p, q, A, B. There is a similar estimate for $\Omega^{(2)}\left(z_1, \zeta\right) - \Omega^{(2)}\left(z_2, \zeta\right)$. Then we calculate

$$\omega\left(z_1\right) - \omega\left(z_2\right) = \omega^{(1)}\left(z_1\right) - \omega^{(1)}\left(z_2\right) + \omega^{(2)}\left(z_1\right) - \omega^2\left(z_2\right), \tag{A.36}$$

where

$$\left|\omega^{(1)}\left(z_1\right) - \omega^{(1)}\left(z_2\right)\right| = -\frac{1}{\pi} \int\int_{|\zeta|\le 1}\left\{t_\zeta(\zeta)\left[\Omega^{(1)}\left(z_1, \zeta\right) - \Omega^{(1)}\left(z_2, \zeta\right)\right]f(\zeta) + \overline{t_\zeta(\zeta)}\left[\Omega^{(2)}\left(z_1, \zeta\right) - \Omega^{(2)}\left(z_2, \zeta\right)\right]\overline{f(\zeta)}\right\}d\xi d\eta. \tag{A.37}$$

By (A.34) and (A.35), we obtain

$$\left|\omega^{(1)}\left(z_1\right) - \omega^{(1)}\left(z_2\right)\right| \le M^{(8)}\left[\left|z_1 - z_2\right|^{\alpha}I^{(1)}\left(z_1\right) + \left|z_1 - z_2\right|I^{(3)}\left(z_1, z_2\right)\right], \tag{A.38}$$

where $M^{(8)}$ is a positive constant,

$$I^{(1)}\left(z_1\right) = \int\int_{|\zeta|\le 1}\frac{|f(\zeta)|}{|\zeta - z_1|}d\xi d\eta \le \left(\frac{2\pi}{\alpha q}\right)^{1/q}2^{\alpha}|f, |z| \le 1|_p,$$

$$I^{(3)}\left(z_1, z_2\right) = \int\int_{|\zeta|\le 1}\frac{|f(\zeta)|}{|\zeta - z_1||\zeta - z_2|}d\xi d\eta$$

$$\le |f, |z| \le 1|_p\left(\int\int_{|\zeta|\le 1}\left|\zeta - z_1\right|^{-q}\left|\zeta - z_2\right|^{-q}d\xi d\eta\right)^{1/q} \tag{A.39}$$

$$\le M^{(9)}|f, |z| \le 1|_p\left|z_1 - z_2\right|^{(2-2q)/q},$$

where $M^{(9)}$ is a positive constant. We use Hadamard estimation again to obtain

$$\left|\omega^{(1)}\left(z_1\right) - \omega^{(1)}\left(z_2\right)\right| \le M^{(10)}|f, |z| \le 1|_p$$
$$\times\left[\left|z_1 - z_2\right|^{\alpha} + \left|z_1 - z_2\right|^{1+((2-2q)/q)}\right] \tag{A.40}$$
$$= 2M^{(10)}|f, |z| \le 1|_p\left|z_1 - z_2\right|^{\alpha},$$

where $M^{(10)}$ is a positive constant. Similarly, we obtain

$$\left|\omega^{(2)}(z_1) - \omega^{(2)}(z_2)\right| \le M^{(8)}\left[\left|z_1 - z_2\right|^\alpha I^{(2)}(z_1) + \left|z_1 - z_2\right| I^{(4)}(z_1, z_2)\right], \tag{A.41}$$

where $I^{(2)}(z_1)$ is estimated in (A.28), and

$$
\begin{aligned}
I^{(4)}(z_1, z_2) &= \iint_{|\zeta| \le 1} \frac{|f(\zeta)|}{|\zeta - z_1||\zeta - z_2|}\, d\xi d\eta \\
&= \iint_{|\zeta| \le 1} \frac{|f(1/\zeta)| d\xi d\eta}{|(1/\zeta) - z_1||(1/\zeta) - z_2||\zeta|^4} \\
&\le M^{(11)}\left|\,|z|^{-2} f\left(\frac{1}{z}\right), |z| \le 1\right|_p |z_1 - z_2|^{\alpha-1},
\end{aligned}
\tag{A.42}
$$

where $M^{(11)}$ is a positive constant. Therefore,

$$\left|\omega^{(2)}(z_1) - \omega^{(2)}(z_2)\right| \le M^{(12)}|z_1 - z_2|^\alpha \left|\,|z|^{-2} f\left(\frac{1}{z}\right), |z| \le 1\right|_p, \tag{A.43}$$

where $M^{(12)}$ is a positive constant. Then, by inequalities (A.40) and (A.43),

$$
\begin{aligned}
\left|(\tilde{K}f)(z_1) - (\tilde{K}f)(z_2)\right| &\le \left|\omega^{(1)}(z_1) - \omega^{(1)}(z_2)\right| \\
&+ \left|\omega^{(2)}(z_1) - \omega^{(2)}(z_2)\right| \le M|z_1 - z_2|^\alpha |f, \theta|_{p,2},
\end{aligned}
\tag{A.44}
$$

where M is a positive constant related to p, q, A, B.

For any $R > 1$, when $|z| \le R$, we utilize estimate equation (A.20) and the boundedness of $|t_z(z)|$ to obtain

$$
\begin{aligned}
\left|\omega^{(1)}(z)\right| &\le M^{(13)} \iint_{|\zeta| \le 1} \frac{|f(\zeta)|}{|\zeta - z|}\, d\xi d\eta \\
&\le \frac{M^{(13)}}{|z| - 1} \iint_{|\zeta| \le 1} |f(\zeta)|\, d\xi d\eta \le \frac{M^{(14)}}{|z| - 1} |f, |z| \le 1|_p,
\end{aligned}
\tag{A.45}
$$

where $M^{(13)}, M^{(14)}$ are positive constants. However,

$$\frac{1}{|z| - 1} = \frac{1}{|z|}\left(1 + \frac{1}{|z| - 1}\right) \le \left(1 + \frac{1}{R - 1}\right)\frac{R^{\alpha-1}}{|z|^\alpha} = \frac{R^\alpha}{R - 1}|z|^{-\alpha}. \tag{A.46}$$

Therefore,

$$\left|\omega^{(1)}(z)\right| \le M^{(15)}|f, |z| \le 1|_p|z|^{-\alpha}, \quad (A.47)$$

where $M^{(15)}$ is a positive constant. Similar to $\omega^{(1)}(z)$, we estimate $\omega^{(2)}(z)$ by Hadamard estimation,

$$
\begin{aligned}
\left|\omega^{(1)}(z)\right| &\le M^{(13)} \int\int_{(\zeta)\le 1} \frac{|f(\zeta)|}{|\zeta - z|} \mathrm{d}\xi \mathrm{d}\eta \\
&= M^{(13)} \int\int_{(\zeta)\le 1} \frac{|f(1/\zeta)|}{|(1/\zeta) - z||\zeta|^4} \mathrm{d}\xi \mathrm{d}\eta \\
&\le M^{(13)} \left||z|^{-2} f\left(\frac{1}{z}\right), |z| \le 1\right|_p \times \left(\int\int_{(\zeta)\le 1} |\zeta|^{-q}|1 - \zeta z|^{-q} \mathrm{d}\xi \mathrm{d}\eta\right)^{1/q} \\
&\le M^{(16)} \left||z|^{-2} f\left(\frac{1}{z}\right), |z| \le 1\right|_p |z|^{-\alpha},
\end{aligned}
\quad (A.48)
$$

where $M^{(16)}$ is a positive constant. When $|z| \ge R > 1$,

$$|(\widetilde{K}f)(z)| \le M|f, |z| \le 1|_{p,2}|z|^{-\alpha}, \quad (A.49)$$

where M is a positive constant related to p, q, A, B, R.

We then want to prove that for all support hypercomplex functions $\phi(z)$, it holds that

$$\mathrm{Re} \int\int_{\mathrm{supp}\phi} t_z(z)[\omega(z)\widetilde{D}\phi(z) + f(z)\phi(z)]\mathrm{d}x\mathrm{d}y = 0, \quad (A.50)$$

i.e.,

$$
\begin{aligned}
&\int\int_{\mathrm{supp}\phi} t_z(z)[\omega(z)\widetilde{D}\phi(z) + f(z)\phi(z)]\mathrm{d}x\mathrm{d}y \\
&+ \int\int_{\mathrm{supp}\phi} \overline{t_z(z)}[\overline{\omega(z)\widetilde{D}\phi(z)} + \overline{f(z)\phi(z)}]\mathrm{d}x\mathrm{d}y = 0,
\end{aligned}
\quad (A.51)
$$

where $\widetilde{D}$ is the correlation operator of D. Assume that hypercomplex functions $A(z), B(z) \in L^{p,2}(\theta)$, $f(z) \in L^p(G)$, $p > 2$, where G is a bounded simply connected region on plane θ. Let $\omega(z)$ be a continuous solution in $\overline{G}$ for

$$\mathfrak{D}\omega \equiv D\omega + A\omega + B\overline{\omega} = F. \quad (A.52)$$

Then

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial G} \Omega^{(1)}(z,\zeta)\omega(\zeta)\mathrm{d}t(\zeta) - \Omega^{(2)}(z,\zeta)\overline{\omega(\zeta)\mathrm{d}t(\zeta)} \\
&\quad - \frac{1}{\pi}\int\int_G \left[t_\zeta(\zeta)\Omega^{(1)}(z,\zeta)F(\zeta) + \overline{t_\zeta(\zeta)}\Omega^{(2)}(z,\zeta)\overline{F(\zeta)}\right]\mathrm{d}\xi\mathrm{d}\eta \\
&= \begin{cases} \omega(z), & z \in G, \\ 0, & z \notin \overline{G}, \end{cases}
\end{aligned}
\quad (A.53)
$$

where hyper-complex functions $\Omega^{(1)}(z,\zeta)$, $\Omega^{(2)}(z,\zeta)$ are basic kernels associated with $A(z), B(z), \zeta = \xi + i\eta$. Thus, for support hyper-complex function $\phi(z)$, we have

$$\iint_{\text{supp}\phi} t_z(z)\phi(z)f(z)dxdy = \iint_{\text{supp}\phi} t_z(z)f(z)\left\{\frac{1}{\pi}\iint_{\text{supp}\phi}\left[t_\zeta(\zeta)\Omega^{(1)}(\zeta,z)\tilde{D}\phi(\zeta) + \overline{t_\zeta(\zeta)}\overline{\Omega^{(2)}(\zeta,z)}\overline{\tilde{D}\phi(\zeta)}\right]d\xi d\eta\right\}dxdy$$

$$= \iint_{\text{supp}\phi} t_\zeta(\zeta)\left[\frac{1}{\pi}\iint_{\text{supp}\phi} t_z(z)\Omega^{(1)}(\zeta,z)f(z)dxdy\right]\tilde{D}\phi(\zeta)d\xi d\eta$$

$$+ \iint_{\text{supp}\phi}\overline{t_\zeta(\zeta)}\left[\frac{1}{\pi}\iint_{\text{supp}\phi} t_z(z)\overline{\Omega^{(2)}(\zeta,z)}f(z)dxdy\right]\overline{\tilde{D}\phi(\zeta)}d\xi d\eta.$$

$$(A.54)$$

The validity of the exchange of integration order is not difficult to verify, so we omit it here. Similarly,

$$\iint_{\text{supp}\phi} t_z(z)\phi(z)f(z)dxdy$$

$$= \iint_{\text{supp}\phi}\overline{t_\zeta(\zeta)}\left[\frac{1}{\pi}\iint_{\text{supp}\phi}\overline{t_z(z)}\overline{\Omega^{(1)}(\zeta,z)}\overline{f(z)}dxdy\right]\overline{\tilde{D}\phi(\zeta)}d\xi d\eta \qquad (A.55)$$

$$+ \iint_{\text{supp}\phi} t_\zeta(\zeta)\left[\frac{1}{\pi}\iint_{\text{supp}\phi}\overline{t_z(z)}\Omega^{(2)}(\zeta,z)\overline{f(z)}dxdy\right]\tilde{D}\phi(\zeta)d\xi d\eta.$$

After that, we obtain

$$\iint_{\text{supp}\phi}\left[t_z(z)f(z)\phi(z) + \overline{t_z(z)f(z)\phi(z)}\right]dxdy$$

$$= \iint_{\text{supp}\phi} t_\zeta(\zeta)\left\{\frac{1}{\pi}\iint_{\text{supp}\phi}\left[t_z(z)\Omega^{(1)}(\zeta,z)f(z) + \overline{t_z(z)}\Omega^{(2)}(\zeta,z)\overline{f(z)}\right]dxdy\right\}\tilde{D}\phi(\zeta)d\xi d\eta$$

$$+ \iint_{\text{supp}\phi}\overline{t_\zeta(\zeta)}\left\{\frac{1}{\pi}\iint_{\text{supp}\phi}\left[\overline{t_z(z)}\overline{\Omega^{(1)}(\zeta,z)}\overline{f(z)} + t_z(z)\overline{\Omega^{(2)}(\zeta,z)}f(z)\right]dxdy\right\}\overline{\tilde{C}\phi(\zeta)}d\xi d\eta$$

$$= \iint_{\text{supp}\phi} t_\zeta(\zeta)\left\{\frac{1}{\pi}\iint_D\left[t_z(z)\Omega^{(1)}(\zeta,z)f(z) + \overline{t_z(z)}\Omega^{(2)}(\zeta,z)\overline{f(z)}\right]dxdy\right\}\tilde{C}\phi(\zeta)d\xi d\eta$$

$$+ \iint_{\text{supp}\phi}\overline{t_\zeta(\zeta)}\left\{\frac{1}{\pi}\iint_{\text{supp}\phi}\left[\overline{t_z(z)}\overline{\Omega^{(1)}(\zeta,z)}\overline{f(z)} + t_z(z)\overline{\Omega^{(2)}(\zeta,z)}f(z)\right]dxdy\right\}\overline{\tilde{C}\phi(\zeta)}d\xi d\eta$$

$$= -\iint_{\text{supp}\phi} t_\zeta(\zeta)(Kf)(\zeta)\tilde{C}\phi(\zeta)d\xi d\eta - \iint_{\text{supp}\phi}\overline{t_\zeta(\zeta)(Kf)(\zeta)}\overline{\tilde{C}\phi(\zeta)}d\xi d\eta.$$

$$(A.56)$$

Then, by $(\tilde{K}f)(\zeta) = \omega(\zeta)$, we obtain equation (A.51).

Proof of Lemma 3   □

Proof. Obviously, in $\theta\backslash\Gamma$, $D\varphi = 0$, and $-A\varphi - B\varphi + C \in L^{p,2}(\theta)$; in $\theta\backslash\Gamma$, we use Lemma 2 to obtain

$$
\begin{aligned}
D\omega &= D\varphi + D\widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right) \\
&= -A\varphi - B\overline{\varphi} + C - A\widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right) - B\overline{\widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right)} \\
&= -A\left[\varphi + \widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right)\right] - B\left[\overline{\varphi} + \overline{\widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right)}\right] + C \\
&= -A\omega - B\overline{\omega} + C.
\end{aligned}
\tag{A.57}
$$

Because integral operator $\widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right)$ is continuous in the whole plane θ, $\varphi(z)$ satisfies the boundary value condition of (69). Therefore, hypercomplex function $\omega(z)$ also satisfies the boundary value condition on Γ in problem (69). When $|z| \longrightarrow \infty$, $|\omega(z)|$ goes to zero, i.e.,

$$
\omega^{-}(\infty) = 0. \tag{A.58}
$$

Therefore, we prove that hypercomplex function $\omega(z)$ is the solution for (69).

Proof of Lemma 4 □

Proof. We prove the second estimate equation in (81). According to Lemma 3, the solution $\omega(z)$ of (69) can be expressed as

$$
\omega(z) = \varphi(z) + \widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right). \tag{A.59}
$$

Then we have

$$
\begin{aligned}
\left|\omega^{-}, G^{-} + \Gamma\right|_{0,\alpha} &\le \left|\varphi^{-}, G^{-} + \Gamma\right|_{0,\alpha} \\
&+ \left|\widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right), G^{-} + \Gamma\right|_{0,\alpha}.
\end{aligned}
\tag{A.60}
$$

According to Lemma 2, we have

$$
\begin{aligned}
&\left|\widetilde{K}\left(-A\varphi - B\overline{\varphi} + C\right), G^{-} + \Gamma\right|_{0,\alpha} \le N^{(2)}\left|-A\varphi - B\overline{\varphi} + C, \theta\right|_{p,2} \\
&\le N^{(2)}\left|\varphi^{-}, G^{-} + \Gamma\right|_{0}\left(|A, \theta|_{p,2} + |B, \theta|_{p,2}\right) + |C, \theta|_{p,2},
\end{aligned}
\tag{A.61}
$$

where $N^{(2)}$ is a positive constant. Because $\left|\varphi^{-}, G^{-} + \Gamma\right|_{0} \le \left|\varphi^{-}, G^{-} + \Gamma\right|_{0,\alpha}$,

$$
\left|\omega^{-}, G^{-} + \Gamma\right|_{0,\alpha} \le N^{(3)}\left[\left|\varphi^{-}, G^{-} + \Gamma\right|_{0,\alpha} + |C, \theta|_{p,2}\right], \tag{A.62}
$$

where $N^{(3)}$ is a positive constant. Utilizing the second equation in (73), we can obtain

$$
\left|\omega^{-}, G^{-} + \Gamma\right|_{0,\alpha} \le N\left(|\gamma, \Gamma|_{0,\alpha} + |C, \theta|_{p,2}\right), \tag{A.63}
$$

where N is a positive constant only related to $q(z), p, A(z), B(z)$, and Γ. The first equation of (81) can be proved similarly, so we omit it here. □

Data Availability

The data that support the findings of this study are openly available.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

References

[1] M. Dai, T. Gao, Y. B. Lu, Y. Zheng, and J. Duan, "Detecting the maximum likelihood transition path from data of stochastic dynamical systems," *Chaos*, vol. 30, no. 11, Article ID 113124, 2020.

[2] A. Diaz-Ruelas, H. Jeldtoft Jensen, D. Piovani, and A. Robledo, "Tangent map intermittency as an approximate analysis of intermittency in a high dimensional fully stochastic dynamical system: the tangled nature model," *Chaos*, vol. 26, no. 12, Article ID 123105, 2016.

[3] B. X. Wang, "Mean-square random invariant manifolds for stochastic differential equations," *Discrete & Continuous Dynamical Systems—A*, vol. 41, no. 3, pp. 1449–1468, 2021.

[4] P. Ao, "Laws in Darwinian evolutionary theory," *Physics of Life Reviews*, vol. 2, pp. 117–156, 2005.

[5] Y. A. Ma, Q. J. Tan, R. S. Yuan, B. Yuan, and P. Ao, "Potential function in a continuous dissipative chaotic system: decomposition scheme and role of strange attractor," *International Journal of Bifurcation and Chaos*, vol. 24, no. 2, Article ID 1450015, 2014.

[6] R. S. Yuan and P. Ao, "Beyond Itô versus Stratonovich," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2012, no. 7, Article ID 07010, 2012.

[7] C. Darwin, *On the Origin of Species by Means of Natural Selection*, John Murray, London, UK, 1859.

[8] R. A. Fisher, *The Genetical Theory of Natural Selection*, Oxford University Press, Oxford, UK, 1930.

[9] S. Wright, "The roles of mutation, inbreeding crossbreeding and selection in evolution," in *Proceedings of the 1932 International Congress Genetics*, vol. 1, pp. 356–366, Ithaca, NY, USA, 1932.

[10] C. Waddington, *Organisers and Genes*, Cambrige University Press, Cambridge, UK, 1940.

[11] H. Alhussein and M. F. Daqaq, "Potential well escape in a galloping twin-well oscillator," *Nonlinear Dynamics*, vol. 99, no. 1, pp. 57–72, 2020.

[12] A. N. Kounadis, "Dynamic buckling of autonomous systems having potential energy universal unfoldings of cuspoid catastrophe," *Nonlinear Dynamics*, vol. 18, no. 3, pp. 235–252, 1999.

[13] X. J. Tian, H. Zhang, and J. Xing, "Coupled reversible and irreversible bistable switches underlying TGF β-induced epithelial to mesenchymal transition," *Biophysical Journal*, vol. 105, no. 4, pp. 1079–1089, 2013.

[14] P. Wang, C. Song, H. Zhang, Z. Wu, X. J. Tian, and J. Xing, "Epigenetic state network approach for describing cell phenotypic transitions," *Interface Focus*, vol. 4, no. 3, Article ID 20130068, 2014.

[15] P. Ao, "Potential in stochastic differential equations: novel construction," *Journal of Physics A: Mathematical and General*, vol. 37, no. 3, pp. L25–L30, 2004.

[16] Y. Wang and H. Qian, "Mathematical representation of Clausius' and Kelvin's statements of the second law and irreversibility," *Journal of Statistical Physics*, vol. 179, no. 3, pp. 808–837, 2020.

[17] S. Kachalo, H. M. Lu, and J. Liang, "Protein folding dynamics via quantification of kinematic energy landscape," *Physical Review Letters*, vol. 96, no. 5, Article ID 058106, 2006.

[18] A. P. Kendal, J. C. Galphin, and E. L. Palmer, "Replication of influenza virus at elevated temperatures: production of virus-like particles with reduced matrix protein content," *Virology*, vol. 76, no. 1, pp. 186–195, 1977.

[19] Y. Tang, R. S. Yuan, and Y. A. Ma, "Dynamical behaviors determined by the Lyapunov function in competitive Lotka-Volterra systems," *Physical Review E—Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, vol. 87, no. 1, Article ID 012708, 2013.

[20] I. Prigogine, *The End of Certainty: Time, Chaos, and the New Laws of Nature*, Free Press, New York, NY, USA, 1997.

[21] L. Y. Qiao, Z. G. Zheng, and M. C. Cross, "Minimum-action paths for wave-number selection in nonequilibrium systems," *Physical Review E—Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, vol. 93, no. 4, Article ID 042204, 2016.

[22] P. Rodríguez-Sánchez, E. H. van Nes, and M. Scheffer, "Climbing Eschers stairs: a way to approximate stability landscapes in multidimensional systems," *PLoS Computational Biology*, vol. 16, no. 4, Article ID e1007788, 2020.

[23] V. Valen, "Energy and evolution," *Evolutionary Theory*, vol. 1, pp. 179–229, 1976.

[24] A. Hu and Z. Xu, "Multi-stable chaotic attractors in generalized synchronization," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 8, pp. 3237–3244, 2011.

[25] D. Angeli and E. D. Sontag, "Multi-stability in monotone input/output systems," *Systems & Control Letters*, vol. 51, no. 3-4, pp. 185–202, 2004.

[26] Y. Liu and Z. You, "Multi-stability and almost periodic solutions of a class of recurrent neural networks," *Chaos, Solitons & Fractals*, vol. 33, no. 2, pp. 554–563, 2007.

[27] M. Freidlin and W. Q. Hu, "Smoluchowski-Kramers approximation in the case of variable friction," *Journal of Mathematical Sciences*, vol. 179, no. 1, pp. 184–207, 2011.

[28] C. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Sciences*, Springer, Berlin, Germany, 3rd edition, 2004.

[29] N. V. Kampen, *Stochastic Processes in Physics and Chemistry*, Elsevier, Amsterdam, Netherlands, 3rd edition, 2007.

[30] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications*, Springer, Berlin, Germany, 2nd edition, 1989.

[31] C. Kwon, P. Ao, and D. J. Thouless, "Structure of stochastic dynamics near fixed points," *Proceedings of the National Academy of Sciences*, vol. 102, no. 37, pp. 13029–13033, 2005.

[32] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics*, Cambridge University Press, Cambridge, UK, 1995.

[33] R. Mannella and P. V. E. McClintock, "Itô versus Stratonovich: 30 years later," *Fluctuation and Noise Letters*, vol. 11, no. 1, Article ID 1240010, 2012.

[34] J. Shi, T. Chen, R. Yuan, B. Yuan, and P. Ao, "Relation of a new interpretation of stochastic differential equations to ito process," *Journal of Statistical Physics*, vol. 148, no. 3, pp. 579–590, 2012.

[35] M. Z. Li, Z. Y. Hou, and Z. Y. Xu, *Theory of Elliptic Partial Differential Equations and Boundary Value Problems*, Fudan University Press, Shanghai, China, 1990.

[36] A. Douglis, "A function theoretic approach to elliptic systems of equations in two variables," *Communications on Pure and Applied Mathematics*, vol. 6, no. 2, pp. 259–289, 1953.

[37] R. P. Gilbert and G. N. Hile, "Generalized hypercomplex function theory," *Transactions of the American Mathematical Society*, vol. 195, pp. 1–29, 1974.

[38] H. Begehr and R. P. Gilbert, "On Riemann boundary value problems for certain linear elliptic systems in the plane," *Journal of Differential Equations*, vol. 32, pp. 1–14, 1979.

[39] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, RI, USA, 2nd edition, 2010.

[40] H. Rhee, R. Aris, and N. Amundson, *First-Order Partial Differential Equations*, Vol. 2, Dover Publications Inc, New York, NY, USA, 2001.

[41] J. Plemelj, "Ein fx zur Cauchyschen integraldarstellung analytisher funktionen, randwerte betreffend," *Monatshefte für Mathematik und Physik*, vol. 19, no. 1, pp. 205–210, 1908.

[42] Y. Sokhotski, "On definite integrals and functions used in series expansions," Ph. D. thesis, St. Petersburg, Russia, 1873.