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Delay dynamic equations on isolated time scales and the relevance of one-periodic coefficients

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We are motivated by the idea that certain properties of delay differential and difference equations with constant coefficients arise as a consequence of their one-periodic nature. We apply the recently introduced definition of periodicity for arbitrary isolated time scales to linear delay dynamic equations and a class of nonlinear delay dynamic equations. Utilizing a derived identity of higher order delta derivatives and delay terms, we rewrite the considered linear and nonlinear delayed dynamic equations with one-periodic coefficients as a linear autonomous dynamic system with constant matrix. As the simplification of a constant matrix is only obtained for one-periodic coefficients, dynamic equations with one-periodic coefficients are the simplest form compared to the commonly used constant coefficients.

KEYWORDS

delay, dynamic equations, periodicity, stability, time scales

MSC CLASSIFICATION

34N05; 39A12; 39A06; 39A60; 39A70

1 | INTRODUCTION

When studying differential equations and difference equations, a common simplification is the assumption of constant coefficients. For example, the delay differential equation with constant coefficient

$$y'(t) = cy(t-\tau), \qquad y(t) = \phi(t), \qquad t \in [-\tau, 0],$$

where $c \in \mathbb{R}$ has the solution[1]

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{\frac{1}{\tau} W_k(-c\tau)t},$$

where C_k is determined by $\phi(t)$ and the W_k refer to the Lambert W function. The main reason is that the analysis simplifies significantly under the assumption of a constant coefficient, compared to time-dependent coefficients.

Similarly, in the study of difference equations, higher order recurrences are often introduced by first considering $y_{t+k} = cy_t$, see for example, Kelley and Peterson.[2], Section 3.3 As in the continuous case, the assumption of a constant coefficient simplifies the analysis compared to time-dependent coefficients as the roots λ to the characteristic polynomial can be used to construct a solution. Once the coefficients are time-dependent, this method cannot be applied since the characteristic polynomial will be time-dependent. Thus, the assumption of constant coefficients is commonly associated with simplifying the corresponding analysis of (delayed) differential and difference equations.

Not surprisingly, this is being also expected from dynamic equations on time scales, a theory that aims to unify the discrete and continuous calculus. Introduced by Stefan Hilger in 1988, the theory of time scales gains increasing attention due to its unifying character as well as its applications to real-life processes, see for example, other studies[3–9] and references therein. Due to its unification property, dynamic equations on time scales are a generalization of differential and difference equations. Thus, the study of dynamic equations on time scales replaces the separate study of corresponding models on different time domains, including the discrete and continuous ones. For example, the behavior of solutions of the logistic dynamic equation apply to the logistic differential equation as well as its time-scales discretization, which differs from the logistic map. This example further highlights that time scales can provide a discretization of a continuous model that differs to commonly considered discretizations but exhibits significantly different dynamics. The interested reader is referred to the introductory books.[10,11] Delay dynamic equations are well-studied objects, some prior investigations may be seen in other studies.[12–16] As mentioned, the assumption of constant coefficients is usually motivated by the simplification of the analysis in the case of constant coefficients. As we will argue in this manuscript, the assumption of one-periodic coefficients may be more appropriate since it simplifies the analysis to resemble the expected behavior from constant coefficients.

In this study, we focus on *isolated time scales* $\mathbb{T}_{\mathcal{I}}$, that is, every $t \in \mathbb{T}_{\mathcal{I}}$ has a positive distance to each of its closest neighbors. Popular examples of such isolated time scales are the discrete domain $\mathbb{Z}$ and the quantum time scales $q^{\mathbb{N}_0} = \{1, q, q^2, \dots\}$ with $q > 1$. The restriction to isolated time scales allows the applications of the recently introduced definition of *periodicity* in Bohner et al[17] and avoids the restrictive, but commonly applied, assumption of periodic time scales, see Wang et al[18] and the references therein. In the special case of $\mathbb{T} = \mathbb{Z}$, a function is ω-periodic if $f(n+\omega) = f(n)$ for all $n \in \mathbb{Z}$. Thus, every constant function $f(t) = c$ is also one-periodic, and every one-periodic function is constant in the discrete time scale. The simplified assumption of constant coefficients is therefore equivalent to the assumption of one-periodic coefficients if $\mathbb{T}_{\mathcal{I}} = \mathbb{Z}$. However, for an arbitrary time scale $\mathbb{T}_{\mathcal{I}}$, a constant is not necessarily one-periodic. In fact, if $\mathbb{T}_{\mathcal{I}} = q^{\mathbb{N}_0}$ with $q > 1$, then a one-periodic function $f : \mathbb{T}_{\mathcal{I}} \to \mathbb{R}$ is of the form $f(t) = \frac{c}{t(q-1)}$ with $c \in \mathbb{R}$. Thus, constant functions differ from one-periodic functions on general time scales. While one assumption may simplify the analysis significantly, the other assumption may not. The question we aim to address in this work is whether assuming constant coefficients or one-periodic coefficients simplifies the analysis of delayed dynamic equations.

In Section 2, we give a brief introduction of time scales and the recently introduced periodicity for isolated time scales. We also derive an identity between delay terms and higher order delta derivatives. In Section 3, we first focus on linear delay dynamic equations with one-periodic coefficients and compare the result to the assumption of constant coefficients. We then, in Section 3.2, discuss a class of nonlinear delay dynamic equations with the assumption of one-periodic coefficients. We argue that the periodicity property simplifies the analysis, not the assumption of constant coefficients.

2 | PERIODIC FUNCTIONS ON TIME SCALES

A nonempty closed subset of the real numbers, a so-called time scale $\mathbb{T}$, is *isolated* if $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} > t$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\} < t$ for all $t \in \mathbb{T}$. Henceforth, $\mathbb{T}_{\mathcal{I}}$ represents an isolated time scale so that the following periodicity definition is well-defined.

Definition 1 (see Bohner et al[17, Definition 4.1 *]). A function $f : \mathbb{T}_{\mathcal{I}} \to \mathbb{C}$ is ω-periodic provided

$$(\sigma^{\omega})^{\Delta} p^{\sigma^{\omega}} = p, \tag{1}$$

or equivalently by using the definition of the Δ-derivative,

$$(\mu p)^{\sigma^{\omega}} = \mu p, \tag{2}$$

*We note that although the original definition in Bohner et al[17] only considered $f : \mathbb{T}_{\mathcal{I}} \to \mathbb{R}$, the definition can be extended to $f : \mathbb{T}_{\mathcal{I}} \to \mathbb{C}$. We furthermore modify the original definition to consider $\mathbb{T}_{\mathcal{I}}$ with finite cardinality by requiring (1) to hold for all $t \in \mathbb{T}_{\mathcal{I}}$ with $t \leq \rho^{\omega}(t_m)$, where $t_m = \max\{t \in \mathbb{T}_{\mathcal{I}}\}$ if such t_m exists.

where $\sigma^{\omega} = \underbrace{\sigma \circ \sigma \circ \ldots \circ \sigma}_{\omega}$. The set of all ω-periodic functions $p : \mathbb{T}_{\mathcal{I}} \to \mathbb{C}$ is denoted by $\mathcal{P}_{\omega}$.

By Bohner et al,[17, Theorem 4.9] we have for $t, t_0 \in \mathbb{T}_{\mathcal{I}}$ and $p \in \mathcal{P}_{\omega}$ regressive,

$$e_p(t, \rho^{\omega}(t)) = e_p(t_0, \rho^{\omega}(t_0)). \tag{3}$$

In Bohner et al,[17, Lemma 4.6] the authors show that for all $\omega \in \mathbb{N}_1 = \{1, 2, \ldots\}$, $\mathcal{P}_{\omega} \subset \mathcal{P}_{2\omega}$, that is, all ω-periodic functions are 2ω-periodic. We generalize this result in the proceeding lemma.

Lemma 2.1. *If $m|n$ and p is m-periodic, then p is n-periodic.*

Proof. Since $m|n$, we may write $n = md$ for some $d \in \mathbb{N}_1$. If $d = 1$, then the statement is trivially true. Now suppose the statement is true for $d > 1$, and compute for $n = m(d+1)$,

$$(\mu p)^{\sigma^n} = \left((\mu p)^{\sigma^{md}}\right)^{\sigma^m} \overset{(2)}{=} (\mu p)^{\sigma^m} \overset{(2)}{=} \mu p,$$

completing the proof. □

One of the results we will use is the generalization of the well-known discrete identity

$$f(t+n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(t). \tag{4}$$

We generalize (4) to arbitrary isolated time scales in Lemma 2.2 and will see that it simplifies the study of delay dynamic equations significantly. It also provides the functional structure of periodic functions, see Theorem 2.3, extending a result in Bohner et al[17] to higher periods.

Lemma 2.2. *If $\mathbb{T}_{\mathcal{I}}$ is an isolated time scale, $n \in \mathbb{N}_0$, and $f : \mathbb{T}_{\mathcal{I}} \to \mathbb{C}$, then*

$$f^{\sigma^n} = \sum_{k=0}^{n} \binom{n}{k} F_k, \tag{5}$$

where $F_0 = f$ and $F_{k+1} = \mu F_k^{\Delta}$ for all $k \in \mathbb{N}_0$.

Proof. The delta derivative of f on $\mathbb{T}_{\mathcal{I}}$ is $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$. We rewrite this as $\hat{\sigma} f = \left(\mu \hat{\Delta} + \mathrm{id}\right) f$, where $(\hat{\sigma} f)(t) = f(\sigma(t))$ and $\left(\hat{\Delta} f\right)(t) = f^{\Delta}(t)$. Using an operator version of the binomial theorem, compute $\hat{\sigma}^n = \left(\mu \hat{\Delta} + \mathrm{id}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\mu \hat{\Delta}\right)^k$. The formula (5) follows after multiplying by f on the right, completing the proof. □

Equation (5) can be used to identify the structure of periodic functions, generalizing Theorem 5.1 in Bohner et al.,[17] which classified one-periodic functions as

$$f \in \mathcal{P}_1 \text{ if and only if there exists } c \in \mathbb{R} \text{ such that } f(t) = \frac{c}{\mu(t)} \text{ for all } t \in \mathbb{T}_{\mathcal{I}}. \tag{6}$$

This implies in particular that for $\mathbb{T}_{\mathcal{I}} = \mathbb{Z}$, f is one-periodic if $f(t) \equiv C \in \mathbb{R}$, that is, f is a constant function. This is consistent with our understanding of periodic functions on $\mathbb{Z}$. Consider now the isolated time scale $\mathbb{T}_{\mathcal{I}} = \{t_i\}_{i \in \mathbb{N}_0}$, with $t_0 = 0$ and $t_{i+1} = t_i + 2 + (-1)^i$ for $i \in \mathbb{N}$. The natural choice of a constant function as one-periodic does not coincide with the introduced definition of periodicity on this time scales. Instead, a function $f : \mathbb{T}_{\mathcal{I}} \to \mathbb{R}$ such that $f(t_i) = 1$ for even $i \in \{0, 2, 4, \ldots\}$ and $f(t_j) = 3$ for odd $j \in \{1, 3, 5, \ldots\}$ is one-periodic as it satisfies $f(t) = \frac{3}{\mu(t)}$ for $t \in \mathbb{T}_{\mathcal{I}}$. This function, in contrast to any constant function, is invariant with respect to integration in the sense that $\int_{t_i}^{t_i+1} f(s)\,\Delta s = \int_{t_j}^{t_j+1} f(s)\,\Delta s$ for any $t_i, t_j \in \mathbb{T}_{\mathcal{I}}$. For more examples of periodic functions on isolated time scales, we refer the reader to Section 5 in Bohner et al.[17]

Theorem 2.3. *$f \in \mathcal{P}_\omega$ if and only if there exists $\mathbf{c} = (c_1, \dots, c_\omega)^T \in \mathbb{C}^\omega$ such that $f(t) = \frac{c_{i(t)}}{\mu(t)}$, where $i(t) = 1 + (k \bmod \omega)$ for $t = \sigma^k(t_0)$.*

Proof. Let $\mathbf{c} = (c_1, \dots, c_\omega)^T \in \mathbb{C}^\omega$ and suppose $f(t) = \frac{c_{i(t)}}{\mu(t)}$. Then

$$\mu(\sigma^\omega(t)) f(\sigma^\omega(t)) = c_{i\left(\sigma^{\omega(t)}\right)} = c_{i(t)} = \mu(t) f(t).$$

Thus, by (2), $f \in \mathcal{P}_\omega$. To prove the converse, let f be ω-periodic. Then, f satisfies (2), to which we apply (5) and obtain

$$\mu(\sigma^\omega(t)) \sum_{i=0}^{\omega} \binom{\omega}{i} F_i(t) = \mu(t) f(t). \tag{7}$$

If $F = (F_0, F_1, \dots, F_{\omega-1})^T$, then (7) together with the definition of F_i becomes

$$F^\Delta(t) = A(t)F(t) \qquad \text{with} \qquad A(t) := \begin{bmatrix} \mathbf{0}_{(\omega-1)\times 1} & \frac{1}{\mu(t)} I_{\omega-1} \\ \frac{1}{\mu(\sigma^\omega(t))} - \frac{1}{\mu(t)} & -\frac{1}{\mu(t)}\mathbf{s} \end{bmatrix}, \tag{8}$$

where $\mathbf{s} = \left(\binom{\omega}{1}, \binom{\omega}{2}, \dots, \binom{\omega}{\omega-1} \right)$. Thus, (8) is a first-order linear dynamic system. We note that

$$B(t) := I_\omega + \mu(t)A(t) = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \frac{1}{(\sigma^\omega)^\Delta(t)} - 1 & -\binom{\omega}{1} & -\binom{\omega}{2} & -\binom{\omega}{3} & \dots & 1 - \binom{\omega}{\omega-1} \end{bmatrix}.$$

We have

$$\begin{aligned} \det(B) &= (-1)^{\omega+1} \left\{ \sum_{j=1}^{\omega} (-1)^{j-1} B_{\omega,j} \right\} \\ &= (-1)^{\omega+1} \left\{ \frac{1}{(\sigma^\omega)^\Delta(t)} - 1 + \sum_{j=1}^{\omega-1} -(-1)^j \binom{\omega}{j} + (-1)^{\omega-1} \right\} \\ &= (-1)^{\omega+1} \left\{ \frac{1}{(\sigma^\omega)^\Delta(t)} + (-1) \sum_{j=0}^{\omega} (-1)^j \binom{\omega}{j} \right\} \\ &= \frac{(-1)^{\omega+1}}{(\sigma^\omega)^\Delta(t)} \neq 0, \end{aligned}$$

where, by the binomial expansion formula, $\sum_{j=0}^{\omega} (-1)^j \binom{\omega}{j} = 0$. Therefore, B is invertible, which implies that A is regressive. Based on Theorem 5.8 in Bohner et al,[11] (8) with initial value $F(0)$ has a unique solution. This means, that if ω consecutive values of a ω-periodic function are known, then there is only one function $f : \mathbb{T}_\mathcal{I} \to \mathbb{R}$ that satisfies (1). This completes our claim, since $f(t) = \frac{c_{i(t)}}{\mu(t)}$ is a solution and satisfies the initial value $\mathbf{c} = F(0)$. □

Thus, the identity (5) was utilized in Theorem 2.3 to generalize the result in Theorem 5.1 in Bohner et al,[17] where one-periodic functions were classified, to periodic functions of higher periods. This useful identity will also be applied in the proceeding section to simplify the expression of delay dynamic equations on isolated time scales.

3 | DELAY DYNAMIC EQUATIONS WITH ONE-PERIODIC COEFFICIENTS

In this section, we apply the definition of periodicity as well as (5) to discuss the behavior of solutions to delay dynamic equations. We will first focus on linear delay dynamic equations before considering a class of nonlinear delay equations, also known as the delay logistic dynamic equation.

3.1 | Linear delay dynamic equations

We begin our analysis with a class of linear dynamic equations of the form

$$y^{\Delta}(t) = p(t)y(\rho^k(t)), \tag{9}$$

with $p : \mathbb{T}_{\mathcal{I}} \to \mathbb{R}$ and $k \in \mathbb{N}_1$. We argue in this section that (9) with $p \in \mathcal{P}_1$ should be considered as the (isolated) time scales analog of the difference equation

$$\Delta y_t = cy_{t-k}, \tag{10}$$

where $c \in \mathbb{R}$, or equivalently, $y_{t+1} - y_t = cy_{t-k}$. We note that y_{t+1} depends not only on y_t but also the k-th previous step y_{t-k}, making (10) a delay difference equation.[19] Since $\mathbb{T}_{\mathcal{I}}$ is an isolated time scale, we express (9) as

$$\frac{y^{\sigma}(t) - y(t)}{\mu(t)} = p(t)y(\rho^k(t)), \qquad t \in [t_0, \infty) \cap \mathbb{T}_{\mathcal{I}},$$

and by algebra, we obtain

$$y^{\sigma^{k+1}}(t) - y^{\sigma^k}(t) = \mu^{\sigma^k}(t)p^{\sigma^k}(t)y(t), \qquad t \in [\sigma^k(t_0), \infty) \cap \mathbb{T}_{\mathcal{I}}.$$

Thus, by (5),

$$\sum_{j=0}^{k+1} \binom{k+1}{j} Y_j - \sum_{j=0}^{k} \binom{k}{j} Y_j = \mu^{\sigma^k} p^{\sigma^k} y,$$

that is,

$$Y_{k+1} = \mu^{\sigma^k} p^{\sigma^k} Y_0 - \sum_{j=0}^{k} \left(\binom{k+1}{j} - \binom{k}{j} \right) Y_j = \mu^{\sigma^k} p^{\sigma^k} Y_0 - \sum_{j=1}^{k} \binom{k}{j-1} Y_j,$$

that is,

$$Y^{\Delta} = A(t)Y \qquad \text{with} \qquad A(t) := \frac{1}{\mu(t)} \begin{bmatrix} \mathbf{0}_k & I_k \\ \mu^{\sigma^k}(t)p^{\sigma^k}(t) & -\mathbf{s} \end{bmatrix}, \tag{11}$$

where $\mathbf{0}_k = (0, 0, \dots, 0)^T \in \mathbb{R}^{1\times k}$ and $\mathbf{s} = \left(\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k-1} \right) \in \mathbb{R}^{k\times 1}$. Define

$$B(t) := I_{k+1} + \mu(t)A(t) = \begin{bmatrix} \mathbf{e}_1 & I_k + L_k \\ \mu^{\sigma^k}(t)p^{\sigma^k}(t) & \mathbf{e}_k^T - \mathbf{s} \end{bmatrix}, \tag{12}$$

where L_k is a matrix in $\mathbb{R}^{k\times k}$ with zeros except for ones in the subdiagonal and $\mathbf{e}_i^T = \left(0, 0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0 \right) \in \mathbb{R}^{k\times 1}$.

Expanding around the last row of (12), we obtain

$$\det(I_{k+1} + \mu(t)A(t)) = (-1)^k \mu^{\sigma^k}(t)p^{\sigma^k}(t). \tag{13}$$

Thus, if A is regressive, then the explicit solution to (11) is given by

$$Y(t) = e_A(t, t_0)Y(t_0) = \prod_{\tau \in [t_0, t)\cap \mathbb{T}_{\mathcal{I}}} (I_{k+1} + \mu(\tau)A(\tau))\, Y(t_0).$$

If $p \in \mathcal{P}_1$, $p \neq 0$, then $\mu^{\sigma^k} p^{\sigma^k} \in \mathbb{R}\backslash\{0\}$, from which we immediately conclude that A is regressive. Since $p \in \mathcal{P}_1$, $p \neq 0$ implies by Theorem 2.3 $p = \frac{c}{\mu}$ for $c \in \mathbb{R}\backslash\{0\}$, and the expression of B in (12) simplifies to a constant matrix. Thus, the following theorem is obtained.

Theorem 3.1. *Let $p \in \mathcal{P}_1$, $p \neq 0$. The unique solution to (9) with initial values $y(t_0), y(\sigma(t_0)), \ldots, y(\sigma^k(t_0))$ is given by*

$$y(t) = \mathbf{e}_1^T e_A(t, t_0) F(t_0) = \prod_{\tau \in [t_0, t) \cap \mathbb{T}_\mathcal{I}} \mathbf{e}_1^T \left(I_{k+1} + \mu(\tau) A(\tau)\right) F(t_0) = \mathbf{e}_1^T B^{\mathcal{N}(t,t_0)} Y(t_0), \tag{14}$$

where $\mathcal{N}(t, t_0) = card\left(\mathbb{T}_\mathcal{I} \cap [t_0, t)\right)$, $B(t) = I_{k+1} + \mu(t)A(t)$ is constant, and $Y(t_0) = (Y_0(t_0), Y_1(t_0), \ldots, Y_k(t_0))^T$, where $Y_0 = y$ and $Y_{i+1} = \mu Y_i^\Delta$.

Remark 3.2. In the case of $\mathbb{T}_\mathcal{I} = \mathbb{Z}$, the investigation of a delay difference equation is usually initiated with the assumption of constant coefficients, that is, $p \in \mathbb{R}$. In general, this simplifies the analysis and the corresponding expression of the solution. For arbitrary time scales, it is however not immediately clear whether the same type of simplification is to be expected from constant or one-periodic coefficients. We note that if p were chosen to be constant in (9), then $e_A(t, t_0) = I_{k+1} + \mu(t)A(t)$ is not constant because the first entry in the $(k+1)$st row, $A_{k+1,1}$, would be constant and $(\mu(t)A)_{k+1,1}$ would depend on t (unless $\mu(t)$ is independent of t, that is, $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$). Thus, the solution given in (14) would not simplify to powers of a constant matrix. As this type of simplification, here the powers of a constant matrix, is however a desirable property, the choice of a one-periodic coefficient seems arguably the preferred choice compared to constant coefficients.

By Theorem 3.1, the solution to the delay dynamic equation (9) reduces to the powers of a constant matrix B. An explicit expression for such a matrix can be obtained by realizing that B can be brought into a companion matrix.[20] If the calculation of powers of matrices is undesirable, then the convenient property of a constant matrix B allows alternatively the calculation of the solution via the roots of a polynomial with constant coefficients.

We recall the trigonometric functions on time scales as defined in Bohner and Peterson[10, Definition 3.25] as

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2}, \quad \sin_p = \frac{e_{ip} - e_{-ip}}{2i},$$

and they consequently obey Euler's formula

$$e_{ip} = \cos_p + i \sin_p. \tag{15}$$

Additionally, from the definition, it is immediately clear that $\sin_{-p} = -\sin_p$.

Theorem 3.3. *Let $p \in \mathcal{P}_1$ and let $x^* \in \mathbb{C}$ be a root of*

$$G(x) = x(1 + x)^k - \mu p. \tag{16}$$

(i) If $x^* \in \mathbb{R}$, then

$$y(t) = C e_q(t, t_0), \quad q(t) = \frac{x^*}{\mu(t)},$$

with arbitrary $C \in \mathbb{R}$, solves (9) for $t \geq \sigma^k(t_0)$.

(ii) If $x^* \notin \mathbb{R}$ (i.e., $x^* = x_1^* + i x_2^*$ with $x_2^* \neq 0$), then

$$y_1(t) = C e_\alpha(t, t_0) \cos_\beta (t, t_0) \qquad \text{and} \qquad y_2(t) = D e_\alpha(t, t_0) \sin_\beta (t, t_0)$$

with arbitrary $C, D \in \mathbb{R}$, $\alpha = \frac{x_1^*}{\mu(t)}$, and $\beta = \frac{x_2^*}{\mu(t)(1+x_1^*)}$, solve (9). For $C, D \neq 0$, y_1, y_2 are two independent solutions of (9).

Proof. Let $p \in \mathcal{P}_1$. By (6), there exists $c \in \mathbb{R}$ such that $\mu p = c$.

(i) Let $x^* \in \mathbb{R}$ be a solution of (16). Define $y(t) = C e_q(t, t_0)$ with $q(t) = \frac{x^*}{\mu(t)}$ for arbitrary $C \in \mathbb{R}$. Then, by (6), $q \in \mathcal{P}_1$ and so, by (3), $e_q(t, \rho^k(t)) = e_q(t_0, \rho^k(t_0))$. Thus,

$$y^\Delta(t) = C q(t) e_q(t, \rho^k(t)) e_q(\rho^k(t), t_0) \stackrel{(3)}{=} q(t) e_q(t_0, \rho^k(t_0)) y(\rho^k(t))$$

and since, by (16),

$$q(t)e_q(t_0,\rho^k(t_0)) = \frac{x^*}{\mu(t)} \prod_{s\in[\rho^k(t_0),t_0)\cap\mathbb{T}_I} \left(1+\mu(s)\frac{x^*}{\mu(s)}\right) = \frac{c}{\mu(t)}\frac{x^*(1+x^*)^k}{c} = \frac{c}{\mu(t)} = p(t),$$

the proof of (i) is complete.

(ii) Let now $x^* \notin \mathbb{R}$. Then $x^* = x_1 + ix_2$ for $x_1, x_2 \in \mathbb{R}$ is also a root of (16). Let $q(t) = \frac{x^*}{\mu(t)}$ and $y(t) = e_q(t,t_0)$. Then,

$$y^\Delta = Cq(t)e_q(t,t_0) = q(t)e_q(t,\rho^k(t))e_q(\rho^k(t),t_0) = q(t)\prod_{s\in[\rho^k(t),t)\cap\mathbb{T}_I}(1+\mu(s)q(s))\,y(\rho^k(t)) = p(t)y(\rho^k(t))$$

because

$$q(t)\prod_{s\in[\rho^k(t),t)\cap\mathbb{T}_I}(1+\mu(s)q(s)) = \frac{x^*}{\mu(t)}\prod_{s\in[\rho^k(t),t)\cap\mathbb{T}_I}(1+x^*) = \frac{x^*(1+x^*)^k}{\mu(t)} \overset{(16)}{=} \frac{c}{\mu(t)} = p(t).$$

Since $x^* = x_1^* + ix_2^*$ is a root of (16), $\overline{x^*} = x_1^* - ix_2^*$ is also a root of (16). Similarly, one can show that $\bar{y}(t) = De_{\bar{q}}(t,t_0)$ also solves the delay dynamic equation, where $\bar{q} = \frac{\overline{x^*}}{\mu(t)}$. Both of these solutions can be simplified by realizing that $q(t) = \frac{x^*}{\mu(t)} = \frac{x_1^*}{\mu(t)} + i\frac{x_2^*}{\mu(t)} = \alpha \oplus i\beta$ for $\alpha = \frac{x_1^*}{\mu(t)}$ and $\beta = \frac{x_2^*}{\mu(1+x_1^*)}$. Thus, $\bar{q}(t) = \alpha - i\beta$, and solutions of the delay dynamic equation follow from (15) as

$$y(t) = Ce_q(t,t_0) = Ce_\alpha(t,t_0)e_{i\beta}(t,t_0) = Ce_\alpha(t,t_0)\left(\cos_\beta(t,t_0) + i\,\sin_\beta(t,t_0)\right)$$

and

$$\bar{y}(t) = De_{\bar{q}}(t,t_0) = De_\alpha(t,t_0)e_{-i\beta}(t,t_0) = De_\alpha(t,t_0)\left(\cos_\beta(t,t_0) - i\,\sin_\beta(t,t_0)\right).$$

To obtain two real solutions z_1, z_2, we apply the superposition principle with $C = D \in \mathbb{R}$ to get

$$z_1(t) = y(t) + \bar{y}(t) = 2Ce_\alpha(t,t_0)\cos_\beta(t,t_0)$$

and for $C = D = \frac{A}{i}$ with $A \in \mathbb{R}$,

$$z_2(t) = y(t) + \bar{y}(t) = 2Ae_\alpha(t,t_0)\sin_\beta(t,t_0).$$

It is easy to verify that z_1, z_2 are real solutions and are elements of the set of fundamental solutions, completing the claim.

This completes the proof. □

Remark 3.4. If p were chosen to be constant, then μp depends on t. In this case, the roots of (16) are functions of t, increasing the complexity of the analysis outlined in Theorem 3.3. Thus, as argued for Theorem 3.1, the type of simplification expected from the analysis of equations with constant coefficients is obtained for one-periodic coefficients.

The explicit solution to (9) given in Theorem 3.1 may simplify the investigation of the global dynamics of solutions. The following result regarding the stability of the only equilibrium of (9), namely the trivial one, can be immediately concluded from Theorem 3.3.

Theorem 3.5. *If* $\sup\mathbb{T} = \infty$, $p \in \mathcal{P}_1$, *and there exists* $s \in \mathbb{T}$ *with* $p(s) > 0$, *then the trivial equilibrium of (9) is unstable.*

Proof. If $p(s) > 0$, then $p(t) = \frac{c}{\mu(t)}$ with $c > 0$. By Descartes' rule of signs, there exists exactly one positive real root, say $\hat{x} > 0$. By Theorem 3.3, $y(t) = e_q(t,t_0)$ with $q(t) = \frac{\hat{x}}{\mu(t)}$ is a solution of (9), and since $\hat{x} > 0$,

$$y(t) = e_q(t,t_0) = \prod_{s\in[t_0,t)\cap\mathbb{T}}(1+\mu(s)q(s)) = (1+\hat{x})^{\mathcal{N}(t,t_0)} \overset{t\to\infty}{\to} \infty,$$

where $\mathcal{N}(t, t_0) = \text{card}([t_0, t) \cap \mathbb{T})$, completing the proof. □

To study the stability of the only equilibrium, namely, the trivial equilibrium, one may apply classical methods, such as the Jury condition.

Theorem 3.6. *Let $p : \mathbb{T} \to \mathbb{R}, p \in \mathcal{P}_1$. If $\mu p \in (-1, 0)$ is such that all roots of $H(z) = z(z+1)^{\omega} - P$ are within the unit-circle, that is, the Jury coefficients a_k^i satisfy $|a_0^i| > |a_{n'}^i|$, $i = 1, \dots, n-2$, where*

$$a_k^{(i)} := \det \begin{bmatrix} a_0^{(i-1)} & a_{n'-k}^{(i-1)} \\ a_{n'}^{(i-1)} & a_k^{(i-1)} \end{bmatrix}, \quad n' = n - i + 1,$$

and $a_k^{(0)} := a_k$ for $k = 1, \dots, n$, then the trivial solution of (9) is globally asymptotically stable.

Proof. We recall that solutions to the delay-dynamic equation under investigation (9) are of the form $e_q(t, t_0)$ with $q(t) = \frac{x^*}{\mu(t)}$, x^* a root of G defined in (16). This function $e_q(t, t_0)$ converges to 0 if $x^* \in (-2, -1)$. Using the fact that the roots of $G(x-1)$ are in the unit-circle if and only if the roots of $G(x)$ are in $(-2, -1)$, enables the application of the Jury stability criterion[21] to

$$H(x) := G(x-1) = (x-1)x^{\omega} - \mu p = x^{\omega+1} - x^{\omega} - \mu p. \tag{17}$$

For $\mu p \in (-1, 0)$, the first three conditions of the Jury conditions are satisfied, that is,

$$|a_0| = |-\mu p| = |\mu p| < 1 = a_{\omega+1}, \ H(1) = -\mu p > 0, \ H(-1) = (-1)^{\omega+1} - (-1)^{\omega} - P = (-2)^{\omega+1} - \mu p,$$

which is positive if $\omega + 1$ is even and negative if $\omega + 1$ is odd. Thus, the first three Jury conditions are satisfied for $\mu p \in (-1, 0)$, which leaves the required assumption to satisfy all Jury conditions. □

To numerically investigate the values $\mu p \in \mathbb{R}$ for which the Jury conditions are satisfied, we divide the interval $(-1, 0)$ into segments of length 0.01 and test if the Jury conditions are satisfied. The simulations reveal an exponential relation, see Figure 1. Figure 1 illustrates that as the delay ω increases, the set of values μp that satisfy all Jury conditions shrinks.

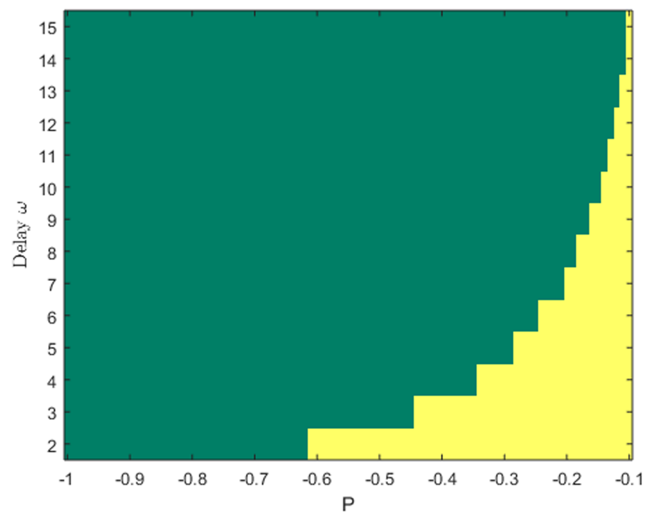

FIGURE 1 Investigation of Jury stability criteria for different values of $P = \mu p \in \mathbb{R}$ and the delay ω. For parameter combinations in the yellow area, the Jury conditions are satisfied, and the trivial equilibrium of $y^{\Delta}(t) = p(t)y(\rho^{\omega}(t))$ with $p \in \mathcal{P}_1$ is globally asymptotically stable. For parameter combinations within the green area, the Jury condition is violated so that the trivial equilibrium is not stable [Colour figure can be viewed at wileyonlinelibrary.com]

Given the relation between the delay and the smallest possible P value that satisfies all Jury conditions indicated in Figure 1, we fit a logistic curve of the form

$$\varphi(\omega) = \frac{1}{1 + e^{a+b\omega+c\omega^2}} - 1,$$

where $\varphi(\omega)$ is the smallest P value that satisfies all Jury conditions for a delay of ω. This can be transformed to the linear problem

$$\log\left(\frac{-\varphi(\omega)}{1+\varphi(\omega)}\right) = a + b\omega + c\omega^3.$$

The parameters found via the Matlab function "`polyfit`" led to the values

$$a = 1.0769, \quad b = -0.4506, \quad c = 0.0163.$$

It will be of interest to verify this exponential relation analytically.

In Wu and Zhou,[22] the condition securing uniform stability of the trivial solution reads in our case as

$$P(\omega+1) \geq -\left\{\frac{3}{2} + \frac{\min_{t\in\mathbb{T}} \mu(t)}{2\max_{t\in\mathbb{T}}(\mu(t) + \rho^{\omega}(0))}\right\}$$

and is therefore dependent on the choice of the time scale.

Based on Anderson et al,[12] sufficient conditions for stability of the trivial equilibrium are provided in the proceeding statements.

Theorem 3.7. *Let $\mathbb{T}$ be unbounded above. If $P \in (-1, 0)$ and there exists $\alpha \in (0, 1)$ such that*

$$|P|^{\omega}\frac{2+|P|}{1+|P|} \leq \alpha, \quad t \in \mathbb{T}, \tag{18}$$

where $N(t, s)$ is the number of (isolated) points in the set $[s, t) \cap \mathbb{T}$, then all solutions to (9) converge to zero.

Proof. This is a consequence from applying Theorem 3.1 in Anderson et al[12] by realizing that the delay-function in Anderson et al[12] is here $\delta = \rho^{\omega}$ and (18) is Condition (3.4) in Anderson et al.[12] □

Corollary 3.7.1. *Let $\mathbb{T}$ be unbounded above. If $P \in (-1, 0)$ such that there exists $\alpha \in (0, 1)$ such that (18) is satisfied for ω, then all solutions to (9) with $p = \frac{P}{\mu}$ and $k \geq \omega$ converge to zero.*

Proof. Let $P \in (-1, 0)$ satisfy (18). Then,

$$\begin{aligned}
\alpha &\geq |P|^{N(\rho^{\omega}(t),t)} + \frac{|P|}{1+|P|}\sum_{\tau\in[t_0,t)\cap\mathbb{T}} |P|^{N(\rho^{\omega}(\tau),t)} \\
&= \frac{1}{|P|}|P|^{N(\rho^{\omega+1}(t),t)} + \frac{|P|}{1+|P|}\sum_{\tau\in[t_0,t)\cap\mathbb{T}} \frac{1}{|P|}|P|^{N(\rho^{\omega+1}(\tau),t)} \\
&\geq |P|^{N(\rho^{\omega+1}(t),t)} + \frac{|P|}{1+|P|}\sum_{\tau\in[t_0,t)\cap\mathbb{T}} |P|^{N(\rho^{\omega+1}(\tau),t)},
\end{aligned}$$

and hence, the condition (18) is satisfied for $\omega+1$. □

Theorem 3.8. *Let $p \in \mathcal{P}_1$. If $\mu p < 0$ and $(-\mu p)k < 1$, then the trivial solution of (9) is globally asymptotically stable.*

Proof. Define $\alpha(t) = \rho^k(t)$ and $A(\eta) = -p(\eta) = \frac{-C}{\mu(\eta)}$, where $C \in \mathbb{R}$. Then, $\alpha(t) \leq t$ for all $t \in \mathbb{T}$ and Condition (A1) in Braverman and Karpuz[23] holds. Furthermore, $\int_{t_0}^{\infty} A(\eta)\,\Delta\eta = \sum_{n=0}^{\infty}(-P) = \infty$ and Condition (24) in Braverman and Karpuz [23] is satisfied. We note that

$$\alpha_*(\eta) = \inf\{\alpha(\tau) : \tau \in [t,\infty) \cap \mathbb{T}\} = \rho^k(t),$$

so that

$$\alpha_{-1}(t) = \sup\{\eta : \alpha_*(\eta) \leq t\} = \sigma^k(t).$$

Thus,

$$\int_{\alpha_*(t)}^{\sigma(t)} A(\eta)\,\Delta\eta = \sum_{n=t}^{\sigma^{k-1}(t)} (-P) = (-P)k,$$

satisfying condition (5). Finally, since

$$\int_{\alpha_*(t)}^{\sigma(t)} A(\eta)\,\Delta\eta = \sum_{n=\rho^k(t)}^{t} (-P) = (-P)k < 1,$$

condition (13) is satisfied and therefore, by Theorem 3.5 in Braverman and Karpuz,[23] the trivial equilibrium is globally asymptotically stable. □

Clearly, one could apply other stability theorems for time scales models, such as presented in other studies.[24–28]

While we may understand the assumption of one-periodic coefficients on isolated time scales to replicate the constant coefficient case in the continuous time scale, we may still want to consider the case of periodic coefficients with higher prime period. In fact, Theorem 3.3 can be generalized to periodic coefficients of any period, but the corresponding (16) becomes more elaborate. For example, if $p \in \mathcal{P}_2$, that is, $\mu p \in \{c_1, c_2\}$ with $c_i \in \mathbb{R}$ for $i = 1, 2$, then (16) turns into

$$x_i^*(1 + x_i^*)^{\lfloor \frac{k}{2} \rfloor}(1 + x_{3-i}^*)^{\lfloor \frac{k+1}{2} \rfloor} = c_i, \tag{19}$$

for $i = 1, 2$. Let x_1^*, x_2^* be solutions of (19). Then, $y = Ce_q(t, t_0)$ with $q(\sigma^{2k}(t_0)) = \frac{x_1^*}{\mu(t)}$ and $q(\sigma^{2k+1}(t_0)) = \frac{x_2^*}{\mu(t)}$ is a solution to (9) with $p \in \mathcal{P}_2$.

An analog of Theorem 3.5 for two-periodic coefficients is formulated below.

Theorem 3.9. *If $\sup \mathbb{T} = \infty$, $p \in \mathcal{P}_2$, and $p(t_0), p(\sigma(t_0)) > 0$, then the trivial solution is unstable.*

Proof. If $p \in \mathcal{P}_2$ with $p(t_0), p(\sigma(t_0)) > 0$, then $c_1,\ c_2 > 0$ where $p(\sigma^{2n}(t_0)) = \frac{c_1}{\mu(t)}$ and $p(\sigma^{2n+1}(t_0)) = \frac{c_2}{\mu(t)}$ for all $n \in \mathbb{N}$ and therefore $p(t) > 0$ for all $t \in \mathbb{T}_{\mathcal{I}}$.

We distinguish between two cases dependent on the delay k.

(i) k even, that is, there exists $n \in \mathbb{N}$ such that $k = 2n$. Let x_1^*, x_2^* be a pair of solutions to the equations in (19), that is,

$$\begin{aligned} x_1^*(1 + x_1^*)^n(1 + x_2^*)^n &= c_1 \\ x_2^*(1 + x_2^*)^n(1 + x_1^*)^n &= c_2. \end{aligned}$$

Since $c_1, c_2 \neq 0$, $x_1^*, x_2^* \neq 0$ and the division of both equations results in

$$\frac{x_1^*}{x_2^*} = \frac{c_1}{c_2} \qquad \Leftrightarrow \qquad x_2^* = \frac{c_2}{c_1}x_1^*. \tag{20}$$

Plugging this into the first equation of (19) yields

$$x_1^*(1 + x_1^*)^n(1 + x_2^*)^n = c_1 \qquad \Rightarrow \qquad \underbrace{x_1^*(1 + x_1^*)^n\left(1 + \frac{c_2}{c_1}x_1^*\right)^n - c_1}_{f(x_1^*)} = 0.$$

For $c_1, c_2 > 0$, the coefficients of $f(x)$ have only one sign change, and thus, by Descartes' rule of signs, there exists one positive root x_1^*. By (20), this results also in $x_2^* > 0$. Following the same argument as in the proof of Theorem 3.5, the solution is unbounded.

(ii) k odd, that is, there exists $n \in \mathbb{N}$ such that $k = 2n + 1$. Then, by (19), we have

$$x_1^*(1 + x_1^*)^n(1 + x_2^*)^{n+1} = c_1$$
$$x_2^*(1 + x_2^*)^n(1 + x_1^*)^{n+1} = c_2.$$

Define $u_i = 1 + x_i^*$ for $i = 1, 2$. Then, the above reads as

$$(u_1 - 1)u_1^n u_2^{n+1} = c_1 \tag{21}$$

$$(u_2 - 1)u_2^n u_1^{n+1} = c_2. \tag{22}$$

By (21), we have

$$u_2^{n+1} = \frac{c_1}{(u_1 - 1)u_1^n} \tag{23}$$

and plugging this into (22), we have

$$c_2 = u_2^{n+1}u_1^{n+1} - u_2^n u_1^{n+1} = \frac{c_1}{(u_1 - 1)u_1^n}u_1^{n+1} - \frac{c_1}{(u_1 - 1)u_1^n}\frac{u_1^{n+1}}{u_2},$$

that is,

$$u_2 = \frac{c_1 u_1}{(c_1 - c_2)u_1 + c_2}. \tag{24}$$

By (24) and (23), we have

$$\frac{c_1^{n+1}u_1^{n+1}}{((c_1 - c_2)u_1 + c_2)^{n+1}} = \frac{c_1}{(u_1 - 1)u_1^n},$$

that is,

$$c_1^n u_1^{2n+1} = \frac{((c_1 - c_2)u_1 + c_2)^{n+1}}{(u_1 - 1)}.$$

That is, u_1 is a root of

$$f(u) := c_1^n u^{2n+2} - c_1^n u^{2n+1} - \sum_{i=0}^{n+1} \binom{n+1}{i} (c_1 - c_2)^i u^i c_2^{n+1-i}.$$

Since $c_1 > 0$, $\lim_{u \to \infty} f(u) = \infty$. Further, since

$$\begin{aligned} f(1) &= c_1^n - c_1^n - \sum_{i=0}^{n+1} \binom{n+1}{i} (c_1 - c_2)^i c_2^{n+1-i} \\ &= -(c_1 - c_2 + c_2)^{n+1} < 0, \end{aligned}$$

there exists a positive root $u_1 > 1$, resulting in a positive root $x_1^* > 0$. Since $u_1 > 1$, $c_2(1 - u_1) < 0$ and therefore $c_1 u_1 > (c_1 - c_2)u_1 + c_2$, hence by (24), $u_2 > 1$, resulting in a positive root $x_2^* > 0$. Thus, there exists an unbounded solution also in this case.

This completes the proof. □

3.2 | Nonlinear dynamic equation: delay Beverton–Holt model

We now apply the results of Section 2 to a class of nonlinear dynamic equations, more precisely, a formulation in Bohner and Warth[29] of the logistic dynamic equation as defined in Bohner and Peterson,[10]

$$x^\Delta = \alpha x^\sigma \left(1 - \frac{x}{K}\right), \qquad \text{where} \qquad \alpha \in (0, 1).$$

This logistic dynamic equation is a time-scale generalization of the popular logistic differential equation. The logistic differential equation is mathematically of interest since it can be solved analytically despite its nonlinearity and biologically due to its applications in population dynamics. The same reason, besides the purpose of generalizing it to time scales, motivates a corresponding study on time scales.

Following the argument described in Bohner and Warth[29] for the discrete case, the logistic dynamic equation on isolated time scales can also be expressed as a Beverton–Holt model in the form

$$x^{\sigma} = \frac{\nu K x}{K + (\nu - 1)x}, \qquad \text{where} \qquad \nu := \frac{1}{1-\alpha} > 1. \tag{25}$$

This can be understood as generalization of the popular logistic differential equation to isolated time scales. We note that for the special case of the integers as isolated time scales, (25) reads as

$$x_{t+1} = \frac{\nu K x_t}{K + (\nu - 1)x_t},$$

which is the well-known Beverton–Holt model that is often considered as metered model of the logistic differential equation, see Brauer and Castillo-Chavez.[30] In fact, the Beverton–Holt recurrence was derived from the logistic differential equation, as outlined in Beverton and Holt.[31] In Bohner et al,[32] a generalization via delay of (25) was considered for the discrete case, namely,

$$x_{t+k} = \frac{\nu K x_t}{K + (\nu - 1)x_t}$$

was considered with $K : \mathbb{Z} \to \mathbb{R}^+$ and $\nu > 1$. Note that if $k = 1$, then, the delay equation is identical to the Beverton–Holt model.[29] Our aim is to generalize this delay model to arbitrary isolated time scales to formulate a delay logistic dynamic equation on isolated time scales.

The class of nonlinear delay dynamic equations on isolated time scales we will consider is of the same structure, namely,

$$x^{\sigma^k} = \frac{\nu K x}{K + (\nu - 1)x},$$

where $k \in \mathbb{N}_1$, $\nu > 1$ is the proliferation rate, and $K > 0$ is the carrying capacity. We define the growth rate by $\alpha(t) = \frac{\nu-1}{\mu\nu}$ so that $\alpha \in \mathcal{P}_1$ by (6), and consequently $\mu\alpha \in (0, 1)$. Now, we rewrite the previous equation as

$$x^{\sigma^k} = \frac{Kx}{(1-\mu\alpha)K + \mu\alpha x}. \tag{26}$$

We investigate (26) with initial values $\vec{x}_0 = (x_0, x_1, \dots, x_{k-1}) \in (0, \infty)^k$. We note that if $x_i = 0$ for all $i = 0, \dots, k-1$, then $x_t = 0$ for all $t \geq k$. If $x_i = 0$ for some $\mathcal{I} \subsetneq \{0, 1, \dots, k-1\}$, then $x_{i+nk} = 0$ for all $n \in \mathbb{N}$ and if $x_i > 0$, then $x_{i+nk} > 0$ for all $n \in \mathbb{N}$ and the solution of x_{i+nk} can be obtained using the same procedure as described below.

Define $z := \frac{x}{K}$. Then, (26) becomes

$$z^{\sigma^k} = \frac{z}{\frac{1}{\nu} + \frac{(\nu-1)}{\nu} z} = \frac{z}{(1-\mu\alpha) + \mu\alpha z}. \tag{27}$$

Using $y := \frac{1}{z}$ (for $z \neq 0$) yields the linear delay dynamic equation

$$y^{\sigma^k} = (1-\mu\alpha)y + \mu\alpha. \tag{28}$$

Apply (5) to the left-hand side of (28) to get

$$\sum_{i=0}^{k} \binom{k}{i} Y_i = (1-\mu\alpha)y + \mu\alpha,$$

and isolate Y_k to obtain

$$Y_k = -\mu\alpha Y_0 + \mu\alpha - \sum_{i=1}^{k-1} \binom{k}{i} Y_i. \tag{29}$$

We see that $Y^{\Delta}_{i-1} = \frac{1}{\mu} Y_i$ so that (29) is

$$Y^{\Delta} = A(t)Y + \mathbf{b}(t) \qquad \text{with} \qquad A(t) := \frac{1}{\mu(t)} \begin{bmatrix} \mathbf{0}_{k-1} & I_{k-1} \\ -\mu\alpha & -\mathbf{s} \end{bmatrix} \text{ and } \mathbf{b}(t) = \begin{pmatrix} \mathbf{0}_{k-1} \\ \alpha \end{pmatrix}, \tag{30}$$

where $\mathbf{s} = \left(\binom{k}{1}, \binom{k}{2}, \binom{k}{3}, \ldots, \binom{k}{k-1}\right)$ and $\mathbf{0}_{k-1} \in \mathbb{R}^{k-1\times 1}$ is vector of zeros.

Lemma 3.10. *The matrix A defined in (30) is regressive.*

Proof. By Definition 5.5 in Bohner et al,[11] the matrix A is regressive if $\det(A + \mu I) \neq 0$, where I is the identity matrix. First compute

$$B := I + \mu A = \begin{bmatrix} \mathbf{e}_1 & I_{k-1} + L_{k-1} \\ -\mu\alpha & \mathbf{e}^T_{k-1} - \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & I_{k-1} + L_{k-1} \\ C & \mathbf{e}^T_{k-1} - \mathbf{s} \end{bmatrix}, \tag{31}$$

where $\mathbf{e}^T_{k-1} = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{1\times(k-1)}$, $C = -\frac{\nu-1}{\nu} \in (-1, 0)$, and $L_{k-1} \in \mathbb{R}^{k-1\times k-1}$ is the matrix that is zero except for ones on the subdiagonal. Thus, expanding along the bottom row, we get

$$\begin{aligned}\det(B) &= (-1)^{k+1}\left(C + \binom{k}{1} - \binom{k}{2} + \ldots + (-1)^{k+1}\left(-\binom{k}{k-1} + 1\right)\right) = 1 + (-1)^{k+1}\left(C + \sum_{i=1}^{k-1}(-1)^{i+1}\binom{k}{i}\right) \\ &= 1 + (-1)^{k+1}\left(C + (-1)^k + 1\right) = (-1)^{k+1}(C+1) \neq 0,\end{aligned}$$

since $C \in (-1, 0)$. □

Remark 3.11. The assumption of $\alpha \in \mathcal{P}_1$ results in a constant matrix B, which would not be the case if α were chosen to be constant. As we will see in the proceeding results, the solution significantly simplifies for constant matrices B. Note that these type of simplifications are usually known to arise in the case of constant coefficients for differential and difference equations. Thus, it gives reason to believe that one-periodic coefficients replace the purpose of constant coefficients in dynamic equations of time scales.

By Theorem 5.24 in Bohner et al[11] and Lemma 3.10, the solution of (30) is

$$Y(t) = e_A(t, t_0)Y(t_0) + \left(\sum_{s\in[t_0,t)\cap\mathbb{T}} \mu(s)e_A(t, \sigma(s))\mathbf{b}(s)\right) = B^{\mathcal{N}(t,t_0)}Y(t_0) + \left(\sum_{s\in[t_0,t)\cap\mathbb{T}} \mu(s)B^{\mathcal{N}(t,\sigma(s))}\mathbf{b}(s)\right), \tag{32}$$

where $\mathcal{N}(t, t_0) = \operatorname{card}(\mathbb{T} \cap [t_0, t))$. The second identity holds because $B = A + \mu I$ has only constant entries.

To simplify the expression (32) and avoid the calculation of matrix powers, we use the following lemma.

Lemma 3.12. *Let $B \in \mathbb{R}^{k\times k}$ be given by (31). Then*

$$\mathbf{e}^T_1 B^m = (1 + C)^{\ell} \sum_{j=0}^{r} \binom{r}{j} \mathbf{e}^T_{j+1}, \tag{33}$$

for $m = \ell k + r$, $0 \leq r < k$ with $C = \frac{1-\nu}{\nu}$.

Lemma 3.12 is proven via induction. Since it is not difficult to prove but is rather long, the details are moved to the appendix. Lemma 3.12 now significantly simplifies the expression of the solution to the delay logistic dynamic equation.

Theorem 3.13. *If $\alpha \in \mathcal{P}_1$ with $\mu\alpha \in (0,1)$ and $K > 0$, then the solution to (26) with initial values $(x(t_0), x(\sigma(t_0)), \dots, x(\sigma^{k-1}(t_0)))$ is given by*

$$x(t) = \frac{K}{(1+C)^{\ell} K \sum_{j=0}^{r} \binom{r}{j} (Y(t_0))_{j+1} + 1 - (1+C)^{\ell+1}}, \tag{34}$$

where $t = \sigma^m(t_0)$ for some $m = \ell k + r$, $0 \le r < k$ and $C = \frac{1-v}{v}$ and $Y(t_0) = (Y_0(t_0), Y_1(t_0), \dots, Y_{k-1}(t_0))$ with $Y_0 = y = \frac{K}{x}$ and $Y_{i+1} = \mu Y_i^{\Delta}$.

Proof. Recall that the solution of (30) is given by (32). Given the construction of (30), the first component, $y = \mathbf{e}_1^T Y$ solves (28). Note that for $\mathcal{N}(t, t_0) = \ell k + r$ and $\mathcal{N}(t, \sigma(\tau)) = \ell_\tau k + r_\tau$ and therefore

$$\begin{aligned} y(t) = \mathbf{e}_1^T Y &= \mathbf{e}_1^T e_A(t, t_0) Y(t_0) + \sum_{\tau \in [t_0, t) \cap \mathbb{T}} \mu(\tau) \mathbf{e}_1^T e_A(t, \sigma(\tau)) \mathbf{b}(\tau) = \mathbf{e}_1^T B^{\mathcal{N}(t,t_0)} Y(t_0) + \sum_{\tau \in [t_0, t) \cap \mathbb{T}} \mu(\tau) \mathbf{e}_1^T B^{\mathcal{N}(t,\sigma(\tau))} \mathbf{b}(\tau) \\ &= (1+C)^{\ell} \sum_{j=0}^{r} \binom{r}{j} \mathbf{e}_{j+1}^T Y(t_0) + \sum_{\tau \in [t_0, t) \cap \mathbb{T}} (1+C)^{\ell_\tau} \sum_{j=0}^{r_\tau} \mu(\tau) \mathbf{e}_{j+1}^T \mathbf{b}(\tau). \end{aligned}$$

Since $\mu(\tau)\mathbf{b}(\tau) = (0, 0, \dots, 0, -C)^T$, we have

$$y(t) = (1+C)^{\ell} \sum_{j=0}^{r} \binom{r}{j} \mathbf{e}_{j+1}^T Y(t_0) + \sum_{\tau \in [t_0, t) \cap \mathbb{T}} (1+C)^{\ell_\tau} \chi_\tau,$$

where $\chi_\tau = \begin{cases} 0 & \text{if } r_\tau < k-1 \\ -C & \text{if } r_\tau = k-1 \end{cases}$. Thus, if $m := \mathcal{N}(t, t_0) = \ell k + r$ with $r < k-1$, then

$$\begin{aligned} y(\sigma^m(t_0)) &= (1+C)^{\ell} \sum_{j=0}^{r} \binom{r}{j} (Y(t_0))_{j+1} - C \sum_{j=0}^{\ell} (1+C)^j = (1+C)^{\ell} \sum_{j=0}^{r} \binom{r}{j} (Y(t_0))_{j+1} - C \frac{1 - (1+C)^{\ell+1}}{1 - (1+C)} \\ &= (1+C)^{\ell} \sum_{j=0}^{r} \binom{r}{j} (Y(t_0))_{j+1} + 1 - (1+C)^{\ell+1}, \end{aligned}$$

where $(Y(t_0))_j$ is the jth component of $Y(t_0)$. Since $w_1 = y = \frac{1}{z} = \frac{K}{x}$, we have with initial values $x(\sigma^i(t_0)) > 0$ for $i = 0, 1, \dots, k-1$,

$$x(t) = x(\sigma^{\ell k + r}(t_0)) = \frac{K}{(1+C)^{\ell} \sum_{j=0}^{r} \binom{r}{j} (Y(t_0))_{j+1} + 1 - (1+C)^{\ell+1}}.$$

To show that this is in fact a solution to (26), if $t = \sigma^{\ell k + r}(t_0)$, then $\sigma^k(t) = \sigma^{(\ell+1)k+r}(t_0)$ and therefore,

$$\begin{aligned} x^{\sigma^k}(t) &= \frac{K}{(1+C)^{\ell+1} \sum_{j=0}^{r} \binom{r}{j} (Y(t_0))_{j+1} + 1 - (1+C)^{\ell+2}} = \frac{K}{(1+C)^{\ell+1} \sum_{j=0}^{r} \binom{r}{j} (Y(t_0))_{j+1} + 1 - (1+C)^{\ell+2}} \\ &= \frac{K}{(1+C)^{\ell+1} \sum_{j=0}^{r} \binom{r}{j} (Y(t_0))_{j+1} + 1 - (1+C)^{\ell+2}} \end{aligned}$$

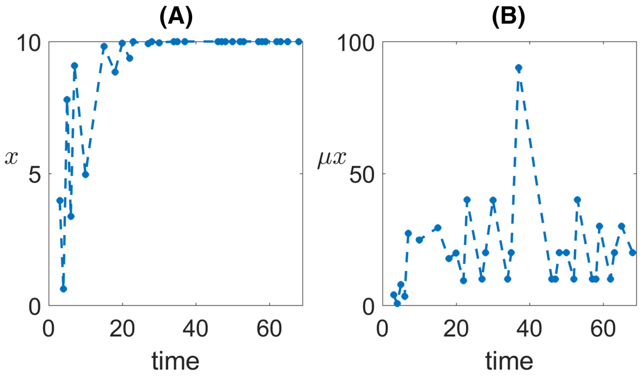

FIGURE 2 Behavior of the solution of (26) with parameter values $K = 10, k = 4, \nu = 5$ on a randomly chosen time scale $\mathbb{T} = \{2, 3, 5, 9, 12,13, 16,18, 21,22, 23,26, 29,32, 36,37, 39,42, 43,46, 47,48, 50,53, 54,56, 58,59, 61,68, 70\}$ and initial values (1.0976, 9.3376, 1.8746, 2.6618). Panel (A) displays the behavior of the solution x, and (B) visualizes the behavior of μx. Recall that if μx is constant, then x is one-periodic [Colour figure can be viewed at wileyonlinelibrary.com]

and, similarly,

$$\begin{aligned}\frac{Kx}{(1-\mu\alpha)K+\mu\alpha x} &= K\frac{\frac{K}{(1+C)^{\ell}\sum_{j=0}^{r}\binom{r}{j}(Y(t_0))_{j+1}+1-(1+C)^{\ell+1}}}{(1+C)K+\mu\alpha\frac{K}{(1+C)^{\ell}\sum_{j=0}^{r}\binom{r}{j}(Y(t_0))_{j+1}+1-(1+C)^{\ell+1}}}\\ &= \frac{K}{(1+C)(1+C)^{\ell}\sum_{j=0}^{r}\binom{r}{j}(Y(t_0))_{j+1}+(1+C)-(1+C)^{\ell+2}+\mu\alpha}\\ &= \frac{K}{(1+C)^{\ell+1}\sum_{j=0}^{r}\binom{r}{j}(Y(t_0))_{j+1}+1-(1+C)^{\ell+2}}.\end{aligned}$$

Comparing both expressions and recalling that $C = -\mu\alpha$, yields the desired equation $x^{\sigma^k} = \frac{Kx}{(1-\mu\alpha)K+\mu\alpha x}$, which completes the proof. □

The expression of the solution simplifies the study of the dynamics, and we obtain the following theorem regarding the global asymptotic stability of K.

Theorem 3.14. *If* $\sup \mathbb{T}_{\mathcal{I}} = \infty$ *and* $\alpha \in \mathcal{P}_1$ *with* $\mu\alpha \in (0, 1)$ *and* $K > 0$*, then solutions to (26) with positive initial values converge to* K.

Proof. By Theorem 3.13, the solution of (26) is given by (34). Since $1 + C = \frac{1}{\nu} < 1$ for $\nu > 1$, we have $\lim_{\ell\to\infty}(1+C)^{\ell} = 0$. Thus,

$$\frac{K}{\sum_{j=0}^{r}\binom{r}{j}(Y(t_0))_{j+1}\lim_{\ell\to\infty}(1+C)^{\ell}+1-\lim_{\ell\to\infty}(1+C)^{\ell+1}} = K,$$

completing the claim. □

As described in Theorem 3.14, solutions converge to the carrying capacity K for one-periodic parameters α, see Figure 2A. Interestingly, the solution is not necessarily one-periodic, that is, $\mu x \notin \mathbb{R}$, as panel (B) suggests.

4 | CONCLUSION

We applied the recently introduced definition of periodicity for arbitrary isolated time scales[17] to delay dynamic equations. In Section 2, an identity relating delays to a combination of higher order delta derivatives is formulated, see Theorem 2.2. This formula also reveals the structure of periodic functions, see Theorem 2.3, and simplifies the analysis of delay dynamic equations. In Section 3, we analyzed first linear and then a class of nonlinear dynamic equations with one-periodic coefficients. The class of nonlinear dynamic equations we consider is related to the logistic differential equation, a popular nonlinear model that is commonly used in population dynamics. Thus, the discussion of the corresponding model on isolated time scales is the first step towards extending this model to time scales and to provide a wider range of applications as the underlying time domain does not have to be continuous nor entirely discrete with equidistant points. Although constant coefficients are commonly used as simplification, we argued that it is indeed the one-periodicity trait inherited by constant parameters that simplifies the analysis significantly. Formulae for the exact solution to the linear and nonlinear dynamic equations with delay were provided. We furthermore provide sufficient conditions to discuss the global asymptotic stability of equilibria for both the linear and nonlinear models. For the linear case, we use for example the Jury conditions to discuss global asymptotic stability of the trivial equilibrium, and apply stability theorems to the specific delay system introduced in this manuscript. To discuss stability of the positive equilibrium for the nonlinear model, we use the explicit solution. It will be interesting to extend this study to periodic coefficients with higher periods. As outlined in Section 3.1, the analysis of two-periodic coefficients is already more complicated as hinted upon. It will also be interesting to extend the study to systems of dynamic equations on isolated time scales. We further believe that stability analysis can be intensified, which may be subject of future work.

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CONFLICT OF INTEREST

No potential conflict of interest was reported by the authors.

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APPENDIX A

Proof of Lemma 3.12. We begin the induction with the base case, that is, $m = 0$. In this case, $\ell = 0$ and $r = 0$. Then,

$$\mathbf{e}_1^T B^0 = \mathbf{e}_1^T I = (1 + C)^0 \mathbf{e}_1,$$

confirming (33) for $m = 0$. We write

$$\mathbf{e}_1^T B^{m+1} = \mathbf{e}_1^T B^m B = \left(\mathbf{e}_1^T B^m\right) B,$$

where $\mathbf{e}_1^T B^m$ is the first row of B^m. We now write the induction hypothesis

$$\mathbf{e}_1^T B^m = \begin{cases} (1 + C)^\ell \mathbf{e}_1^T & \text{if } m = \ell k \\ (1 + C)^\ell \left(\binom{r}{0}, \binom{r}{1}, \dots, \binom{r}{r}, 0, \dots 0\right)^T & \text{if } m = \ell k + r \ (1 \le r < k). \end{cases}$$

Case 1. There exists $\ell \in \mathbb{N}$ such that $m = \ell k$ and $r = 0$. We then have

$$\begin{aligned} \mathbf{e}_1^T B^{\hat{m}} &= \mathbf{e}_1^T B^{m+1} = \mathbf{e}_1^T B^m B \\ &\overset{(33)}{=} (1 + C)^\ell \sum_{j=0}^{0} \binom{0}{j} \mathbf{e}_{j+1}^T B = (1 + C)^\ell \binom{0}{0} \mathbf{e}_1^T B = (1 + C)^\ell \mathbf{e}_1^T B = (1 + C)^\ell (\mathbf{e}_1^T + \mathbf{e}_2^T), \end{aligned}$$

which coincides with (33) for $\hat{m} = m + 1 = \ell k + 1$.

Case 2. $m = \ell k + r$ with $0 < r < k-1$, thus $\hat{m} = m+1 = \ell k + (r+1) = \ell k + \hat{r}$ where $\hat{r} = r+1 < k$. We therefore have

$$\begin{aligned}
(1+C)^{-\ell}\mathbf{e}_1^T B^{\hat{m}} &= (1+C)^{-\ell}\mathbf{e}_1^T B^m B \overset{(33)}{=} \sum_{j=0}^{r}\binom{r}{j}\mathbf{e}_{j+1}^T B \\
&= \mathbf{e}_1^T B + \sum_{j=1}^{r}\binom{r}{j}\mathbf{e}_{j+1}^T B = \mathbf{e}_1^T + \mathbf{e}_2^T + \sum_{j=1}^{r}\binom{r}{j}(\mathbf{e}_{j+1}^T + \mathbf{e}_{j+2}^T) \\
&= \mathbf{e}_1^T + \binom{r}{0}\mathbf{e}_2^T + \sum_{j=1}^{r}\binom{r}{j}\mathbf{e}_{j+1}^T + \sum_{j=2}^{r+1}\binom{r}{(j-1)}\mathbf{e}_{j+1}^T \\
&= \mathbf{e}_1^T + \left(\binom{r}{0} + \binom{r}{1}\right)\mathbf{e}_2^T + \sum_{j=2}^{r}\left(\binom{r}{j} + \binom{r}{j-1}\right)\mathbf{e}_{j+1}^T + \mathbf{e}_{r+1}^T \\
&= \mathbf{e}_1^T + \binom{r+1}{1}\mathbf{e}_2^T + \sum_{j=2}^{r}\binom{r+1}{j}\mathbf{e}_{j+1}^T + \binom{r+1}{r+1}\mathbf{e}_{r+1}^T \\
&= \sum_{j=0}^{r+1}\binom{r+1}{j}\mathbf{e}_{j+1}^T.
\end{aligned}$$

This is consistent with (33) for $\hat{m} = m+1 = \ell k + r + 1$ with $r+1 < k$, after multiplying both sides with $(1+C)^{\ell}$.

Case 3. $m = \ell k + (k-1)$, that is, $r = k-1$. In this case, $\hat{m} = (\ell+1)k$. Thus, we have

$$\begin{aligned}
(1+C)^{-\ell}\mathbf{e}_1^T B^{\hat{m}} &= (1+C)^{-\ell}\mathbf{e}_1^T B^m B \overset{(33)}{=} \sum_{j=0}^{k-1}\binom{k-1}{j}\mathbf{e}_{j+1}^T B \\
&= \mathbf{e}_1^T B + \sum_{j=1}^{k-2}\binom{k-1}{j}\mathbf{e}_{j+1}^T B + \mathbf{e}_k^T B \\
&= \mathbf{e}_1^T + \mathbf{e}_2^T + \sum_{j=1}^{k-2}\binom{k-1}{j}(\mathbf{e}_{j+1}^T + \mathbf{e}_{j+2}^T) + \mathbf{e}_k^T B \\
&= \mathbf{e}_1^T + \mathbf{e}_2^T + \binom{k-1}{1}\mathbf{e}_2^T + \binom{k-1}{k-2}\mathbf{e}_k^T + \sum_{j=2}^{k-2}\binom{k}{j}\mathbf{e}_{j+1}^T + C\mathbf{e}_1^T - \sum_{j=1}^{k-1}\binom{k}{j}\mathbf{e}_{j+1}^T + \mathbf{e}_k^T \\
&= \mathbf{e}_1^T + \binom{k}{1}\mathbf{e}_2^T + \binom{k-1}{k-2}\mathbf{e}_k^T + C\mathbf{e}_1^T - \binom{k}{1}\mathbf{e}_2^T - \binom{k}{k-1}\mathbf{e}_k^T + \binom{k-1}{k-1}\mathbf{e}_k^T \\
&= \mathbf{e}_1^T + \binom{k}{k-1}\mathbf{e}_k^T + C\mathbf{e}_1^T - \binom{k}{k-1}\mathbf{e}_k^T = (1+C)\mathbf{e}_1^T,
\end{aligned}$$

which is consistent with (33), after multiplying both sides with $(1+C)^{\ell}$.

This completes the proof.