01 Feb 2022

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Recommended Citation

M. Bohner et al., "Periodicity on Isolated Time Scales," *Mathematische Nachrichten*, vol. 295, no. 2, pp. 259 - 280, Wiley; Wiley-VCH Verlag, Feb 2022.
The definitive version is available at https://doi.org/10.1002/mana.201900360

Received: 18 August 2019 | Revised: 8 May 2020 | Accepted: 27 May 2020

DOI: 10.1002/mana.201900360

MATHEMATISCHE NACHRICHTEN

ORIGINAL PAPER

Periodicity on isolated time scales

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Dedicated to the memory of Professor Dr. Reinhard Mennicken (March 16, 1935 – June 13, 2019)

Abstract
In this work, we formulate the definition of periodicity for functions defined on isolated time scales. The introduced definition is consistent with the known formulations in the discrete and quantum calculus settings. Using the definition of periodicity, we discuss the existence and uniqueness of periodic solutions to a family of linear dynamic equations on isolated time scales. Examples in quantum calculus and for mixed isolated time scales are presented.

KEYWORDS
dynamic equation, first-order equation, periodic, time scales

MSC (2020)
34N05, 39A06, 39A10, 39A13, 39A23

1 | INTRODUCTION

The theory of dynamic equations on time scales is recent, it was introduced by Stefan Hilger in 1988 in his PhD thesis. Since then, this theory has been attracting the attention of many researchers, due to its power of unification, extension and discretization. It is a known fact that this theory can unify discrete and continuous analysis, as well as the cases “in between”.

However, despite its potential for unification, for instance, it is still an open problem how to define periodicity on time scales in a unified way. The first studies concerning periodicity on time scales appeared in the literature by requiring a very restricted periodicity condition on the time scale $\mathbb{T}$. This condition is described as follows: *A time scale* $\mathbb{T}$ is called ω-periodic if for every $t \in \mathbb{T}$, we have $t + \omega \in \mathbb{T}$ and $\sigma(t + \omega) = \sigma(t)$, where σ is the forward jump operator of $\mathbb{T}$. Notice that this definition only makes sense if we ensure that the time scale has such additive property, which is a strong hypothesis. For instance, this definition does not include the quantum scale $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$ with $q > 1$, which plays an important role for applications since it gives rise to quantum calculus, which is a crucial tool in the study of phenomena in quantum physics (see [22, 25, 26] and the references therein). Only in 2012, M. Bohner and R. Chieochan introduced in the literature the concept of periodicity in quantum calculus for the first time (see [8]). Since then, many important results were proved for this case (see [9, 10, 12, 13, 16–18]). However, all studies and investigations for quantum calculus were made separately. For the large literature concerning periodic time scales and alternate concepts of periodicity, we refer to [1–7, 11, 19–21, 23, 24, 27–30] and the references therein. The connection between Adıvar’s [1] periodicity concept and ours is discussed in the appendix, but the material presented there is not needed in any way to understand the results given in this paper.

The goal of this paper is to present a unified definition of periodicity for all isolated time scales and to prove many interesting and relevant results in this direction. The definition presented here is consistent with the known formulations in the discrete and quantum calculus settings. The definition says that a function $f : \mathbb{T} \to \mathbb{R}$ is called *ω-periodic* provided

$$p(t) = \left(\sigma^{\omega}\right)^{\Delta}(t)p\left(\sigma^{\omega}(t)\right) \quad \text{for all} \quad t \in \mathbb{T},$$

where $\mathbb{T}$ is an isolated time scale. Using this concept, we prove surprising and interesting properties of this class of periodic functions. For instance, we prove that the space of all periodic and regressive functions $p : \mathbb{T} \to \mathbb{R}$ with the operation $\oplus$ is a subgroup of $(\mathcal{R}, \oplus)$. We also give a characterization of 1-periodic functions and prove useful properties concerning the delta integral of an ω-periodic function. Another surprising property is that the chain rule for the composition of a function f with σ^{ω} holds for isolated time scales, although such a chain rule does not hold for general time scales. Further, we investigate the existence and uniqueness of periodic solutions of the linear dynamic equations

$$x^{\Delta} = a(t)x + b(t) \quad \text{and} \quad x^{\Delta} = -c(t)x^{\sigma} + d(t),$$

where the coefficient functions a, b, c, d satisfy certain conditions related to periodicity. We give explicit formulations of the periodic solutions of both equations and show that the found results are consistent with the known ones for difference equations and q-difference equations.

The paper is organized as follows. Section 2 is devoted to some fundamentals of the theory of time scales. We only state the definitions and results for *isolated time scales*, as only they are considered in this paper. In Section 3, we consider "iterated shifts" and prove some auxiliary results about these fundamental objects. Section 4 then introduces our concept of periodicity on any isolated time scale. Several important properties of periodic functions on isolated time scales are given. In Section 5, examples are presented to illustrate our new definition. Finally, in Section 6 and Section 7, we investigate existence and uniqueness of periodic solutions of homogeneous and inhomogeneous linear dynamic equations on isolated time scales.

2 | TIME SCALES ESSENTIALS

We first introduce some fundamentals of time scales that we will use. A *time scale* $\mathbb{T}$ is a nonempty closed subset of the real numbers.

Definition 2.1 (See [14, Chapter 1]). For $t \in \mathbb{T}$, the *forward and backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\sigma(t) > t$, then t is called *right-scattered*. Otherwise, t is called *right-dense*. Similarly, if $\rho(t) < t$, then t is said to be *left-scattered*, while if $\rho(t) = t$, then t is called *left-dense*. In this paper, from now on until the end, we only consider *isolated time scales*, i.e., all points are right-scattered and all points are left-scattered. For any function $f : \mathbb{T} \to \mathbb{R}$, we put

$$f^{\sigma} = f \circ \sigma.$$

The *graininess* function $\mu : \mathbb{T} \to (0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t \quad \text{for all} \quad t \in \mathbb{T}.$$

If $t \in \mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{M\}$, while otherwise, we put $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 2.2 (See [14, Definition 1.10]). For $f : \mathbb{T} \to \mathbb{R}$, the derivative of f at $t \in \mathbb{T}^{\kappa}$ is defined as

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

If $F : \mathbb{T} \to \mathbb{R}$ is an *antiderivative* of f, i.e., $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}$, then we define the *integral*

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Theorem 2.3. *For $f : \mathbb{T} \to \mathbb{R}$, the "simple useful formula"*

$$f^{\sigma} = f + \mu f^{\Delta} \tag{2.1}$$

holds, and for $f, g : \mathbb{T} \to \mathbb{R}$, the product rule and, if $g \neq 0$, the quotient rule

$$(fg)^{\Delta} = f^{\Delta}g^{\sigma} + fg^{\Delta} \quad \text{and} \quad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}$$

hold.

Definition 2.4 (See [14, Definition 2.25]). A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

The set of regressive functions on $\mathbb{T}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.5 (See [15, p. 13]). Define the "circle plus" addition and the "circle minus" subtraction on $\mathcal{R}$ as

$$p \oplus q = p + q + \mu pq \quad \text{and} \quad p \ominus q = \frac{p - q}{1 + \mu q}.$$

Theorem 2.6 (See [14, Theorem 2.33]). *Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. Then*

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1$$

possesses a unique solution, called the exponential function *and denoted by $e_p(\cdot, t_0)$.*

Some properties of the exponential function that are used in this paper are given next.

Theorem 2.7 (See [14, Chapter 2]). *If $p \in \mathcal{R}$, then*

1. $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$.
2. $e_p(t, s) = \frac{1}{e_p(s,t)}$.
3. *The semigroup property holds:* $e_p(t, r)e_p(r, s) = e_p(t, s)$.
4. $e_{p\oplus q}(t, s) = e_p(t, s)e_q(t, s)$.
5. $e_{\ominus p}(t, s) = e_p(s, t) = \frac{1}{e_p(t,s)}$.
6. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$.

Theorem 2.8 (Variation of Constants, see [14, Theorem 2.77]). *Suppose $a \in \mathcal{R}$ and $b : \mathbb{T} \to \mathbb{R}$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the IVP*

$$y^{\Delta} = a(t)y + b(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_a(t, t_0)y_0 + \int_{t_0}^{t} e_a(t, \sigma(s))b(s)\Delta s.$$

3 | ITERATED SHIFTS

We let $\omega \in \mathbb{N}$ and define the iterated shift $\nu : \mathbb{T} \to \mathbb{T}$ by

$$\nu := \sigma^{\omega} := \underbrace{\sigma \circ \sigma \circ \ldots \circ \sigma}_{\omega \text{ times}}.$$

Let $f : \mathbb{T} \to \mathbb{R}$. In analogy to the notation $f^{\sigma} = f \circ \sigma$, we use the notation

$$f^{\nu} = f \circ \nu.$$

Note that this notation implies

$$f^{\nu\sigma} := \left(f^{\nu}\right)^{\sigma} = \left(f^{\sigma}\right)^{\nu} =: f^{\sigma\nu}. \tag{3.1}$$

Moreover, σ and ν commute, i.e.,

$$\sigma \circ \nu = \nu \circ \sigma, \quad \text{i.e.,} \quad \sigma^{\nu} = \nu^{\sigma}. \tag{3.2}$$

The derivative of the function ν is given next.

Lemma 3.1. *We have*

$$\nu^{\Delta} = \frac{\mu^{\nu}}{\mu}. \tag{3.3}$$

Proof. Let $t \in \mathbb{T}$. Then the short calculation

$$\begin{aligned} \nu^{\Delta}(t) &= \frac{\nu(\sigma(t)) - \nu(t)}{\mu(t)} \\ &\overset{(3.2)}{=} \frac{\sigma(\nu(t)) - \nu(t)}{\mu(t)} \\ &= \frac{\mu(\nu(t))}{\mu(t)} \end{aligned}$$

confirms (3.3). □

The chain rule now reads as follows.

Lemma 3.2. *For $f : \mathbb{T} \to \mathbb{R}$, we have*

$$f^{\nu\Delta} = \nu^{\Delta} f^{\Delta\nu}. \tag{3.4}$$

Proof. The short calculation

$$\begin{aligned} f^{\nu\Delta} &= \frac{f^{\nu\sigma} - f^{\nu}}{\mu} \\ &\overset{(3.1)}{=} \frac{f^{\sigma\nu} - f^{\nu}}{\mu} \\ &\overset{(3.3)}{=} \nu^{\Delta} \frac{f^{\sigma\nu} - f^{\nu}}{\mu^{\nu}} \end{aligned}$$

$$= \nu^\Delta \left(\frac{f^\sigma - f}{\mu} \right)^\nu$$
$$= \nu^\Delta f^{\Delta\nu}$$

confirms (3.4). □

The second derivative of ν will be needed as well.

Lemma 3.3. *We have*

$$\nu^{\Delta\Delta} = \nu^\Delta \frac{\sigma^{\Delta\nu} - \sigma^\Delta}{\mu^\sigma}. \tag{3.5}$$

Proof. We use the quotient rule for (3.3) and the chain rule (3.4) applied to μ to find

$$\nu^{\Delta\Delta} = \frac{\mu^{\nu\Delta}\mu - \mu^\Delta \mu^\nu}{\mu\mu^\sigma}$$
$$= \frac{\nu^\Delta \mu^{\Delta\nu}\mu - \mu^\Delta \nu^\Delta \mu}{\mu\mu^\sigma}$$
$$= \nu^\Delta \frac{\mu^{\Delta\nu} - \mu^\Delta}{\mu^\sigma}.$$

Since $\mu(t) = \sigma(t) - t$, we get

$$\mu^\Delta = \sigma^\Delta - 1, \tag{3.6}$$

and hence (3.5) is established. □

Remark 3.4. Note also that, by using the "simple useful formula" (2.1), we have

$$\mu^\sigma = \mu + \mu\mu^\Delta = \mu\left(1 + \mu^\Delta\right) \overset{(3.6)}{=} \mu\sigma^\Delta. \tag{3.7}$$

Example 3.5. If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$, $\sigma(t) = t + 1$, $\nu(t) = t + \omega$, $\nu^\Delta(t) = 1$, and $\nu^{\Delta\Delta}(t) = 0$ for all $t \in \mathbb{T}$.

Example 3.6. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then $\mu(t) = h$, $\sigma(t) = t + h$, $\nu(t) = t + h\omega$, $\nu^\Delta(t) = 1$, and $\nu^{\Delta\Delta}(t) = 0$ for all $t \in \mathbb{T}$.

Example 3.7. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then $\mu(t) = (q-1)t$, $\sigma(t) = qt$, $\nu(t) = q^\omega t$, $\nu^\Delta(t) = q^\omega$, and $\nu^{\Delta\Delta}(t) = 0$ for all $t \in \mathbb{T}$.

We next give a result for the derivative of an integral from t to $\nu(t)$.

Lemma 3.8. *For $f : \mathbb{T} \to \mathbb{R}$, define*

$$F_\nu(t) := \int_t^{\nu(t)} f(\tau)\Delta\tau.$$

Then

$$F_\nu^\Delta = \nu^\Delta f^\nu - f.$$

Proof. Letting $t_0 \in \mathbb{T}$ and defining

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau,$$

we obtain $F^\Delta = f$ and $F_\nu = F^\nu - F$. Hence, using the chain rule (3.4) applied to F, we get

$$F_\nu^\Delta = F^{\nu\Delta} - F^\Delta = \nu^\Delta F^{\Delta\nu} - F^\Delta = \nu^\Delta f^\nu - f,$$

confirming the claim. □

We conclude with two formulas for the exponential function.

Lemma 3.9. *Let $t_0 \in \mathbb{T}$. For $f \in \mathcal{R}$, we have*

$$h(t) := e_f(\nu(t), t) \quad \textit{implies} \quad h^\Delta(t) = \left(\left(\nu^\Delta f^\nu\right) \ominus f\right)h(t) \tag{3.8}$$

and

$$e_f(\nu(t), t) = e_f\left(\nu\left(t_0\right), t_0\right)\frac{e_{\nu^\Delta f^\nu}\left(t, t_0\right)}{e_f\left(t, t_0\right)} \quad \textit{for all} \quad t \in \mathbb{T}. \tag{3.9}$$

Proof. Defining

$$h(t) := e_f(\nu(t), t)$$

and noticing that the semigroup property implies

$$h(t) = e_f\left(\nu(t), t_0\right)e_f\left(t_0, t\right) = e_f\left(\nu(t), t_0\right)e_{\ominus f}\left(t, t_0\right),$$

we may use the product rule and the chain rule to obtain

$$\begin{aligned}
h^\Delta(t) &= \nu^\Delta(t)f(\nu(t))e_f\left(\nu(t), t_0\right)(1 + \mu(t)(\ominus f)(t))e_{\ominus f}\left(t, t_0\right) + e_f\left(\nu(t), t_0\right)(\ominus f)(t)e_{\ominus f}\left(t, t_0\right) \\
&= \left[\nu^\Delta(t)f(\nu(t))(1 + \mu(t)(\ominus f)(t)) + (\ominus f)(t)\right]h(t) \\
&= \frac{\nu^\Delta(t)f(\nu(t)) - f(t)}{1 + \mu(t)f(t)}h(t) \\
&= \left(\left(\nu^\Delta f^\nu\right) \ominus f\right)(t)h(t),
\end{aligned}$$

confirming (3.8) and hence (3.9). □

Lemma 3.10. *Let $t_0 \in \mathbb{T}$. For $f \in \mathcal{R}$, we have*

$$e_f(\nu(t), \nu(s)) = e_{\nu^\Delta f^\nu}(t, s) \quad \textit{for all} \quad s, t \in \mathbb{T}. \tag{3.10}$$

Proof. Using the semigroup property and Lemma 3.9, we get

$$\begin{aligned}
e_f(\nu(t), \nu(s)) &= e_f(\nu(t), t)e_f(t, s)e_f(s, \nu(s)) \\
&= \frac{e_f(\nu(t), t)}{e_f(\nu(s), s)}e_f(t, s)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e_{\nu^\Delta f^\nu}(t,s)}{e_f(t,s)} e_f(t,s) \\
&= e_{\nu^\Delta f^\nu}(t,s),
\end{aligned}
$$

confirming the required formula. □

4 | PERIODICITY

This work is based upon the formulation of periodicity for isolated time scales, which we introduce in the following definition.

Definition 4.1. A function $p : \mathbb{T} \to \mathbb{R}$ is called ω-*periodic* provided

$$\nu^\Delta p^\nu = p. \tag{4.1}$$

The set of all ω-periodic functions $p : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{P}_\omega = \mathcal{P} = \mathcal{P}(\mathbb{T}, \mathbb{R})$.

Remark 4.2. Because of (3.3), it is easy to see that $p \in \mathcal{P}$ if and only if

$$(\mu p)^\nu = \mu p. \tag{4.2}$$

Example 4.3. If $\mathbb{T} = \mathbb{Z}$, then $p \in \mathcal{P}$ provided

$$p(t) = p(t + \omega) \quad \text{for all} \quad t \in \mathbb{T},$$

which is the usual definition of ω-periodicity.

Example 4.4. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then $p \in \mathcal{P}$ provided

$$p(t) = p(t + h\omega) \quad \text{for all} \quad t \in \mathbb{T}.$$

Example 4.5. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then $p \in \mathcal{P}$ provided

$$p(t) = q^\omega p\big(q^\omega t\big) \quad \text{for all} \quad t \in \mathbb{T},$$

which is the periodicity condition from quantum calculus introduced in [8, Definition 3.1], see also [12, 16].

Lemma 4.6. *We have* $\mathcal{P}_\omega \subset \mathcal{P}_{2\omega}$.

Proof. Let us define $\tilde{\nu} : \mathbb{T} \to \mathbb{T}$ by

$$\tilde{\nu}(t) = \sigma^{2\omega}(t) = \nu(\nu(t)).$$

Assuming that $p : \mathbb{T} \to \mathbb{R}$ is ω-periodic and using the chain rule (3.4) applied to ν, we obtain

$$
\begin{aligned}
\tilde{\nu}^\Delta(t) p(\tilde{\nu}(t)) &= \nu^\Delta(t) \nu^\Delta(\nu(t)) p(\nu(\nu(t))) \\
&\overset{(4.1)}{=} \nu^\Delta(t) p(\nu(t)) \\
&\overset{(4.1)}{=} p(t),
\end{aligned}
$$

which shows that p is also 2ω-periodic. □

A crucial property of periodicity is obtained next.

Theorem 4.7. *If $p \in \mathcal{P}$, then the integral*

$$\int_t^{\nu(t)} p(\tau)\Delta\tau \quad \text{is independent of} \quad t \in \mathbb{T}.$$

Proof. This follows now immediately from Lemma 3.8 and Definition 4.1. □

Theorem 4.8. *If $p \in \mathcal{P}$, then*

$$\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau = \int_s^t p(\tau)\Delta\tau \quad \text{for all} \quad s, t \in \mathbb{T}.$$

Proof. Let $p \in \mathcal{P}$. Since

$$\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau = \int_{\nu(s)}^{s} p(\tau)\Delta\tau + \int_s^t p(\tau)\Delta\tau + \int_t^{\nu(t)} p(\tau)\Delta\tau$$

and the first and last integrals cancel out due to Theorem 4.7, the proof is complete. □

Finally, two results about the exponential function are given.

Theorem 4.9. *If $p \in \mathcal{P} \cap \mathcal{R}$, then*

$$e_p(\nu(t), t) \quad \text{is independent of} \quad t \in \mathbb{T} \tag{4.3}$$

and

$$e_p(\nu(t), \nu(s)) = e_p(t, s) \quad \text{for all} \quad s, t \in \mathbb{T}. \tag{4.4}$$

Proof. While (4.3) follows now immediately from Lemma 3.9 and Definition 4.1, (4.4) follows from Lemma 3.10 and Definition 4.1. □

5 | EXAMPLES

We first characterize 1-periodic functions on an arbitrary isolated time scale. Note that these play the rôle that is assumed by constant functions in the classical discrete ($\mathbb{T} = \mathbb{Z}$) case.

Theorem 5.1. *Let $f : \mathbb{T} \to \mathbb{R}$. Then f is 1-periodic if and only if there exists $c \in \mathbb{R}$ such that*

$$f(t) = \frac{c}{\mu(t)} \quad \text{for all} \quad t \in \mathbb{T}. \tag{5.1}$$

Proof. First suppose there exists $c \in \mathbb{R}$ such that $f : \mathbb{T} \to \mathbb{R}$ is given by (5.1). Then

$$\sigma^\Delta(t) f(\sigma(t)) \overset{(3.3)}{=} \frac{\mu(\sigma(t))}{\mu(t)} \frac{c}{\mu(\sigma(t))} = \frac{c}{\mu(t)} = f(t).$$

Therefore, f is 1-periodic. Assume now that f is 1-periodic. By Remark 4.2, we have $(\mu f)(\sigma(t)) = (\mu f)(t)$ for all $t \in \mathbb{T}$, and hence $(\mu f)(t)$ is independent of $t \in \mathbb{T}$, equal to a constant c, so that f is in the form (5.1). □

Remark 5.2. A consequence of Theorem 5.1 is that any 1-periodic function $f : \mathbb{T} \to \mathbb{R}$ for a given isolated time scale $\mathbb{T}$ can be described uniquely by the area between two consecutive time points, since

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \int_t^{\sigma(t)} \frac{c}{\mu(\tau)}\Delta\tau = \mu(t)\frac{c}{\mu(t)} = c.$$

It follows that a 1-periodic function with unit area 1, i.e., the area between two consecutive time points, is of the form

$$f(t) = \frac{1}{\mu(t)} \quad \text{for all} \quad t \in \mathbb{T}.$$

Now we present three examples of ω-periodic functions on an isolated time scale.

Example 5.3. Consider any time scale

$$\mathbb{T} = \{t_i : i \in \mathbb{Z}\} \quad \text{with} \quad \sigma(t_i) = t_{i+1} > t_i \quad \text{for all} \quad i \in \mathbb{Z}.$$

Define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(t_i) = \frac{(-1)^i}{\mu(t_i)} \quad \text{for all} \quad i \in \mathbb{Z}.$$

If $i \in \mathbb{Z}$, then

$$\begin{aligned}(\sigma^2)^\Delta(t_i) f(\sigma^2(t_i)) &\overset{(3.3)}{=} \frac{\mu(\sigma^2(t_i))}{\mu(t_i)} f(\sigma^2(t_i)) = \frac{\mu(t_{i+2})}{\mu(t_i)} f(t_{i+2}) \\ &= \frac{\mu(t_{i+2})}{\mu(t_i)} \frac{(-1)^{i+2}}{\mu(t_{i+2})} = \frac{(-1)^i}{\mu(t_i)} = f(t_i).\end{aligned}$$

Hence, f is 2-periodic on $\mathbb{T}$.

Example 5.4. Consider any time scale

$$\mathbb{T} = \{t_i : i \in \mathbb{Z}\} \quad \text{with} \quad \sigma(t_i) = t_{i+1} > t_i \quad \text{for all} \quad i \in \mathbb{Z}.$$

Define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(t_i) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ \dfrac{1}{\mu(t_i)} & \text{if } i \text{ is even.} \end{cases}$$

If i is even, then

$$(\sigma^2)^\Delta(t_i) f(\sigma^2(t_i)) \overset{(3.3)}{=} \frac{\mu(\sigma^2(t_i))}{\mu(t_i)} f(\sigma^2(t_i)) = \frac{\mu(t_{i+2})}{\mu(t_i)} \frac{1}{\mu(t_{i+2})} = \frac{1}{\mu(t_i)} = f(t_i),$$

while if i is odd, then

$$(\sigma^2)^\Delta(t_i) f(\sigma^2(t_i)) = 0 = f(t_i).$$

Hence, f is 2-periodic on $\mathbb{T}$.

Example 5.5. Consider any time scale

$$\mathbb{T} = \{t_i : i \in \mathbb{Z}\} \quad \text{with} \quad \sigma(t_i) = t_{i+1} > t_i \quad \text{for all} \quad i \in \mathbb{Z}.$$

Let $g : \mathbb{Z} \to \mathbb{R}$ be a "normal" periodic function of period ω, i.e., $g(i+\omega) = g(i)$ for all $i \in \mathbb{Z}$. Define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(t_i) = \frac{g(i)}{\mu(t_i)} \quad \text{for all} \quad i \in \mathbb{Z}.$$

If $i \in \mathbb{Z}$, then

$$\begin{aligned}(\sigma^\omega)^\Delta(t_i) f(\sigma^\omega(t_i)) &\overset{(3.3)}{=} \frac{\mu(\sigma^\omega(t_i))}{\mu(t_i)} f(\sigma^\omega(t_i)) = \frac{\mu(t_{i+\omega})}{\mu(t_i)} f(t_{i+\omega}) \\ &= \frac{\mu(t_{i+\omega})}{\mu(t_i)} \frac{g(i+\omega)}{\mu(t_{i+\omega})} = \frac{g(i)}{\mu(t_i)} = f(t_i).\end{aligned}$$

Hence, f is ω-periodic on $\mathbb{T}$.

In the following, we give examples how one, with given periodic functions, can construct more periodic functions.

Theorem 5.6. *Assume $p, q \in \mathcal{P}$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\alpha p + \beta q \in \mathcal{P} \quad \textit{and} \quad \mu p q \in \mathcal{P}.$$

Moreover, if $\alpha + \mu(t)q(t) \neq 0$ for all $t \in \mathbb{T}$, then

$$\frac{p}{\alpha + \mu q} \in \mathcal{P}.$$

Proof. Assuming that $\alpha, \beta \in \mathbb{R}$ and $p, q \in \mathcal{P}$, the formulas

$$[\mu(\alpha p + \beta q)]^\nu = \alpha(\mu p)^\nu + \beta(\mu q)^\nu = \alpha(\mu p) + \beta(\mu q) = \mu(\alpha p + \beta q),$$

$$[\mu(\mu p q)]^\nu = (\mu p)^\nu (\mu q)^\nu = (\mu p)(\mu q) = \mu(\mu p q),$$

and, if $\alpha + \mu(t)q(t) \neq 0$ for all $t \in \mathbb{T}$,

$$\left[\mu \frac{p}{\alpha + \mu q}\right]^\nu = \frac{(\mu p)^\nu}{\alpha + (\mu q)^\nu} = \frac{\mu p}{\alpha + \mu q} = \mu \frac{p}{\alpha + \mu q}$$

together with Remark 4.2 verify all claims. □

Remark 5.7. Theorem 5.6 together with Theorem 5.1 shows that $p(t) \neq 0$ for all $t \in \mathbb{T}$ implies that

$$p \in \mathcal{P} \quad \text{if and only if} \quad \frac{1}{\mu^2 p} \in \mathcal{P}.$$

The next result shows that the set of all ω-periodic and regressive functions is a subgroup of the set of regressive functions.

Corollary 5.8. *If $p \in \mathcal{P} \cap \mathcal{R}$, then $\ominus p \in \mathcal{P}$. If $p, q \in \mathcal{P}$, then $p \oplus q \in \mathcal{P}$.*

Proof. These claims follow from

$$p \oplus q = p + q + \mu pq \quad \text{and} \quad \ominus p = -\frac{p}{1+\mu p}$$

and the results given in Theorem 5.6. □

Remark 5.9. For the notation in this remark, we refer to [15, Definition 2.35]. Using Theorem 5.6, we can show that for $\alpha \in \mathbb{R}$ and $p \in \mathcal{P} \cap \mathcal{R}(\alpha)$, we have $\alpha \odot p \in \mathcal{P}$. In particular, e.g.,

$$2p + \mu p^2 = 2 \odot p \in \mathcal{P} \quad \text{and} \quad \frac{p}{1+\sqrt{1+\mu p}} = \frac{1}{2} \odot p \in \mathcal{P}.$$

6 | HOMOGENEOUS LINEAR DYNAMIC EQUATIONS

In this section, we apply our definition of periodicity to homogeneous linear dynamic equations on isolated time scales.

Theorem 6.1. *Let $a \in \mathcal{R}$. If*

$$x^{\Delta} = a(t)x \tag{6.1}$$

has a nontrivial ω-periodic solution, then

$$\left(a + \frac{1}{\mu}\right)\sigma^{\Delta} \in \mathcal{P}. \tag{6.2}$$

Proof. Assume that (6.1) has a nontrivial ω-periodic solution $\bar{x}$. Then, by Theorem 2.8, we have

$$\bar{x}(\nu(t)) = e_a(\nu(t), t)\bar{x}(t) \quad \text{for all} \quad t \in \mathbb{T}.$$

Thus, by Definition 4.1, we get

$$\bar{x}(t) = \nu^{\Delta}(t)\bar{x}(\nu(t)) = \nu^{\Delta}(t)e_a(\nu(t), t)\bar{x}(t) \quad \text{for all} \quad t \in \mathbb{T}. \tag{6.3}$$

If $\bar{x}(t_0) = 0$ for some $t_0 \in \mathbb{T}$, then Theorem 2.8 yields that $\bar{x}$ is identically zero, which is not possible. Hence, $\bar{x}(t) \neq 0$ for all $t \in \mathbb{T}$, so that (6.3) implies

$$\nu^{\Delta}(t)e_a(\nu(t), t) = 1 \quad \text{for all} \quad t \in \mathbb{T}. \tag{6.4}$$

Now applying the product rule while taking the derivative of (6.4) and using (3.8), we obtain

$$\nu^{\Delta\Delta}(t)e_a(\nu(t), t) + \nu^{\Delta\sigma}(t)\left(\left(\nu^{\Delta}a^{\nu}\right) \ominus a\right)(t)e_a(\nu(t), t) = 0 \quad \text{for all} \quad t \in \mathbb{T},$$

i.e.,

$$\nu^{\Delta\Delta} + \nu^{\Delta\sigma}\left(\left(\nu^{\Delta}a^{\nu}\right) \ominus a\right) = 0. \tag{6.5}$$

By Lemma 6.2 below, (6.5) is equivalent to (6.2). □

We now give several conditions that are equivalent to (6.2).

Lemma 6.2. *If $a \in \mathcal{R}$, then a satisfies (6.2) if and only if*

$$\nu^{\Delta\Delta} + \nu^{\Delta}\nu^{\Delta\sigma}a^{\nu} = \nu^{\Delta}a \tag{6.6}$$

holds, and (6.6) is also equivalent to

$$\left(\nu^{\Delta}a^{\nu}\right) \ominus a = -\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}}. \tag{6.7}$$

Proof. Let $a \in \mathcal{R}$. By Remark 4.2, (6.2) is equivalent to

$$\left[\mu\left(a + \frac{1}{\mu}\right)\sigma^{\Delta}\right]^{\nu} = \mu\left(a + \frac{1}{\mu}\right)\sigma^{\Delta},$$

which is, by (3.7), equivalent to

$$\sigma^{\Delta\nu} + \left(\mu^{\sigma}a\right)^{\nu} = \sigma^{\Delta} + \mu^{\sigma}a,$$

which is, as ν^{Δ} and μ^{σ} are never zero, equivalent to

$$\begin{aligned}
\nu^{\Delta}a &= \frac{\nu^{\Delta}}{\mu^{\sigma}}\mu^{\sigma}a \\
&= \frac{\nu^{\Delta}}{\mu^{\sigma}}\left\{\sigma^{\Delta\nu} - \sigma^{\Delta} + (\mu^{\sigma}a)^{\nu}\right\} \\
&\overset{(3.1)}{=} \frac{\nu^{\Delta}}{\mu^{\sigma}}\left\{\sigma^{\Delta\nu} - \sigma^{\Delta} + \mu^{\nu\sigma}a^{\nu}\right\} \\
&\overset{(3.3)}{=} \nu^{\Delta}\left\{\frac{\sigma^{\Delta\nu} - \sigma^{\Delta}}{\mu^{\sigma}} + \nu^{\Delta\sigma}a^{\nu}\right\} \\
&\overset{(3.5)}{=} \nu^{\Delta\Delta} + \nu^{\Delta\sigma}\nu^{\Delta}a^{\nu},
\end{aligned}$$

which is (6.6). Now (6.6) is equivalent to

$$\begin{aligned}
\nu^{\Delta}a^{\nu} &= \frac{\nu^{\Delta}a - \nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} \\
&= \frac{\left(\nu^{\Delta\sigma} - \mu\nu^{\Delta\Delta}\right)a}{\nu^{\Delta\sigma}} - \frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} \\
&= -\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} + a - \mu\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}}a \\
&= \left(-\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}}\right) \oplus a,
\end{aligned}$$

which is equivalent to (6.7). □

Next, we give two results about the exponential function.

Theorem 6.3. *Let $a \in \mathcal{R}$ and assume (6.2). For $t_0 \in \mathbb{T}$, we have*

$$e_a(\nu(t), t) = e_a(\nu(t_0), t_0)\frac{\nu^\Delta(t_0)}{\nu^\Delta(t)} \quad \text{for all} \quad t \in \mathbb{T}. \tag{6.8}$$

Moreover, we have

$$e_a(\nu(t), \nu(s)) = e_a(t, s)\frac{\nu^\Delta(s)}{\nu^\Delta(t)} \quad \text{for all} \quad s, t \in \mathbb{T}. \tag{6.9}$$

Proof. Suppose $f : \mathbb{T} \to \mathbb{R}$ is such that $f(t) \neq 0$ for all $t \in \mathbb{T}$. Then $-f^\Delta/f^\sigma \in \mathcal{R}$ since

$$1 - \mu\frac{f^\Delta}{f^\sigma} = \frac{f^\sigma - \mu f^\Delta}{f^\sigma} \overset{(2.1)}{=} \frac{f}{f^\sigma}.$$

Now fix $s \in \mathbb{T}$ and define $y : \mathbb{T} \to \mathbb{R}$ by $y(t) = f(s)/f(t)$. Then $y(s) = 1$ and

$$y^\Delta(t) = -\frac{f(s)f^\Delta(t)}{f(t)f^\sigma(t)} = -\frac{f^\Delta(t)}{f^\sigma(t)}y(t).$$

This shows that $e_{-f^\Delta/f^\sigma}(t, s) = y(t) = f(s)/f(t)$. Applying this fact to $f = \nu^\Delta$ and using (6.7) together with (3.9) shows (6.8), while (3.10) yields (6.9). □

The next theorem supplements Theorem 6.1 to a complete characterization of periodic solutions of (6.1).

Theorem 6.4. *Let $a \in \mathcal{R}$ and assume (6.2). If*

$$\nu^\Delta(t_0)e_a(\nu(t_0), t_0) = 1,$$

then all solutions of (6.1) are ω-periodic. Otherwise, no nontrivial solution of (6.1) is ω-periodic.

Proof. Let $a \in \mathcal{R}$, assume (6.2), and let x be any solution of (6.1). Then, by Theorem 2.8, we have

$$x(\nu(t)) = e_a(\nu(t), t)x(t) \quad \text{for all} \quad t \in \mathbb{T}.$$

Thus, (6.8) gives

$$\nu^\Delta(t)x(\nu(t)) = \nu^\Delta(t)e_a(\nu(t), t)x(t) = \nu^\Delta(t_0)e_a(\nu(t_0), t_0)x(t),$$

from which all claims follow. □

7 | INHOMOGENEOUS LINEAR DYNAMIC EQUATIONS

In this section, we apply our definition of periodicity to linear dynamic equations on isolated time scales. We prove the existence and uniqueness of a periodic solution for two families of linear dynamic equations.

Theorem 7.1. *Let $a \in \mathcal{R}$ and $b : \mathbb{T} \to \mathbb{R}$. Assume (6.2). If*

$$x^\Delta = a(t)x + b(t) \tag{7.1}$$

has a nontrivial ω-periodic solution, then

$$b\mu^{\sigma} \in \mathcal{P}. \tag{7.2}$$

Proof. Assume that (7.1) has a nontrivial ω-periodic solution $\bar{x}$. Define $g := \nu^{\Delta}\bar{x}^{\nu} - \bar{x} = 0$. Now we use the product rule to calculate

$$\begin{aligned} 0 &= g^{\Delta} = \nu^{\Delta\Delta}\bar{x}^{\nu} + \nu^{\Delta\sigma}\nu^{\Delta}\bar{x}^{\Delta\nu} - \bar{x}^{\Delta} \\ &= \nu^{\Delta\Delta}\bar{x}^{\nu} + \nu^{\Delta\sigma}\nu^{\Delta}\left(a^{\nu}\bar{x}^{\nu} + b^{\nu}\right) - a\bar{x} - b \\ &= \left(\nu^{\Delta\Delta} + \nu^{\Delta\sigma}\nu^{\Delta}a^{\nu}\right)\bar{x}^{\nu} - a\bar{x} + \nu^{\Delta\sigma}\nu^{\Delta}b^{\nu} - b \\ &\overset{(6.6)}{=} \nu^{\Delta}a\bar{x}^{\nu} - a\bar{x} + \nu^{\Delta\sigma}\nu^{\Delta}b^{\nu} - b \\ &= ag + \nu^{\Delta\sigma}\nu^{\Delta}b^{\nu} - b \\ &= \nu^{\Delta\sigma}\nu^{\Delta}b^{\nu} - b. \end{aligned}$$

By Lemma 7.2 below, $\nu^{\Delta}\nu^{\Delta\sigma}b^{\nu} = b$ is equivalent to (7.2). □

We now give a condition that is equivalent to (7.2).

Lemma 7.2. *If $b : \mathbb{T} \to \mathbb{R}$, then b satisfies (7.2) if and only if*

$$\nu^{\Delta}\nu^{\Delta\sigma}b^{\nu} = b. \tag{7.3}$$

Proof. Let $b : \mathbb{T} \to \mathbb{R}$. Then (7.2) holds if and only if

$$\nu^{\Delta}\left(b\mu^{\sigma}\right)^{\nu} = b\mu^{\sigma},$$

which is equivalent to

$$b = \nu^{\Delta}\frac{b^{\nu}\mu^{\sigma\nu}}{\mu^{\sigma}} \overset{(3.2)}{=} \nu^{\Delta}b^{\nu}\frac{\mu^{\nu\sigma}}{\mu^{\sigma}} \overset{(3.3)}{=} \nu^{\Delta}b^{\nu}\nu^{\Delta\sigma},$$

i.e., to (7.3). □

Theorem 7.3. *Let $a \in \mathcal{R}$ and $b : \mathbb{T} \to \mathbb{R}$. Assume (6.2) and (7.2). If (7.1) has a solution x that satisfies*

$$\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0) \quad \textit{for some} \quad t_0 \in \mathbb{T},$$

then $x \in \mathcal{P}$.

Proof. Define $g := \nu^{\Delta}x^{\nu} - x$ and use the same calculation as in the proof of Theorem 7.1 to find $g^{\Delta} = ag$. Therefore, we have $g(t) = e_a(t, t_0)g(t_0) = 0$ for all $t \in \mathbb{T}$, so $x \in \mathcal{P}$. □

We use the following result in the proofs of our main theorems.

Lemma 7.4. *Let $a \in \mathcal{R}$ and $b : \mathbb{T} \to \mathbb{R}$. Assume (6.2) and (7.2). If*

$$H(t) := \int_t^{\nu(t)} e_{\ominus a}(\sigma(s), t_0)b(s)\Delta s, \tag{7.4}$$

then

$$H^{\Delta}(t) = \left(\frac{1}{\nu^{\Delta}(t_0)e_a(\nu(t_0), t_0)} - 1\right)b(t)e_{\ominus a}(\sigma(t), t_0). \tag{7.5}$$

Proof. We use Lemma 3.8 to find

$$\begin{aligned}
H^{\Delta}(t) &= \nu^{\Delta}(t)e_{\ominus a}(\sigma(\nu(t)), t_0)b(\nu(t)) - e_{\ominus a}(\sigma(t), t_0)b(t) \\
&= \left\{\frac{\nu^{\Delta}(t)}{e_a(\sigma(\nu(t)), \sigma(t))}b(\nu(t)) - b(t)\right\}e_{\ominus a}(\sigma(t), t_0) \\
&\overset{(3.2)}{=} \left\{\frac{\nu^{\Delta}(t)\nu^{\Delta}(\sigma(t))b(\nu(t))}{\nu^{\Delta}(\sigma(t))e_a(\nu(\sigma(t)), \sigma(t))} - b(t)\right\}e_{\ominus a}(\sigma(t), t_0) \\
&\overset{(6.8)}{=} \left\{\frac{\nu^{\Delta}(t)\nu^{\Delta}(\sigma(t))b(\nu(t))}{\nu^{\Delta}(t_0)e_a(\nu(t_0), t_0)} - b(t)\right\}e_{\ominus a}(\sigma(t), t_0) \\
&\overset{(7.3)}{=} \left\{\frac{b(t)}{\nu^{\Delta}(t_0)e_a(\nu(t_0), t_0)} - b(t)\right\}e_{\ominus a}(\sigma(t), t_0),
\end{aligned}$$

verifying (7.5). □

The next theorem is the main result of this paper.

Theorem 7.5. *Let $t_0 \in \mathbb{T}$, $a \in \mathcal{R}$, and $b : \mathbb{T} \to \mathbb{R}$. Assume (6.2) and (7.2). If*

$$\nu^{\Delta}(t_0)e_a(\nu(t_0), t_0) \neq 1, \tag{7.6}$$

then (7.1) has a unique ω-periodic solution $\bar{x}$ given by

$$\bar{x}(t) = \lambda \int_t^{\nu(t)} e_{\ominus a}(\sigma(s), t)b(s)\Delta s, \tag{7.7}$$

where

$$\lambda := \frac{\nu^{\Delta}(t_0)}{e_{\ominus a}(\nu(t_0), t_0) - \nu^{\Delta}(t_0)}. \tag{7.8}$$

Proof. Let $t_0 \in \mathbb{T}$, $a \in \mathcal{R}$, and $b : \mathbb{T} \to \mathbb{R}$. Assume (6.2), (7.2), and (7.6). First, we assume that (7.1) has an ω-periodic solution $\bar{x}$. Then, by Theorem 2.8, we have

$$\bar{x}(\nu(t)) = e_a(\nu(t), t)\bar{x}(t) + \int_t^{\nu(t)} e_a(\nu(t), \sigma(s))b(s)\Delta s. \tag{7.9}$$

Since $\bar{x}$ is ω-periodic, we obtain

$$\begin{aligned}
\bar{x}(t) &\overset{(4.1)}{=} \nu^{\Delta}(t)\bar{x}(\nu(t)) \\
&\overset{(7.9)}{=} \nu^{\Delta}(t)\left\{e_a(\nu(t), t)\bar{x}(t) + \int_t^{\nu(t)} e_a(\nu(t), \sigma(s))b(s)\Delta s\right\}
\end{aligned}$$

$$= \nu^{\Delta}(t)e_a(\nu(t),t)\left\{\bar{x}(t)+\int_t^{\nu(t)} e_a(t,\sigma(s))b(s)\Delta s\right\}$$

$$\stackrel{(6.8)}{=} \nu^{\Delta}(t_0)e_a\big(\nu\big(t_0\big),t_0\big)\left\{\bar{x}(t)+\int_t^{\nu(t)} e_a(t,\sigma(s))b(s)\Delta s\right\},$$

i.e., due to (7.6) and (7.8), (7.7) holds. Conversely, assume that $\bar{x}$ is given by (7.7). Hence,

$$\bar{x}(t) = \lambda e_a\big(t,t_0\big)H(t), \tag{7.10}$$

where H is defined in (7.4). By (7.5) and (7.8), we have

$$H^{\Delta}(t) = \frac{1}{\lambda}b(t)e_{\ominus a}\big(\sigma(t),t_0\big). \tag{7.11}$$

Now we get

$$\begin{aligned} H(\nu(t)) - H(t) &= \int_t^{\nu(t)} H^{\Delta}(s)\Delta s \\ &\stackrel{(7.11)}{=} \int_t^{\nu(t)} \frac{1}{\lambda}b(s)e_{\ominus a}\big(\sigma(s),t_0\big)\Delta s \\ &= \frac{1}{\lambda}H(t), \end{aligned}$$

i.e.,

$$H(\nu(t)) = \frac{\lambda+1}{\lambda}H(t). \tag{7.12}$$

Next, we use the product rule in (7.10) to find

$$\begin{aligned} \bar{x}^{\Delta}(t) &= \lambda a(t)e_a\big(t,t_0\big)H(t) + \lambda e_a\big(\sigma(t),t_0\big)H^{\Delta}(t) \\ &\stackrel{(7.11)}{=} a(t)\bar{x}(t) + b(t), \end{aligned}$$

so $\bar{x}$ indeed solves (7.1). It is left to show that $\bar{x}$ is ω-periodic. To this end, we calculate

$$\begin{aligned} \nu^{\Delta}(t)\bar{x}(\nu(t)) &= \nu^{\Delta}(t)\lambda e_a\big(\nu(t),t_0\big)H(\nu(t)) \\ &\stackrel{(7.12)}{=} \nu^{\Delta}(t)e_a(\nu(t),t)\lambda e_a\big(t,t_0\big)\frac{\lambda+1}{\lambda}H(t) \\ &\stackrel{(6.8)}{=} \nu^{\Delta}\big(t_0\big)e_a\big(\nu\big(t_0\big),t_0\big)(\lambda+1)e_a\big(t,t_0\big)H(t) \\ &\stackrel{(7.8)}{=} \lambda e_a\big(t,t_0\big)H(t) = \bar{x}(t), \end{aligned}$$

confirming that $\bar{x}$ is ω-periodic. □

Remark 7.6. The unique solution to (7.1) with the initial condition

$$x(t_0) = \lambda\int_{t_0}^{\nu(t_0)} e_a(t_0,\sigma(s))b(s)\Delta s$$

results in the unique ω-periodic solution.

Example 7.7. Let $\mathbb{T} = \mathbb{Z}$. Then (6.2) reads $a + 1 \in \mathcal{P}$, i.e.,

$$a(t+\omega) + 1 = a(t) + 1 \quad \text{for all} \quad t \in \mathbb{T},$$

i.e., $a(t+\omega) = a(t)$ for all $t \in \mathbb{T}$, i.e., $a \in \mathcal{P}$. Moreover, (7.2) reads $b \in \mathcal{P}$. Hence, Theorem 7.5 reads as follows. If $a \in \mathcal{R}$ and $a, b \in \mathcal{P}$, then, provided (7.6) holds,

$$x^\Delta = a(t)x + b(t)$$

has a unique ω-periodic solution given by

$$\bar{x}(t) = \lambda \sum_{k=t}^{t+\omega-1} e_{\ominus a}(k+1, t)b(k).$$

This is consistent with known results for difference equations.

Example 7.8. Let $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$. Then (6.2) reads

$$\left[a(t) + \frac{1}{(q-1)t}\right]q = q^\omega\left[a\left(q^\omega t\right) + \frac{1}{(q-1)q^\omega t}\right]q \quad \text{for all} \quad t \in \mathbb{T},$$

i.e., $q^\omega a\left(q^\omega t\right) = a(t)$ for all $t \in \mathbb{T}$, i.e., $a \in \mathcal{P}$. Moreover, (7.2) reads

$$b(t)(q-1)qt = q^\omega\left[b\left(q^\omega t\right)(q-1)qq^\omega t\right] \quad \text{for all} \quad t \in \mathbb{T},$$

i.e., $tb(t) = q^\omega\left[\left(q^\omega t\right)b\left(q^\omega t\right)\right]$ for all $t \in \mathbb{T}$, i.e., $c \in \mathcal{P}$, where $c(t) = tb(t)$. Therefore, Theorem 7.5 ensures that the dynamic equation

$$x^\Delta = a(t)x + \frac{c(t)}{t},$$

where $a \in \mathcal{R}$ and $a, c \in \mathcal{P}$, has a unique ω-periodic solution provided (7.6) is satisfied. This is consistent with known results for q-difference equations. See [13, Remark 3.11].

Example 7.9. Consider the dynamic equation on an isolated time scale

$$x^\Delta = \frac{\left(5\sigma(t) - \sigma^2(t) - 4t\right)x + 2}{\mu(\sigma(t))\mu(t)}. \tag{7.13}$$

Notice that (7.13) is in the form (7.1) with

$$a(t) = \frac{4}{\mu(\sigma(t))} - \frac{1}{\mu(t)} \quad \text{and} \quad b(t) = \frac{2}{\mu(t)\mu(\sigma(t))}.$$

Clearly, $a \in \mathcal{R}$ since

$$1 + \mu(t)a(t) = \frac{4\mu(t)}{\mu(\sigma(t))} \neq 0$$

because of $\mu(t) > 0$ for all $t \in \mathbb{T}$. Also, due to Theorem 5.1, we have that

$$\left(a + \frac{1}{\mu}\right)\sigma^\Delta = 4\frac{\sigma^\Delta}{\mu^\sigma} \overset{(3.3)}{=} \frac{4}{\mu} \in \mathcal{P} \quad \text{and} \quad b\mu^\sigma = \frac{2}{\mu} \in \mathcal{P},$$

so (6.2) and (7.2) are satisfied. Thus, from Theorem 7.5, (7.13) possesses a unique ω-periodic solution provided (7.6) is satisfied, for any $\omega \in \mathbb{N}$.

The next theorem supplements Theorem 7.5 to a complete characterization of periodic solutions of (7.1).

Theorem 7.10. *Let $t_0 \in \mathbb{T}$, $a \in \mathcal{R}$, and $b : \mathbb{T} \to \mathbb{R}$. Assume* (6.2), (7.2), *and*

$$\nu^\Delta(t_0)e_a\big(\nu(t_0), t_0\big) = 1. \tag{7.14}$$

If

$$\int_{t_0}^{\nu(t_0)} e_{\ominus a}\big(\sigma(s), t_0\big)b(s)\Delta s = 0,$$

then all solutions of (7.1) are ω-periodic. Otherwise, no nontrivial solution of (7.1) is ω-periodic.

Proof. Under the stated conditions, let x be any solution of (7.1). As in the proof of Theorem 7.5, we obtain

$$\nu^\Delta(t)x(\nu(t)) = \nu^\Delta(t_0)e_a\big(\nu(t_0), t_0\big)\left\{x(t) + \int_t^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s\right\},$$

i.e., using (7.14),

$$\nu^\Delta(t)x(\nu(t)) = x(t) + \int_t^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s,$$

i.e., with H defined as in (7.4),

$$\nu^\Delta(t)x(\nu(t)) = x(t) + e_a\big(t, t_0\big)H(t). \tag{7.15}$$

Due to (7.5) and (7.14), we get $H^\Delta = 0$ and hence (7.15) yields

$$\nu^\Delta(t)x(\nu(t)) = x(t) + e_a\big(t, t_0\big)H\big(t_0\big). \tag{7.16}$$

From (7.16), we can see that x is ω-periodic if and only if $H\big(t_0\big) = 0$, which concludes the proof. □

Finally, we consider "the other" linear dynamic equation.

Theorem 7.11. *Let $t_0 \in \mathbb{T}$, $c \in \mathcal{R}$ and $d : \mathbb{T} \to \mathbb{R}$. Assume*

$$\frac{c + \frac{1}{\mu}}{\sigma^\Delta} \in \mathcal{P} \tag{7.17}$$

and

$$\mu d \in \mathcal{P}. \tag{7.18}$$

If

$$\nu^{\Delta}(t_0)e_{\ominus c}(\nu(t_0), t_0) \neq 1,$$

then

$$x^{\Delta} = -c(t)x^{\sigma} + d(t) \tag{7.19}$$

has a unique ω-periodic solution $\bar{x}$ given by

$$\bar{x}(t) = \lambda \int_t^{\nu(t)} e_c(s,t)d(s)\Delta s,$$

where

$$\lambda := \frac{\nu^{\Delta}(t_0)}{e_c(\nu(t_0), t_0) - \nu^{\Delta}(t_0)}.$$

Proof. Besides the assumptions, assume also that x is a solution of (7.19). Then, using the "simple useful formula" (2.1), we get

$$(1 + \mu c)x^{\Delta} = -cx + d,$$

so (7.19) is equivalent to

$$x^{\Delta} = \tilde{a}x + \tilde{b}, \quad \text{where} \quad \tilde{a} = \ominus c \quad \text{and} \quad \tilde{b} = \frac{d}{1 + \mu c}.$$

Now (6.2) and (7.2) (use also (3.7)) read

$$\frac{1}{\mu^2 p} = \left(\tilde{a} + \frac{1}{\mu}\right)\sigma^{\Delta} \in \mathcal{P} \quad \text{and} \quad \frac{d}{p} = \mu^{\sigma}\tilde{b} \in \mathcal{P}, \quad \text{where} \quad p = \frac{c + \frac{1}{\mu}}{\sigma^{\Delta}},$$

which is, by Theorem 5.6 and Remark 5.7, equivalent to

$$p \in \mathcal{P} \quad \text{and} \quad \mu d \in \mathcal{P},$$

i.e., to (7.17) and (7.18). Hence, all claims follow from Theorem 7.5. □

ACKNOWLEDGEMENTS

The authors would like to thank the organizers of the workshop "Dynamic Equations on Time Scales" at the Banach Center in Będlewo, Poland, May 30 – June 3, 2018, where all three authors participated and initiated the study that led to the presented results. This research was also supported through the programme "Oberwolfach Leibniz Fellows" by the *Mathematisches Forschungsinstitut Oberwolfach* in 2018. Further support came from FEMAT (Fundação de Estudos em Ciências Matemáticas) Proc. 042/2018. Finally, the authors thank both referees and the handling Editor, who has asked us to compare our periodicity concept with that of Adıvar [1], which has been included as an appendix.

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How to cite this article: Bohner M, Mesquita J, Streipert S. *Periodicity on isolated time scales*. Mathematische Nachrichten. 2022;**295:**259–280. https://doi.org/10.1002/mana.201900360

APPENDIX A: ADIVAR'S PERIODICITY CONCEPT

In this Appendix, we point out the connection between our periodicity definition and Murat Adıvar's [1] concept. We start by recalling the three relevant definitions from [1]. First, the so-called shift operators are defined.

Definition A.1 (See [1, Definition 3]). Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_\pm : [t_0, \infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:

P.1. The functions $\delta_\pm$ are strictly increasing with respect to their second arguments, i.e., if

$$(T_0, t), (T_0, u) \in \mathcal{D}_\pm := \left\{(s, t) \in [t_0, \infty)_\mathbb{T} \times \mathbb{T}^* : \ \delta_\pm(s, t) \in \mathbb{T}^*\right\},$$

then

$$T_0 \le t < u \quad \text{implies} \quad \delta_\pm\left(T_0, t\right) < \delta_\pm\left(T_0, u\right).$$

P.2. If $(T_1, u), (T_2, u) \in \mathcal{D}_-$ with $T_1 < T_2$, then

$$\delta_-\left(T_1, u\right) > \delta_-\left(T_2, u\right),$$

and if $\left(T_1, u\right), \left(T_2, u\right) \in \mathcal{D}_+$ with $T_1 < T_2$, then

$$\delta_+\left(T_1, u\right) < \delta_+\left(T_2, u\right).$$

P.3. If $t \in [t_0, \infty)_\mathbb{T}$, then $\left(t, t_0\right) \in \mathcal{D}_+$ and $\delta_+\left(t, t_0\right) = t$. Moreover, if $t \in \mathbb{T}^*$, then $\left(t_0, t\right) \in \mathcal{D}_+$ and $\delta_+\left(t_0, t\right) = t$.
P.4. If $(s, t) \in \mathcal{D}_\pm$, then $(s, \delta_\pm(s, t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s, \delta_\pm(s, t)) = t$.
P.5. If $(s, t) \in \mathcal{D}_\pm$ and $(u, \delta_\pm(s, t)) \in \mathcal{D}_\mp$, then $(s, \delta_\mp(u, t)) \in \mathcal{D}_\pm$ and $\delta_\mp(u, \delta_\pm(s, t)) = \delta_\pm(s, \delta_\mp(u, t))$.

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be *backward and forward shift operators* on the set $\mathbb{T}^*$. The variable $s \in [t_0, \infty)_\mathbb{T}$ is called the *shift size*.

Next, the notion of a time scale that is periodic in shifts is defined.

Definition A.2 (See [1, Definition 4]). Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is called *periodic in these shifts* if there exists $p \in \left(t_0, \infty\right)_{\mathbb{T}^*}$ such that $(p, t) \in \mathcal{D}_\mp$ for all $t \in \mathbb{T}^*$. Furtheremore, if

$$P := \inf\left\{p \in \left(t_0, \infty\right)_{\mathbb{T}^*} : \ (p, t) \in \mathcal{D}_\mp \quad \text{for all} \quad t \in \mathbb{T}^*\right\} \neq t_0,$$

then P is called the *period* of the time scale $\mathbb{T}$.

The notion of periodic functions in shifts [1, Definition 5] is not relevant to our considerations and thus is not recalled here. However, the following notion of Δ-periodic functions in shifts is relevant.

Definition A.3 (See [1, Definition 6]). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with period P. Then we say that a function $f : \mathbb{T}^* \to \mathbb{R}$ is called Δ-*periodic in shifts* $\delta_\pm$ if there exists $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T, t) \in \mathcal{D}_\pm \quad \text{for all} \quad t \in \mathbb{T}^*,$$

the shifts $\delta_\pm^T$ are Δ-differentiable with rd-continuous derivatives, and

$$f\left(\delta_\pm^T(t)\right)\left(\delta_\pm^T\right)^\Delta(t) = f(t) \quad \text{for all} \quad t \in \mathbb{T}^*, \tag{A.1}$$

where $\delta_\pm^T(t) := \delta_\pm(T, t)$. The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ with these properties is called the period of f.

Remark A.4. Note that [1, (3.10)] displays the undefined symbol $\delta_{\pm}^{\Delta T}$, which we have changed in (A.1) to its probable meaning $\left(\delta_{\pm}^{T}\right)^{\Delta}$.

For some isolated time scales, we can find shift functions δ_+ such that the above coincides with our presented (in our opinion, "simpler") periodicity concept, and in those cases (A.1) matches (4.1). For those cases, the result [1, Theorem 2] matches our Theorem 4.7, while none of the other results presented in Sections 3–7 above are proved for the setting in [1].

Example A.5. If $\mathbb{T} = \mathbb{Z}$, then we can pick $\delta_+(s,t) = t + s$ for $t, s \in \mathbb{T}$, and then

$$\nu(t) = \sigma^{\omega}(t) = \delta_+(\omega, t) \quad \text{for} \quad t \in \mathbb{T} \quad \text{and} \quad \omega \in \mathbb{N}.$$

If $\mathbb{T} = q^{\mathbb{N}_0}$, then we can pick $\delta_+(s,t) = ts$ for $t, s \in \mathbb{T}$, and then

$$\nu(t) = \sigma^{\omega}(t) = q^{\omega} t = \delta_+\left(q^{\omega}, t\right) \quad \text{for} \quad t \in \mathbb{T} \quad \text{and} \quad \omega \in \mathbb{N}.$$

If $\mathbb{T} = \mathbb{N}^2$, then we can pick $\delta_+(s,t) = \left(\sqrt{t} + \sqrt{s}\right)^2$ for $t, s \in \mathbb{T}$, and then

$$\nu(t) = \sigma^{\omega}(t) = \left(\sqrt{t} + \omega\right)^2 = \delta_+\left(\omega^2, t\right) \quad \text{for} \quad t \in \mathbb{T} \quad \text{and} \quad \omega \in \mathbb{N}.$$

Also, there is a slight difference in terminology. For example, in the last case, e.g., if $\omega = 5$, a function would be called 5-periodic according to Definition 4.1, while Adıvar's Definition A.3 would call this function periodic with period 25.

However, for many isolated time scales, we cannot find shift functions δ_+ such that the above coincides with our presented periodicity concept,

Example A.6. None of the isolated time scales

$$\mathbb{T} = \left\{ \sum_{k=1}^{n} \frac{1}{k} \ : \ n \in \mathbb{N} \right\}, \quad \mathbb{T} = \mathbb{N}! = \{n! \ : \ n \in \mathbb{N}\},$$

$$\mathbb{T} = \{n^n \ : \ n \in \mathbb{N}\}, \quad \mathbb{T} = \left\{ \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \ : \ n \in \mathbb{N} \right\},$$

$$\mathbb{T} = \{2^n + 3^n \ : \ n \in \mathbb{N}\}, \quad \mathbb{T} = \left\{ \exp(n) + \ln(n) - \frac{1}{n} \ : \ n \in \mathbb{N} \right\}$$

is periodic in shifts in the sense of Definition A.2, using a *fixed* $P \in \mathbb{T}$. For none of the above time scales, it is possible to find shift functions δ_+ such that there exists a *fixed* $T \in \mathbb{T}$ with

$$\nu(t) = \sigma^{\omega}(t) = \delta_+(T, t) \quad \text{for all} \quad t \in \mathbb{T}.$$