Theory and Applications of Graphs

Volume 10 | Issue 1 Article 5

February 2023

Outer Independent Double Italian Domination of Some Graph Products

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Recommended Citation

Jalaei, Rouhollah and Mojdeh, Doost Ali (2023) "Outer Independent Double Italian Domination of Some Graph Products," *Theory and Applications of Graphs*: Vol. 10: Iss. 1, Article 5.
DOI: 10.20429/tag.2023.100105
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol10/iss1/5

Cover Page Footnote
Our thanks to the Editor in Chief and Reviewers for valuable suggestions and comments.

Abstract

An outer independent double Italian dominating function on a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ for which each vertex $x \in V(G)$ with $f(x) \in \{0, 1\}$ then $\sum_{y \in N[x]} f(y) \geqslant 3$ and vertices assigned 0 under f are independent. The outer independent double Italian domination number $\gamma_{oidI}(G)$ is the minimum weight of an outer independent double Italian dominating function of graph G. In this work, we present some contributions to the study of outer independent double Italian domination of three graph products. We characterize the Cartesian product, lexicographic product and direct product of custom graphs in terms of this parameter. We also provide the best possible upper and lower bounds for these three products for arbitrary graphs.

Keywords:(Outer independent) double Italian domination number, Cartesian product, lexicographic product, direct product.
Mathematics Subject Classification (2010): 05C69

1 Introduction and preliminaries

Roman domination, Italian domination, double Roman domination and double Italian domination are now the major areas in graph theory. Their steady and rapid growth during the past ten years may be due to the diversity of their applications to both theoretical and real-world problems, such as facility location problems, strategy of defence of cities and etc. The initial studies of Roman domination [19, 21] have been motivated by a historical application. In the 4th century, Emperor Constantine decreed that for all cities in the Roman Empire, at most two legions should be stationed. Further, if a location having no legions was attacked, then it must be within the vicinity of at least one city at which two legions were stationed, so that one of the two legions could be sent to defend the attacked city. A new version of Roman domination namely Italian domination has been defined by Chellali et al. [5] says that if a location having no legions was attacked, then it must be within the vicinity of at least one city at which two legions or two cities at each of which one legion was stationed, so that one of the legion within the vicinity could be sent to defend the attacked city. Mojdeh et al. [16] have defined double Italian domination. What they propose, is a stronger version of Roman and Italian domination that support the protection by ensuring that any attack can be defended by at least three or more legions from one or more other locations. Jalaei et al. [14] have defined outer independent double Italian domination, with one more property that, any two attacked locations are not adjacent. In this work we study OIDIDF on three different product of graphs
Throughout this paper, we consider G as a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We follow [23] as a reference for terminology and notation which are not explicitly defined here. As a common, we let $N(v)$ be the *open neighborhood* of a vertex v, and $N[v] = N(v) \cup \{v\}$ as *closed neighborhood* of v. We denote by $\delta(G)$ and $\Delta(G)$, respectively for the *minimum* and *maximum degrees* of G. A set $S \subseteq V(G)$ of G is an independent set if no two vertices in S are adjacent. The independence number, denoted by $\alpha(G)$, is the maximum cardinality of an independent set in G. Any maximum independent set S of size α is called an α-set. A vertex cover of a graph G is a set $F \subseteq V(G)$ such that each edge in $E(G)$ has at least one end point in F. The minimum cardinality of a vertex

cover is denoted by $\beta(G)$. A graph G is a non empty graph if it has at least one edge. For two vertices a, b we use $a \sim b$ ($a \nsim b$) for showing that the vertices a and b are adjacent (nonadjacent).

A set $S \subseteq V(G)$ of G is called a *dominating set* if every vertex in $V(G) - S$ has a neighbor in S. The minimum cardinality among all dominating sets of G is called the *domination number* $\gamma(G)$ of G.

A *Roman dominating function* of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that if $f(x) = 0$ for some $x \in V(G)$, then there exists $y \in N(x)$ for which $f(y) = 2$. The *Roman domination number* of G, denoted by $\gamma_R(G)$ is the minimum weight of a Roman dominating function f of G. This concept was formally defined by Cockayne *et al.* [6] as originally defined by I. Stewart entitled "Defend the Roman Empire!" ([21]).

The concept of double Roman domination introduced by Beeler *et al.* [4]. More formally, a *double Roman dominating function* (DRDF) of a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ such that the following criteria are fulfilled.

(a) If $f(v) = 0$, then the vertex v must have at least two neighbors in V_2 or one neighbor in V_3.

(b) If $f(v) = 1$, then the vertex v must have at least one neighbor in $V_2 \cup V_3$.

This parameter was also verified in [1, 13, 25]. For simplicity's sake, suppose that $V_i^f = V_i = \{v \in V : f(v) = i\}$ for $i \geq 0$. The concept of independent (total) double Roman dominating function have been investigated in [12, 16]. The double Roman dominating function is *independent (total) double Roman dominating function* if the induced subgraph $\langle V_1 \cup V_2 \cup V_3 \rangle$ has no edges (isolated vertices) and the double Roman dominating function is said to be *an outer independent double Roman dominating function* if the induced subgraph $\langle V_0 \rangle$ has no edges [17].

Chellali et al. [5] have introduced a Roman $\{2\}$-dominating function (and now is called Italian dominating function) f as follows. A Roman $\{2\}$-dominating function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex $v \in V$, with $f(v) = 0$, $f(N(v)) \geq 2$. Once the original paper [5] was published, many researchers attended to this topics. Some of these variations of Italian domination have been outlined in [9, 10, 15]. For instance, an *Italian domination in trees* version of such parameter which was presented in [10], a *perfect Italian domination in trees* version which was presented in [9]. *Covering Italian domination or outer independent Italian domination* which was presented in [15]. Recently, Mojdeh and Volkmann. [18] introduced the concept of double Italian domination (Roman $\{3\}$-domination). A *double Italian dominating function* (DID function for short) of a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ for which the following conditions are satisfied.

(a) If $f(v) = 0$, then the vertex v must have at least three neighbors in V_1, or one neighbor in V_1 and one neighbor in V_2, or two neighbors in V_2, or one neighbor in V_3.

(b) If $f(v) = 1$, then the vertex v must have at least two neighbors in V_1, or one neighbor in $V_2 \cup V_3$.
In the other words, if $v \in V_0 \cup V_1$, then $f(N[v]) = \sum_{x \in N_G[v]} f(x) \geq 3$.

DOI: 10.20429/tag.2023.100105

This parameter was also outlined in [2, 3, 11, 20]. For instance *Total double Italian domination*, which was investigated in [20], *outer independent double Italian domination* which was studied in [2, 3, 14] and *Perfect double Italian domination,* which was studied in [11].

Consequently, an *outer independent double Italian dominating function* (OIDIDF) is a DID function for which V_0^f is independent. The minimum weight of an OIDIDF of G, denoted by $\gamma_{oidI}(G)$, is called *outer independent double Italian domination number*. An OIDID function f of a graph G with weight $\gamma_{oidI}(G)$ is called a $\gamma_{oidI}(G)$-function. The behavior of domination parameter in Cartesian graph product have investigate in [22]. *Roman domination in Cartesian product graphs and strong product graphs* was studied in [24].

This paper is organized as follows: In section 2, we study the γ_{oidI} of Cartesian product of graphs and give several upper bounds. We also investigate the γ_{oidI} of Lexicographic product of graphs and establish various upper bounds in Section 3. Finally in Section 4, we study the OIDID number of the direct product of graphs.

2 Cartesian product in graphs

In this section, we characterize the OIDIDN of cartesian product of some graphs and present sharp upper bounds. We begin with the following definition.

Definition 2.1. *The Cartesian product of G and H denoted by $G\Box H$ where $V = V(G) \times V(H)$ and two vertices (u_i, v_j) and (u_s, v_t) are adjacent if and only if:*

1. *$u_i = u_s$ and v_j is adjacent to v_t in H, or*
2. *$v_j = v_t$ and u_i is adjacent to u_s in G.*

Proposition 2.1. *Let G be a graph with $\delta(G) \geq 2$. Then for every $\gamma_{oidI}(G)$-function $f = (V_0, V_1, V_2, V_3)$, $V_3 = \emptyset$.*

Proof. Suppose that $f = (V_0, V_1, V_2, V_3)$ is a $\gamma_{oidI}(G)$-function on G. Let $V_3 \neq \emptyset$ and $u \in V(G)$ with $f(u) = 3$. If all neighbors of u have positive weight, then the function g defined by $g(u) = 2$ and $g(x) = f(x)$ otherwise is an OIDIDF on G of weight less than $w(f)$, a contradiction. Let v be a neighbor of u with $f(v) = 0$. Since $\delta(G) \geq 2$, v has at least one neighbor (other than u) of weight 1 or 2. Therefore the function g defined as before is an OIDIDF on G of weight less than $w(f)$, that is a contradiction. Thus $V_3 = \emptyset$. □

The following observation has routine proof.

Observation 2.1. *Suppose that G and H are two graphs.*
(i) *Any vertex $(u_k, v_s) \in V(G\Box H)$ can be adjacent to (u_i, v_s), for some $1 \leq i \leq m$, $i \neq k$ or (u_k, v_j), for some $1 \leqslant j \leqslant n$, $j \neq s$.*
(ii) *If G and H are two graphs without isolated vertex, then $V_1(G\Box H) \cup V_2(G\Box H)$ is both an outer independent double Italian dominating set and a vertex cover in $G\Box H$.*

For complete graph K_n and $\overline{K_m}$, the Cartesian product of $\overline{K_m}$ and K_n ($\overline{K_m}\Box K_n$) is the union of m components of K_n. Also the Cartesian product of K_m and K_n ($K_m\Box K_n$) is consisted of m complete graph K_n and n complete graph K_m. Now we can easily investigate that.

Observation 2.2. (i) $\gamma_{oidI}(\overline{K_m}\Box K_n) = m(n-1),\ n \geqslant 4$.
(ii) $\gamma_{oidI}(K_m\Box K_n) = \min\{m,n\}(\max\{m,n\}-1)$.

Suppose that $C_k,\ P_k,\ K_{1,k-1}$ are cycle, path and star of order k respectively.

Proposition 2.2. $\gamma_{oidI}(P_m\Box K_{1,n}) = \begin{cases} \frac{3m+2}{2} + \lfloor \frac{m}{2} \rfloor n, & \textit{if } m \textit{ is even} \\ \frac{3m+3}{2} + \lfloor \frac{m}{2} \rfloor n, & \textit{if } m \textit{ is odd} \end{cases}$

Proof. Let $V(P_m) = \{u_1, u_2, \cdots, u_m\}$ and $V(K_{1,n}) = \{v_1, v_2, \cdots, v_n, v_{n+1}\}$ where v_1 is the center of $K_{1,n}$. For m even, we assign:

$$f((u_i,v_j)) = \begin{cases} 2, & \text{if } i=1, i \text{ is even, and } j=1 \\ 0, & \text{if } i \text{ is odd, and } 2 \leq j \leq n \\ 1, & otherwise \end{cases}$$

For m odd, we assign:

$$f((u_i,v_j)) = \begin{cases} 2, & \text{if } i=1, \text{or } i=m, \text{or } i \text{ is even, and } j=1 \\ 0, & \text{if } i \text{ is odd, and } 2 \leq j \leq n \\ 1, & otherwise \end{cases}$$

From the first assignment, we deduce for m even

$$\gamma_{oidI}(P_m\Box K_{1,n}) \leq \frac{3m+2}{2} + \left\lfloor \frac{m}{2} \right\rfloor n$$

and the second assignment, we deduce for m odd

$$\gamma_{oidI}(P_m\Box K_{1,n}) \leq \frac{3m+3}{2} + \left\lfloor \frac{m}{2} \right\rfloor n.$$

We now investigate the converse of each of which. In each row $2 \leq j \leq n$ we must have at most $\lceil \frac{m}{2} \rceil$ vertices of weight 0 and the other vertices are of positive weight. Each vertex in the place of columns one and m, and row $2 \leq j \leq n$ is of degree 2 and any vertex in the place of columns $2 \leq i \leq m-1$ and row $2 \leq j \leq n$ is of degree 3. Let v be the specified vertex. Then the weight of $N[v]$ must be at least 3. On the other hand with the assignment in the case of m odd or even, for each vertex (u_i, v_j) in the row $2 \leq j \leq n$ and column i, if $f((u_i,v_j)) = 0$, then (u_i,v_j) has one neighbor in row one of weight 2 or has three neighbors, in which one of neighbors belongs to the row one of weight 1, and two neighbors in the row j of weight 1. Therefore, these assignments show that the given graphs must have at least $\frac{3m+2}{2} + \lfloor \frac{m}{2} \rfloor n$ for m even and $\frac{3m+3}{2} + \lfloor \frac{m}{2} \rfloor n$ for m odd. □

For positive integers m, n maybe, one cannot say anything about the outer independent double Italian domination numbers of the Cartesian product two graphs G, H of orders m, n respectively. If H is restricted to the complete graph K_n with $n \geq 3$, then can the following discussions.

Proposition 2.3. *Let $n \geq 4$ be an integer and G be a graph of order $m \leq n$. Then* $\gamma_{oidI}(G\Box K_n) = |V(G)|(n-1)$.

Proof. Since $\delta(G\Box K_n) \geq 3$ and we must exactly assign one 0 to each row, so we must assign value 1 to the other vertices. These assignments give us a minimum OIDID of $G\Box K_n$. Therefore $\gamma_{oidI}(G\Box K_n) = |V(G)|(n-1)$. □

If we restrict G to the path P_m or cycle C_m, we have.

Proposition 2.4. *For complete graph K_n, path P_m and cycle C_m where $n \geq 3$ and $m \geq 4$,*

1. $\gamma_{oidI}(P_m\Box K_n) = \begin{cases} 2(m+1), & \text{if } n=3 \\ m(n-1), & \text{if } n \geq 4 \end{cases}$
2. $\gamma_{oidI}(C_m\Box K_n) = \begin{cases} 2m, & \text{if } n=3 \text{ and } 3 \mid m \\ 2m+1, & \text{if } n=3 \text{ and } 3 \nmid m \\ m(n-1), & \text{if } n \geq 4 \end{cases}$

Proof. 1. Let $n=3$ and m be a positive integer. It is clear that each row is assigned with at most one 0. If the first row has a vertex of weight 0, then the sum of the first and second rows must be at least 4. Also if the mth row has a vertex of weight 0, then the sum of the mth and $(m-1)$th rows must be at least 5. Therefore without loss of generality that we can assign at most one 0 to all rows other than first and mth rows. Thus $\gamma_{oidI}(P_m\Box K_3) \leq 2(m+1)$. On the other hand if we assign 0 to vertices $(2+3k,1)$, $(3+3k,2)$ and $(4+3k,3)$ subject to $2+3k$, $3+3k$ or $4+3k$ be at most $m-1$. Thus $\gamma_{oidI}(P_m\Box K_3) \geq 2(m+1)$.
Let $n \geq 4$, it is easy to see that assigning one 0 to each vertex in each row with the suitable column, we obtain $\gamma_{oidI}(P_m\Box K_n) = m(n-1)$.

2. Let $n=3$, $m \geq 4$. If $3 \mid m$, then assigning 0 to the vertices $(1+3k,1)$, $(2+3k,2)$ and $(3+3k,3)$ for $k \geq 0$ with $3+3k \leq m$, and 1 to the other vertices. These assignments give us a minimum outer independent double Italian domination number of $C_m\Box K_3$ of weight $2m$.
If $3 \mid m-1$, then assigning 0 to the vertices $(1+3k,1)$, $(2+3k,2)$ and $(3+3k,3)$ for $k \geq 0$ with $3+3k \leq m-1$, and 1 to the other vertices. These assignments give us a minimum outer independent double Italian domination number of $C_m\Box K_3$ of weight $2m+1$.
If $3 \mid m-2$, then assigning 0 to the vertices $(1+3k,1)$, $(2+3k,2)$ and $(3+3k,3)$ for $k \geq 0$ with $1+3k \leq m-1$, and 1 to the other vertices. These assignments give us a minimum outer independent double Italian domination number of $C_m\Box K_3$ of weight $2m+1$.
Let $n \geq 4$ and m be a positive integer with $m = tn+r$ where $0 \leq r \leq n-1$. If $r=1$, then we assign 0 to the vertices $(kn+i,i)$ where $0 \leq k \leq t-1$ and $1 \leq i \leq n$ and the vertex $(m,2)$, and 1 to the other vertices.
If $r=0$, then we assign 0 to the vertices $(kn+i,i)$ where $0 \leq k \leq t-1$ and $1 \leq i \leq n$ and 1 to the other vertices.
If $r \notin \{0,1\}$, then we assign 0 to the vertices $(kn+i,i)$ where $0 \leq k \leq t$ and $1 \leq i \leq n$ and 1 to the other vertices. All in all for $n \geq 4$ and positive integer m , above assignments give us a minimum outer independent double Italian domination number of $C_m\Box K_n$ of weight $m(n-1)$. □

The following is a nontrivial sharp upper bound.

Proposition 2.5. *Suppose that G and H are two connected graphs without isolated vertices and $G\Box H$ is not isomorphic of $P_n\Box P_2$ where $n \leq 4$. Then $\gamma_{oidI}(G\Box H) \leq |V(G)||V(H)| - 1$. This bound is sharp.*

Proof. Since $G\Box H$ is not isomorphic with $P_n\Box P_2$ where $n \leq 4$, $G\Box H$ has at least six vertices of degree at least 3 like v_1, v_2, v_3 and u_1, u_2, u_3 where v_i is adjacent to u_i or has two adjacent vertices of degree at least 4, like v, u where each of which has at least three neighbors of degree at least 2. In the first case we assign 0 to v_2 and 1 to the other vertices. In the second case, we assign value 2 to two vertices v, u, value 1 to the neighbors of v and value 0 to the neighbors of u. In the third case there maybe some neighbors of a vertex in $N(v) \setminus \{u\}$ such as v' of degree at least two, which is adjacent to a vertex u' of degree at least two in $N(u) \setminus \{v\}$. We assign 2 to the vertex u' and 0 to v' and 1 to the other vertices. Now in any case $\gamma_{oidI}(G\Box H) \leq |V(G)||V(H)| - 1$.
This bound is sharp for the graphs $P_2\Box P_5$, $P_3\Box P_3$ and $P_3\Box P_4$. □

Proposition 2.6. *Suppose that G and H are two graphs such that $\delta(G), \delta(H) \geq 2$ and $V_0(G)$, $V_0(H)$ are set of vertices of weight 0 in the graphs G and H respectively. Then*

$$\gamma_{oidI}(G\Box H) \leq |V(G)||V(H)| - |V_0(G)||V_0(H)|.$$

Proof. Let $g = (A_0, A_1, A_2, A_3)$ and $h = (B_0, B_1, B_2, B_3)$ be $\gamma_{oidI}(G)$ and $\gamma_{oidI}(H)$ functions of G and H respectively. Since $\delta(G), \delta(H) \geqslant 2$, Proposition 2.1 implies that $A_3 = B_3 = \emptyset$. We define function f on $G\Box H$ as follow. If $(x, y) \in A_0 \times B_0$, then $f((x, y)) = 0$; and $f((x, y)) = 1$ otherwise. If $(x, y) \notin A_0 \times B_0$, then $g(x) \geq 1$ or $h(y) \geq 1$. Let $g(x) \geq 1$ and $h(y) = 0$. Then y has at least two neighbors with positive weight like b_1, b_2 such that (x, b_1) and (x, b_2) are in $N_{G\Box H}(x, y)$ of weight 1. Let $g(x) = 0$ and $h(y) \geq 1$. Then as above there exist vertices $a_1, a_2 \in N_G(x)$ such that (a_1, y) and (a_2, y) are in $N_{G\Box H}(x, y)$ of weight 1. Let $g(x) \geq 1$ and $h(y) \geq 1$. Assume that $a \in N_G(x)$ and $b \in N_H(y)$. Then $(a, y), (x, b) \in N_{G\Box H}(x, y)$ of weight 1. This shows that in any way, when $f((x, y)) = 1$, then (x, y) is adjacent to at least two vertices in $G\Box H$ of weight 1.
If $(x, y) \in A_0 \times B_0$, then $g(x) = 0 = h(y)$ and there exist vertices $a_1, a_2 \in A_1 \cup A_2$ and $b_1, b_2 \in B_1 \cup B_2$ such that $a_1, a_2 \in N_G(x)$ and $b_1, b_2 \in N_H(y)$. Therefore $(x, b_1), (x, b_2), (a_1, y), (a_2, y)$ are in $N_{G\Box H}(x, y)$ of weight 1. We deduce that f is an OIDIDF on $G\Box H$. Thus

$$\gamma_{oidI}(G\Box H) \leqslant w(f) = |V(G)||V(H)| - |A_0||B_0| = |V(G)||V(H)| - |V_0(G)||V_0(H)|.$$

□

Proposition 2.7. *Let G and H be two graphs of order m, n respectively with $\delta(G), \delta(H) \geq 2$. Then $\gamma_{oidI}(G\Box H) \leq |V(G)||V(H)| - min\{|V(G)|, |V(H)|\}$ and this bound is sharp.*

Proof. Let $V(G) = \{u_1, u_2, \cdots, u_m\}$, $V(H) = \{v_1, v_2, \cdots, v_n\}$ and (u_i, v_j) be a vertex in $V(G\Box H)$. We define the function f as $f((u_i, v_j)) = 0$ for $i = j$ and $f((u_i, v_j)) = 1$ otherwise. By Observation 2.1(i), the vertices in V_0^f are independent and each vertex in V_0^f is adjacent to at least three vertices in V_1^f and each vertex in V_1^f is adjacent to at least two vertices in V_1^f. Thus f is an OIDIDF on $V(G\Box H)$ and $\gamma_{oidI}(G\Box H) \leq |V(G)||V(H)| - min\{|V(G)|, |V(H)|$.
For the graph $G\Box K_n, n \geq |V(G)|$ this bound is sharp. □

DOI: 10.20429/tag.2023.100105

Theorem 2.3. *Let G be a graph and $\delta(G) > 0$. Then $\gamma_{oidI}(G\Box K_n) = \beta(G\Box K_n)$ for $n \geq 3$.*

Proof. Suppose that S is maximum independent set on $V(G\Box K_n)$. Since $\delta(G\Box K_n) \geq 3$, each vertex of S is adjacent to at least three vertices of $V - S$, so the function f defined by $f(u) = 0, u \in S$ and $f(u) = 1$ otherwise is an OIDIDF on $G\Box K_n$ and thus $\gamma_{oidI}(G\Box K_n) \leq |V(G\Box K_n) - S| = n|V(G)| - \alpha(G\Box K_n) = \beta(G\Box K_n)$. Therefore $\gamma_{oidI}(G\Box K_n) \leq \beta(G\Box K_n)$. In the other hand $\gamma_{oidI}(G\Box K_n) \geq \beta(G\Box K_n)$, so $\gamma_{oidI}(G\Box K_n) = \beta(G\Box K_n)$. □

3 Lexicographic product of graphs

In this section we investigate outer independent double Italian domination for lexicographic product of graphs. For this end, first we need the below definition.

Definition 3.1. *The Lexicographic product of G and H denoted by $G \circ H$ where $V = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if:*
1. *$u_1u_2 \in E(G)$ or,*
2. *$u_1 = u_2$ and $v_1v_2 \in E(H)$.*

This definition shows that for two graphs G and H, $G\Box H$ is a subgraph of $G \circ H$.

Observation 3.1. *Let G and H be two graphs without isolated vertices. Then*
(i) *There is a γ_{oidI}-function $f = (V_0, V_1, V_2, V_3)$ on $G \circ H$, in which $V_2 = V_3 = \emptyset$.*
(ii) *$V_1(G \circ H)$ is both an outer independent double Italian dominating set and a vertex cover in $G \circ H$.*

Proof. (i) Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{oidI}- function on $G\circ H$. It follows from Proposition 2.1 that $V_3 = \emptyset$. Now assume that $V_2 \neq \emptyset$. We notice that $\delta(G \circ H) = \delta(G)|V(H)| + \delta(H) \geq 3$. By assigning value 0 to a vertex and 1 to the other vertices we obtain an OIDIDF for $G \circ H$. On the other hand the minimum clique in $G \circ H$ is K_4, since for $a \sim b$ in G and $x \sim y$ in H, the vertices $(a, x), (b, x), (a, y), (b, y)$ forms the clique K_4 in $G\circ H$. Now let f be γ_{oidI}-function with $f((a, x)) = 2$ in $G \circ H$. So at most one neighbor of (a, x) must be assigned 0 and the other neighbors of (a, x) have positive weight under f. Without loss of generality, we can define the function g on $G \circ H$ with $g((a, x)) = 1$ and $g((z, t)) = f((z, t))$ otherwise, is an OIDIDF of weight less than $w(f)$, a contradiction. Therefore $V_2 = \emptyset$.
(ii) Let $|V_0(G \circ H)| = \alpha(G \circ H)$. Then by part (i), $|V_1(G \circ H)|$ is the γ_{oidI}-set. Since $|V(G \circ H)| = |V_1(G \circ H)| + |V_0(G \circ H)|$, so $|V_1(G \circ H)| = \beta(G \circ H)$. Thus the proof desired. □

We have the following classic results.

Lemma 3.2. ([7] Lemma 3.1.21) *For any graph G, $\alpha(G) + \beta(G) = |V(G)|$.*

Theorem 3.3. ([8] Theorem 1) *For two graphs G and H, $\alpha(G \circ H) = \alpha(G)\alpha(H)$.*

Theorem 3.4. *Let G and H be two graphs of order at least two and $\delta(G), \delta(H) \geq 1$. Then $\gamma_{oidI}(G \circ H) = \beta(G \circ H) = |V(G)|\beta(H) + |V(H)|\beta(G) - \beta(G)\beta(H)$.*

Proof. Let $S \subseteq V(G \circ H)$ be a maximum independent set in $G \circ H$. According to Observation 3.1(i) there is a γ_{oidI} function $f = (V_0, V_1, V_2, V_3)$ in which, $V_2 = V_3 = \emptyset$ in $G \circ H$, $V_0 = S$ and $V_1 = V \setminus S$. We also deduce each vertex in $V_0 = S$ is adjacent to at least three vertices in $V_1 = V(G \circ H) - S$ and each vertex in V_1 is adjacent to at least two other vertices in V_1. Thus $\gamma_{oidI}(G \circ H) = |V(G \circ H) - S| = |V(G \circ H)| - \alpha(G \circ H) = \beta(G \circ H)$. On the other hand from Lemma 3.2 and Theorem 3.3, $\gamma_{oidI}(G \circ H) = \beta(G \circ H) = V(G)V(H) - \alpha(G)\alpha(H) = V(G)V(H) - (V(G) - \beta(G))(V(H) - \beta(H)) = |V(G)|\beta(H) + |V(H)|\beta(G) - \beta(G)\beta(H)$. □

From Theorem 3.4, we can have the following examples.

Example 3.5. *Let P_m, K_n and C_n be path, complete and circle graphs respectively. Then:*

1. $\gamma_{oidI}(P_m \circ C_n) = mn - \lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor$.
2. $\gamma_{oidI}(P_m \circ K_n) = mn - \lceil \frac{m}{2} \rceil$.
3. $\gamma_{oidI}(K_m \circ C_n) = mn - \lfloor \frac{n}{2} \rfloor$.
4. $\gamma_{oidI}(P_m \circ P_n) = mn - \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$.
5. $\gamma_{oidI}(C_m \circ C_n) = mn - \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$.
6. $\gamma_{oidI}(K_{1,m} \circ P_n) = mn - m\lceil \frac{n}{2} \rceil$.

Proof. We just prove case 1 and 2, the other cases are easily proven from Theorem 3.4 and from Theorem 3.3.
1. If $a \sim b$ in P_m then the vertex (a, x) is adjacent to all vertices of second column in $P_m \circ C_n$. So in the first column we can assign 0 and 1 alternatively, the second column must be assign 1, the third column assign 0 and 1 alternatively and so on. Let S be the maximum independent set of vertices in $P_m \circ C_n$. Thus $\alpha(P_m \circ C_n) = |S| = \lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor$ and so $\gamma_{oidI}(P_m \circ C_n) \leq |P_m \circ C_n - S| = mn - \alpha(P_m \circ C_n) = mn - \lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor$.
2. In the graph $P_m \circ K_n$, each vertex of P_m is adjacent to all vertices of K_n and we choose one of the P_m's graph and assign with 0 and 1. Thus we can assign 0 to $\lceil \frac{m}{2} \rceil$ vertices of the path graph and 1 to other vertices which gives an $OIDIDF$ and so $\gamma_{oidI}(P_m \circ K_n) \leq mn - \lceil \frac{m}{2} \rceil$. Because in the graph $P_m \circ K_n$, the other section is complete graph, we cannot reduce the weight, so the proof holds. □

Theorem 3.6. *Suppose that G and H are two graphs and $\delta(G), \delta(H) > 0$. Then $\gamma_{oidI}(G \circ H) \leq |V(G)||V(H)| - max\{\alpha(G), \alpha(H)\}$ and this bound is sharp.*

Proof. Let S be an α-set of G and D be α-set of H. If $a, b \in S$, then $(a, x) \nsim (b, y)$ in $G \circ H$, in the other word any two vertices in S, are also independent corresponding vertices in $G \circ H$ and we can assign at least one zero in column related to it. Let $|D| \geq |S|$, since $\delta(G \circ H) \geq 3$ we can assign zero to at least $|D|$ vertices of $G \circ H$ and assign 1 to other vertices, so proof is hold. This bound is sharp for the graph $G \circ K_n$, mentioned as follow. □

Proposition 3.1. *Let G be a graph and $\delta(G) > 0$. Then $\gamma_{oidI}(G \circ K_n) = \beta(G \circ K_n)$.*

Proof. In the graph $G \circ K_n$, the rows are G and the columns are K_n and in each column we can assign at most one zero and other vertices assign 1. If vertices u and v are adjacent in G, the corresponding vertices in $G \circ K_n$ are also adjacent. So we can not assign zero to that column. Conversely, if two vertices in G are not adjacent, the corresponding vertices

in $G \circ K_n$ are not also adjacent. Let S be maximum independent set of vertices of graph $G \circ K_n$. Then there is exactly $|S|$ columns such that in each column we have one vertex of weight zero and the other vertices assign one. So
$\gamma_{oidI}(G \circ K_n) \leq (n-1)|S| + n(|V(G)| - |S|) = n|V(G)| - \alpha(G \circ K_n) = \beta(G \circ K_n)$. □

Theorem 3.7. *Let G and H be two connected graphs and $\delta(G), \delta(H) \geq 1$, then $\gamma_{oidI}(G \circ H) \geqslant |V(G)| + |V(H)| - 1$, the equality holds if G, H are stars.*

Proof. Let $|V(G)| = m$ and $|V(H)| = n$. By definition, there are at least two rows in $G \circ H$ such that the vertices are one to one adjacent. As well as, there are at least two columns which have this property. So at least one row and one column must be assigned with 1 and the other vertices can be assigned 1 or 0. So $\gamma_{oidI}(G \circ H) \geqslant m+n+1 > m+n-1$.

Now if $G \circ H = K_{1,m-1} \circ K_{1,n-1}$, with centers a and b and leaves $v_1, v_2, \cdots, v_{m-1}$ and $u_1, u_2, \cdots, u_{n-1}$ respectively. The vertex (a, b) is adjacent to all vertices of $G \circ H$. The vertex (a, u_i) is adjacent to each vertex (v_k, u_i) and (v_i, b). Also (v_i, b) is adjacent to all (v_i, u_j). Now by assigning 1 to vertices $(a, b), (a, u_i), (v_j, b)$ for $1 \leq i \leq n-1$ and $1 \leq j \leq m-1$, and 0 to the other vertices is an OIDIDF of weight $m+n-1$. Therefore $\gamma_{oidI}(G \circ H) = m+n-1$. □

4 Direct product graph

In this section we study the outer independent double Italian domination of direct product of graphs. First, we define the following.

Definition 4.1. *The Direct product of G and H denoted by $G \times H$ where $V = V(G) \times V(H)$ and two vertices (u, v) and (x, y) are adjacent if and only if $ux \in E(G)$ and $vy \in E(H)$.*

From definition, if $v \in V(G)$ of degree k and $u \in V(H)$ of degree l, then it is clear that $(v, u) \in G \times H$ is of degree kl. Therefore we have.

Observation 4.1. *Let G and H be two regular graphs of order m, n respectively. Then $G \times H$ is a regular graph of order mn.*

Proposition 4.1. *Let G, H be two graphs of order m, n respectively, where $m \geq n$.*
(i) *If $\delta(G \times H) \geq 2$, then $\gamma_{oidI}(G \times H) \leq 2m(n-1)$.*
(ii) *If $\delta(G) \geq 2, \delta(H) \geq 2$, then $\gamma_{oidI}(G \times H) \leq m(n-1)$.*
These bounds are sharp.

Proof. (i) By definition, the vertices of any row or any column in $G \times H$ are independent and we can assign 0 to m vertices in the first row and assign 2 to other vertices. Since $\delta(G \times H) \geq 2$, each vertex in $V_0(G \times H)$ is adjacent to at least two vertices in $V_2(G \times H)$. Thus $\gamma_{oidI}(G \times H) \leq 2mn - 2m = 2m(n-1)$.
For seeing the sharpness, since $C_m \times P_2 = 2C_m = C_m \cup C_m$ and $\gamma_{oidI}(C_m) = m$ [14]. Therefore, this bound is sharp for $C_m \times P_2$.

(ii) It is clear $\delta(G \times H) \geq 4$. Also if $a \in G$ and $b \in H$, then $N_G(a)$ has at least two vertices x, y and $N_H(b)$ has at least two vertices u, v such that $N_{G\times H}(a, b)$ contains $(x, u), (x, v), (y, u), (y, v)$ from two different rows. If we assign 0 to the vertices of the first row and 1 to the other vertices. Every vertex of weight 0 is adjacent to at least 4 vertices of weight 1 and each vertex of weight 1 is adjacent to at least 2 vertices of weight 1. Therefore $\gamma_{oidI}(G \times H) \leq mn - m = m(n-1)$.
This bound is sharp for $G = K_4 - e$ and $H = C_4$.

□

Some application of Proposition 4.1, we have now the exact value of outer independent double Italian domination number (γ_{oidI}) of direct product of some custom graphs.

Theorem 4.2. *For the complete graphs, stars, cycles and paths we have.*

1. *Let $m \geq n$. Then $\gamma_{oidI}(K_m \times K_n) = m(n-1)$.*
2. $\gamma_{oidI}(K_{1,m} \times P_n) = \begin{cases} 7, & \textit{if } m \geq 2 \ \textit{ and } n = 3 \\ 2n+2, & \textit{if } m \geq 2 \ \textit{ and } n \geqslant 4. \end{cases}$

.

3. $\gamma_{oidI}(K_{1,m} \times C_n) = 2n, m \geqslant 2, n \geqslant 3.$
4. $\gamma_{oidI}(P_m \times K_n) = \begin{cases} (n+2)\lfloor \frac{m}{2} \rfloor + 1, & \textit{if } m \geqslant 2, m \neq 5, n \geqslant 3, \ \textit{or } (m,n) = (5,3), \\ 2n+4, & \textit{if } m = 5, n \geqslant 4. \end{cases}$

Proof. For two graphs G and H with $V(G) = \{v_1, v_2, \cdots, v_m\}$ and $V(H) = \{u_1, u_2, \cdots, u_n\}$, suppose without loss of generality that the vertices of the $V(G\times H)$ is a matrix table of $m\times n$ such that the i^{th} row is $\{v_iu_1, v_iu_2, \cdots, v_iu_n\}$ and the j^{th} column $\{v_1u_j, v_2u_j, \cdots, v_mu_j\}$.

1. In the construction of $K_m \times K_n$ any two vertices of the position (i, j), (l, k) with $i \neq l$ and $j \neq k$ are adjacent. Therefore in the assignment of we must consider all vertices of one row is 0 and the others 1 or all vertices of one column is 0 and the others 1. Since $m \geq n$, $\gamma_{oidI}(K_m \times K_n) = m(n-1)$.

2. Let $V(K_{1,m}) = \{a, a_1, a_2, \cdots, a_m\}$ and $V(P_n) = \{b_1, b_2, \cdots, b_n\}$. For $n = 3$ and $m \geq 2$ we have 2 vertices of degree m which have m common neighbors of degree 2, and one support vertex of degree $2m$ which has $2m$ neighbors of degree 1. Thus the proof is now trivial. Let $n \geq 4$. The graph $K_{1,m} \times P_n$ has 2 support vertices (a, b_2), (a, b_{n-1}) of degree $2m$, in which m neighbors of each of them are end vertices. The vertices (a, b_1), (a, b_n) are of degree m and the other vertices (a, b_j) in the first column are of degree $2m$. All vertices like (a_i, b_j) for $j \notin \{1, n\}$ are of degree 2 and for $j \in \{1, n\}$ are of degree 1. The minimum values that one can assign to the vertices of $K_{1,m} \times P_n$ are as follows: We assign 3 to the two support vertices (a, b_2), (a, b_{n-1}), assign 2 to the vertices (a, b_j) for $j \neq 2, n-1$ and 0 otherwise. Therefore $\gamma_{oidI}(K_{1,m} \times P_n) = 2(3) + (n-2)2 = 2n + 2$.

3. Let $V(K_{1,m}) = \{a, a_1, a_2, \cdots, a_m\}$ and $V(C_n) = \{b_1, b_2, \cdots, b_n\}$. The graph $K_{1,m}\times C_n$ has n vertices (a, b_j) for $1 \leq j \leq n$ of degree $2m$ in the first column and the vertex (a_i, b_j) are of degree 2 for $i, j \geq 1$. The vertex of degree 2 like (a_i, b_j) is adjacent to exactly two vertices (a, b_{j-1}) and (a, b_{j+1}). The minimum values that one can assign to the vertices of $K_{1,m} \times C_n$ is as follows: We should assign 2 to the each vertex (a, b_j) and assign 0 to the

other vertices. Therefore $\gamma_{oidI}(K_{1,m} \times C_n) = 2n$.

4. First, we bring up $P_m \times K_n$ where $m \geqslant 2, m \neq 5, n \geqslant 3$, or $(m,n) = (5,3)$. Let $V(P_m) = \{a_1, a_2, \cdots, a_m\}$ and $V(K_n) = \{x_1, x_2, \cdots, x_n\}$. Consider two positions.
Let m be odd. Then assigning value 2 to the vertex (a_{2i}, x_1), value 1 to the vertex (a_{2i}, x_j) for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$ and $2 \leq j \leq n$ and also value 1 to the vertex (a_{2i-1}, x_1) for $1 \leq i \leq \lceil \frac{m}{2} \rceil$, and 0 otherwise.
Let m be even. Then assigning value 2 to the vertex (a_{2i}, x_1), and (a_{m-1}, x_1), value 1 to the vertex (a_{2i}, x_j) for $1 \leq i \leq \frac{m}{2}$ and $2 \leq j \leq n$ and also value 1 to the vertex (a_{2i-1}, x_1) for $1 \leq i \leq \frac{m}{2} - 1$, and 0 otherwise. Both assignments offer us $\gamma_{oidI}(P_m \times K_n) \leq (n+2)\lfloor \frac{m}{2} \rfloor + 1$. Since each vertex of weight 0 must be adjacent to the at least three vertices of weight 1 or at least one vertex of weight 2 and one vertex of weight 1, and since any vertex of weight 1 must be adjacent to at least two vertices of weight 1 or at least one vertex of weight 2, and since the set of vertices of weight 0 must be independent. Hence the above assignments are the best possible and obviously observe that $\gamma_{oidI}(P_m \times K_n) \geq (n+2)\lfloor \frac{m}{2} \rfloor + 1$. Therefore $\gamma_{oidI}(P_m \times K_n) = (n+2)\lfloor \frac{m}{2} \rfloor + 1$.

Now we consider $P_5 \times K_n$ where $n \geqslant 4$ with $V(P_5) = \{a_1, a_2, a_3, a_4, a_5\}$ and $V(K_n) = \{x_1, x_2, \cdots, x_n\}$. We assign value 2 to the vertices $(a_2, x_1), (a_3, x_1), (a_4, x_1)$ of the first row, value 1 to the vertices (a_k, x_i) for $k \in \{2, 4\}$ and for $2 \leq i \leq n$ and value 0 to another vertices. These assignments give us an OIDID function of weight $2n+4$. Thus $\gamma_{oidI}(P_m \times K_n) \leq 2n+4$. Same as the first case, these assignments are the best possible. Therefore $\gamma_{oidI}(P_m \times K_n) = 2n + 4$. □

Theorem 4.3. *Let G, H be two graphs and $\delta(G), \delta(H) \geqslant 2$. Let $g = (A_0, A_1, A_2, A_3)$ be a $\gamma_{oidI}(G)$-function and $h = (B_0, B_1, B_2, B_3)$ be a $\gamma_{oidI}(H)$-function, Then*
$\gamma_{oidI}(G \times H) \leqslant |V(G)|\gamma_{oidI}(H) + \alpha(H)\gamma_{oidI}(G) + |A_2||B_1|$.

Proof. Since $\delta(G), \delta(H) \geqslant 2$, Proposition 2.1 implies that $A_3 = B_3 = \emptyset$. We define function f on $G \times H$ as follow. If $(x,y) \in (A_1 \times (B_0 \cup B_1)) \cup (A_0 \times B_1)$, then $f((x,y)) = 1$; if $(x,y) \in (A_2 \times (B_0 \cup B_1 \cup B_2)) \cup (A_0 \cup A_1 \times B_2)$, then $f((x,y)) = 2$; and $f((x,y)) = 0$ otherwise. Now let $f((x,y)) = 0$, then $g(x) = 0$ and $h(y) = 0$. These show that, there exist at least three vertices v_1, v_2, v_3 in $A_1 \cap N_G(x)$ or one vertex v in $A_1 \cap N_G(x)$ and one vertex w in $A_2 \cap N_G(x)$, also there exist at least three vertices u_1, u_2, u_3 in $B_1 \cap N_H(y)$ or one vertex u in $B_1 \cap N_H(y)$ and one vertex z in $B_2 \cap N_H(y)$. Therefore there exist three vertices in $A_1 \times B_1 \cap N_{G\times H}(x,y)$, or there exist three vertices in $A_2 \times B_1 \cap N_{G\times H}(x,y)$, or there exist three vertices in $A_1 \times B_2 \cap N_{G\times H}(x,y)$, or there exist one vertex in $A_1 \times B_1 \cap N_{G\times H}(x,y)$ and one vertex in $A_2 \times B_2 \cap N_{G\times H}(x,y)$. In any of four cases it is not hard to see that $f(N_{G\times H}(x,y)) \geqslant 3$.
If $f((x,y)) = 1$, then $g(x) = 1$ and $0 \leq h(y) \leq 1$, or $g(x) = 0$ and $h(y) = 1$. If $g(x) = 1$ and $0 \leq h(y) \leq 1$, then there exist at least two vertices v_1, v_2 in $A_1 \cap N_G(x)$, or one vertex w in $A_2 \cap N_G(x)$, also at least two vertices u_1, u_2 in $B_1 \cap N_H(y)$ or one vertex z in $B_2 \cap N_H(y)$. If $g(x) = 0$ and $h(y) = 1$, then there exist at least three vertices v_1, v_2, v_3 in $A_1 \cap N_G(x)$ or one vertex v in $A_1 \cap N_G(x)$ and one vertex w in $A_2 \cap N_G(x)$, also there exist at least two vertices u_1, u_2 in $B_1 \cap N_H(y)$ or one vertex z in $B_2 \cap N_H(y)$. In any of three cases, $f(N_{G\times H}(x,y)) \geqslant 3$. Thus we deduce f is an OIDID function on $G \times H$, which leads to

$\gamma_{oidI}(G \times H) \leqslant w(f) = |A_1||B_0| + |A_1||B_1| + |A_0||B_1| + 2|A_2||B_0| + 2|A_2||B_1| + 2|A_2||B_2| + 2|A_0||B_2| + 2|A_1||B_2|$
$= |A_0|(|B_1| + 2|B_2|) + |A_1|(|B_1| + 2|B_2|) + |A_2|(2|B_1| + 2|B_2|) + |B_0|(|A_1| + 2|A_2|)$
$= (|B_1| + 2|B_2|)(|A_0| + |A_1| + |A_2|) + |A_2||B_1| + |B_0|(|A_1| + 2|A_2|)$
$\leqslant |V(G)|\gamma_{oidI}(H) + \alpha_H \gamma_{oidI}(G) + |A_2||B_1|.$ □

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