A MODIFIED CLASS OF COMPOSITE DESIGNS FOR THE RESPONSE MODEL APPROACH WITH NOISE FACTORS

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A class of composite designs involves factorial, axial, and center points. Factorial points are with a variance-optimal design for a first-order or interaction model, and axial points provide information about the existence of curvature. The center points allow for efficient estimation of the pure quadratic terms. From these properties, a class of composite designs is recommended if resources are readily available and a high degree of precision of parameter estimate is expected and evolves from their use in sequential experimentation. However, there are often cost constraints imposed on experiments. Previous studies show that resolution, orthogonal quadratic effect property, and saturated or near-saturated design reduce the number of experiments. This study extends the response model approach with noise factors to composite designs satisfying these properties. These modified composite designs are further discussed and examined in terms of scaled prediction error variance and extended scaled prediction variance, which provides a good distribution of the prediction variance of the response. Based on these criteria, the best performance design is suggested according to the number of control and noise factors. As a result, we show that the modified designs showing robustness to noise factors and stability of predictive variance are a class of modified small composite designs and modified augmented-pair designs.

Keywords: Class of Composite Designs, Resolution, Orthogonal Quadratic Effect Property, Prediction Variance, Small Composite Design

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1. INTRODUCTION

A set of properties that should be considered when choosing a response surface design is a good fit of the model to the data, a good distribution of prediction variance of the response, and cost constraints imposed on experiments. Besides these, many other important characteristics are suggested (Myers *et al*. 2016). Some properties conflict with each other, so trade-offs almost always exist when choosing an appropriate design. Generally, a larger experiment can consider that a high degree of precision of parameter estimate is expected and evolves from their use in sequential experimentation. However, there are often cost constraints imposed on experiments.

Following previous studies (Hartley, 1959; Westlake, 1965; Draper, 1985; Shoemaker *et al*., 1991; Morris, 2000; Box and Draper, 2007; Angelopoulos *et al*., 2009; Nguyen and Lin, 2011; Georgiou *et al*., 2014), several proposals and algorithms were shown to reduce the number of experiments in a class of composite designs for the response surface methodology.

Some well-known composite designs from the literature are the central composite designs (CCDs, Box and Wilson, 1951), the small composite designs (SCDs, Draper, 1985; Draper and Lin, 1990), Type 1 SCD, Type 2 SCD (Nguyen and Lin, 2011), the augmented-pair designs (APDs, Morris, 2000), and the modified mean orthogonal composite designs (Georgiou *et al*., 2014). These studies introduce a method for generating efficient and economical response surface designs using and combining available designs.

There are several response surface alternatives for solving the robust parameter designs (RPDs) problem and for conducting process robustness studies (Taguchi and Wu, 1980; Taguchi, 1986, 1987; Welch *et al*., 1990; Shoemaker *et al*., 1991; Park and Antony, 2008; Bingham and Nair, 2012). RPDs are a principle that emphasizes the proper choice of levels of control factors in a system. The term RPD entails designing the system to achieve robustness to inevitable changes in the noise factors. The modeling of both control factor (x) and noise factor (z) in the same model has been called a response model approach (Myers *et al*., 2016). The response model approach models both the x and z in the same model. For this approach, the combined array in which a design is chosen to allow estimability of a reasonable model in x and z is often less costly and reasonably sufficient.

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This study extends the response model approach to well-known composite designs. It considers a situation in which these designs have control and noise factors. Designs constructed in this manner shall be called modified composite designs. The modified composite designs are further discussed and examined in terms of scaled prediction error variance (SPEV, Borror *et al*., 2002) and extended scaled prediction variance (ESPV, Oh *et al*., 2017, 2018; Oh, 2022), which provides a good distribution of prediction variance of the response. Based on these criteria, the best performance design is suggested according to the number of control and noise factors. The remainder of this paper is organized as follows: Section 2 shows a class composite design and response model approach with control and noise factors. Section 3 extends a response model approach to composite designs, which proposes modified composite designs and the choice of candidate designs in modified composite designs. Concluding remarks are presented in section 4.

2. A CLASS OF COMPOSITE DESIGNS AND ROBUST PARAMETER DESIGN

2.1 Composite designs for second-order response surfaces

If the supposed response surface model suffers from a lack of fit due to some surface curvature, then the model needs to be more complex than a simple first-order or first-order plus interactions model. In this case, a second-order response surface model (RSM) is a reasonable choice. In this study, the following second-order model can be used:

$$\mathrm{y} = \beta_0 + \mathrm{x}'\beta + \mathrm{x}'\mathrm{B}\mathrm{x} + \varepsilon, \tag{1}$$

where y is the response, $\mathrm{n} \times 1$ vector, x is $\mathrm{n} \times p_1$ vector, β is a $p_1 \times 1$ vector containing the regression coefficients of the controllable factors, B is $p_1 \times p_1$ a matrix whose main diagonals exhibit the regression coefficients associated with the pure quadratic effects of the control factors and whose off-diagonal is one-half of the mixed quadratic (interaction) effects of the controllable factors. In Eq. (1), we assumed that ε is $\mathrm{NID}(0, \sigma^2)$.

Some well-known classes of designs in second-order RSM are the CCDs, the SCDs, Type 1 and 2 SCD, the APDs, and the modified mean orthogonal composite designs.

Much of the motivation for the CCD evolves from its use in sequential experimentation. This study divides the sequential experimentation process into the first and second stages. It involves the use of a two-level factorial or fraction combined with the following $2p_1$ or n_0 center points. As a result, the CCD has factorial points, $2p_1$ axial points, n_0 center runs. The factorial points represent a variance-optimal design for a first-order model or first-order + two-factor interaction model. Center points provide information about the existence of curvature in the system. If the curvature is found in the system, the additional axial points allow for efficient estimation of the pure quadratic terms (Myers *et al*. 2016).

The APDs consist of a first-order two-level orthogonal design with n_1 runs and n_0 center points in the first stage. This design is then augmented by $n_2 = \binom{n_1}{2}$ runs. For each pair of runs x_u and x_v in n_1, a run in n_2 is generated as $x_{uv} = -0.5(x_u + x_v)$. This procedure adds one new design run, x_{uv}, for each pair of runs (x_u, x_v) in x.

The run size of the APD design in the first stage is minimal, and the quadratic effects of APDs are always orthogonal to all main effects and interaction effects. It is called an orthogonal quadratic effect (OQE) property by Nguyen and Lin (2011).

The SCDs get their name from the idea of the CCD, but the factorial portion is a special resolution III fraction in which no four-letter word is among the defining relations. The total run size is reduced from that of the CCD. Since not all SCDs have the OQE property, Nguyen and Lin (2011) have provided a new algorithm that can augment any first-order design with additional design points to form a good design for fitting second-order models. For Type 1 SCD (SCD_1), the $2p_1$ axial points are fixed in the second stage, and an algorithm is used to search for the best first-order design, which should be used in the first stage. For Type 2 SCD (SCD_2), a small first-order design is used at the first stage. The $2p_1$ axial points are anticipated in the second stage.

The modified mean orthogonal composite designs give a general construction method. The axial points of traditional CCDs are replaced by some edge points of the hypercube that circumscribes the sphere of zero center and radius. Georgiou *et al*. (2014) showed that these designs satisfy the OQE property if their factorial part is of resolution IV or higher.

Table 1 displays the run sizes of different composite designs with n_0 for sequential experimentations regarding the number of factors, p_1, and parameters, p. It can be seen that no class of design in Table 1 is the best design based on runs. Because all designs except CCD should be considered if the runs are expensive, while CCDs are highly recommended if resources are readily available and a high degree of precision of parameter estimates is expected.

Table 1. Comparison of run sizes for SCD, SCD_1, SCD_2, CCD, APD, and Georgiou's design

p_1	p	SCD	SCD_1	SCD_2	CCD	APD	Georgiou's design
3	10	10	10	10	14	10	10
4	15	16	16	16	24	36	16, 20
5	21	21	22	26	26	36	21, 22, 24, 26, 36
6	28	28	28	36	44	36	28, 36
7	36	36	38	38	78	36	36, 38
8	45	46	48	48	80	78	45, 46
9	55	56	58	58	146	78	55, 56, 58
10	66	66	68	68	148	78	66, 68

2.2 Response model approach with control and noise factors

In many experimental situations, the model of the control and noise variables may involve second-order or quadratic terms in the control variables. Therefore, we consider a plausible model between the response variable y and the p_1 controllable variables where $\mathrm{x} = (x_1, x_2, \cdots, x_{p_1})'$ and the p_2 noise variables where $\mathrm{z} = (z_1, z_2, \cdots, z_{p_2})'$ which can be described in a matrix form as:

$$\mathrm{y}(\mathrm{x},\mathrm{z}) = \beta_0 + \mathrm{x}'\beta + \mathrm{x}'\mathrm{B}\mathrm{x} + \mathrm{z}'\gamma + \mathrm{x}'\Delta\mathrm{z} + \varepsilon, \tag{2}$$

where γ is a $p_2 \times 1$ vector of regression coefficients for the main effects of the noise variables and Δ is a $p_1 \times p_2$ matrix of the control factor induced by noise factor interaction effect. In Eq. (2), we assume that ε is $\mathrm{NID}(0, \sigma^2)$ and that the noise factors have been scaled so that they have mean zero and covariance matrix $\mathrm{Var}(\mathrm{z}) = \sigma_z^2\mathrm{V}$. V is an $p_2 \times p_2$ symmetric positive definitive matrix. We also assume that $\mathrm{V} = \mathrm{I}$ so that the noise factors are uncorrelated and have identical, constant variances. As in Borror *et al*. (2002) and Myers *et al*. (2016), there are many scenarios where these assumptions are certainly reasonable. These could include situations when the noise factors are process variables that are difficult to control or raw material properties. On the other hand, if the noise factors are certain environmental variables, such as temperature and relative humidity, then they are most likely correlated. It is also customary to assume that σ_z^2 and the elements of V are known based on knowledge of and experience with the noise factors under RPDs.

According to Myers *et al*. (1992), concerning the noise variables (z) and the random error (ε), the model for the response mean is obtained by considering the conditional expectation of $\mathrm{y}(\mathrm{x},\mathrm{z})$ in Eq. (2). They show:

$$\int\int \mathrm{y}(\mathrm{x},\mathrm{z})p(\mathrm{z},\varepsilon)d\mathrm{z}d\varepsilon = E_{\mathrm{z},\varepsilon}[\mathrm{y}(\mathrm{x},\mathrm{z})] = \beta_0 + \mathrm{x}'\beta + \mathrm{x}'\mathrm{B}\mathrm{x}, \tag{3}$$

where $p(\mathrm{z},\varepsilon)$ is the joint conditional probability density function of z and ε, given x. The notation $E_{\mathrm{z},\varepsilon}[\mathrm{y}(\mathrm{x},\mathrm{z})]$ denotes the expectation $\mathrm{E}[\mathrm{y}(\mathrm{x},\mathrm{z})]$ for z and ε,

Similarly, the model for the response variance is

$$Var_{\mathrm{z},\varepsilon}[\mathrm{y}(\mathrm{x},\mathrm{z})] = Var_{\mathrm{z},\varepsilon}[(\gamma' + \mathrm{x}'\Delta)\mathrm{z}] + \sigma^2 = \sigma_z^2(\gamma' + \mathrm{x}'\Delta)(\gamma' + \mathrm{x}'\Delta)' + \sigma^2, \tag{4}$$

where $\gamma' + \mathrm{x}'\Delta$ is the vector of partial derivatives of $y(\mathrm{x},\mathrm{z})$ considering noise variables z. Thus, this is the slope of the response surface in the direction of the noise variables z. Equations (3) and (4) represent the mean and variance response surfaces developed from the model containing both control and noise variables. The estimated response surfaces are obtained by replacing parameters with the fitted model's ordinary least square (OLS). The estimated process mean and variance response surfaces are given by,

$$\hat{E}_{\mathrm{z},\varepsilon}[\mathrm{y}(\mathrm{x},\mathrm{z})] = \hat{\beta}_0 + \mathrm{x}'\hat{\beta} + \mathrm{x}'\hat{\mathrm{B}}\mathrm{x} = \mathrm{x}^{(2)\prime}\hat{\beta}_1 \tag{5}$$

$$\widehat{Var}_{\mathrm{z},\varepsilon}[\mathrm{y}(\mathrm{x},\mathrm{z})] = \hat{\sigma}_z^2\left(\hat{\gamma}' + \mathrm{x}'\hat{\Delta}\right)\left(\hat{\gamma}' + \mathrm{x}'\hat{\Delta}\right)' + \widehat{\sigma^2}\ , \tag{6}$$

where the regression coefficient $\hat{\beta}_0$, the elements of the vector $\hat{\beta}$, and the elements of the matrix $\hat{\beta}$ are contained in the vector $\hat{\beta}_1$. And $\mathrm{x}^{(2)\prime}$ is a vector of the controllable variables expanded to a second-order model containing the constant 1, the first-order terms, the second-order terms, and the control-by-control factor interactions. $\hat{\sigma}^2$ is the residual mean square error from the fitted response model. Most authors formulate the standard robust parameter design problems using Equations (5) and (6).

Borror *et al*. (2002) developed a SPEV for the mean model in Eq. (5). The form for the variance of the prediction error is

$$Var_{\mathrm{z},\varepsilon}\left[\mathrm{y}(\mathrm{x},\mathrm{z}) - \hat{E}_{\mathrm{z},\varepsilon}[\mathrm{y}(\mathrm{x},\mathrm{z})]\right] = Var_{\varepsilon}\left(\hat{E}_{\mathrm{z},\varepsilon}[\mathrm{y}(\mathrm{x},\mathrm{z})]\right) + Var_{\mathrm{z},\varepsilon}[(\gamma' + \mathrm{x}'\Delta)\mathrm{z}] + \sigma^2 \, . \tag{7}$$

The SPEV is desirable to evaluate designs for experiments involving noise factors and is an appropriate measure for making comparisons among competing RPDs. The SPEV is found by multiplying an expanded form of Eq. (7) by the number of runs and dividing by σ^2 to give

$$\frac{NVar_{\mathrm{z},\varepsilon}[\mathrm{y}(\mathrm{x},\mathrm{z}) - \mathrm{x}^{(m)\prime}\hat{\beta}^*]}{\sigma^2} = \mathrm{N}\left[\mathrm{x}^{(m)\prime}P^{11}\mathrm{x}^{(m)}\right] + N(\gamma' + \mathrm{x}'\Delta)(\gamma' + \mathrm{x}'\Delta)' \, , \tag{8}$$

where $\mathrm{x}^{(m)}$ represents the control variables expanded to the form of the model used for those variables, P^{11} represents the variance-covariance matrix, $\hat{\gamma} + \mathrm{x}'\hat{\Delta}$ is the mean of $\gamma' + \mathrm{x}'\Delta$.

To remove σ^2 from $(\gamma' + \mathrm{x}'\Delta)(\gamma' + \mathrm{x}'\Delta)'/\sigma^2$, the elements of γ and Δ are constants, and x is a design point in the region of interest. It will subsequently be convenient to define the elements of the vector γ, denoted by γ_i, and the elements of the matrix Δ, δ_{ij}, as multiples of the process standard deviation σ. That is $\gamma_i = t_i\sigma$ and $\delta_{ij} = t_{ij}\sigma$. For instance, if it is believed that the noise variables equally influence the response, then $t_i = t_1$ for each noise variable where t_1 can take on any non-negative value (Borror *et al*., 2002). Finally, SPEV is as follows:

$$N\left[\mathrm{x}^{(m)\prime}P^{11}\mathrm{x}^{(m)} + p_2\left(t_i^2 + 2t_it_j\mathrm{x}'1 + t_j^2\mathrm{x}'\mathrm{J}\mathrm{x}\right)\right] , \tag{9}$$

where 1 is an $\gamma_i \times 1$ vector of $1'$s and J is an $\gamma_i \times \gamma_i$ matrix of $1'$s.

Oh *et al*. (2017) proposed a measure for the prediction variance of future values (z_f) of z as a prediction method for examining design robustness and showed an ESPV for the response surface approach to RPDs, and adopted this idea for the extended scaled quantity as follows:

$$\frac{\mathrm{N}\widehat{Var}_{\mathrm{z}_f,\hat{\beta}^*_{\mathrm{OLS}}}[\hat{y}(\mathrm{x},\mathrm{z}_f)]}{\sigma^2} = \mathrm{N}\left[\mathrm{x}^{(m)\prime}P^{11}\mathrm{x}^{(m)} + tr(C) + p_2\left(t_i^2 + 2t_it_j\mathrm{x}'1 + t_j^2\mathrm{x}'\mathrm{J}\mathrm{x}\right)\right] . \tag{10}$$

Also, we assume that σ_z^2 is known, and the high and low levels of z_j are at $\pm\sigma_z$ in coded form. Thus, $\sigma_z^2 = 1$. In Eq. (10), $\hat{\beta}^*_{\mathrm{OLS}}$ denotes the vector containing all estimates of the parameters in Eq. (2). Because the z_f in the expectation refer to future values of the noise factors; after running the experiment and fitting the model, z is a random variable. Although the OLS estimates of $\mathrm{y}(\mathrm{x},\mathrm{z})$ contain noise factor information, these z values refer to past noise factor values obtained when z can be controlled at the research or development level. C is the matrix for linear, quadratic, and interaction terms involving only the control variables. The value of C is simply $\frac{\mathrm{Var}[\hat{\gamma}' + \mathrm{x}'\hat{\Delta}]}{\sigma^2}$, and $\mathrm{Cov}_{\mathrm{z}_f,\hat{\beta}^*_{\mathrm{OLS}}}[\hat{\beta}_0 + \mathrm{x}'\hat{\beta} + \mathrm{x}'\hat{\mathrm{B}}\mathrm{x}, (\widehat{\gamma'} + \mathrm{x}'\hat{\Delta})\mathrm{z}_f] = 0$. Note that, if $t_i = t_1$ and $t_{ij} = t_2$ (where $t_1 \geq 0$ and $t_2 \geq 0$), then Equation (10) becomes

$$N\left[\mathrm{x}^{(m)\prime}P^{11}\mathrm{x}^{(m)} + tr(C) + p_2(t_1^2 + 2t_1t_2\mathrm{x}'1 + t_2^2\mathrm{x}'\mathrm{J}\mathrm{x})\right] . \tag{11}$$

When computing the RPDs, we consider a situation where the noise variables influence the response equally. To obtain the general form of Eq. (10), we can modify the Eq. (11) in terms of t_1 and t_2 as follows:

$$N\left[\mathrm{x}^{(m)\prime}P^{11}\mathrm{x}^{(m)} + tr(C) + p_2(t_1^2 + 2t_1^2r\mathrm{x}'1 + t_1^2r^2\mathrm{x}'\mathrm{J}\mathrm{x})\right] , \tag{12}$$

where r is the appropriate constant. Also, it can be seen that Eq. (9) and Eq. (12) hold the relationship SPEV = ESPV − $Ntr(C)$. Oh *et al*. (2107) proposed that $Ntr(C)$ represents a property of the variability in the noise variables themselves. A detailed review and comments on Eq. (11) and Eq. (12) can be found in Oh *et al*. (2017) and Oh (2022).

Based on SPEV and ESPV values for each modified composite design, we will investigate which design provides a good distribution of prediction variance of the response. Therefore, we illustrate a comparison of well-known composite designs in the order of ESPV in terms of $(t_1 = t_2)$, $(t_1 > t_2, r < 1)$, and $(t_1 < t_2, r > 1)$. In addition, we will illustrate the fraction of the design space plots (FDS plots, Zahran *et al.*, 2003) of ESPV that are in order of ESPV in terms of $(t_1 = t_2)$, $(t_1 > t_2, r < 1)$, and $(t_1 < t_2, r > 1)$ in part of the application of graphical evaluation.

FDS plots give the researcher more detailed information by quantifying the fraction of the design space where scale prediction variance (SPV) is less than or equal to any pre-specified value.

Zahran *et al*. (2003) noticed how the design's maximum and minimum SPV values occur at different radii of the variance dispersion graphs (VDGs; Giovannitti-Jensen and Myers, 1989) with other associated volumes. Zahran *et al*. (2003) developed FDS plots to complement the VDGs. While the VDGs show where various value SPV are observed in the design space, the FDS plots summarise the prediction performance for the entire design space with a single curve.

3. EXTEND A RESPONSE MODEL APPROACH TO COMPOSITE DESIGNS

3.1 Response model approach with control and noise factors

Suppose one focuses on the response model approach in which a single model is constructed for both control, x, and noise factor, z, with the simultaneous development of Eq. (3) and Eq. (4). In that case, the product array is not needed. For this approach, the combined array, in which a design is chosen to allow estimability of a single model in x and z, is often less costly. Also, combined arrays offer more flexibility in estimating effects or regression coefficients and savings in run size (Borkowski and Lucas, 1997; Myers *et al*., 2016). Because the response surface approach emphasizes efficient estimation of the appropriate response model in x and z, the combined array is suitable with the flexibility needed in the selected model terms. From this point of view, the response model approach and the combined array are pretty compatible. Many authors have suggested combined array designs (Welch *et al*., 1990; Shoemaker *et al*., 1991; Montgomery, 1991; Lucas, 1994; Myers *et al*., 1992, Box and Jones, 1989).

The candidate designs to examine are well-known composite designs such as the CCDs, SCDs, APDs, and Georgiou's designs. The modified composite designs proposed in this study are constructed from a class composite design by removing the axial points for the noise factors. For example, D_1 is the design matrix of CCD for three control factors with $n_0 = 4$ and

$$D_1 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ -\alpha & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -\alpha \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ -\alpha & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

axial point α, and D_2 is the design matrix of modified CCD that deleted the axial runs associated with the one noise factor.

The values of axial points were chosen based on the designs' cuboidal, rotatability, and spherical properties. Note that the pure quadratic terms for the noise factors are typically unnecessary and omitted in this study. The modified central composite designs (MCCDs), modified small composite designs (MSCDs), modified augmented-pair designs (MAPDs), and modified Georgiou's designs (MGDs) are constructed from standard CCDs, SCDs, APDs, and Georgiou's design which are by deleting the axial runs associated with the noise variables, respectively. This study considers a process robustness study involving a few control and noise factors, and a design region of interest is cuboidal.

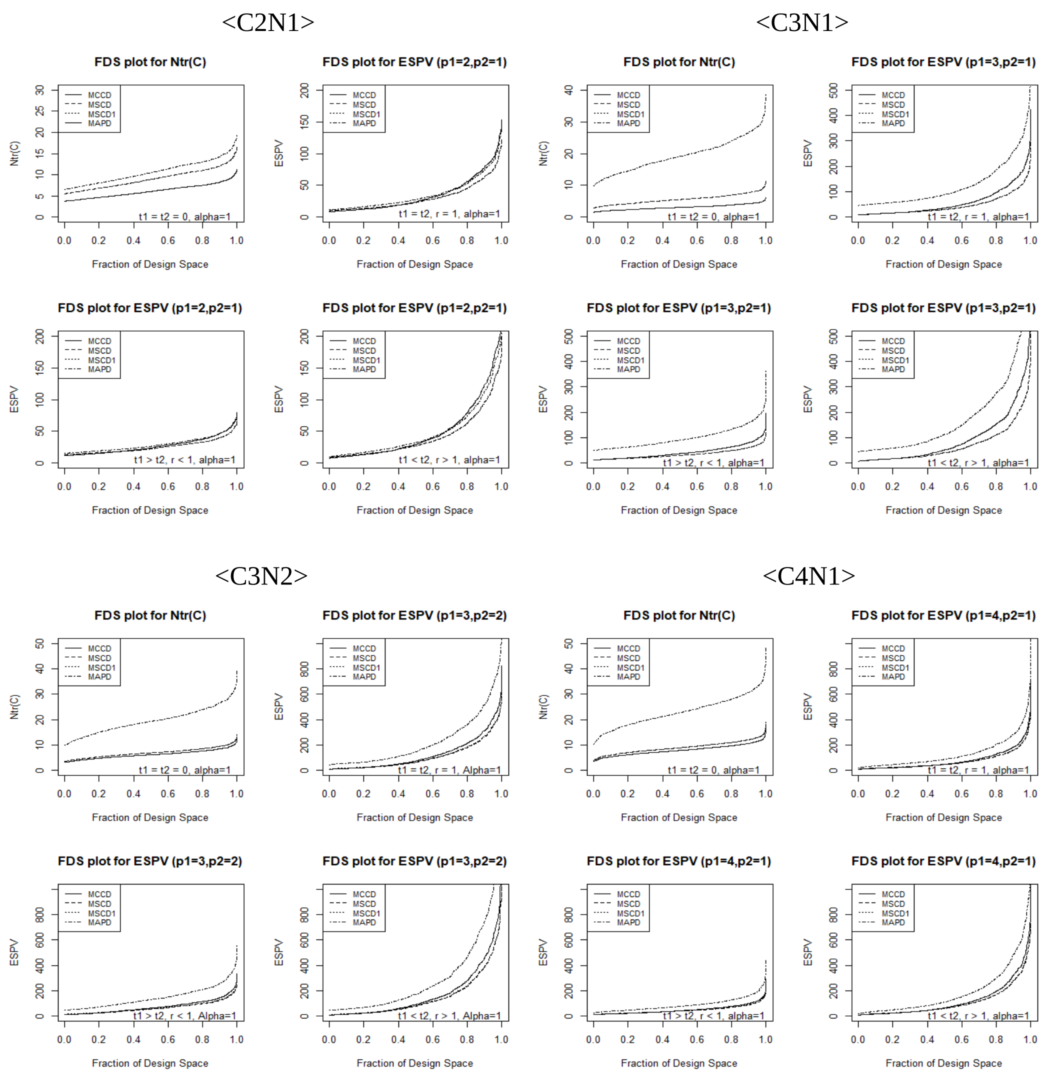
3.2 Numerical comparisons and the choice of candidate designs in modified composite designs

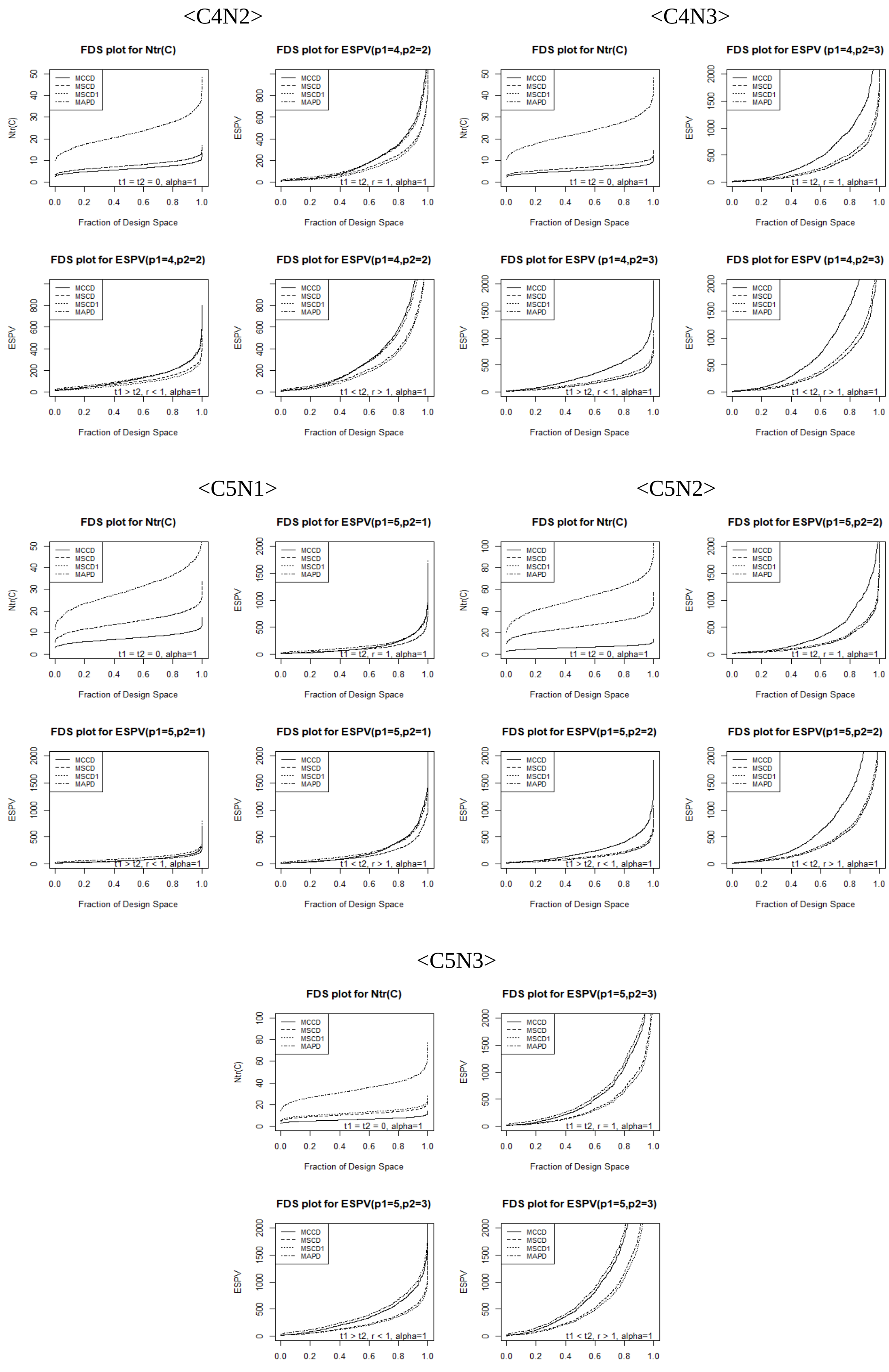
This study aims to extend a response model approach to well-known candidate composite designs and propose the best performance design considering the number of control and noise factors under SPEV and ESPV. To this end, we compare candidate designs C2N1, C3N1, C3N2, C4N1, C4N2, C4N3, C5N1, C5N2, and C5N3 for several composite designs except for MGDs in Table 1. C2 and N2 indicate the number of control and noise variables, respectively. For instance, C2N1 is a combined array with two control factors and one noise factor. In the case of MGDs, it is not easy to select a candidate group because all design runs except for $p_1 = 3$ are two or more. Oh *et al*. (2017) propose a measure of ESPV for evaluating RPDs on MCCDs. This paper is on how ESPV of work can be reduced by changing control and noise variables under MCCDs. Oh (2022) shows a graphical evaluation of ESPV when the number of noise variable increase in RPDs and has sufficiently dealt with situations where the noise factor is equal to or greater than the control factor.

Therefore, this study considers the MCCDs, MSCDs, MSCD1s, and MAPDs as candidate designs. Generally, the axial point distance should be chosen between 1 and $\sqrt{p_1 + p_2}$ but rarely outside this range (Wu and Hamada, 2000). But, since MAPDs are designed to consider only three levels, α is deemed to be 1 in this paper.

Here, we illustrate the graphical evaluation for the MCCDs, MSCDs, MSCD1s, and MAPDs regarding SPEV or ESPV as the number of control and noise factors. And we compare candidate designs using FDS plots to evaluate the relative stability performances of the design space. The FDS plots of ESPV are shown in terms of t_1, t_2 and r which ESPV are in terms of $t_1 = t_2, t_1 > t_2, r < 1$, and $t_1 < t_2, r > 1$ in Figure 1.

Figure 1 shows several FDS plots of ESPV that are in order of $Ntr(C)$, ESPV in terms of $(t_1 = t_2, r = 0, \alpha = 1), (t_1 > t_2, r < 1, \alpha = 1)$, and $(t_1 < t_2, r > 1, \alpha = 1)$. As in Myers *et al*. (2016), the FDS plots for an ideal design will have a large fraction of the design space with small SPEV or ESPV values and be relatively flat, which corresponds to the stability of SPEV or ESPV throughout the region.





Note: The first plot in each FDS plots is on a very different ESPV scale than those. SPEV shows the same pattern as ESPV because there is a SPEV=ESPV- $Ntr(C)$ relationship.

Figure 1. FDS plots for ESPV from C2N1 to C5N3 in MCCD, MSCD, $MSCD_1$, and MAPD

Table 2. Comparison of candidate designs for SPEV and ESPV with $n_0 = 3$

CnNn	Design (runs)	Scaled Prediction Error Variance			Extended Scaled Prediction Variance		
		$t_1 > t_2$	$t_1 = t_2$	$t_1 < t_2$	$t_1 > t_2$	$t_1 = t_2$	$t_1 < t_2$
C2N1 $p_1(2)$ $p_2(1)$	MSCD (11) MSCD1 (11) MCCD (15) MAPD (13)	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$
C3N1 $p_1(3)$ $p_2(1)$	MSCD (17) MSCD1 (17) MCCD (25) MAPD (39)	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$
C3N2 $p_1(3)$ $p_2(2)$	MSCD (21) MSCD1 (21) MCCD (25) MAPD (39)	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$
C4N1 $p_1(4)$ $p_2(1)$	MSCD (23) MSCD1 (23) MCCD (27) MAPD (39)	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$
C4N2 $p_1(4)$ $p_2(2)$	MSCD (27) MSCD1 (27) MCCD (43) MAPD (39)	MCCD >MAPD >MSCD >MSCD$_1$	MCCD, MAPD >MSCD >MSCD$_1$	MCCD >MAPD >MSCD >MSCD$_1$	MCCD >MAPD >MSCD >MSCD$_1$	MCCD, MAPD >MSCD >MSCD$_1$	MCCD >MAPD >MSCD >MSCD$_1$
C4N3 $p_1(4)$ $p_2(3)$	MSCD (35) MSCD1 (35) MCCD (75) MAPD (39)	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$
C5N1 $p_1(5)$ $p_2(1)$	MSCD (29) MSCD1 (29) MCCD (45) MAPD (39)	MAPD >MCCD >MSCD, MSCD$_1$	MAPD, MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$	MAPD, MCCD >MSCD, MSCD$_1$	MAPD >MCCD >MSCD, MSCD$_1$
C5N2 $p_1(5)$ $p_2(2)$	MSCD (37) MSCD1 (37) MCCD (77) MAPD (39)	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$	MCCD >MAPD >MSCD, MSCD$_1$
C5N3 $p_1(5)$ $p_2(3)$	MSCD (49) MSCD1 (45) MCCD (77) MAPD (81)	MAPD >MCCD >MSCD >MSCD$_1$	MAPD >MCCD >MSCD >MSCD$_1$	MAPD >MCCD >MSCD >MSCD$_1$	MAPD >MCCD >MSCD >MSCD$_1$	MAPD >MCCD >MSCD >MSCD$_1$	MAPD >MCCD >MSCD >MSCD$_1$

Note: 'n' of 'CnNn' represents the number of control and noise factor. This result is to generate a random variable in uniform distribution for some point $(x_1, x_2, \cdots, x_{p_1})$. p_1 and p_2 represent the number of control factors and the number of noise factors.

From this viewpoint, we can identify relatively good or poor design performance among the candidate designs, and the results are summarized in Table 2. In Table 2, '>' indicates the order ESPV non-stability. For example, if MSCD is more stable than MCCD, we note MSCD<MCCD. And if MSCD and MSCD$_1$s are the same regular pattern in design space, we note 'MSCD, MSCD$_1$s'. Table 1 shows the order of SPEV and ESPV throughout the design space. From Eq. (9) and Eq (10), SPEV is ESPV-$Ntr(C)$.

MSCD$_1$s and MSCD are stable curves in all FDS plots. Although MCCD is relatively flat in small-size design, $p_1 + p_2 \leq 5$, when the noise factors enter and increase in a combined array, MCCD and MAPDs are relatively poor designs with a steep curve. As for the control and noise factors, the number of runs rapidly outgrows the resources of most experiments. The number of experimental runs shows next to the design name in Table 2. In Table 2, it can be seen that the number of experiments for MCCDs and MAPD relatively increases when it is $p_1 + p_2 \geq 7$.

The MSCDs and MSCD$_1$s have a relatively larger number of experiments than the MCCDs and MAPDs. In particular, in the MCCD, considering seven factors requires 32 more runs of an experiment than six factors. The MAPD experiment considering eight factors, needs 42 more runs than runs of an experiment for seven factors. When looking at the FDS plots of C4N2, C4N3, C5N1, and C5N2, it can be seen that the MAPDs are less sensitive to noise factors in terms of ESPV or SPEV than the MCCDs.

Therefore, examining both runs of the experiment and SPEV or ESPV indicates the $MSCD_1$s and MSCD are all relatively good performance designs. In particular, the $MSCD_1$s are designs that have even the OQE property.

4. CONCLUSIONS

As previously indicated, the number of experimental runs should be one of the design criteria for the cost-effectiveness of the design. A larger experiment can often provide the improved fit of the model to the data and reasonable model parameter estimates but at the expense of driving up the total cost of the experiment.

This study introduces a class of composite designs to reduce the number of experimental runs to which resolution, OQE property, and fractional factorial design techniques are applied. Moreover, it expands these designs to a response model approach with noise factors. These modified designs evaluated design performance with SPEV and ESPV indicators to judge the stability of predicted variance.

The modified designs showing robustness to noise factors and stability of predictive variance are classes of modified SCD (MSCDs, $MSCD_1$s, and $MSCD_2$s) and MAPDs. As for the control and noise factors, the number of runs rapidly outgrows the resources of most experiments. These designs have fewer experiments than MCCDs, and $MSCD_1$s and $MSCD_2$s have OQE properties. When looking at the FDS plots of $6 \leq p_1 + p_2 \leq 7$, it can be seen that the MAPDs are less sensitive to noise factors in terms of ESPV or SPEV than the MCCDs. Therefore, selecting a design that provides small experimental runs and stability for prediction variance is essential in the response model approach with the noise factor.

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