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Nonoscillatory Solutions of Higher-Order Fractional Differential Equations

Martin Bohner, Said R. Grace, Irena Jadlovská and Nurten Kılıç

Abstract. This paper deals with the asymptotic behavior of the nonoscillatory solutions of a certain forced fractional differential equations with positive and negative terms, involving the Caputo fractional derivative. The results obtained are new and generalize some known results appearing in the literature. Two examples are also provided to illustrate the results.

Mathematics Subject Classification. 34A08, 34E10, 34A34.

Keywords. Fractional differential equations, integro-differential equations, nonoscillatory solutions, boundedness, Caputo derivative.

1. Introduction

Consider the forced fractional differential equation with positive and negative terms of the form

$$ {}^{\mathrm{C}}D_c^{\alpha} y(t) + f_1(t, x(t)) = b(t) + k(t)x^{\beta}(t) + f_2(t, x(t)), \tag{1.1} $$

where

$$ y = \left(a\left(x'\right)^{\beta}\right)^{(n-1)} \quad \text{with} \quad n \in \mathbb{N}, $$

β is the ratio of two odd positive integers, and ${}^{\mathrm{C}}D_c^{\alpha} y$ denotes the Caputo fractional derivative defined as

$$ {}^{\mathrm{C}}D_c^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_c^t \frac{y'(s)}{(t-s)^{\alpha}} \mathrm{d}s \quad \text{with} \quad \alpha \in (0,1), \quad c > 1, $$

where

$$ \Gamma(x) = \int_0^{\infty} s^{x-1} e^{-s} \mathrm{d}s, \quad x > 0. $$

Throughout this paper, we assume:

(H_1) $a, k \in \mathrm{C}([c,\infty),(0,\infty))$, $b \in \mathrm{C}([c,\infty),\mathbb{R})$.

(H_2) $f_1, f_2 \in \mathrm{C}([c,\infty)\times\mathbb{R},\mathbb{R})$ and there exist

$$g_1, g_2 \in \mathrm{C}([c,\infty),(0,\infty)) \quad \text{and} \quad \lambda_1, \lambda_2 > 0$$

with $\lambda_1 > \lambda_2$ and

$$xf_1(t,x) \geq g_1(t)|x|^{\lambda_1+1} \quad \text{and} \quad 0 \leq xf_2(t,x) \leq g_2(t)|x|^{\lambda_2+1}.$$

A function $x : [c,\infty) \to \mathbb{R}$ is called a solution of (1.1) if $x \in \mathrm{C}^1([c,\infty),\mathbb{R})$, $a(x')^\beta \in \mathrm{C}^{n-1}([c,\infty),\mathbb{R})$, and x satisfies (1.1). Oscillation and nonoscillation of such solutions is defined in the usual way.

Integro-differential and fractional differential equations have recently received attention due to their potential applications in many disciplines such as engineering, mechanics, physics, chemistry, aerodynamics, mathematical biology, electrodynamics, and others. For more details, we refer the reader to the monographs [1–4]. (See also the papers [5–7] for specific results from mathematical biology and physics, where the models are formulated by means of differential equations with forces idealized by nonlocal and/or taxis-driven terms.) Oscillation and other asymptotic results for solutions of such equations are relatively scarce in the literature. We refer to [8–28] for corresponding results. Except the recent papers [17,19], there are no such results for forced fractional differential equations of the form (1.1) In [17,19], (1.1) was considered in the cases $n = 1$, $n = 2$, and $\beta = 1$, while the remaining cases were left as open problems. The main objective of this paper is to present a solution to these open problems and to generalize the results in [17,19] to the case of arbitrary $n \in \mathbb{N}$. We also refer to Remark 2.5 at the end of Sect. 2 that compares the results given in [17,19] to the ones presented in this paper. We note that (1.1) is equivalent to the Volterra-type equation

$$y(t) = y(c) + \frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1}[b(s) + k(s)x^\beta(s) + F(s,x(s))]\mathrm{d}s \\ \text{with} \quad F = f_2 - f_1, \tag{1.2}$$

see Medveď [22, Lemma 2.5] and Medveď and Pospíšil [23, Lemma 1]. In the proofs of our main results in Sect. 2, we use the equivalence (1.2) as well as Young's inequality. The paper concludes in Sect. 3 with two examples illustrating the applicability of our two main results.

2. Main Results

To obtain our results in this paper, we shall use the following two auxiliary results, which are also used in [17,19].

Lemma 2.1. (See [11, Lemma 2.3]) *Let $\alpha, p > 0$ satisfy $p(\alpha-1)+1 > 0$. Then,*

$$\int_0^t (t-s)^{p(\alpha-1)} e^{ps}\mathrm{d}s \leq Qe^{pt}, \quad t \geq 0,$$

where

$$Q = \frac{\Gamma(1+p(\alpha-1))}{p^{1+p(\alpha-1)}}.$$

Lemma 2.2. (Young's inequality [29]) *If $X, Y \geq 0$ and $\delta > 1$, then*

$$XY \leq \frac{X^\delta}{\delta} + \frac{Y^\eta}{\eta} \quad \textit{with} \quad \eta = \frac{\delta}{\delta-1}. \tag{2.1}$$

For notational purpose, it is convenient to set

$$g(t) := \left(\frac{g_2^{\lambda_1}(t)}{g_1^{\lambda_2}(t)}\right)^{1/(\lambda_1-\lambda_2)} \quad \text{and} \quad A(t,c) := \int_c^t a^{-1/\beta}(s)\mathrm{d}s.$$

We now give sufficient conditions under which any nonoscillatory solution x of (1.1) satisfies

$$|x(t)| = \mathrm{O}\left(t^{(n-1)/\beta}e^{t/\beta}A(t,c)\right) \quad \text{as} \quad t \to \infty.$$

Theorem 2.3. *Assume (H_1)–(H_2). Let $p > 1$ and $\alpha \in (0,1)$ be such that $p(\alpha-1)+1 > 0$ holds. If*

$$\int_c^\infty k^q(s)s^{(n-1)q}A^{\beta q}(s,c)\mathrm{d}s < \infty, \qquad \textit{where} \quad q = \frac{p}{p-1}, \tag{2.2}$$

$$\lim_{t\to\infty}\int_c^t (t-s)^{\alpha-1}|b(s)|\mathrm{d}s < \infty, \tag{2.3}$$

and

$$\lim_{t\to\infty}\int_c^t (t-s)^{\alpha-1}g(s)\mathrm{d}s < \infty \tag{2.4}$$

hold, then every nonoscillatory solution x of (1.1) *satisfies*

$$\limsup_{t\to\infty}\frac{|x(t)|}{t^{(n-1)/\beta}e^{t/\beta}A(t,c)} < \infty. \tag{2.5}$$

Proof. Let x be a nonoscillatory solution of (1.1). As the case of eventually negative x can be treated similarly, we assume in this proof that x is eventually positive, i.e., there exists a $t_1 \geq c$ such that $x(t) > 0$ for all $t \geq t_1$. From now on in this proof, let $t \geq t_1$. Applying (2.1) with

$$\delta = \frac{\lambda_1}{\lambda_2} > 1, \quad X = x^{\lambda_2}(t), \quad Y = \frac{\lambda_2 g_2(t)}{\lambda_1 g_1(t)}, \quad \text{and} \quad \eta = \frac{\lambda_1}{\lambda_1-\lambda_2},$$

we obtain

$$
\begin{aligned}
g_2(t)x^{\lambda_2}(t)-g_1(t)x^{\lambda_1}(t) &= \frac{\lambda_1}{\lambda_2}g_1(t)\left(x^{\lambda_2}(t)\frac{\lambda_2 g_2(t)}{\lambda_1 g_1(t)}-\frac{\lambda_2}{\lambda_1}\left(x^{\lambda_2}(t)\right)^{\lambda_1/\lambda_2}\right)\\
&= \frac{\lambda_1}{\lambda_2}g_1(t)\left(XY-\frac{X^\delta}{\delta}\right)\\
&\le \frac{\lambda_1}{\lambda_2}g_1(t)\left(\frac{Y^\eta}{\eta}\right)\\
&= \frac{\lambda_1-\lambda_2}{\lambda_2}\left(\frac{\lambda_2 g_2(t)}{\lambda_1}\right)^{\lambda_1/(\lambda_1-\lambda_2)}(g_1(t))^{\lambda_2/(\lambda_2-\lambda_1)}\\
&= (\lambda_1-\lambda_2)\left(\frac{\lambda_2^{\lambda_2}}{\lambda_1^{\lambda_1}}\right)^{1/(\lambda_1-\lambda_2)} g(t).
\end{aligned}
\tag{2.6}
$$

By (H_1)–(H_2) and (1.1), we get

$$
\begin{aligned}
y(t) &\overset{(1.2)}{=} y(c)+\frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1}[b(s)+k(s)x^\beta(s)+F(s,x(s))]\mathrm{d}s\\
&\le |y(c)|+\frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1}|b(s)|\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t-s)^{\alpha-1}k(s)\left|x^\beta(s)\right|\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}k(s)x^\beta(s)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t-s)^{\alpha-1}|F(s)|\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\left(g_2(s)x^{\lambda_2}(s)-g_1(s)x^{\lambda_1}(s)\right)\mathrm{d}s\\
&\overset{(2.6)}{\le} |y(c)|+\frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1}|b(s)|\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t_1-s)^{\alpha-1}k(s)\left|x^\beta(s)\right|\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}k(s)x^\beta(s)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t_1-s)^{\alpha-1}|F(s)|\mathrm{d}s\\
&\quad+\frac{\lambda_1-\lambda_2}{\Gamma(\alpha)}\left(\frac{\lambda_2^{\lambda_2}}{\lambda_1^{\lambda_1}}\right)^{1/(\lambda_1-\lambda_2)}\int_{t_1}^{t}(t-s)^{\alpha-1}g(s)\mathrm{d}s,
\end{aligned}
$$

i.e.,

$$
\left(a\left(x'\right)^\beta\right)^{(n-1)}(t)\le M_{n-1}+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}k(s)x^\beta(s)\mathrm{d}s, \tag{2.7}
$$

where, in view of (2.3) and (2.4), $M_{n-1} > 0$ is defined by

$$\begin{aligned} M_{n-1} := |y(c)| &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t_1 - s)^{\alpha-1} k(s) \left| x^\beta(s) \right| \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t_1 - s)^{\alpha-1} |F(s)| \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \lim_{t\to\infty} \int_c^t (t-s)^{\alpha-1} |b(s)| \mathrm{d}s \\ &+ \frac{\lambda_1 - \lambda_2}{\Gamma(\alpha)} \left(\frac{\lambda_2^{\lambda_2}}{\lambda_1^{\lambda_1}} \right)^{1/(\lambda_1-\lambda_2)} \lim_{t\to\infty} \int_{t_1}^t (t-s)^{\alpha-1} g(s) \mathrm{d}s. \end{aligned}$$

Integrating (2.7), unless $n = 1$, $(n-1)$ times from t_1 to t, each time reversing the order of integration in the double integral and evaluating the inner integral and using the recursion formula for the Gamma function, yields

$$\begin{aligned} a(t) \left(x'(t)\right)^\beta \le \sum_{\nu=0}^{n-1} M_\nu \frac{(t-t_1)^\nu}{\nu!} \\ + \frac{1}{\Gamma(\alpha+n-1)} \int_{t_1}^t (t-s)^{\alpha+n-2} k(s) x^\beta(s) \mathrm{d}s, \end{aligned} \tag{2.8}$$

where

$$M_\nu := \left| \left(a \left(x'\right)^\beta \right)^{(\nu)} (t_1) \right| \ge 0 \quad \text{for all} \quad 0 \le \nu < n-1.$$

Note that (2.8) is also correct when $n = 1$, compare (2.7). Hence, (2.8) holds for all $n \in \mathbb{N}$ and for all $t \ge t_1$. We continue to estimate (2.8) as

$$\begin{aligned} & a(t) \left(x'(t)\right)^\beta \\ &\le \sum_{\nu=0}^{n-1} M_\nu \frac{t^\nu}{\nu!} + \frac{t^{n-1}}{\Gamma(\alpha+n-1)} \int_{t_1}^t (t-s)^{\alpha-1} k(s) x^\beta(s) \mathrm{d}s \\ &= t^{n-1} \left(\sum_{\nu=0}^{n-1} M_\nu \frac{t^{\nu-n+1}}{\nu!} + \frac{1}{\Gamma(\alpha+n-1)} \int_{t_1}^t (t-s)^{\alpha-1} k(s) x^\beta(s) \mathrm{d}s \right) \\ &\le t^{n-1} \left(\sum_{\nu=0}^{n-1} \frac{M_\nu}{\nu!} + \frac{1}{\Gamma(\alpha+n-1)} \int_{t_1}^t (t-s)^{\alpha-1} k(s) x^\beta(s) \mathrm{d}s \right) \end{aligned}$$

for all $t \ge t_1 \ge c > 1$, and thus

$$a(t) \left(x'(t)\right)^\beta \le t^{n-1} \left(C_1 + C_2 \int_{t_1}^t (t-s)^{\alpha-1} k(s) x^\beta(s) \mathrm{d}s \right), \tag{2.9}$$

where $C_1, C_2 > 0$ are defined by

$$C_1 := \sum_{\nu=0}^{n-1} \frac{M_\nu}{\nu!} \quad \text{and} \quad C_2 := \frac{1}{\Gamma(\alpha+n-1)}.$$

Utilizing now Hölder's inequality as well as Lemma 2.1 for the integral on the right-hand side of (2.9), we obtain

$$\begin{aligned}
&\int_{t_1}^{t}(t-s)^{\alpha-1}k(s)x^{\beta}(s)\mathrm{d}s\\
&\quad=\int_{t_1}^{t}\left[(t-s)^{\alpha-1}e^{s}\right]\left[e^{-s}k(s)x^{\beta}(s)\right]\mathrm{d}s\\
&\quad\le\left(\int_{t_1}^{t}(t-s)^{p(\alpha-1)}e^{ps}\mathrm{d}s\right)^{1/p}\left(\int_{t_1}^{t}e^{-qs}k^{q}(s)x^{\beta q}(s)\mathrm{d}s\right)^{1/q}\\
&\quad\le\left(\int_{0}^{t}(t-s)^{p(\alpha-1)}e^{ps}\mathrm{d}s\right)^{1/p}\left(\int_{t_1}^{t}e^{-qs}k^{q}(s)x^{\beta q}(s)\mathrm{d}s\right)^{1/q}\\
&\quad\le(Qe^{pt})^{1/p}\left(\int_{t_1}^{t}e^{-qs}k^{q}(s)x^{\beta q}(s)\mathrm{d}s\right)^{1/q}\\
&\quad=Q^{1/p}e^{t}\left(\int_{t_1}^{t}e^{-qs}k^{q}(s)x^{\beta q}(s)\mathrm{d}s\right)^{1/q},
\end{aligned}$$

where $Q>0$ is given in Lemma 2.1, and using this in (2.9) yields

$$a(t)\left(x'(t)\right)^{\beta}\le t^{n-1}e^{t}\omega(t), \tag{2.10}$$

where

$$\omega(t):=C_1+C_3\left(\int_{t_1}^{t}e^{-qs}k^{q}(s)x^{\beta q}(s)\mathrm{d}s\right)^{1/q}\quad\text{with}\quad C_3:=C_2Q^{1/p}>0.$$

We rewrite (2.10) as

$$x'(t)\le\left(\frac{t^{n-1}e^{t}\omega(t)}{a(t)}\right)^{1/\beta}\quad\text{for}\quad t\ge t_1. \tag{2.11}$$

Noting that t^{n-1}, e^{t}, and $\omega(t)$ are all increasing, integrating (2.11) from t_1 to t yields that

$$\begin{aligned}
x(t)&\le x(t_1)+\int_{t_1}^{t}s^{(n-1)/\beta}e^{s/\beta}\omega^{1/\beta}(s)a^{-1/\beta}(s)\mathrm{d}s\\
&\le x(t_1)+t^{(n-1)/\beta}e^{t/\beta}\omega^{1/\beta}(t)\int_{t_1}^{t}a^{-1/\beta}(s)\mathrm{d}s\\
&=x(t_1)+t^{(n-1)/\beta}e^{t/\beta}\omega^{1/\beta}(t)A(t,t_1)\\
&=\left(\frac{x(t_1)}{t^{(n-1)/\beta}e^{t/\beta}A(t,t_1)}+\omega^{1/\beta}(t)\right)t^{(n-1)/\beta}e^{t/\beta}A(t,t_1)\\
&\le\left(\frac{x(t_1)}{t_2^{(n-1)/\beta}e^{t_2/\beta}A(t_2,t_1)}+\omega^{1/\beta}(t)\right)t^{(n-1)/\beta}e^{t/\beta}A(t,t_1)
\end{aligned}$$

holds for $t\ge t_2$ with $t_2>t_1$, and thus

$$\frac{x(t)}{t^{(n-1)/\beta}e^{t/\beta}A(t,t_1)}\le C_4+\omega^{1/\beta}(t)\quad\text{for}\quad t\ge t_2, \tag{2.12}$$

where

$$C_4 := \frac{x(t_1)}{t_2^{(n-1)/\beta} e^{t_2/\beta} A(t_2, t_1)} > 0.$$

Applying one of the elementary inequalities

$$(A+B)^\beta \le \begin{cases} 2^{\beta-1}(A^\beta + B^\beta) & \text{if} \quad \beta \ge 1 \\ A^\beta + B^\beta & \text{if} \quad 0 < \beta < 1 \end{cases} \tag{2.13}$$

with $A, B \ge 0$ to (2.12) gives

$$\left(\frac{x(t)}{t^{(n-1)/\beta} e^{t/\beta} A(t, t_1)} \right)^\beta \le C_5 + C_6 \omega(t) \quad \text{for} \quad t \ge t_2, \tag{2.14}$$

where $C_5, C_6 > 0$ are defined by

$$C_5 := \begin{cases} 2^{\beta-1} C_4^\beta & \text{if} \quad \beta \ge 1 \\ C_4^\beta & \text{if} \quad 0 < \beta < 1 \end{cases}$$

and

$$C_6 := \begin{cases} 2^{\beta-1} & \text{if} \quad \beta \ge 1 \\ 1 & \text{if} \quad 0 < \beta < 1. \end{cases}$$

Recalling the definition of $\omega(t)$, (2.14) implies that

$$\left(\frac{x(t)}{t^{(n-1)/\beta} e^{t/\beta} A(t, t_1)} \right)^\beta \le C_7 + C_8 \left(\int_{t_1}^t e^{-qs} k^q(s) x^{\beta q}(s) \mathrm{d}s \right)^{1/q}$$

holds for $t \ge t_2$, where

$$C_7 := C_5 + C_1 C_6 > 0 \quad \text{and} \quad C_8 := C_3 C_6 > 0,$$

from which, by employing again (2.13), we get

$$\left(\frac{x(t)}{t^{(n-1)/\beta} e^{t/\beta} A(t, t_1)} \right)^{\beta q} \le C_9 + C_{10} \int_{t_1}^t e^{-qs} k^q(s) x^{\beta q}(s) \mathrm{d}s \tag{2.15}$$

for $t \ge t_2$, where

$$C_9 := 2^{q-1} C_7^q > 0 \quad \text{and} \quad C_{10} := 2^{q-1} C_8^q.$$

Denoting the left-hand side of (2.15) by $w(t)$, (2.15) yields

$$w(t) \le C_9 + C_{10} \int_{t_1}^t k^q(s) s^{(n-1)q} A^{\beta q}(s, t_1) w(s) \mathrm{d}s$$

for $t \ge t_2$, and this can be rewritten as

$$w(t) \le C_{11} + C_{10} \int_{t_2}^t k^q(s) s^{(n-1)q} A^{\beta q}(s, t_1) w(s) \mathrm{d}s$$

for $t \ge t_2$, where

$$C_{11} := C_9 + C_{10} \int_{t_1}^{t_2} k^q(s) s^{(n-1)q} A^{\beta q}(s, t_1) w(s) \mathrm{d}s > 0.$$

Thanks to Gronwall's inequality and (2.2), $w(t)$ is bounded. Thus, (2.5) is established. □

We now give conditions that ensure that any nonoscillatory solution of (1.1) with $\beta = 1$ is bounded.

Theorem 2.4. *Assume (H_1)–(H_2) and $\beta = 1$. Let $p > 1$, $\alpha \in (0,1)$, $S > 0$, and $\sigma > 1$ be such that $p(\alpha - 1) + 1 > 0$ and*

$$\frac{t^{n-1}}{a(t)} \leq Se^{-\sigma t} \tag{2.16}$$

hold. If

$$\int_c^\infty k^q(s)e^{-qs}\mathrm{d}s < \infty, \quad \text{where} \quad q = \frac{p}{p-1}, \tag{2.17}$$

(2.3), and (2.4) hold, then every nonoscillatory solution of (1.1) is bounded.

Proof. Let x be a nonoscillatory solution of (1.1). As the case of eventually negative x can be treated similarly, we assume in this proof that x is eventually positive, i.e., there exists a $t_1 \geq c$ such that $x(t) > 0$ for all $t \geq t_1$. Exactly as in the proof of Theorem 2.3, we obtain (2.11) with $\beta = 1$. Recalling that $\omega(t)$ is increasing, integrating (2.11) from t_1 to t yields

$$\begin{aligned}
x(t) &\leq x(t_1) + \int_{t_1}^t \frac{u^{n-1}e^u\omega(u)}{a(u)}\mathrm{d}u \\
&\overset{(2.16)}{\leq} x(t_1) + \int_{t_1}^t Se^{(1-\sigma)u}\omega(u)\mathrm{d}u \\
&\leq x(t_1) + \int_{t_1}^t Se^{(1-\sigma)u}\omega(t)\mathrm{d}u \\
&= x(t_1) + S\omega(t)\left(\frac{e^{(1-\sigma)t_1}}{\sigma-1} - \frac{e^{(1-\sigma)t}}{\sigma-1}\right) \\
&\leq x(t_1) + S\omega(t)\frac{e^{(1-\sigma)t_1}}{\sigma-1},
\end{aligned}$$

and hence, by the definition of $\omega(t)$, we have

$$x(t) \leq C_{12} + C_{13}\left(\int_{t_1}^t e^{-qs}k^q(s)x^q(s)\mathrm{d}s\right)^{1/q} \tag{2.18}$$

for $t \geq t_1$, where

$$C_{12} := x(t_1) + C_1S\frac{e^{(1-\sigma)t_1}}{\sigma-1} > 0 \quad \text{and} \quad C_{13} := C_3S\frac{e^{(1-\sigma)t_1}}{\sigma-1} > 0.$$

Employing (2.13), (2.18) yields

$$x^q(t) \leq C_{14} + C_{15}\int_{t_1}^t e^{-qs}k^q(s)x^q(s)\mathrm{d}s \tag{2.19}$$

for $t \geq t_1$, where

$$C_{14} := 2^{q-1}C_{12}^q > 0 \quad \text{and} \quad C_{15} := 2^{q-1}C_{13}^q > 0.$$

From Gronwall's inequality and (2.17), we conclude that x is bounded. ∎

Remark 2.5. In this remark, we would like to compare our presented results with the results offered in [17,19]. Just like in our work, Lemmas 2.1 and 2.2 are utilized in both [17,19].

1. First, in 2019, Grace, Graef, and Tunç considered in [17] the same equation (1.1) with $\beta = 1$. In the definition of y, the cases $n = 1$ and $n = 2$ were allowed. They proved two results, [17, Theorem 2.1] in the case $n = 2$ and [17, Theorem 2.2] in the case $n = 1$, both of which are special cases of the above Theorem 2.4. Note that our Theorem 2.4 also has the restriction $\beta = 1$, but it allows $n \in \mathbb{N}$ to be arbitrary.
2. Next, in 2020, the same authors considered in [19] the same equation (1.1) with $\beta \geq 1$. In the definition of y, again the cases $n = 1$ and $n = 2$ were allowed. [19, Theorem 2.3] in the case $n = 2$ and [19, Theorem 2.4] in the case $n = 1$, both of which are special cases of the above Theorem 2.3. Note that our Theorem 2.3 allows for $\beta > 0$ and $n \in \mathbb{N}$ be arbitrary. Note also that our proof of Theorem 2.3 is, in the special cases $n = 1$ and $n = 2$, slightly different from the proofs in [19], which is mainly due to the fact that we could not verify that the function φ in [19, (2.25)] was indeed an increasing function (case $n = 1$), while our function ω presented in the proof of Theorem 2.3 is increasing for any $n \in \mathbb{N}$, also for $n = 1$.

3. Examples

We conclude this paper with two examples to illustrate our results.

Example 3.1. Consider the equation

$$\begin{aligned} &{}^{\mathrm{C}}D_8^{1/2}\left(t(x'(t))^3\right)''' + f_1(t,x(t)) \\ &\quad = e^{-4t}\sin t + \frac{1}{1+t^6}x^3(t) + f_2(t,x(t)), \quad t \geq 8. \end{aligned} \tag{3.1}$$

Hence, (3.1) is in the form (1.1) with

$$\begin{aligned} &y(t) = \left(t(x'(t))^3\right)''', \quad n = 4, \quad \alpha = \frac{1}{2}, \quad c = 8, \\ &\beta = 3, \quad a(t) = t, \quad b(t) = e^{-4t}\sin t, \quad k(t) = \frac{1}{1+t^6}, \end{aligned}$$

and

$$A(t,c) = A(t,8) = \int_8^t s^{-1/3}\mathrm{d}s = \frac{3}{2}\left(t^{2/3} - 4\right).$$

Then, it is easy to see that (H_1) holds. Putting $p = 3/2$, we get $q = 3$ and $p(\alpha - 1) + 1 = 1/4 > 0$. Letting

$$f_1(t,x) = g_1(t)|x|^{\lambda_1 - 1}x \quad \text{and} \quad f_2(t,x) = g_2(t)|x|^{\lambda_2 - 1}x$$

with $\lambda_1 > \lambda_2$ and $g_1(t) = g_2(t) = e^{-4t}$, we see that (H_2) holds. Moreover,

$$g(t) = \left(\frac{g_2^{\lambda_1}(t)}{g_1^{\lambda_2}(t)}\right)^{1/(\lambda_1 - \lambda_2)} = e^{-4t}.$$

Since

$$\int_c^\infty k^q(s)s^{(n-1)q}A^{\beta q}(s,c)\mathrm{d}s \le \left(\frac{3}{2}\right)^9 \int_8^\infty \frac{s^{15}}{(1+s^6)^3}\mathrm{d}s \le \left(\frac{3}{2}\right)^9 \int_8^\infty \frac{\mathrm{d}s}{s^3} < \infty,$$

(2.2) holds. Applying the substitution $u = t - s + 8$, we find

$$\begin{aligned}
&\int_c^t (t-s)^{\alpha-1}|b(s)|\mathrm{d}s \\
&\quad = \int_8^t (t-s)^{-1/2}\left|e^{-4s}\sin s\right|\mathrm{d}s \\
&\quad \le \int_8^t (t-s)^{-1/2}e^{-4s}\mathrm{d}s \\
&\quad = \int_8^t (u-8)^{-1/2}e^{4u-4t-32}\mathrm{d}u \\
&\quad \le \frac{1}{e^{4t+32}}\int_8^t (u-8)^{-1/2}e^{4u}\mathrm{d}u \\
&\quad = \frac{1}{e^{4t+32}}\left(\int_8^{16}(u-8)^{-1/2}e^{4u}\mathrm{d}u + \int_{16}^t (u-8)^{-1/2}e^{4u}\mathrm{d}u\right) \\
&\quad = \frac{1}{e^{4t+32}}\left(\lim_{b\to 8^+}\int_b^{16}(u-8)^{-1/2}e^{4u}\mathrm{d}u + \int_{16}^t (u-8)^{-1/2}e^{4u}\mathrm{d}u\right) \\
&\quad \le \frac{e^{64}}{e^{4t+32}}\lim_{b\to 8^+}\int_b^{16}(u-8)^{-1/2}\mathrm{d}u + \frac{(16-8)^{-1/2}}{e^{4t+32}}\int_{16}^t e^{4u}\mathrm{d}u \\
&\quad = \frac{2^{5/2}e^{64}}{e^{4t+32}} + \frac{2^{-7/2}}{e^{4t+32}}\left(e^{4t}-e^{64}\right) < \infty \quad \text{as} \quad t\to\infty,
\end{aligned}$$

so (2.3) holds, and similarly

$$\int_c^t (t-s)^{\alpha-1}g(s)\mathrm{d}s = \int_8^t (t-s)^{-1/2}e^{-4s}\mathrm{d}s < \infty,$$

so (2.4) holds. Since all assumptions of Theorem 2.3 are satisfied, we may conclude that every nonoscillatory solution x of (3.1) satisfies (2.5), i.e.,

$$\limsup_{t\to\infty}\frac{|x(t)|}{e^{t/3}t\left(t^{2/3}-4\right)} < \infty.$$

Example 3.2. Consider the equation

$$\begin{aligned}
&{}^{\mathrm{C}}D_2^{3/4}\left(2t^4e^{5t}x'(t)\right)'''' + f_1(t,x(t)) \\
&\quad = e^{-t}\cos t + e^{t/4}x(t) + f_2(t,x(t)), \quad t \ge 2. \qquad (3.2)
\end{aligned}$$

Hence, (3.2) is in the form (1.1) with

$$y(t) = \left(2t^4e^{5t}x'(t)\right)'''', \quad n = 5, \quad \alpha = \frac{3}{4}, \quad c = 2,$$
$$\beta = 1, \quad a(t) = 2t^4e^{5t}, \quad b(t) = e^{-t}\cos t, \quad k(t) = e^{t/4}.$$

Clearly, (H_1) is satisfied. With $S = 1/2$ and $\sigma = 5$, we see that (2.16) holds. Putting $p = 2$, we get $q = 2$ and $p(\alpha - 1) + 1 = 1/2 > 0$. Moreover,

$$\int_c^\infty k^q(s)e^{-qs}\mathrm{d}s = \int_2^\infty e^{s/2}e^{-2s}\mathrm{d}s = \int_2^\infty e^{-3s/2}\mathrm{d}s < \infty,$$

so (2.17) holds. Letting

$$f_1(t,x) = g_1(t)|x|^{\lambda_1-1}x \quad \text{and} \quad f_2(t,x) = g_2(t)|x|^{\lambda_2-1}x$$

with $\lambda_1 > \lambda_2$ and $g_1(t) = g_2(t) = e^{-t}$, so that $g(t) = e^{-t}$, we see that (H_2), (2.3), and (2.4) hold. Since all assumptions of Theorem 2.4 are satisfied, we may conclude that every nonoscillatory solution of (3.2) is bounded.

4. Conclusions

In this paper, we considered a higher-order fractional differential equation of Caputo type. By employing an equivalent representation in form of a Volterra-type equation as well as Young's inequality, we derived some new oscillation criteria. These criteria contain some previously published results for special cases of our general equation. We presented also two examples, which cannot be treated by the methods available in the literature thus far.

References

[1] Diethelm, K.: The analysis of fractional differential equations. Lecture Notes in Mathematics, vol. 2004. Springer, Berlin (2010). https://doi.org/10.1007/978-3-642-14574-2

[2] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)

[3] Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)

[4] Podlubny, I.: Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)

[5] Frassu, S., van der Mee, C., Viglialoro, G.: Boundedness in a nonlinear attraction-repulsion Keller-Segel system with production and consumption. J. Math. Anal. Appl. **504**(2), 20 (2021). https://doi.org/10.1016/j.jmaa.2021.125428

[6] Li, T., Pintus, N., Viglialoro, G.: Properties of solutions to porous medium problems with different sources and boundary conditions. Z. Angew. Math. Phys. **70**(3), 18 (2019). https://doi.org/10.1007/s00033-019-1130-2

[7] Li, T., Pintus, N., Viglialoro, G.: Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime. Differ. Integral Equa. **34**(5/6), 315–336 (2021)

[8] Aslıyüce, S., Güvenilir, A.F., Zafer, A.: Oscillation criteria for a certain class of fractional order integro-differential equations. Hacet. J. Math. Stat. **46**(2), 199–207 (2017)

[9] Benchohra, M., Hamani, S., Nieto, J.: Oscillation and nonoscillation for Caputo-Hadamard impulsive fractional differential equations. Nonlinear Dyn. Syst. Theory **19**(4), 474–485 (2019)

[10] Bohner, M., Grace, S., Sultana, N.: Asymptotic behavior of nonoscillatory solutions of higher-order integro-dynamic equations. Opuscula Math. **34**(1), 5–14 (2014). https://doi.org/10.7494/OpMath.2014.34.1.5

[11] Brestovanská, E., Medved, M.: Asymptotic behavior of solutions to second-order differential equations with fractional derivative perturbations. Electron. J. Differ. Equ. **2014**, 201 (2014)

[12] Feng, L., Han, Z.: Oscillation behavior of solution of impulsive fractional differential equation. J. Appl. Anal. Comput. **10**(1), 223–233 (2020). https://doi.org/10.11948/20190133

[13] Feng, L., Sun, Y., Han, Z.: Philos-type oscillation criteria for impulsive fractional differential equations. J. Appl. Math. Comput. **62**(1–2), 361–376 (2020). https://doi.org/10.1007/s12190-019-01288-5

[14] Grace, S.R.: On the oscillatory and asymptotic behavior of solutions of certain integral equations. Mediterr. J. Math. **13**(5), 2721–2732 (2016). https://doi.org/10.1007/s00009-015-0649-5

[15] Grace, S.R.: On the asymptotic behavior of nonoscillatory solutions of certain fractional differential equations. Mediterr. J. Math. **15**(2), 11 (2018). https://doi.org/10.1007/s00009-018-1120-1

[16] Grace, S.R., Agarwal, R.P., Wong, P.J.Y., Zafer, A.: On the oscillation of fractional differential equations. Fract. Calc. Appl. Anal. **15**(2), 222–231 (2012). https://doi.org/10.2478/s13540-012-0016-1

[17] Grace, S.R., Graef, J.R., Tunç, E.: On the boundedness of nonoscillatory solutions of certain fractional differential equations with positive and negative terms. Appl. Math. Lett. **97**, 114–120 (2019). https://doi.org/10.1016/j.aml.2019.05.032

[18] Grace, S.R., Zafer, A.: On the asymptotic behavior of nonoscillatory solutions of certain fractional differential equations. Eur. Phys. J. Spec. Top. **226**, 3657–3665 (2017)

[19] Graef, J.R., Grace, S.R., Tunç, E.: On the asymptotic behavior of nonoscillatory solutions of certain fractional differential equations with positive and negative terms. Opuscula Math. **40**(2), 227–239 (2020). https://doi.org/10.7494/opmath.2020.40.2.227

[20] Medved, M.: A new approach to an analysis of Henry type integral inequalities and their Bihari type versions. J. Math. Anal. Appl. **214**(2), 349–366 (1997). https://doi.org/10.1006/jmaa.1997.5532

[21] Medved, M.: Integral inequalities and global solutions of semilinear evolution equations. J. Math. Anal. Appl. **267**(2), 643–650 (2002). https://doi.org/10.1006/jmaa.2001.7798

[22] Medved, M.: Asymptotic integration of some classes of fractional differential equations. Tatra Mt. Math. Publ. **54**, 119–132 (2013). https://doi.org/10.2478/tmmp-2013-00010

[23] Medved, M., Pospíšil, M.: Asymptotic integration of fractional differential equations with integrodifferential right-hand side. Math. Model. Anal. **20**(4), 471–489 (2015). https://doi.org/10.3846/13926292.2015.1068233

[24] Meng, Q., Jin, Z., Liu, G.: Sufficient and necessary conditions for oscillation of linear fractional-order delay differential equations. Adv. Differ. Equ. **89**, 9 (2021). https://doi.org/10.1186/s13662-021-03223-0

[25] Saker, S.H., Logaarasi, K., Sadhasivam, V.: Forced oscillation of conformable fractional partial delay differential equations with impulses. Ann. Univ. Mariae Curie-Skłodowska Sect. A **74**(2), 61–80 (2020)

[26] Tunç, C., Tunç, O.: New results on the stability, integrability and boundedness in Volterra integro-differential equations. Bull. Comput. Appl. Math. **6**(1), 41–58 (2018)

[27] Uzun, T.Y., Büyükçavuşoğlu Erçolak, H., Yıldız, M.K.: Oscillation criteria for higher order fractional differential equations with mixed nonlinearities. Konuralp J. Math. **7**(1), 203–207 (2019)

[28] Zeng, W., Li, D.: Oscillation of a class of nonlinear fractional differential equations with damping term. J. Zhejiang Univ. Sci. Ed. **48**(1), 35–40 (2021)

[29] Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge mathematical library. Cambridge University Press, Cambridge (1988). (**Reprint of the 1952 edition**)

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