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Fundamental Theorem of Algebra

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FUNDAMENTAL THEOREM OF ALGEBRA

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CHAPTER I

INTRODUCTION

The Fundamental Theorem of Algebra states that every equation with complex coefficients

$$f(z) \equiv z^n + a_1 z^{n-1} + \text{-----------} + a_n = 0$$

has a complex (real or imaginary) root.

Altho this theorem is the basis of algebra, it was comparatively recently that it was proved. Albert Girard, in 1629, asserted that "every algebraic equation has as many solutions as the exponent of the highest term" but added the exception that no powers could be omitted, but he pointed out that if there were fewer roots than the degree, it is useful to introduce as many "impossible solutions" as will make the total number of roots and impossible solutions equal to the degree. The mathematicians of the eighteenth century were convinced of the existence of a root for ever algebraic equation from contemplation of particular equations: the binomial, those of uneven degree, and those of even degree with sign $(a_0 a_n) = -1$. Many mathematicians, such as d'Alembert, Euler, Daviet de Foncenex, and Lagrange, tried vainly to prove it. Some proofs were given, but these were later proved to be not strictly rigorous.

The first rigorous proof of this theorem was given by Gauss in 1797, and was published in 1799 in his dissertation avoiding of the use of complex quantities.

Gauss states his purpose is to demonstrate a new theorem that all integral, rational, algebraic functions in one variable can be resolved into real factors of first or second degree. He also states in the introduction to the fourth proof: "(the first proof) ......had a double purpose, first to show that all the proofs previously attempted of this most important theorem of the theory of algebraic equations are unsatisfactory and illusory, and secondly to give a newly constructed rigorous proof." Since that time so many proofs have been given that it is now necessary to group them and then characterize the groups.

CHAPTER II

CLASSIFICATION OF PROOFS

A classification of the proofs avoiding the use of the function theory and integral calculus is given by Von E. Netto.* He divides the proofs into two main groups: those making use of the analytical or geometric means, and those making use of the fact that an equation can be proved to have a root since odd powered equations are known to have roots.

The proofs of the first group can be divided into those which use geometric continuity and those which approach a root asymptotically.

The characteristic geometric proofs include the first proof of Gauss, the first and second proofs of Cauchy. The first proof of Gauss will be given in complete detail in Chapter III. The first proof of Cauchy, with some additions by Dickson,' will be given in Chapter IV. Another method of this same proof is given by Macnie," and will be given with a criticism in Chapter VI.

The second subdivision of proofs of the first group gives an analytical rule for the approximation of a root, z, for the f(z) to vanish. A series of proofs of the

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* Encyclopadie of Mathematics, Druck und Von B.G. Teubner, 1898-1904, Vol. 1, Part 1.

' L.E. Dickson, Elementary Theory of Equations, John Wiley & Sons, 1917.

" John Macnie, Algebraical Equations, A.S.Barnes & Company, 1876.

first subdivision can be altered to bring the proof to this form.

The d'Alembert proof, the first which was attempted for the fundamental theorem belonged here; it rested on the analytical inversion of $F(y)=f(x)$. Gauss showed that this proof was not rigorous, and then later showed how it could be changed to obtain full rigour.

R. Lipschitz gave a proof in analytic form. Two others which are of the same train of thought but have completely different methods are those of F. Mertens and C. Weierstrass, both of which make use of the approximation formula of Newton as the method of calculation.

Under the proofs of the second group, the second proof of Gauss is the most important. A summary of this proof will be given later in the thesis.

E. Phragmen gave a proof, using a new representation, in which he derived an algebraic congruence

$F(z)\equiv f(w,r)z+G(w,r)$ modulus $z^2-2\omega z+\rho^2$ where $r=\rho$ by an equation with reduced degree solubility determined so that $f(\omega,\rho)=0$, $G(\omega,\rho)=0$ posses a common root $w=\omega$: therefore $f(z)$ is divisible by $z^2-2z\omega+\rho^2$

CHAPTER III

FIRST PROOF OF GAUSS

Every equation with complex coefficients

$$f(z) \equiv z^n + a_1 z^{n-1} + \text{-----------------} + a_n = 0$$

has a complex (real or imaginary) root.

We shall consider all of the coefficients as real, then the general case in which some of the coefficients are complex easily follows. For, if $f_1(z)$ is a function of z, whose coefficients are, respectively, the conjugate imaginaries of the second function, $f_2(z)$, then we may write $f_1(z)=A+iB$, $f_2(z)=A-iB$, and $f_1(z)f_2(z)=A^2+B^2=f(z)$ where $f(z)$ has only real coefficients. Now, if $f(z)=0$ can be shown to have a root α_1, then we must have either $f_1(\alpha_1)=0$, or $f_2(\alpha_1)=0$. Suppose $f_1(\alpha_1)=0$, then it follows that $f_2(\alpha_2)=0$, where α_2 is the conjugate of α_1. Hence $f_1(z)=0$ and $f_2(z)=0$ have at least one root.

We are to prove that there always exists at least one value of z which causes the polynomial $f(z)$ to vanish.

Let $z=x+iy$, then we may write $f(z)=X+iY$, where X and Y are polynomials in x and y with real coefficients. To find expressions for X and Y, let $x=r\cos\phi$, $y=r\sin\phi$, and then $z=r(\cos\phi+i\sin\phi)$ where $0 \leq \phi < 2\pi$. By De Moivre's

$$z^m = r^m(\cos\phi + i\sin\phi)^m = r^m(\cos m\phi + i\sin m\phi).$$

Substituting for z in $f(z)$ we get,

$$X = r^{n}\cos n\phi + a_1 r^{n-1}\cos(n-1)\phi + a_2 r^{n-2}\cos(n-2)\phi + \text{------} + a_n$$

$$Y = r^{n}\sin n\phi + a_1 r^{n-1}\sin(n-1)\phi + a_2 r^{n-2}\sin(n-2)\phi + \text{----}$$

$$\text{----} + a_{n-1}\, r\sin\phi$$

A second expression for X and Y is obtained by setting $t = \tan\frac{1}{2}\phi$. Then

$$\frac{2t}{1+t^2} = \frac{2\tan\frac{1}{2}\phi}{\sec\frac{1}{2}\phi} = 2\sin\tfrac{1}{2}\phi\cos\tfrac{1}{2}\phi = \sin\phi$$

$$\tan\phi = \frac{2t}{1-t^2}, \qquad \cos\phi = \frac{\sin\phi}{\tan\phi} = \frac{1-t^2}{1+t^2}$$

Thus $z = \dfrac{r(1+ti)^2}{1+t^2}$

$$(1+t^2)^n(X+Yi) = r^n(1+ti)^{2n} + a r^{n-1}(1+ti)^{2n-2}(1+t^2) + \text{-------}$$

$$\text{----} + a_n(1+t^2)^n$$

If we expand the terms on the right and arrange results according to powers of t, we get

$$X = \frac{G(t)}{(1+t^2)^n} \qquad Y = \frac{F(t)}{(1+t^2)^n}$$

where F(t) is a polynomial in t of degree less that 2n, and G(t) a polynomial in t of degree 2n, each with coefficients involving r integrally.

Each point (x,y) representing a complex number $z = x + iy$ having the modulus, r, lies on the circle $x^2 + y^2 = r^2$ with radius r, and center at the origin of coordinates. To find the points on this circle for which X=0, or Y=0, we solve F(t)=0, or G(t)=0, for a given value of r, and note that to each real root, t, corresponds a single real value of $\cos\phi$, consistent with that of $\sin\phi$, and hence a single point ($x = r\cos\phi$, $y = r\sin\phi$). Elementary algebra proves that no equation can have more roots than its power, so if $F(t)=0$ and $G(t)=0$ have any roots at all they cannot have more than $2n$. From this it follows that neither X or Y can be equal to zero at all points of an area in a plane, for in that event we could select r such that the circle would pass through that area and X and Y would vanish at an infinite number of points on this circle.

The value of Y may be written

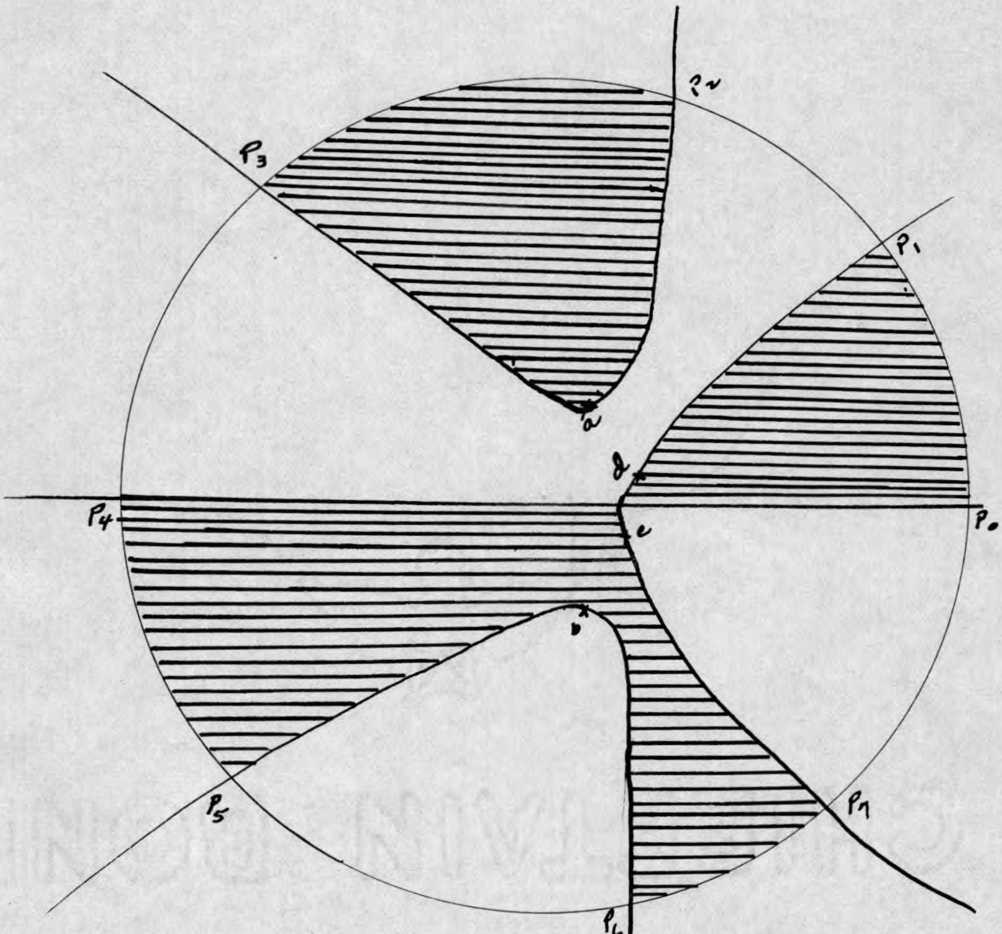
$$Y=r^N(\sin n\phi + \frac{a_1}{r}\sin(n-1)\phi + \frac{a_2}{r^2}\sin(n-2)\phi \text{------})$$

From this expression it is readily seen that r may be taken so large that Y has the same sign as $\sin n\phi$ on all points on the circle where $\sin n\phi$ is numerically larger than some value ϵ, which may be as small as we please but not zero. Mark on the circle the points

$$0,\ \pi/n,\ 2\pi/n,\ \ldots\ldots\ldots\ 2(n-1)\pi/n$$

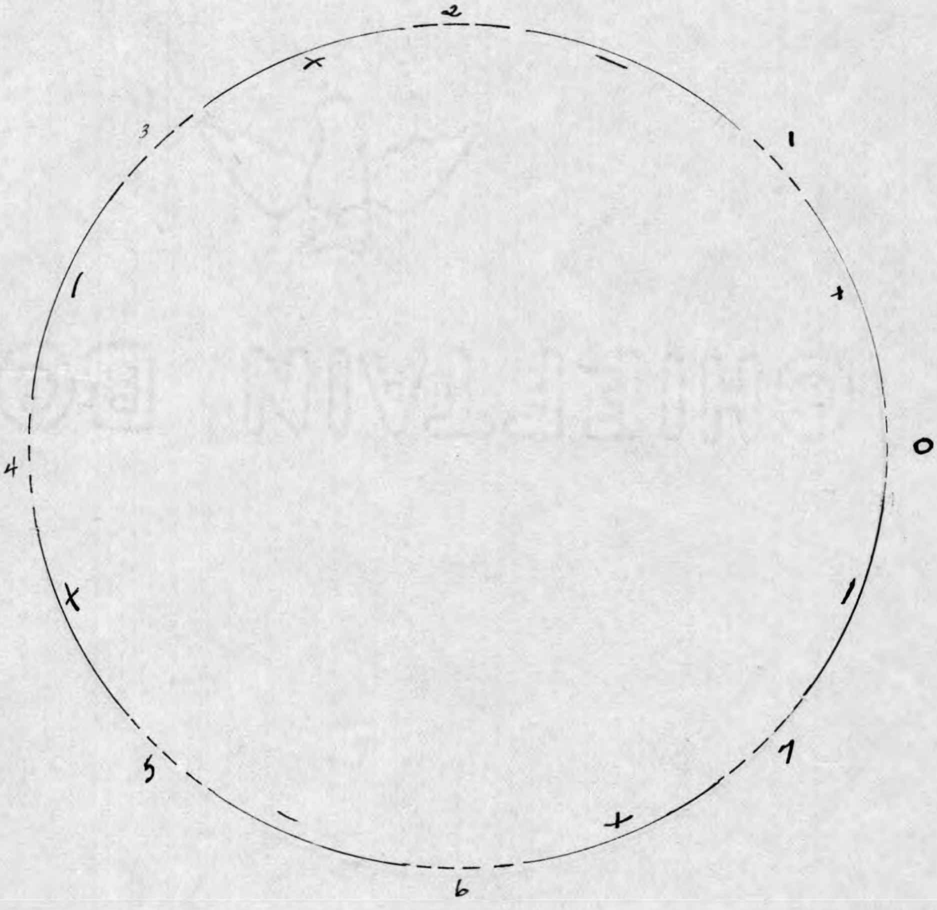
and designate them, respectively, by 0, 1, 2, ...2n-1. Thus the circle is divided into 2n arcs (01), (12), (23), ...(2n-1,0) in which $\sin n\phi$ is alternately + and −.

$\sin\phi$, $\sin 2\phi$,........$\sin n\phi$ are continuous functions of ϕ. Since r is now a constant Y is therefore a continuous function of ϕ, and has a single value for each value of ϕ. But Y had opposite signs at the two ends of the region of any one of our points of division on the circle. Hence Y is zero for some point within each region, and at just one such point, since Y was shown to vanish at not more than 2n points of a circle with center at the



The graph is of the equation

$$f(z) = z^4 - 4z^3 + 9z^2 - 16z + 20.$$



The + and − regions of

$$Y = r^4 \sin 4\Theta - 4 r^3 \sin 3\Theta + 9 r^2 \sin 2\Theta - 16 r \sin \Theta$$

$$\Theta = 7^\circ\, 36'$$

origin. We shall denote the points on the circle at which Y is zero by

$$P_0, P_1, \ldots\ldots\ldots P_{2n-1}$$

Let the constant r be chosen so large that X also has the same sign as its first term, r cos n , for ϕ not too near one of the values $\pi/2n$, $3\pi/2n$, $5\pi/2n$,...... for which cos $n\phi=0$. Since these values correspond to the middle points of the arcs (01), (12),........ no one of them lies in a neighborhood of a division point 0, 1, ... Now cos $n\phi- \pm 1$ when ϕ is an even or an odd multiplier of π/n respectively. Hence X is positive in the neighborhood of the division points 0, 2, 4, ..., 2n-2 and thus at P_0, P_2, P_4, ... but negative in that of 1, 3, 5,... .. 2n-1, and thus at P_1, P_3, P_5,........

We saw that Y is not zero throughout a region of the plane. Hence there is a region in which Y is everywhere positive, and perhaps regions in which Y is everywhere negative, while Y is zero on the boundary lines.

Let R be the part inside our circle of a positive region having the points P_{2h} and P_{2h+1} on its boundary. The points of arc $P_{2h}P_{2h+1}$ may be the only boundary points of R lying on the circle (as for P_2P_3a and P_0P_1d in the figure on page 8) or else its boundary includes at least another such arc $P_{2k}P_{2k+1}$ (as shaded region $P_4P_5bP_6P_7c$ in the figure). In the first case, X and Y are both zero at some point (a or d) on the inner boundary, since X is negative at P_{2h+1} and positive at P_{2h} and hence zero at some intermediate point. In the second case, a point moving from P_{2h} to P_{2h+1} along the smaller included arc and then along the inner boundary of R until it first returns to the circle arrives at a point P_{2k} of even subscript (as in the case of $P_4P_5bP_6$). Indeed if a person travels as did the point, he will always have the region R at his left and hence will pass from P_{2k} to P_{2k+1} and not vice versa. Since X is negative at $P_{2h\ 1}$ and positive at P_{2k}, it (as also Y) is zero at some point b on the part of the inner boundary of R joining these two points. Hence b represents a root of $f(z)=0$. Thus in either of the possible cases, the equation has a root, real or imaginary.*

* This proof is a combination of the proofs given in F. Cajori, Theory of Equations, The Macmillan Co., 1928, and L.E. Dickson, Elementary Theory of Equations, John Wiley & Sons, 1917.

CHAPTER IV

FIRST PROOF OF CAUCHY

Theorem: An equation of degree n with any complex coefficients

$$f(z)=z^{N}+a_1 z^{N-1}+\text{-------------}+a_N=0$$

has a complex (real or imaginary) root.

Write $z=x+iy$ where x and y are real and similarly $a_1=c_1+id_1$ etc. By means of the binomial theorem, we may express any power of z in the form $X+iY$. Hence

$$f(z)=\phi(x,y)+i\,\psi(x,y) \qquad (1)$$

where ϕ and ψ are polynomials with real coefficients.

Lemma 1: $F(h)=a_1h+a_2h^2+\text{---------}+a_Nh^N$ is less in absolute value than any assigned positive number p for all complex values of h sufficiently small in absolute value.

Denote by g the greatest absolute value of $a_1,\text{----}a_N$. If $|h|$ is less than $|k|$, where $|k|<1$, we see that F is less in absolute value than

$$g(k+k^2+\text{-----------}k^N)<g\frac{k}{1-k}<p \text{ if } |k|<\left|\frac{p}{p+g}\right| \text{ which}$$

is obtained from the expression $g\frac{k}{1-k}<p$, thus

$|gk|<|p-pk|$ then $|k(g+p)|<|p|$, and $|k|<\left|\frac{p}{p+g}\right|$.

Lemma 2: Given any positive number P, we can find a positive number R such that $|f(z)|>P$ if $|z|\geq R$.

We have

$$f(z)=z^N(1+D),\ D\equiv a_1(1/z)+\text{---------}+a_N(1/z)^N.$$

Since the absolute value of a sum of two complex numbers is equal to or greater than the difference of their absolute values, we have

$$|f(z)| \geq |z|^n (1-|D|)$$

Let p be any assigned positive number <1. Applying Lemma 1 with h replaced by $1/z$, we see that $|D|<p$ if $|1/z|$ is sufficiently small, i.e., if $\rho \equiv |z|$ is sufficiently large. Then

$$|f(z)| > \rho^n(1-p) \geq P$$

if $\rho^n \geq P/(1-p)$, which is true if

$$\rho \geq \sqrt[n]{\frac{P}{1-p}} = R$$

This proves Lemma 2.

Lemma 3: Given a complex number a such that $f(a) \neq 0$, we can find a complex number z for which $|f(z)| < |f(a)|$.

Write $z=a+h$. By Taylor's theorem

$$f(a+h)=f(a)+f'(a)h+\text{--------}+f^{(r)}(a)\frac{h^r}{r!}\text{----}+f^{(n)}(a)\ \frac{h^n}{n!}$$

Not all of the values $f'(a)$, $f''(a)$,------ are zero since $f^{(n)}(a)=n!$. Let $f^{(r)}(a)$ be the first one of these values which is not zero. Then

$$\frac{f(a+h)}{f(a)} = 1+\frac{f^{(r)}(a)\ h}{f(a)r!}+\cdots+\frac{f^{(n)}(a)\ h^n}{f(a)\ n!}$$

Writing the second member in the simpler notation

$$g(h) \equiv 1+bh^r+ch^{r+1}+\ldots\ldots\ldots+mh^n, b \neq 0$$

we shall prove that complex value of h may be found such that $|g(h)|<1$. Then the absolute value of $f(z)/f(a)$ will be <1 and Lemma 3 proved. To find such a value of h, write h and b in their trigonometric forms

$h=\rho(\cos\theta + i\sin\theta)$, $b=|b|(\cos\beta + i\sin\beta)$

Then

$$bh^r = |b|\rho^r\{\cos(\beta + r\theta) + i\sin(\beta + r\theta)\}$$

Since h is at our choice, ρ and angle θ are at our choice. We choose θ so that $\beta + r = 180^\circ$. Then the quantity in braces reduces to -1, whence

$$g(h)=(1-|b|\rho^r)+h^r(ch+\ldots\ldots+mh^{n-r}).$$

By Lemma 1, we may choose so small that

$$|ch + \ldots\ldots\ldots\ldots + mh^{n-r}| < |b|$$

By taking ρ still smaller of necessary, we may assume at the same time that $|b|\rho^r<1$. Then

$$|g(h)|<(1-|b|\rho^r)+\rho^r|b|, \qquad |g(h)|<1.$$

Minimum Value of a Continuous Function: Let $F(x)$ be any polynomial with real coefficients. Among the real values of x for which $1\leq x\leq 2$ there is at least one value x_1 for which $F(x)$ takes its minimum value $F(x_1)$, i.e., for which $F(x_1)=F(x)$ for all real values of x such that $1\leq x\leq 2$. This becomes intuitive geometrically. The portion of the graph of $y=F(x)$ which extends from its point with the abscissa 1 to its point with the abscissa 2 either has a lowest point or else has several equally low points, each lower than all the remaining points. The arithmetic proof depends upon the fact that $F(x)$ is continuous for each x between 1 and 2 inclusive.

We are interested in the analogous question for

$$G(x,y) = \phi^2(x,y) + \psi^2(x,y),$$

which, by (1), is the square of $|f(z)|$. As in the elements of solid analytic geometry, consider the surface represented by $Z=G(x,y)$ and the right circular cylinder $x^2+y^2=R^2$ Of these points on the first surface and on or within their curve of intersection there is a lowest point or there are several equally low lowest points, possibly an infinite number of them. Expressed arithmetically, among all the pairs of real numbers x,y for which $x^2+y^2=R^2$, there is at least one pair x_1, y_1 for which the polynomial $G(x,y)$ takes a minimum value $G(x_1,y_1)$, i.e., for which $G(x_1,y_1) \leq G(x,y)$ for all pairs of real numbers x,y for which $x^2+y^2 \leq R^2$

Proof of the Fundamental Theorem: Let z' denote any complex number for which $f(z')=0$. Let P denote any complex number exceeding $|f(z')|$. Determine R as in Lemma 2. In it the condition $|z| \geq R$ may be interpreted geometrically to imply that the point (x,y) representing $z=x+iy$ is outside or on the circle C having the equation $x^2+y^2=R^2$. Lemma 2 states that, if z is represented by any point outside or on the circle C, then $|f(z)|>P$. In other words, if $|f(z)| \leq P$, the point representing z is inside circle C. In particular, the point representing z' is inside circle C.

In view of the preceding section on minimum value, we have

$$G(x_1, y_1) \leq G(x, y)$$

for all pairs of real numbers x, y for which $x^2 + y^2 \leq R^2$, where x_1, y_1 is one such pair. Write z_1 for $x_1 + iy_1$. Since $|f(z)|^2 = G(x,y)$, we have

$$|f(z_1)| \leq |f(z)|$$

for all z's represented by points on or within circle C. Since z' is represented by such a point,

$$|f(z_1)| \leq |f(z')| < P \qquad (2)$$

This number z is a root of f(z)=0. For, if $f(z_1) \neq 0$, Lemma 3 shows that there would exist a complex number z for which

$$|f(z)| < |f(z_1)| \qquad (3)$$

Then $|f(z)| < P$ by a preceding statement, so that the point representing z is inside circle C, as shown above. By the statements preceding, (2),

$$|f(z_1)| \leq |f(z)|$$

But this contradicts (3). Hence the fundamental theorem is proved.*

* L.E. Dickson, First Course in Theory of Equations, John Wiley & Sons, 1922.

CHAPTER V

SUMMARY OF GAUSS' SECOND PROOF

In his second proof, Gauss makes use systematically of an important expedient. A series of properties of an equation of nth degree with n roots $g(z)=0$ can be expressed by relations, which are themselves expressible rationally in the symmetric roots, and thus in terms of coefficients of $g(z)$ one can then replace these coefficients by the corresponding coefficients of an equation $f(z)=0$ concerning the existence of whose roots nothing is known, and can conclude from the unchangeablemess of the relations, these properties also for $f(z)$.*

Gauss himself sums up this proof as follows:**
"The solution of the equation

$$Y=x^{M}-L'x^{M-1}+L''x^{M-2}-\ \ldots\ldots\ldots\ldots=0$$

that is, the determination of a particular value of x which satisfies the equation and is either real or of the form $g+h\sqrt{-1}$, may be made to depend upon the solution of the equation $F(u,X)=0$ provided the discriminant of the function Y is not zero. It may be remarked that if all coefficients of Y are real and if as is permissible we take a real value for X, all the coefficients in $F(u,X)$

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* Encyclopædie of Mathematics, Druck und Von B.G. Teubner, 1898-1904, Vol. 1, Part 1.
** Source Book in Mathematics, D.E. Smith, McGraw-Hill Book Company, Inc., 1929

are also real. The degree of the auxiliary equation $F(u,X)=0$ is expressed by the number $\frac{1}{2}m(m-1)$; if then m is an even number of the form $2^{\mu}k$, k designating an odd number, the degree of the second equation is expressed by a number of the form $2^{\mu-1}k$.

In case the discriminant of the function Y is zero, it will be possible to find another function Δ which is a divisor of Y, whose discriminant is not zero, and whose degree is expressed by a number $2^{\nu}k$, where $\nu<\mu$. Every solution of the equation $\Delta=0$ is again made to depend upon the solution of another equation whose degree is expressed by a number $2^{\nu-1}k$.

From this we conclude that in general the solution of every equation whose degree is expressed by an even number of the form $2^{\mu}k$ can be made to depend upon the solution of another equation whose degree is expressed by a number of the form $2^{\mu'}k$ with $\mu'<\mu$. In case this number is also even, i.e., if μ' is not zero, this method can be applied again, and so we proceed until we come to an equation whose degree is expressed by an odd number, the coefficients of this equation are all real if all the coefficients of the original equation are real. It is known, however, that such an equation of odd degree is surely solvable and indeed has a real root. Hence each of the proceeding equations is solvable, having either real or roots of the form $g+h\sqrt{-1}$.

Thus it has been proved that every function Y of the form $x^{M}-L'x^{M-1}+L''x^{M-2}-\ldots\ldots\ldots\ldots$, in which L', L",... are particular real numbers, has a factor x-A where A is real or of form $g+h\sqrt{-1}$. In the second case it is easily seen that Y is also zero for $z=g-h\sqrt{-1}$ and therefore divisible by $x-(g-h\sqrt{-1})$ and so by the product $xx-2gx+gg+hh$. Consequently every function Y certainly has a real factor of first or second degree. Since the same is true of the quotient (of Y by this factor) it is clear that Y can be reduced to real factors of the first or second degree. To prove this fact was the object of this paper."

CHAPTER VI

TWO OTHERS PROOFS OF THE THEOREM

This proof is of a type which is not mentioned in the classification given in Chapter **II**.*

It is shown in the theory of complex variables that in any region of the complex plane, simply connected except for excised points, ~~that~~ if $\phi(x,y)$ doesn't return to its original value when changing continuously as (x,y) describes the boundary of a region, it may be inferred that there must be points in the region for which R=0. $(R=\sqrt{X^2+Y^2})$

Consider the function

$F(z)=z^N+a_1z^{N-1}+ \text{----------} +a_{N-1}z+a_N=X(x,y)+iY(x,y)$

where X and Y are found by setting z=x+iy and expanding and rearranging. The functions X and Y will be polynomials in (x,y) and will therefore by continuous everywhere in(x,y). Consider the angle ϕ of F. Then

$$\phi = \text{ang. of } F = \text{ang of } z^N\left(1+\frac{a}{z}+\text{-----------}+\frac{a_{N-1}}{z^{N-1}}+\frac{a_N}{z^N}\right) =$$

$$\text{ang } z^N + \text{ang of } \left(1+\frac{a_1}{z}+ \text{-----} +\frac{a_{N-1}}{z^{N-1}}+\frac{a_N}{z^N}\right.$$

Next draw about the origin a circle of radius r so large that

$$\left|\frac{a_1}{z}\right|+\cdots\cdots+\left|\frac{a_{N-1}}{z^{N-1}}\right|+\left|\frac{a_N}{z^N}\right|=\frac{|a_1|}{r}+\cdots\cdots+\frac{|a_{N-1}|}{r^{N-1}}+\frac{|a_N|}{r^N} < \epsilon$$

Then for all points of z upon the circumference the

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*E.B. Wilson, Advanced Calculus, Ginn and Company, 1912.

angle of F is

ϕ =ang of F=n(ang of z) + ang of $(1+\eta)$ $|\eta| < \epsilon$

Now let the point (x,y) describe the circumference. The angle of z will change by 2π for the complete circuit, hence ϕ must change by $2n\pi$ and does not return to its original value. Hence there is within the circle at least one point (a,b) for which R(a,b)=0 and consequently for which X(a,b)=0 and Y(a,b)=0, and F(a,b)=0. Thus if α =a+ib, then $F(\alpha)=0$ and the equation F(z)=0 is seen to have at least the one root α. It follows that $(z-\alpha)$ is a factor of F(z) and hence by induction it may be seen that F(z)=0 has just n roots.

The proof which is given by Macnie* is much the same as the first proof of Cauchy given in this paper. In this proof, Macnie makes an assumption which the following criticism** shows is not always true.

To prove that one value of x, in general a complex number, can always be found which causes the rational integral function, $f(x)-P+Q\sqrt{-1}$, to vanish, "it is required to show that some value of $a+b\sqrt{-1}$ must exist, for which $\sqrt{P^2+Q^2}$ becomes zero. For, if it could not become zero, there would be some value below which it could not be diminished. But it will be proved that whatever value of $\sqrt{P^2+Q^2}$, different from zero, can be obtained, a value still smaller can be obtained by making a suitable change in the expression that is substituted for x in the function. $\sqrt{P^2+Q^2}$, therefore, must be capable of becoming zero for some value of $a+b\sqrt{-1}$, that is, the function must become zero for some value of x."

The assumption here is, that if for every value of f(x) an h can be found such that mod $f(x+h) <$ mod $f(x)$, f(x) must necessarily vanish for some value of x. This does not necessarily follow. The inference is not warranted that a function which permits of dimunition

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*J. Macnie, Algebraical Equations, A.S. Barnes & Co., 1876
** R.E. Moritz, On Certain Proofs of the Fundamental Theorem of Algebra, Amer. Math. Monthly, Vol. X.

for every value of the argument possesses necessarily a zero value. If, for example,

$$f(x) = \left(\frac{1+x^2}{x^2}\right)^{x^2+1}$$

f(x) has no zero value, yet for every value of x an h may be found such that

$$f(x+h) < f(x)$$

CHAPTER VII

PROBLEMS

$z^3 = 11+2i$

Let $Z = x+iy$

$x^3 + 3x^2iy - 3xy^2 - y^3 i = 11+2i$. From which

$X = x^3 - 3xy^2 - 11 = 0$

$Y = 3x^2y - y^3 - 2 = 0$

From the equations, we get

$$y = \sqrt{\frac{x^3-11}{3x}}$$

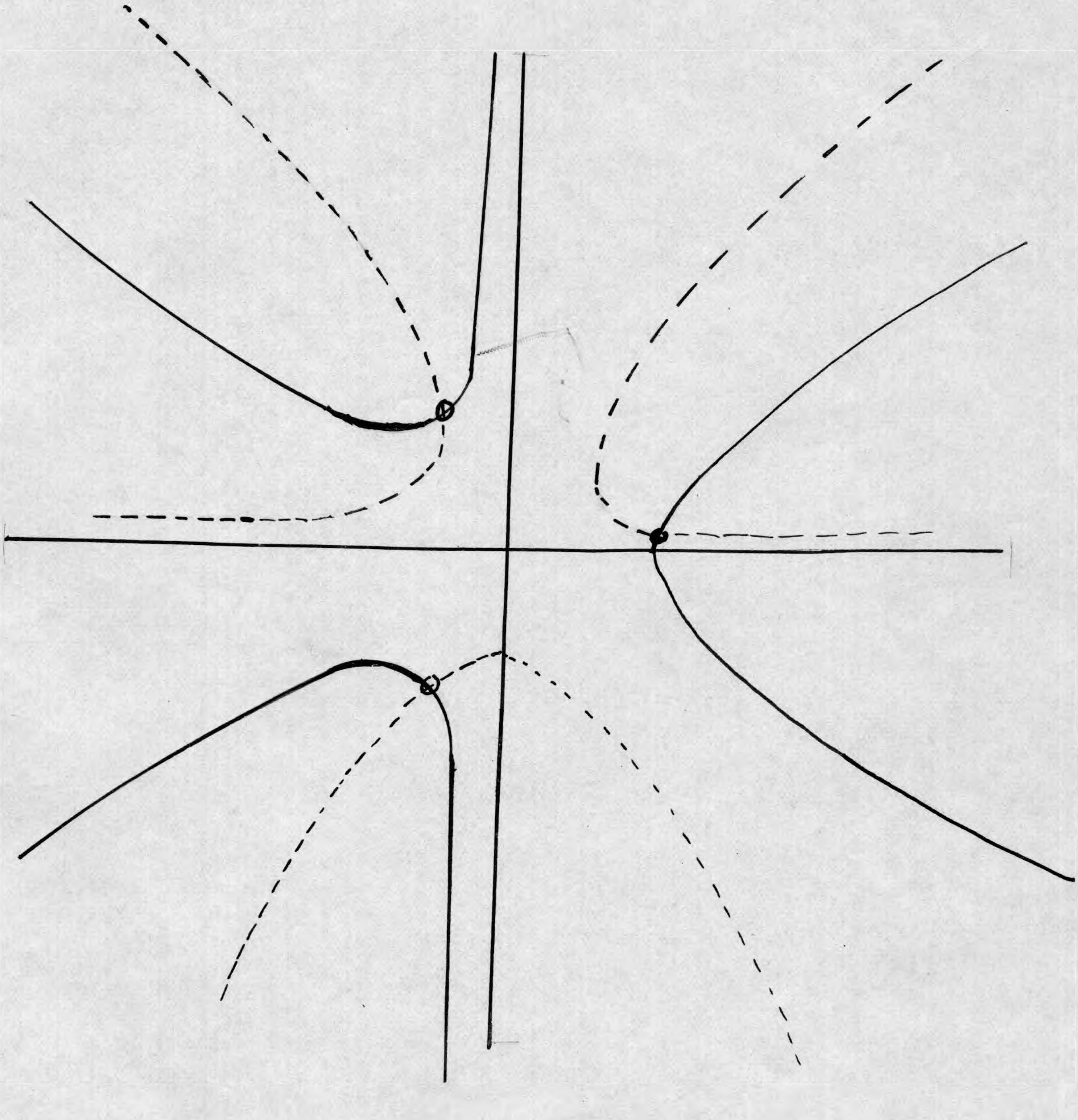
$$x = \sqrt{\frac{y^3+2}{3y}}$$

$\bar{y}$ equation

x	y
+0	imag.
−0	∞
1	imag
2	imag.
$\sqrt[3]{11}$	0
3	$\pm 4/3$
4	$\pm\sqrt{\frac{53}{12}}$
−1	± 2
−2	$\pm\sqrt{\frac{19}{6}}$
−3	$\pm\frac{\sqrt{38}}{3}$
−4	$\pm 5/2$

$\bar{x}$ equation

x	y
∞	0
± 1	1
$\pm\sqrt{\frac{5}{3}}$	2
$\pm\sqrt{\frac{29}{3}}$	3
$\pm\frac{\sqrt{22}}{2}$	4
imag	−1
± 1	−2
$\pm 5/3$	−3
$\pm\sqrt{\frac{31}{6}}$	−4



O roots of equation $= 1 - 2i$; $1.2 + 1.8i$; $2.4 + .5i$.

- - - - $= x = \sqrt{\dfrac{y^3 + 2}{3y}}$

——— $= y = \sqrt{\dfrac{x^3 - 11}{3x}}$

$z^4+2z^3-z^2-2z+10=0$

Let $z=x+iy$

$$Y=2y(x^2-x-1-y^2)(2x+1)$$

$$X=\frac{6x^2+6x-1 \pm \sqrt{(6x^2+6x-1)-4(x^4+2x^3-x^2-2x+10)}}{2}$$

Let $z=r^4 \cos 4\theta + i \sin 4\theta$

$$Y=r^4 \sin 4\theta+2r^3 \sin 3\theta - r^2 \sin 2\theta - 2r \sin\theta$$

$$|D| < \left(\frac{2}{r}+\frac{2}{r^2}+\frac{2}{r^3}\right) \quad \text{Let } r=4$$

$$|D| < \tfrac{1}{2}+1/8+1/32 = .65625$$

Let $c = \text{Sin } 4C = \text{Sin } 41^\circ 0' 52''$

$C=10^\circ 15' 13''$

The positive angles $\theta (\theta < 2\pi)$ for which $\sin 4\theta$ exceeds $\sin 4C$ numerically are those between C & $(\pi/4 - C)$; $(\pi/4 + C)$ & $(\pi/2 - C)$; between $(\pi/2 + C)$ & $(3\pi/4 - C)$; $(\frac{7\pi}{4} + C)$ & $(2\pi - C)$.

For any such angle θ and for $r=4$ Y has the same sign as $\sin 4\theta$ and hence is alternately + and − in these succesive intervals.

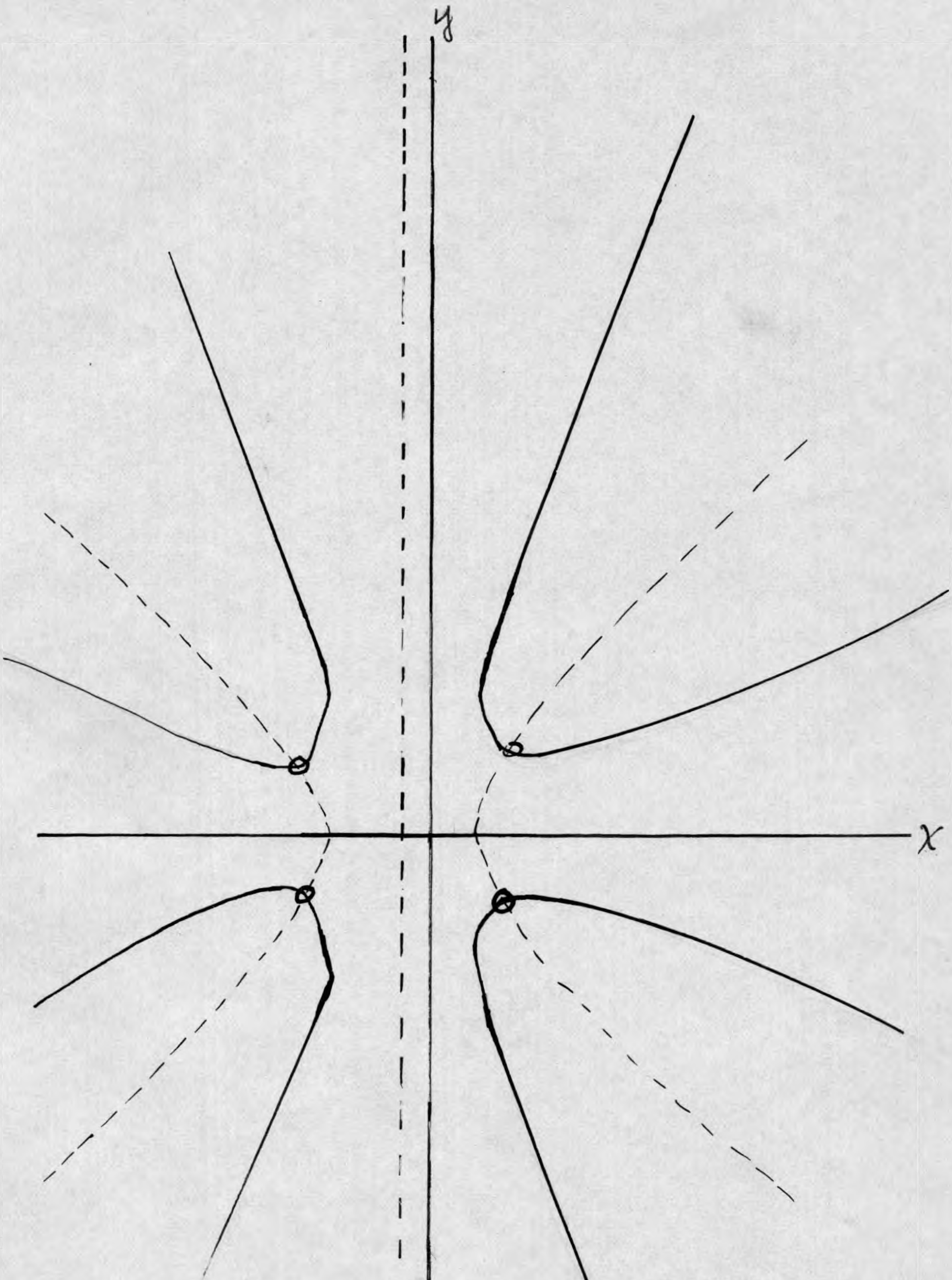
The X equation is

$$X=r^4 \cos 4\theta+2r^3 \cos 3\theta - r^2 \cos 2\theta - 2r \sin\theta + 10$$

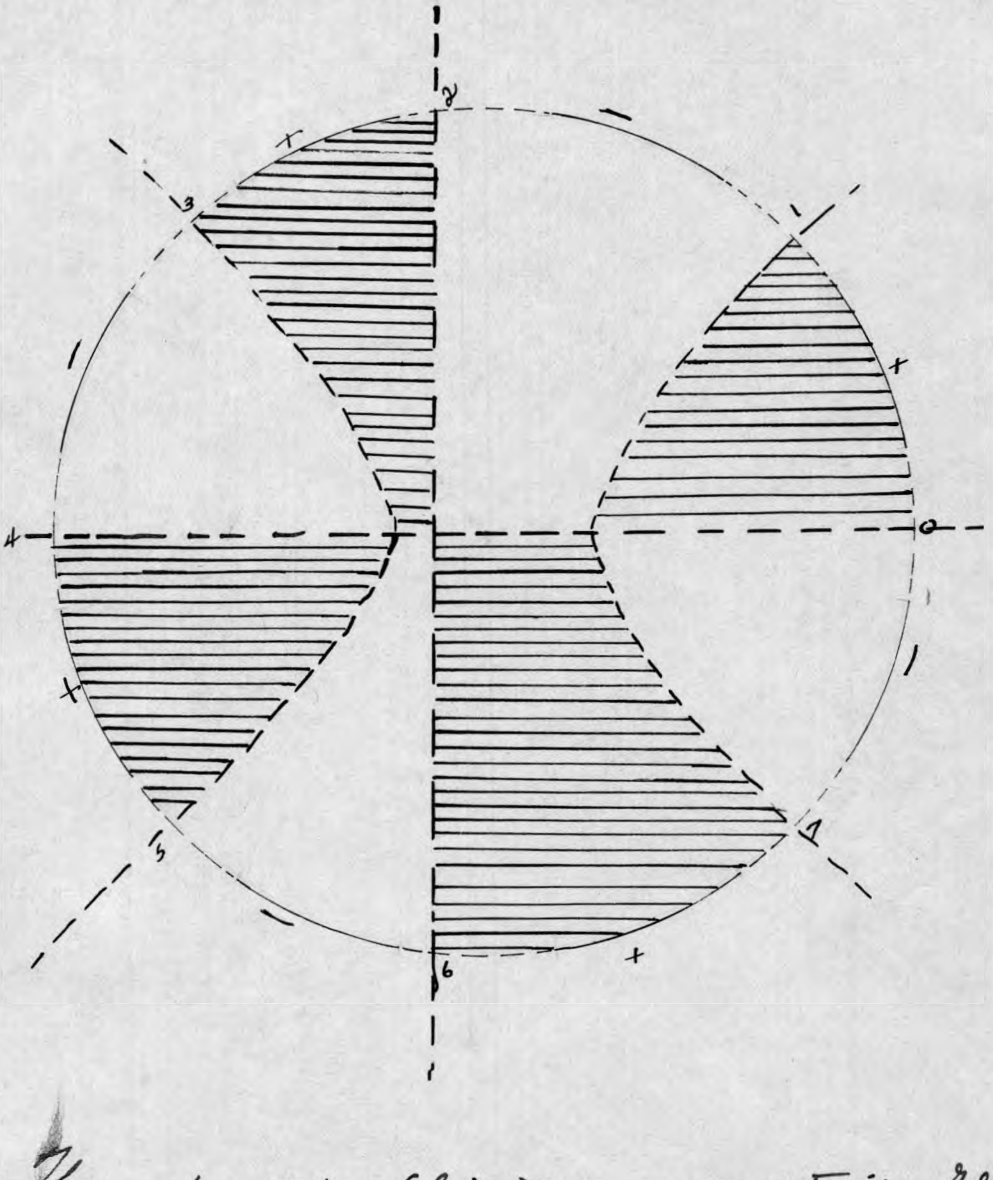
For any such angle θ and for $r=4$, X has the same sign as $\cos 4\theta$.

x	y
.691	±1.7
.7	±1.92 or ±1.56
1	±3.2, ±1
2	±5.8, ±1
3	±8.3, ±1.4
6	±2.6
8	±3.5
−1.7	±2.5
−2	±3.2, ±1.
−3	±6, ±1
−6	±2.5

X equation.



—— = X curve.
- - - - = Y curve also x-axis
o roots $1 \pm i$, $-2 \pm i$.



The positive (lined) + negative regions

$$Y = 2y(x^2 - x - y^2 - 1)(2x+1)$$

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