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Higher Spanier Groups

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Higher Spanier Groups

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Abstract

When non-trivial local structures are present in a topological space X, a common approach to characterizing the isomorphism type of the n-th homotopy group $\pi_n(X, x_0)$ is to consider the image of $\pi_n(X, x_0)$ in the n-th Čech homotopy group $\check{\pi}_n(X, x_0)$ under the canonical homomorphism $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$. The subgroup $\ker \Psi_n$ is the obstruction to this tactic as it consists of precisely those elements of $\pi_n(X, x_0)$, which cannnont be detected by polyhedral approximations to X. In this paper we present a definition of higher dimensional analouges of Thick Spanier groups use higher dimensional Spanier groups to characterize $\ker \Psi_n$. In particular, we prove that if X is paracompact, Hausdroff, and UV^{n-1}, then $\ker \Psi_n$ is equal to the n-th Spanier group of X. We also use the perspective of higher Spanier groups to generalize a theorem of Kozlowski-Segal, which gives conditions to ensure that Ψ_n is an isomorphism.

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Homotopy Groups and Polyhedra

1.0.1 Homotopy & Homotopy Groups

We begin with the notion of homotopy; the idea of homotopy is to define continuous deformation between maps of topological spaces. A **path** in a space X will mean a continuous map $p : I \to X$ where $I = [0, 1]$ is the unit interval. Let us first give the general definition of homotopy and then we will move into the notion of homotopy of paths; Indeed,

Definition 1.1. Suppose that f, g are continuous maps of some space X into a space Y. We will say that f is **homotopic** to g if there exists a continuous mapping $F : X \times I \to Y$ so that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = g(x)$$

For every $x \in X$. We call the mapping F a **homotopy** between the maps f and g(we may also write $f_t = f_t(x) = F(x, t)$. In the case that both f, g are homotopic then we will denote this by writing $f \simeq g$.

1.1 Homotopy Groups and Polyhedra

1.1.1 Homotopy & Homotopy Groups

We begin with the notion of homotopy; the idea of homotopy is to define continuous deformation between maps of topological spaces. A **path** in a space X will mean a continuous map $p : I \to X$ where $I = [0, 1]$ is the unit interval. Let us first give the general definition of homotopy and then we will move into the notion of homotopy of paths; Indeed,

Definition 1.2. Suppose that f, g are continuous maps of some space X into a space Y. We will say that f is **homotopic** to g if there exists a continuous mapping $F : X \times I \to Y$ so that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = g(x)$$

For every $x \in X$. We call the mapping F a **homotopy** between the maps f and g(we may also write $f_t = f_t(x) = F(x, t)$. In the case that both f, g are homotopic then we will denote this by writing $f \simeq g$.

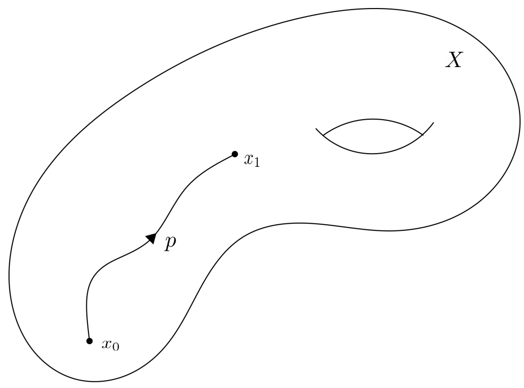

Figure 1.1: A path in a space X

Remark 1.3. In the spcial case that g is the consant mapping and $f \simeq g$; we say that f is **nullhomotopic**.

Example 1.4. Let $X = Y = \mathbb{R}^2$. Let $f(x) = \sin(x), g(x) = \cos(x)$; then $f \simeq g$ by the *straight line homotopy*. That is,

$$F(t, s) = f_t(s) = (1 - t)sin(s) + t\cos(s)$$

Hence, we can "deform" the function $\sin(x)$ to the function $\cos(x)$. In fact, we can do this for any cotinuous maps in the plane. Next, we move away from general continuous functions to the idea of homotopic paths; we define this now.

Definition 1.5. We say that two paths $p_0, p_1 : I \to X$ are **path homotopic** if they have the same initial point x_0 and the same terminal point x_1, and in addition if there exists a continuous mapping $H : I \times I \to X$ so that:

$$H(s, 0) = p_0(s) \quad \text{and} \quad H(s, 1) = p_1(s)$$

$$H(0, t) = x_0 \quad \text{and} \quad H(1, t) = x_1$$

For all $s, t \in I$.

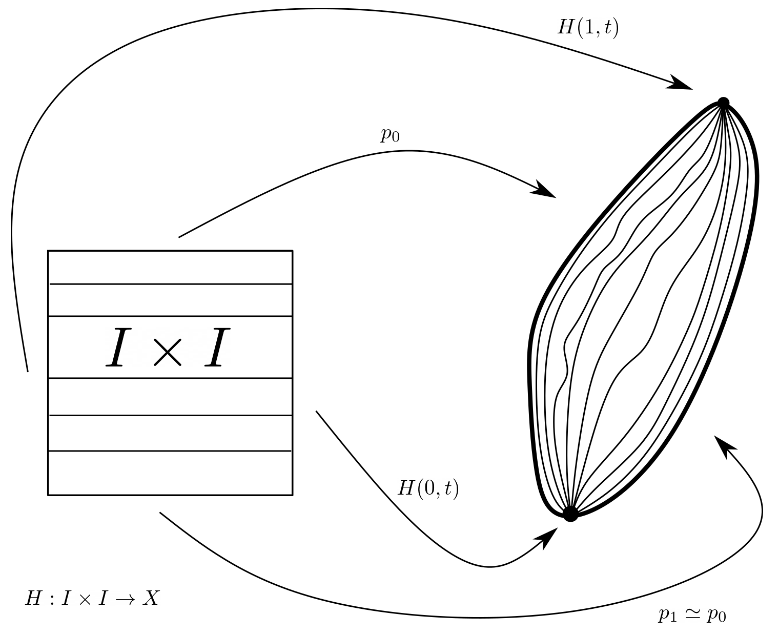

Figure 1.2: Homotopy of paths

Example 1.6. Let $X = \mathbb{R}^n$ for $n \in \mathbb{N}$ and $x_0, x_1 \in X$. If p_1, p_2 are paths of X so that $p_1(0) = p_2(0) = x_0$ and $p_1(1) = p_2(1) = x_1$ then p_1 and p_2 are homotopic by the following homotopy: $p_t(s) = (1-t)p_0(s) + tp_1(s)$. In fact, for any convex subspace X of $\mathbb{R}^n$ all paths with initial point x_0 and terminal point x_1 are homotopic. This is because if p_0, p_1 are paths that lie in X then so does the homotopy p_t.

Proposition 1.7. *The relation of homotopy which we will denote $\simeq$ and path homotopy (denote $\simeq_p$) (with fixed endpoints) is an equivalence relation. If p is a path under the equivalence relation of homotopy we will denote it by* $[p]$.

Proof. It's immediate that the relation is reflexive for $p \simeq p$ by the constant homotopy $p_t = p$. Symmetry is not much harder for suppose that $p_0 \simeq p_1$ for homotopy p_t, then observe that $p_1 \simeq p_0$ through the inverse homotopy p_{1-t}. Transitivity is slightly more involved, but not difficult. For assume that $p_0 \simeq p_1$ through homotopy p_t and if $p_1 \simeq q_0$ where $q_0 \simeq q_1$ through homotopy q_t then we can deduce that $p_0 \simeq q_1$ via homotopy r_t defined so that

$$r_t = \begin{cases} p(2t) & 0 \leq t \leq \frac{1}{2} \\ q(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Note that the definition of r_t will agree with $t = \frac{1}{2}$ as we have assumed that $p_1 = q_0$. Contiuity of $R(s,t) = r_t(s)$ follows from the pasting lemma.

♣

We now define the notion of **finite path concatenation**; that is given two paths $p_0, p_1 : I \to X$ so that $p_0(1) = p_1(0)$, there exists a product path $p_0 \cdot p_1$. This will first traverse p_0 and then p_1; in particular this concatenation can be defined explicitly as follows:

$$p_0 \cdot p_1(s) = \begin{cases} p_0(2s) & 0 \leq s \leq \frac{1}{2} \\ p_1(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Observe that p_0, p_1 by this definition are traversed "twice as fast" as to complete the total path in unit time. Note that this product operation will respect homotopy classes. Indeed, if $p_0 \simeq p_1$ and $q_0 \simeq q_1$ through the homotopies p_t, q_t respective and in addition $p_0(1) = q_0(0)$ so that $p_0 \cdot q_0$ is defined, then $p_t \cdot q_t$ is as well and gives a homotopy $p_0 \cdot q_0 \simeq p_1 \cdot q_1$. Note that there is a dual to the notion of path that we will call the **reverse path**. Suppose that X is a space where $x_0, x_1 \in X$ and p is a path from x_0 to x_1; the reverse path $\overline{p}$ is defined by $\overline{p}(s) = p(1-s)$.

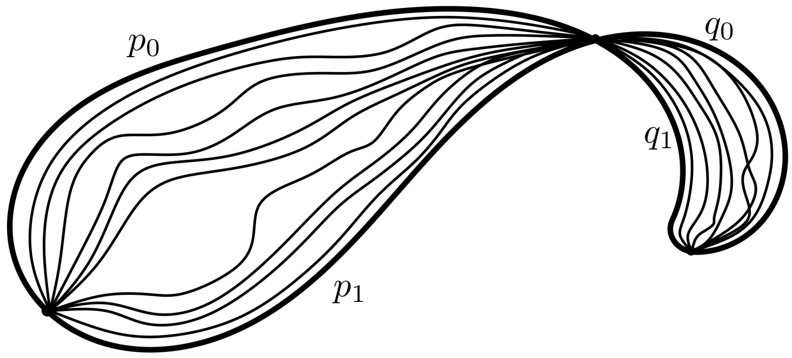

Figure 1.3: Concatenation of homotopic paths

Let us now turn our attention to the notion of **based spaces** and **based maps**:

Definition 1.8. Let X be a topological space with a distinguished point $x_0 \in X$. We call the pair (X, x_0) a **based space** whenever we consider a space with it a corresponding point $x_0 \in X$.

A **based map** is a continuous map $f : (X, x_0) \to (Y, y_0), f(x_0) = y_0$ (i.e basepoints are preserved). Let $\mathbf{Top}_*$ denote the category whose objects are based spaces and morphisms are based maps; this category will appear often in our future definitions. While homotopy of paths is an important concept in its own right, we shall restrict our attention to a particular subset of such paths which we will call **loops**.

Definition 1.9. Let (X, x_0) be a pointed topological space. A **loop** based at x_0, will be a path $\gamma : I \to X$ where $\gamma(0) = \gamma(1) = x_0$.

Note here that the point x_0 will be referred to as the **basepoint**. Loops are of particular interest to us since we may form a special group through the operation of loop concatenation. First let (X, x_0) be a based space and consider the set of all homotopy classes $[\gamma]$ of loops $\gamma : I \to X$ with basepoint x_0; this set is denoted $\pi_1(X, x_0)$ where an element is of the form $[\gamma]$. The set $\pi_1(X, x_0)$ is a group:

Proposition 1.10. *$\pi_1(X, x_0)$ is a group where the product operation is defined by* $[\gamma_0][\gamma_1] = [\gamma_0 \cdot \gamma_1]$

The proof of this fact can be found in (Insert reference to Hatcher). For any homotopy equivalence $[\gamma]$ there is a corresponding identity denoted $[e_{x_0}]$ so that $[\gamma] \cdot [e_{x_0}] = [e_{x_0} \cdot \gamma] = [\gamma \cdot e_{x_0}] = [e_{x_0} \cdot \gamma] = [\gamma]$ (so e_{x_0} is a two sided identity). In addition, there is a notion of inverse; that is given any homotopy equivalence of loops $[\gamma]$ there exists an inverse loop $[\gamma^{-}]$ so that $[\gamma] \cdot [\gamma^{-}] = [\gamma \cdot \gamma^{-}] = [\gamma^{-} \cdot \gamma] = [e_{x_0}]$ (and so γ^{-} is a two sided inverse).

Definition 1.11. Let X, Y be spaces. We will say that X is **homotopic** to Y (or alternativly X and Y have the same homotopy type) if there exists continuous mappings $f : X \to Y$ and $g : Y \to X$ so that the compositions $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are homotopic to the identity maps $\mathrm{id}_X : X \to X$ and $\mathrm{id}_Y : Y \to Y$ respectivly.

Remark 1.12. The above definition generalizes to based maps in an obvious way; so that given two based spaces $(X, x_0), (Y, y_0)$ if there exists maps $f : X \to Y, g : Y \to X$ where $f(x_0) = y_0, g(y_0) = x_0$ then when the compositions $g \circ f : X \to X, f \circ g : Y \to Y$ are homotopic to $\mathrm{id}_X : X \to X$ and id_Y respectivly then the based spaces will be homotopic. In addition there is another generalization we can make to the homotopy theory of pairs (X, A) where A is a distinguished subspace of X. So then we may consider a map between pairs $f : (X, A) \to (Y, B)$ and then the notion of homotopic relative spaces is clear from the previous definition and generalization to based spaces.

We can define the n-th homotopy group in a similar manner to how we have defined the fundamental group. First, let us establish some additional notation that we will need throughout the rest of this exposition:

1. Recall that $I = [0, 1]$ is the unit interval. We then define $I^n = \prod_{i=1}^{n} [0, 1]$ be the n-**cube**.
2. We will let ∂I^n denote the boundary of I^n which is a subspace consisting of points with at least a single coordinate equal to 0 or 1.
3. Let (X, x_0) be a based space; we define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $p : (I^n, \partial I^n) \to (X, x_0)$, where we have that homotopies must satisfy: $p_t(\partial I^n) = x_0$ for every t (In particular, the map p in this case is an **n-loop**).
4. We let $\Omega^n(X, x_0)$ (or $\Omega^n X$ when context is absolute) be the set of n-loops in X based at x_0.

Remark 1.13. Note that for $\pi_n(X, x_0)$ we may extend the definition to the case of $n = 0$; Indeed, then by the above we have that I^0 is just a single point and $\partial I^n = \emptyset$. Thus we can conclude that $\pi_0(X, x_0)$ is just the set of path components of X.

Generalizing the construction of the fundamental group $\pi_1(X, x_0)$ we may define the notion of concatention of n-loops of $\Omega^n(X, x_0)$ for $n \geq 2$ with $n \in \mathbb{N}$.

Definition 1.14. Let (X, x_0) be a based space and $\gamma_0, \gamma_1 : I^n \to X$ be n-loops of X. We define the n-loop concatenation of γ_0, γ_1 as follows:

$$(\gamma_0 \cdot \gamma_1)(s_1, s_2, \ldots, s_n) = \begin{cases} \gamma_0(2s_1, s_2, \ldots, s_n) & s_1 \in [0, \frac{1}{2}] \\ \gamma_1(2s_1 - 1, s_2, \ldots, s_n) & s_1 \in [\frac{1}{2}, 1]. \end{cases}$$

Note that this operation is well defined as only the first coordinate is used in the sum operation. Similar arguements as in the proof of $\pi_1(X, x_0)$ being a group work for $\pi_n(X, x_0)$. Observe here that we mention the product as a "sum" and not just a product. This is because for $n \geq 2$ it's always the case that $\pi_n(X, x_0)$ is an abelian group (unlike the fundamental group). To see why this is the case let γ_0, γ_1 be n-loops and observe that the following homotopy below represents the expression: $\gamma_0 \cdot \gamma_1 \simeq \gamma_1 \cdot \gamma_0$

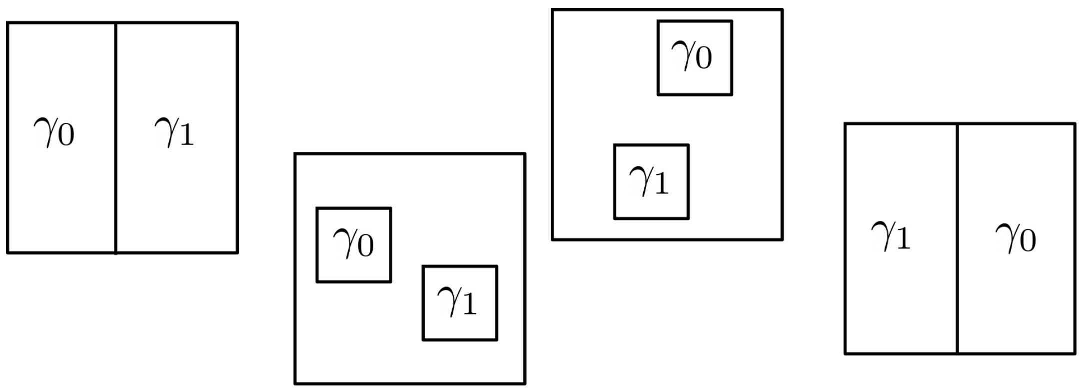

Figure 1.4: The product operation in $\pi_n(X, x_0)$ for $n \geq 2$ is abelian

Hence we see from the above that for $n \geq 2$; $\pi_n(X, x_0)$ is abelian. Note that the maps $(I^n, \partial I^n) \to (X, x_0)$ may be identified with based maps $(S^n, s_0) \to (X, x_0)$ using a choice of homeomorphism $I^n/\partial I^n \cong S^n$ to X. Let $s_0 = \partial I^n/\partial I^n$ be the basepoint of S^n which is mapped to the basepoint x_0 of X. In this way, we can view $\pi_n(X, x_0)$ as a homotopy class of maps $(S^n, s_0) \to (X, x_0)$. With this interpretation of $\pi_n(X, x_0)$ the sum of homotopy classes$[\gamma_0] + [\gamma_1]$ may be realized as the following composition:

$$S^n \xrightarrow{\varphi} S^n \vee S^n \xrightarrow{\gamma_0 \vee \gamma_1} X$$

where φ will collapse the equator S^{n-1} as a subspace of S^n to a point. We may take the basepoint s_0 to lie in this S^{n-1}. This deformation creates a "bouquet" of spheres which is then mapped into the space X via the instructions of γ_0, γ_1.

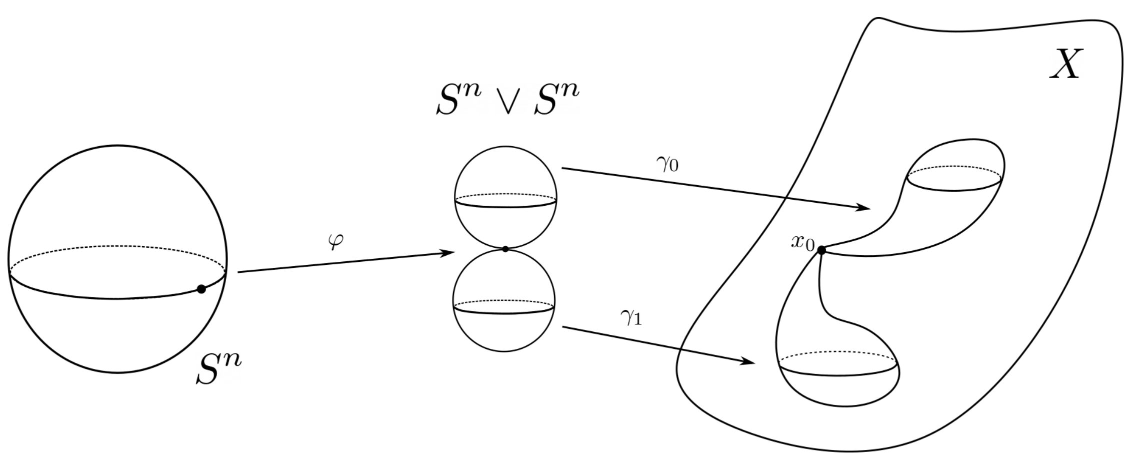

Figure 1.5: The path concatenation of n-loops

We also have the concept of the reverse path; which we give the definition of below:

Definition 1.15. Let $\gamma : I^n \to X$ be an n-loop. By the **reverse** of an n-loop we mean the n-loop $\gamma^{-}(t_1, t_2, \ldots, t_n) = \gamma(1 - t_1, t_2, \ldots, t_n)$ for every $(t_1, t_2, \ldots, t_n) \in I^n$.

Proposition 1.16. *Let $\{X_\alpha\}_{\alpha \in I}$ be a family of path connected spaces. Consider th directe product $\prod_\alpha X_\alpha$; then there exists a canonical isomorphism:*

$$\pi_n\Big(\prod_\alpha X_\alpha\Big) \cong \prod_\alpha \pi_n(X_\alpha).$$

Proof. Observe that given a mapping $f : Y \to \prod_\alpha X_\alpha$ is the same as the family of maps $f_\alpha : Y \to X_\alpha$. Letting Y to be S^n and $S^n \times I$ yields the desired result.

♣

Next, let us consider the set of all n-loops in a based space (X, x_0); we denote this set by $\Omega^n(X, x_0)$.

Definition 1.17. Let (X, x_0) be a based space and $\Omega^n(X, x_0)$ be the set of all n-loops of X based at x_0. We can topologize $\Omega(X, x_0)$ by equipping it with the compact-open topology. When $\Omega^n(X, x_0)$ is defined in this manner we call it the **n-th loop space**.

Observe that the above may be identified with the space of maps $f : (I^n, \partial I^n) \to (X, x_0)$.

Remark 1.18. Note that we can view the homotopy groups $\pi_n(X, x_0)$ of a based space (X, x_0) as the n-loop space under homotopy; that is we have the relationship

$$(\Omega^n(X, x_0)/\simeq) = \pi_n(X, x_0)$$

Next, we define the notion of an induced homomorphism

Definition 1.19. Let $f : (X, x_0) \to (Y, y_0)$ be a based map and $n \geq 1$ for $n \in \mathbb{N}$. Then there exists an **induced homomorphism** $f_\# : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ defined by

$$f_\#([\gamma]) = [\gamma \circ f]$$

The above definition follows directly from the fact that a functor between categories maps morphisms of the source category to the target category. In particular, π_1 is a functor (as well as π_n for $n \geq 2$).

Let us now consider the dependence of the choice of basepoint for $\pi_n(X, x_0)$ for $n \geq 2$. Consider any $p : I \to X$ that joins two distinct base points x_0, x_1 of the based space (X, x_0). Then there exists an induced isomorphism $p_\# : \pi_n(X, x_0) \to \pi_n(X, x_1)$. To consturct $p_\#$ we first construct the map η of S^n onto the wedge sum of the "boquet" $S^n \vee I$ which maps the basepoint of S^n into ∂I. Next, take an n-loop $\gamma : S^n \to X$ which will preserve the basepoint x_0 and in addition construct the mapping

$$p_\# : S^n \xrightarrow{\eta} S^n \vee I \xrightarrow{\gamma \vee p} X$$

preserving the basepoint of S^n into x_1. Lastly, it's not difficult to verify that $p_\#(\gamma_0 + \gamma_1) = p_\#(\gamma_0) + p_\#(\gamma_1)$ and $(p^{-1})_\# = (p_\#)^{-1}$.Observe that the construction of $p_\#$ can be seen in Fig. 6.

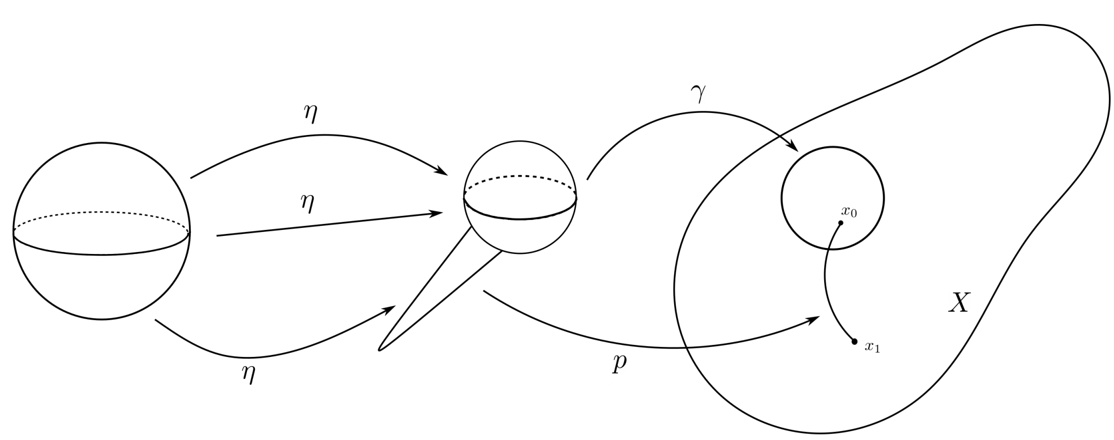

Figure 1.6: Change of base point

This construction will also be valid for the case of $n = 1$ for Fundametnal groups. Hence, in path connected spaces of any dimension n; we have that the choice of basepoint is up to ismorphism equivalent.

Theorem 1.20. *Let **Grp**, **Ab** denote the category of groups and abelian groups respective. Then* $\pi_1 : \textbf{Top}_* \to \textbf{Grp}$ *is a functor. In addition, for* $n \geq 2$ *we have that:* $\pi_n : \textbf{Top}_* \to \textbf{Ab}$ *is a functor.*

A proof of the above can be found in (Hatcher Refrence).An important feature of higher homotopy groups is the fact that the fundamental group naturally acts on $\pi_n(X, x_0)$ in such a way that extends the conjugation action of $\pi_1(X, x_0)$. We shall extend this action to an action of the fundamental groupoid on $\pi_n(X, x_0)$.

Definition 1.21. Suppose $\gamma \in \Omega^n(X, y_0)$ and $p : I \to X$ is a path from x_0 to x_1, then there exists a canonical map $\gamma * p \in \Omega^n(X, x_0)$ that corresponds to the action of the fundamental groupoid on $\coprod_{\gamma \in X} \pi_n(X, a_0)$. In particular, $(\gamma * p)|_{[1/3,2/3]^n} \equiv p$ and $(\gamma * p)(\partial[s, 1-s]^n) = \gamma(3s)$ for $s \in [0, \frac{1}{3}]$. We shall refer to $\gamma * p$ as the **path conjugate** of p by γ.

Note that this is well defined up to homotopy. If $[\gamma_0] = [\gamma_1]$ and $[p] = [q]$, then $[\gamma_0 * p] = [\gamma_1 * q]$. In addition: $[\gamma_0 * (p \cdot q)] = [\gamma_0 * p][\gamma_0 * q]$

Let us now define the notion of n-connectedness; this will be of particular importantce in our case for we may apply this to the Hurewicz theorem to determine certain properties of homotopy through machiney of homology.

Definition 1.22. Let X be a space; we will say that X is **n-connected** if for every $x \in X$ and $0 \leq k \leq n$ we have that $\pi_n(X, x_0)$ is trivial.

Remark 1.23. Note that a based space (X, x_0) is said to be 0**-connected** if and only if X is path connected (i.e $\pi_0(X, x_0) = 1$).

Example 1.24. As an example of n-connected spaces consider the n-sphere as a based space (S^n, s_0). Then S^n is $(n-1)$ connected for every $n \geq 1$.

Let us now turn our attention to the Hurewicz homomorphism and the Hurwicz theorem. First, let us define the homomorphism.

Definition 1.25. Let (X, x_0) be a based path connected space in $n \in \mathbb{N}$. Then there exists a group homomorphism

$$h_\# : \pi_n(X) \to H_n(X)$$

Where $\pi_n(X)$ is the n-th homotopy group and $H_n(X)$ is the n-th singular homology group. It is defined in the following manner: Let $u_n \in H_n(S^n)$ be a canonical generator, then we have that a homotopy class of maps $f \in \pi_n(X, x_0)$ is taken to $f_\#(u_n) \in H_n(X)$.

Alternatively, the Hurewicz homomorphism h_* can be thought of as a natural transformation between the functors $\pi_n(X, x_0)$ and $H_n(X)$. More explicity, we see that given the spaces $(X, x_0), (Y, y_0) \in \text{Ob}(\textbf{Top}_*)$ we have that the following square commutes with components $h_{\#X}, h_{\#Y}$

$$\begin{array}{ccccc} (X, x_0) & \quad & \pi_n(X, x_0) & \xrightarrow{h_{\#X}} & H_n(X) \\ \downarrow f & & \downarrow \pi_n(f) & & \downarrow H_n(f) \\ (Y, y_0) & & \pi(Y, y_0) & \xrightarrow[h_{\#Y}]{} & H_n(Y) \end{array}$$

The Hurewicz homomorphism is of particular importance as in most cases computation of homotopy groups is significantly more difficult than computation of homology groups. The Hurewicz Theorem stipulates under a certain connectedness condition $h_\#$ will be an isomorphism; which we present now.

Theorem 1.26. *Let (X, x_0) be a based space. If X is $(n-1)$-connected for $n \geq 2$ then the Hurewicz homomorphism*

$$h_\# : \pi_n(X, x_0) \to H_n(X)$$

is an isomorphism.

See (Refrence Hatcher 4.32 for proof).

1.1.2 Polyhedra

Let us now turn our attention to polyhedra; we start off with the idea of a simplex which leads to the notion of a geometric simplicial complex.

Definition 1.27. Suppose $\{v_0, v_2, \ldots, v_k\}$ is an affinely independent set of $k+1$ points in $\mathbb{R}^n$. We define the **simplex** spanned by them denoted by $[v_0, \ldots, v_k]$ to be the set

$$[v_0, \ldots, v_k] = \left\{ \sum_{j=0}^{k} \alpha_j v_j : \alpha_j \geq 0, \sum_{j=0}^{k} \alpha_j = 0 \right\}$$

with appropriate subspace topology.

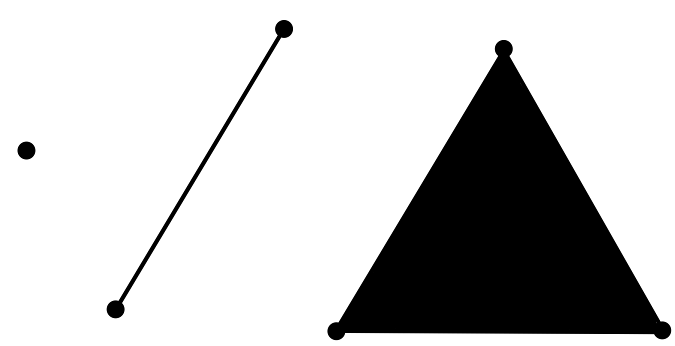

Figure 1.7: A 0-Simplex, 1-Simplex, and 2-Simplex

We will call each v_j a **vertex** of the simplex. In addition, we can define the **dimension** of a simplex to be the integer k as given above. We also will call a simplex a k-simplex if it's of dimension k. To make this more clear, a 0-simplex will simply be a point, a 1-simplex is a segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on for higher dimensions.

Definition 1.28. A **simplicial complex** is a collection K of simplices in some Euclidean subspace $\mathbb{R}^J$ that meets the following stipulations:

1. Given $\sigma \in K$, we have that every face of σ is in K.

2. Given two simplicies of K their intersection is either empty or a face of each other.

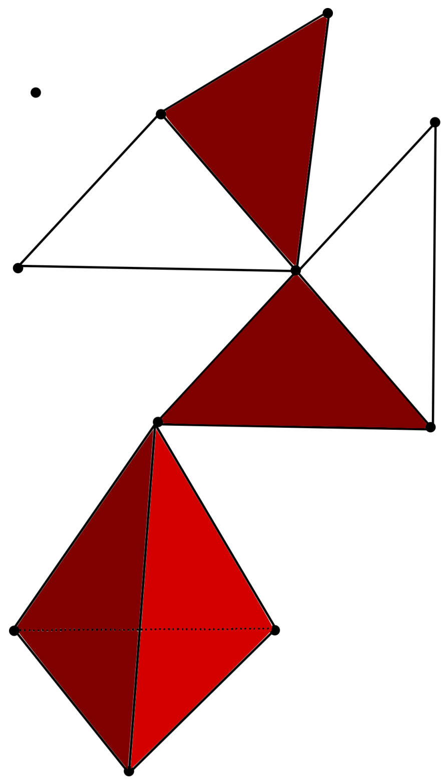

Figure 1.8: A simplicial complex

Remark 1.29. The dimension of a simplicial complex will be the maximum dimension of the simplicies in K. If no such maximum exists then we define the dimension to be infinite which we denote by ∞.

Remark 1.30. Suppose $L \subseteq K$, we will say that L is a **subcomplex** of K if whenever $\sigma \in L$ we have that every face of σ is itself in L. A subcomplex is itself a simplicial complex.

Remark 1.31. For K a simplicial complex of $\mathbb{R}^n$, the union of all simplicies in K with inherited subspace topology from $\mathbb{R}^n$ is a topological space which is denoted by $|K|$ and which we shall call the **polyhderon of** K

Definition 1.32. Let $\mathcal{A}$ be a nonempty collection of finite sets. We will say that $\mathcal{A}$ is an **abstract simplicial complex** precisely when if $S \in \mathcal{A}$ and if $X \subseteq S$ then $X \in \mathcal{A}$.

More succiently, an abstract simplicial complex is a collection of finite sets which is closed under taking subsets. If $\mathcal{A}$ is an abstract simplicial complex, whenever $X \in \mathcal{A}$ we will say that X is a **simplex of** $\mathcal{A}$. There exists a notion of dimension of both a simplicial complex and its simplicies.

Definition 1.33. The **dimension of a simplex** $\dim(X)$ will be defined to be ones less than it's cardinality; $\dim(X) = |X| - 1$ where $|X|$ denotes the cardinality of a set. In a similar spirit, the **dimension of an abstract complex** $\mathcal{A}$ is the maximal cardinality of it's elements:

$$\dim(\mathcal{A}) = \max\{\dim(X) \mid X \in \mathcal{A}\}$$

Note that if no such maximum exists then the dimension will be defined to be infinite.

In an abstract simplicial complex we define the vertex set as the union of all one point sets of a complex $\mathcal{K}$. We may consider a subcollection $\mathcal{S}$ of a complex $\mathcal{A}$; if $\mathcal{S}$ is itself a complex then we shall call $\mathcal{S}$ a **subcomplex**. Let $\mathcal{A}, \mathcal{B}$ be two abstract simplicial complexes and let $V_{\mathcal{A}}, V_{\mathcal{B}}$ denote their vertex sets respectivly. We will say that $\mathcal{A}, \mathcal{B}$ are **isomorphic** if there exists a bijection map $\phi : V_{\mathcal{A}} \to V_{\mathcal{B}}$ so that $\{p_0, p_1, \ldots, p_n\} \in \mathcal{A}$ if and only if $\{\phi(p_0), \phi(p_1), \ldots, \phi(p_n)\} \in \mathcal{B}$.

Given the idea of an abstract simplicial complex; we should want to transmute its abstract structure to something geometrically tractable. Indeed, our first steps towards this construction is given by the following definition.

Definition 1.34. Suppose $\mathcal{A}$ is a simplicial complex and $V_{\mathcal{A}}$ is its corresponding vertex set. Take $\mathcal{G}$ to be the set of all subsets $\{p_0, p_1, \ldots, p_n\}$ of $V_{\mathcal{A}}$ so that $\text{span}(p_0, p_1, \ldots, p_n) = X$ where X is a simplex of $\mathcal{A}$. We call $\mathcal{G}$ the **vertex scheme**.

$\mathcal{G}$ is itself an example of an abstract simplicial complex. We shall now see an important theorem relating to this idea.

Theorem 1.35. *Every abstract simplicial complex $\mathcal{A}$ is isomorphic to the vertex scheme of some simplicial complex K. In addition, two simplicial complexes will be said to be* ***linearly isomorphic*** *if and only if their vertex schemes are isomorphic as abstract simplicial complexes.*

A proof can be found in (insert citation to Munk.). Equipped with the above theorem we can now approach the crux of the matter.

Definition 1.36. If the abstract simplicial complex $\mathcal{A}$ is isomorphic to the vertex scheme of some simplicial complex K, we call K a **geometric realization** of $\mathcal{A}$ (unique up to linear isomorphism).

Definition 1.37. We define a **polyhedron** to be a topological space that is homeomorphic to the geometric realization of some abstract simplicial complex.

Example 1.38. The simplest example of a polyhedron is the standard **Euclidean n-simplex**. This is given by the simplex $\Delta^n \in \mathbb{R}^{n+1}$

$$\Delta^n = \left\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_j \geq 0 \text{ for all } j; \sum_{j=0}^{n} t_j = 1\right\}.$$

Let us now give an example of an abstract simplicial complex that we will make frequent use of throughout this paper.

Example 1.39. Suppose that (X, x_0) is a based path connected space. Let $\mathcal{O}(X)$ denote the set of all open covers of X which are direct by refinement. In addition, let $\mathcal{O}(X, x_0)$ be the set of all open covers with a distinguished element that contains the basepoint. We define the **nerve of an open cover** $(\mathcal{U}, U_0) \in \mathcal{O}(X, x_0)$ denoted by $N(\mathcal{U})$, as the abstract simplicial complex whose vertex set is written $N(\mathcal{U})_0 = \mathcal{U}$ and verticies $A_0, \ldots, A_n \in \mathcal{U}$ span some n simplex given that $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

Definition 1.40. Let v be a vertex of $\mathcal{A}$ a simpicial abstract or geometric complex. We define the **star** of v in $\mathcal{A}$ to be the union of the interiors of those simplices of $\mathcal{A}$ that have v as their vertex; that is

$$\text{St}(v, \mathcal{A}) = \bigcup_{i} \{\text{int}(\sigma) \mid v \subseteq \sigma\}$$

where σ_i is a simplex of $\mathcal{A}$ and i ranges over some index set I.

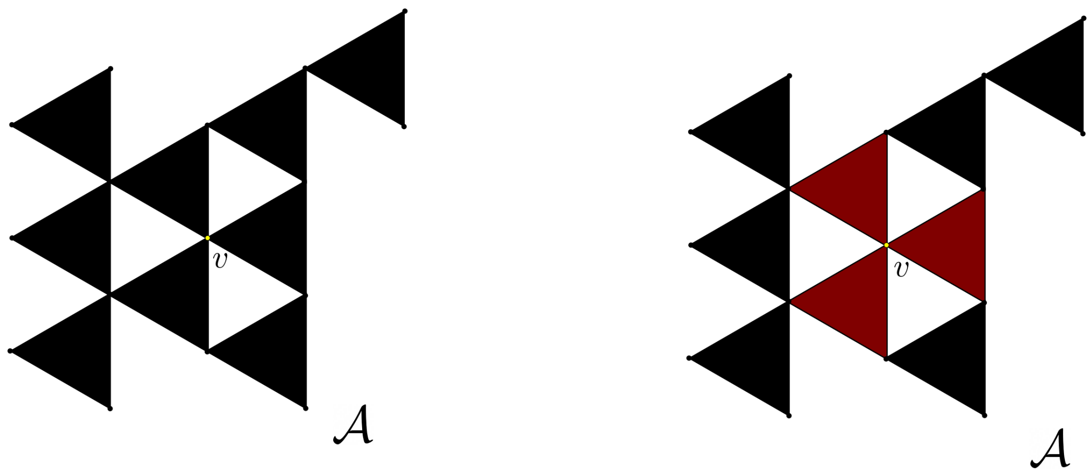

Figure 1.9: The **star** of a simplicial complex $\mathscr{A}$ at a vertex v

Remark 1.41. The open star of a vertex is contractible. One may use the straight line homotopy to see this.

Given the star of a simplicial complex, we may consider it's closure. We give this definition as follows:

Definition 1.42. Let $\mathscr{A}$ be an abstract or geometric simplicial complex and v a vertex of $\mathscr{A}$. We may define the **closed star** denoted by $\overline{\text{St}}(v, \mathscr{A})$ as the union of all simplices σ_i of $\mathscr{A}$ that have v as a vertex, and in adiditon is the polytope of some subcomplex of $\mathscr{A}$.

Definition 1.43. Let σ be a simplex of a simplicial complex $\mathscr{A}$. We define the **star of a simplex** to be the intersection of all stars of its vertices; that is

$$\text{St}(\sigma) = \bigcap_{v \in \sigma} \text{St}(v, \mathscr{A})$$

Definition 1.44. Let X be a space and $|K|$ the geometric realization of some abstract complex. Suppose $f, g : X \to |K|$ are contnuous maps such that for every $x \in X$ there exists a simplex $\sigma \in K$ so that $f(x), g(x) \in |\sigma|$ In this case, f and g are said to be **contiguous**.

In additon to defining the star of a subcomplex we can define the notion of a star of a simplex itself, let us give this definition

Theorem 1.45. *Suppose that f, g are contiguous maps; then f and g are homotopic.*

A proof of this may be seen in [18].

Chapter 2

Shape Theory

2.0.1 Canonical Maps

We begin our discussion with the notion of maps between nervers of covers and "canonical maps".

Let $O(X, x_0)$ denote the set of open covers of a based space (X, x_0) with a distinguished element that contains the basepoint x_0. Since $N(\mathcal{U})$ is an abstract simplicial complex we can consider its geometric realization $|N(\mathcal{U})|$ and take the distinguished element U_0 as its basepoint.

Definition 2.1. Suppose that $(\mathcal{V}, V_0)$ is an open cover that refines $(\mathcal{U}, U_0)$. We can construct a simplicial mapping $p_{\mathcal{U}\mathcal{V}} : N(\mathcal{V}) \to N(\mathcal{U})$, which we call the **projection** (Where $N(\mathcal{V}), N(\mathcal{U})$ are nerves of their respective covers).

In particular, note that $p_{\mathcal{V}\mathcal{U}}$ will send the vertex $V \in N(\mathcal{V})$ to some vertex $U \in \mathcal{V}$ so that $V \subseteq U$. Also, we must have that V_0 is sent to U_0 by the projection. We can consider the induced mapping $|p_{\mathcal{U}\mathcal{V}}| : |N(\mathcal{V})| \to |N(\mathcal{U})|$ which is unique up to based homotopy and hence the homomorphism

$$p_{\mathcal{U}\mathcal{V}_\#} : \pi_n(|N(\mathcal{V})|, V_0) \to \pi_n(|N(\mathcal{U}), U_0)|$$

induced on homotopy groups will be independent of the selection of simplicial map. Next, recall the notion of a normal open cover; an open cover is normal whenever it admits a partition of unity which is subordinated to $\mathcal{U}$. We will let Λ denote a subset of $O(X, x_0)$ which shall consit of the pairs $(\mathcal{U}, U_0)$ with $\mathcal{U}$ a normal open cover of X.

Definition 2.2. Let $(\mathcal{U}, U_0)$ be an open cover of a space X. We will say that a map $p_{\mathcal{U}} : X \to |N(\mathcal{U})|$ is a **based canonical map** if $p_{\mathcal{U}}^{-1}(\mathrm{St}(U, N(\mathcal{U}))) \subseteq U$ for every $U \in \mathcal{U}$ and $p_{\mathcal{U}}(x_0) = U_0$.

Theorem 2.3. *Let X be a space and $\mathcal{V}$ an open cover that refines $\mathcal{U}$. Further, let $p_{\mathcal{U},\mathcal{V}}$ be the canonical map between these covers. Then the induced map $p_{\mathcal{U},\mathcal{V}\#}$ is surjective on all fundamental groups.*

Proof. First observe, that if $N(\mathcal{V})$ and $N(\mathcal{U})$ are homeomorphic, then it's immediate that $p_{\mathcal{U},\mathcal{V}\#}$ is a surjection. In addition, if $\pi_1(N(\mathcal{U}), U_0) \cong 0$ then obviously $p_{\mathcal{U},\mathcal{V}\#}$ is a surjection as well. So assume that the nerves $N(\mathcal{U})$ and $N(\mathcal{U})$ are not homeomorphic. We aim to show, that $p_{\mathcal{U},\mathcal{V}\#}$ is still a surjection; Indeed, let $[\eta] \in \pi_1(N(\mathcal{U}), U_0)$

♣

Definition 2.4. Let $\mathcal{U}$ be an open cover of some based space X with basepoint x_0. Consider the nerve $N(\mathcal{U})$ of this cover and its geometric realization $|N(\mathcal{U})|$. Further, let $p_{\mathcal{U}\mathcal{V}\#}$ be the induced projection between nerves and Λ a subset of open covers of X where those covers $\mathcal{U} \in \Lambda$ are normal with partition of unity subordinated to $\mathcal{U}$. Then we define the n-th homotopy shape group as follows:

$$\check{\pi}_n(X, x_0) = \varprojlim(\pi_n(|N(\mathcal{U})|, U_0), p_{\mathcal{U}\mathcal{V}\#}, \Lambda)$$

where the bonding maps $\pi_n(|N(\mathcal{V})|, V_0) \to \pi_n(|N(\mathcal{U})|, U_0)$ are induced by refinement.

Canonical maps are unique up to based homotopy. In fact, whenever $(\mathcal{V}, V_0)$ is a refinement of $(\mathcal{U}, U_0)$ we have that $p_{\mathcal{U}\mathcal{V}} \circ p_{\mathcal{U}}$ and $p_{\mathcal{U}}$ will be homotopic as based maps. This implies that the homomorphisms $p_{\mathcal{U}\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathcal{U})|, U_0)$ will satisfy $p_{\mathcal{U}\mathcal{V}\#} \circ p_{\mathcal{U}_\#} = p_{\mathcal{U}_\#}$. This induces the following canonical homomorphisms.

Definition 2.5. Let X be a based space with basepoint x_0, $\mathcal{U}$ an open cover of X and consider the family of homomorphisms $p_{\mathcal{U}_\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathcal{U})|, U_0)$ where $p_{\mathcal{U}\mathcal{V}_\#} \circ p_{\mathcal{V}_\#} = p_{\mathcal{U}_\#}$. These homomorphisms induced the **n-th canonical shape homomorphism** given by

$$\Psi^n_X : \pi_n(X, x_0) \to \check{\pi}(X, x_0)$$

where $\Psi^n_X([\gamma]) = ([p_{\mathcal{U}} \circ \gamma])$.

Of particular interest is to examine when $\ker \Psi^n_X = 1$. In this case, the n-th shape group homomorphism will retain all data from the n-th homotopy group. When Ψ^n_X is injective, we will say that the space X is **π_n-shape injective**. From this, one can ascertain a description of the elements of $\pi_n(X, x_0)$ as sequences in the inverse limit of n-th homotopy group of polyhedra.

Chapter 3

n-Spanier groups and Thick n-Spanier groups

3.0.1 n-Spanier groups

Let us begin by constructing the notion of the n-Spanier group. We will let $\widetilde{X}$ denote the set of homotopy classes (rel. endpoints) of paths starting at x_0 (i.e the star of the fundamental groupoid of X at x_0. Multiplication of homotopy classes of paths is taken in the fundamental groupoid of X so that $[p][q] = [p \cdot q]$ with $p(1) = q(0)$.

Definition 3.1. Let (X, x_0) be a based space and $\mathcal{U}$ an arbitrary open cover of X. We define the n**-Spanier group with respect to** $\mathcal{U}$ to be the subgroup of the n-th homotopy group $\pi_n(X, x_0)$ which is generated by elements of the form $[p * \gamma]$ where $p : I \to X$ is a path and $\gamma(S^n) \subseteq U$ for some $U \in \mathcal{U}$. In this sense we can write this subgroup as follows:

$$\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) = \langle [p * \gamma] \mid \gamma(S^n) \subseteq U, U \in \mathcal{U} \rangle.$$

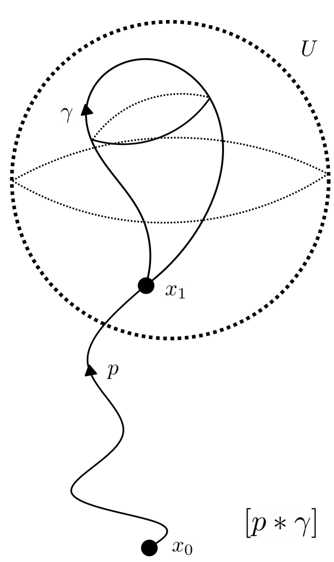

Figure 3.1: A generator of the 2-Spanier group

Remark 3.2. When $n = 1$, the first Spanier group $\pi_1(\mathcal{U}, x_0)$ is generated by the conjugate path concatenation in the following manner:

$$\pi_1^{\mathrm{Sp}}(X, x_0) = \langle [p \cdot \gamma \cdot p^{-}] \mid \exists U \in \mathcal{U} \text{ s.t } \mathrm{Im}(\gamma) \subseteq U \rangle$$

Note that here we use path concatenation instead of path conjugation since when $n = 1$ these operations agree.

Remark 3.3. It should be observed that in our definition of n-Spanier groups we are actually using what is called the "unbased n-Spanier group" while there also exists a notion of "based n-Spanier groups". We omit these designations as the spaces we will work with will at least be locally path connected and these definitions agree for such a space. [2]

Remark 3.4. The first Spanier group is a normal subgroup of the fundamental group. For $n \geq 2$ it's immediate that $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ is normal in the n-th homotopy group (as $\pi_n(X, x_0)$ is abelian for $n \geq 2$).

It's important to note that the generators of $\pi_n^{\mathrm{Sp}}(X, x_0)$ are those homotopy classes of path conjugates $[p * \gamma]$ while a generic element has the form $\prod_{j=1}^{n}[p_j * \gamma_j] \in \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$.

Definition 3.5. The ***n*-Spanier group** with respect to X is given by

$$\pi_n^{\mathrm{Sp}}(X, x_0) = \bigcap_{\mathcal{U}} \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$$

where $\mathcal{U}$ ranges over all open covers of X.

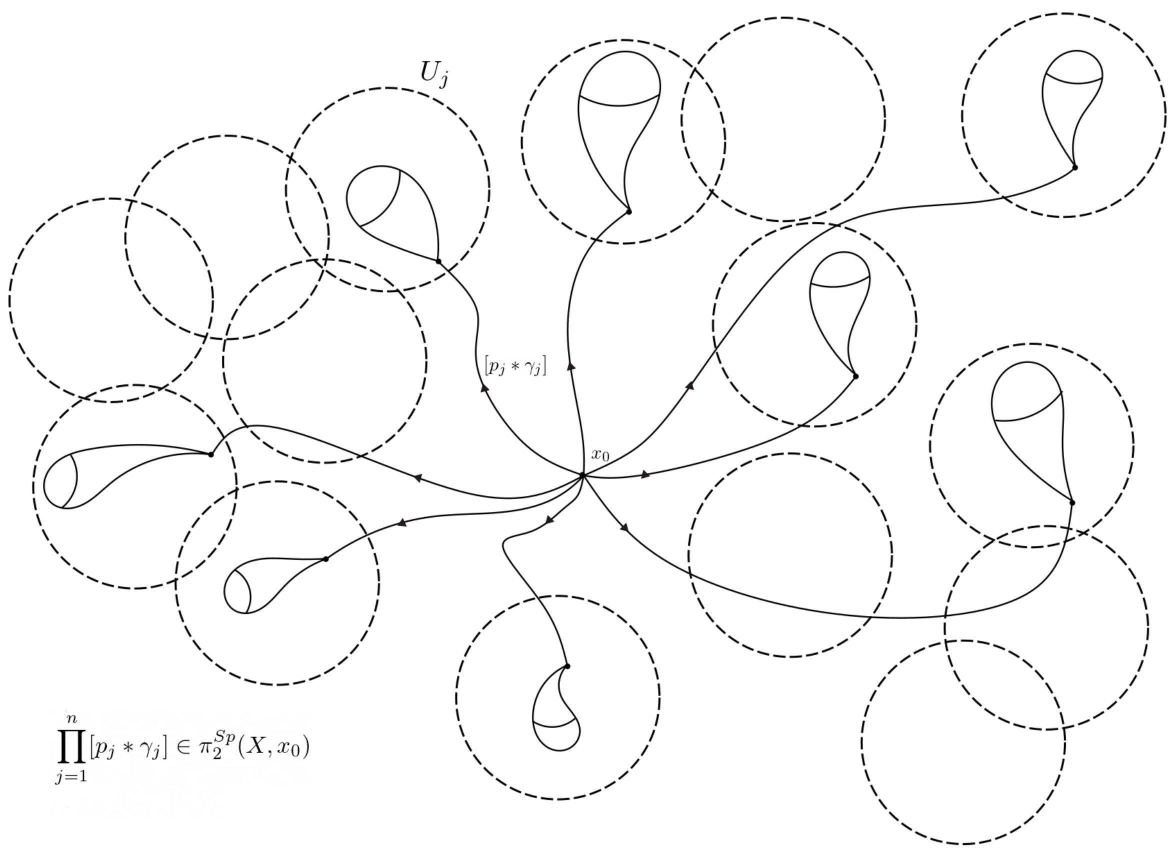

Figure 3.2: A generic element of the 2-Spanier group.

Remark 3.6. Let (X, x_0) be a based space and $\mathcal{U}$ an arbitrary open cover of X. An alternative consturction of the n-Spanier group is to conceptualize it as the inverse limit of a system of refined coverings of X. More explicity, if $\mathcal{V}$ is an open cover of X such that $\mathcal{U}$ refines $\mathcal{V}$, then we have the inclusion relation: $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) \subseteq \pi_n^{\mathrm{Sp}}(\mathcal{V}, x_0)$. This relation gives us the existense of the inverse limit of $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$; that is

$$\pi_n^{\mathrm{Sp}}(X, x_0) = \varprojlim(\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0))$$

where the bonding maps are the inclusion homomorphisms $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) \to \pi_n^{\mathrm{Sp}}(\mathcal{V}, x_0)$.

Remark 3.7. Spanier groups are preserved by basepoint-change in the following sense: If $q : I \to X$ where q is a path from x_0 to x_1, then the base-point change isomorphism $\rho_q : \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $\rho_q([\gamma]) = [q * \gamma]$ will satisfy $\rho_q(\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_1)) = \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ for any open cover $\mathcal{U}$. In fact, it follows that:

$$\rho_q(\pi_n^{\mathrm{Sp}}(X, x_1)) = \pi_n^{\mathrm{Sp}}(X, x_0).$$

Definition 3.8. Suppose that X is a space. We will say that X is ***n*-semilocally simply connected** if and only if for each $x \in X$ there exists an open subset U of X that contains x so that every n-loop in U is null-homotopic in X.

3.0.2 Defining Thick n-Spanier groups

Definition 3.9. The first Thick Spanier group is defined in [2] and is utilized to formula the short exact sequence. An open probelm is to extend this sequence to dimensions of $n \geq 2$; we aim to define the higher Thick n-Spanier groups as a step forwards towards a resolution. Prior to doing this; however, we need to utilize caution in how we define them. Let us first recall the definition of the first thick Spanier for ease of refrence

Definition 3.10. Let $\mathcal{U}$ be an open cover of any space X. The first Thick Spanier group of X with respect to $\mathcal{U}$ is given by

$$\Pi_1^{\mathrm{Sp}}(\mathcal{U}, x_0) = \{p \cdot \gamma \cdot p^{-1} \mid p \in P(X, x_0), \gamma \in \Omega(U_1 \cup U_2, p(1)) \text{ for } U_1, U_2 \in \mathcal{U}\}.$$

It should also be noted that the first Thick=Spanier group with respect to X is given by $\bigcap_{\mathcal{U}} \Pi_1^{\mathrm{Sp}}(\mathcal{U}, x_0)$. Let us now turn our attention to bringing to fruition a suitable definition for the higher thick Spanier groups. In doing so, let us first recall our motivation for such a construction. Our concern for higher thick Spanier groups comes from the fact that the first Thick Spanier group fits into the following short exact sequence

$$0 \longrightarrow \Pi_1^{\mathrm{Sp}}(\mathcal{U}, x_0) \longrightarrow \pi_1(X, x_0) \xrightarrow{p_{\mathcal{U}\#}} \pi_1(|N(\mathcal{U})|, \mathcal{U}_0) \longrightarrow 0$$

In other words, $\Pi_1^{\mathrm{Sp}}(\mathcal{U}, x_0)$ is defined precisely to provide the simplest description of the generating set for ker($p_{\mathcal{U}\#}$. Of primary interest in this paper, is to eastblish such a sequence in the case of the higher homotpy groups π_n and thus any such definition of the Thick n-Spanier shall fit into the analogous sequence under certain stipulations of connectedness properties of the space X and the cover $\mathcal{U}$. Let us first give an example of why this sequence fails to be exact if the regular n-Spanier groups are used

Example 3.11. Fix $n \geq 2$ and let $B^n(\epsilon) = \{(x_1, x_2, \ldots, x_n, 0) \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 < \epsilon\}$ and write B^n for $\overline{B^n(1)}$. Further, let $p : S^n \to B^n(1)$ be the vertical projection $p(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n, 0)$. We identify S^{n-1} with $\partial B^n(1)$. Now, fix $k \geq 5$ and let $\mathcal{F}$ be a finite open cover of S^{n-1} by contractible open sets such that

1. If $F \in \mathcal{F}$ then diam$(F) < \frac{1}{k}$
2. If a single element of $\mathcal{F}$ is removed, then it no longer covers B^n.
3. Any non-empty intersection of elements of $\mathcal{F}$ is homeomorphic to $\mathbb{R}^{n-1}$.

Further, for each $F \in \mathcal{F}$, let $C(F)$ denote the convex hull of $F \cup B^n\left(\frac{1}{2k}\right)$ and set $\mathscr{B} = \{C(F) \mid F \in \mathcal{F}\}$. Now, observe that $\{C(F) \mid F \in \mathcal{U}\}$ is an open cover of B^n and $\mathcal{U} = \{p^{-1}(C(F)) \mid F \in \mathcal{F}\}$

is an open cover of S^n. Note that every element of $\mathcal{U}$ is contractible and contains a "small" neighborhood of both the north pole $(0, 0, 0, \ldots, 0, 1)$ and the south pole $(0, 0, 0, \ldots, 0, -1)$. Hence, for distinct $U_1 = p^{-1}(C(F_1)), U_2 = p^{-1}(C(F_2))$ in $\mathcal{U}$, $U_1 \cap U_2$ is either contractible (if F_1 meets F_2) or homotopy equivalent to S^1 (when $F_1 \cap F_2 = \emptyset$).

Consider that $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) = 0$ since every element of $\mathcal{U}$ is contractible. Note; however, that $\bigcap \mathcal{U} \neq \emptyset$ and so $|N(\mathcal{U})|$ consists of a single $|\mathcal{U}|$-simplex. In particular, $\pi_n(|N(\mathcal{U})|, \mathcal{U}_0) = 0$. Of course, we have that $\pi_n(S^n) \cong \mathbb{Z}$ and the sequence

$$0 \longrightarrow \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) \longrightarrow \pi_n(X, x_0) \xrightarrow{p_{\mathcal{U}\#}} \pi_n(|N(\mathcal{U})|, \mathcal{U}_0) \longrightarrow 0$$

fails to be exact.

Recall that $\Pi_1(\mathcal{U}, x_0)$ is defined so that its generators are path-conjugates of loops having image in a union of two intersecting elements of $\mathcal{U}$, we show that for $\Pi_n(\mathcal{U}, x_0)$ we don't wish to place an upper bound on the number of elements the path-conjugated loop has image in. Consider $m \geq 2$ and define $G_n^{(m)}(\mathcal{U}, x_0)$ to be the subgroup of $\pi_n(X, x_0)$ which consists of all $[p * \gamma] \in \pi(X, x_0)$ where $p \in P(X, x_0), \gamma \in \Omega^n(X, p(1))$ and such that there exists $U_1, U_2, \ldots, U_m \in \mathcal{U}$ such that $\bigcap_{i=1}^m U_i \neq \emptyset$: aditionally, $\mathrm{Im}(\gamma) \subseteq \bigcup_{i=1}^m U_i$. Note that by taking k large enough, we may ensure that the elements of $\mathcal{F}$ are small enough such that $|\mathcal{F}| = |\mathcal{U}| > m$. Now, suppose that $[p * \gamma] \in G_n^{(m)}(\mathcal{U}, x_0)$ where $\mathrm{Im}(\gamma)$ lies in $\bigcup_{i=1}^m U_i$ for $U_1, U_2, \ldots, U_m \in \mathcal{U}$ (recalling that $\bigcap_i U_i$ contains both the north and south pole). Since $|\mathcal{U}| > m$ the sets $U_1, U_2, \ldots, U_m$ do not cover S^n. So we see that γ is null-homotopic in S^n such that $G_n^{(m)}(\mathcal{U}, x_0) = 0$. Observe that it's stil the case that $\pi_n(|N(\mathcal{U})|, \mathcal{U}_0) = 0$ and $\pi_n(S^n, x_0) \cong \mathbb{Z}$. So we observe the following sequence will not be exact:

$$0 \longrightarrow G_n^{(m)}(\mathcal{U}, x_0) \longrightarrow \pi_n(S^n, x_0) \xrightarrow{p_{\mathcal{U}\#}} \pi_n(|N(\mathcal{U})|, \mathcal{U}_0) \longrightarrow 0$$

Let us give one more example before presenting the definition of higher thick Spanier groups

Example 3.12. Let X consist of an arc A with a 2-sphere attached at each endpoint of A. The basepoint x_0 lies in the interior of A. Now, since $X \equiv S^2 \vee S^2$, we have that $\pi_3(X) \cong \mathbb{Z}^3$ which is generated by the Hopf fibrations $g_1, g_2 : S^3 \to S^2$ in each wedge-summand and the Whitehead product $[f_1, f_2]$. Let $\mathcal{U} = \{U_1, U_2\}$ be an open cover of X where U_1 contains the first sphere of X and a small neighborhood of x_0 in the second sphere. In a similar manner, U_2 will contain the second sphere and a small neighborhood of x_0 in the first sphere; we choose these so that $U_1 \cap U_2$ is contractible. Now, $|N(\mathcal{U})|$ is an arc and thus $\pi_3(|N(\mathcal{U}), \mathcal{U}_0) = 0$. However, it's clear that $\pi_n(\mathcal{U}, x_0) = \langle [f_1], [f_2] \rangle \cong \mathbb{Z}^2$. Hence, due to connectivity issues, the following sequence fails to be exact:

$$0 \longrightarrow \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) \longrightarrow \pi_n(S^n, x_0) \xrightarrow{p_{\mathcal{U}\#}} \pi_n(|N(\mathcal{U})|, \mathcal{U}_0) \longrightarrow 0$$

From the above examples we can formulate a definition for the higher thick Spanier groups; we give them as follows.

Definition 3.13. Let (X, x_0) be a based space and let $\mathcal{U}$ be an open cover of X. Then the **The n-th Thick Spanier Group** of X with respect to $\mathcal{U}$ is the subgroup $\Pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ of $\pi_n(X, x_0)$ which consists of all elements $[p * \gamma]$ where $p \in P(X, x_0), \gamma \in \Omega^n(X, \alpha(1))$ and such that there exists $m \in \mathbb{N}$ and $U_1, U_2, \ldots, U_m \in \mathcal{U}$ where $\bigcap_{i=1}^m U_i \neq \emptyset$ and $im(\gamma) \subseteq \bigcup_{i=1}^m U_i$. We may also write:

$$\Pi_n^{\text{Sp}}(\mathcal{U}, x_0) = \{[p * \gamma] \in \pi_n(X, x_0) \mid p \in P(X, x_0), \gamma \in \Omega^n(X, p(1)); \bigcap_{i=1}^{m} U_i \neq \emptyset, \text{Im}(\gamma) \subseteq \bigcup_{i=1}^{m} U_i\}.$$

Definition 3.14. Let (X, x_0) be a based space. We define the **thick n-Spanier group with respect to** X as

$$\Pi_n^{\text{Sp}}(X, x_0) = \bigcap_{\mathcal{U}} \Pi_n^{\text{Sp}}(\mathcal{U}, x_0)$$

where $\mathcal{U}$ ranges over all open covers of X.

We have now defined the higher thick Spanier groups; let us return to an example to observe that without certain assumptions on a given space we still may not have exactness.

Example 3.15. Let X consist of an arc A with a 2-sphere X_1, X_2 attached at each endpoint of A (let the basepoint x_0 lie in the interior of A). Observe that $X \simeq S^2 \vee S^2$ and that $\pi_3(X) \cong \mathbb{Z}^3$ which is generated by Hopf fibrations $g_i : S^3 \to X_i$ and the Whitehead product $[f_1, f_2]$ for the inclusions $f_i : S^2 \to X_i$ (note that we take path conjugates of these so that they are all based at a_0). Construct an open cover $\mathcal{U}$ of X under the following stipulations

- At least three elements of $\mathcal{U}$ cover the arc A
- The restriction of $\mathcal{U}$ to each X_i gives an open cover of the 2-sphere of the same type constructed in (insert example).

With this cover, we have that $\Pi_3^{\text{Sp}}(\mathcal{U}, x_0) \cong \mathbb{Z}^2$ generated by $[g_1], [g_2]$. Note; however, that $|N(\mathcal{U})|$ consists of an arc with two simplicies attached at each end. From this, we see that $\pi_3(|N(\mathcal{U}), \mathcal{U}_0) = 0$. So then

$$0 \longrightarrow \Pi_n^{\text{Sp}}(\mathcal{U}, x_0) \longrightarrow \pi_n(S^n, x_0) \xrightarrow{p_{\mathcal{U}\#}} \pi_n(|N(\mathcal{U})|, \mathcal{U}_0) \longrightarrow 0$$

is not exact. The reasoning here is that $\Pi_3^{\text{Sp}}(\mathcal{U}, x_0)$ need not include Whitehead products of generators of $\Pi_2^{\text{Sp}}(\mathcal{U}, x_0)$.

The last example above tells us that we need to impose connectedness assumptions on both the space X and the open cover $\mathcal{U}$ if we are to find the above sequences to be exact.

3.0.3 Properties of Thick n-Spanier groups

Let us now examine how the n-Spanier group and the thick n-Spanier group relate; first it's important to note that it need not be true that $\pi_n^{\text{Sp}}(\mathcal{U}, x_0) = \Pi_n^{\text{Sp}}(\mathcal{U}, x_0)$.

Example 3.16. Suppose $n \in \mathbb{N}$ with $n \geq 1$ and let $X = S^n$. Now, define $\mathcal{U} = \{U_1, U_2\}$ to be an open cover of X where $U_1 \cap U_2$ is the disjoint union of n-connected hemispheres; this is equivalent to the twice punctured n-sphere (which is homotopic to S^{n-1}). We've assumed that U_1 and U_2 are n-connected; hence, it's clear that their homotopy groups are trivial. On the other hand, observe that $\Pi_n^{\text{Sp}}(\mathcal{U}, x_0) = \mathbb{Z}$ since it contains a generator of $\pi_n(X)$.

Theorem 3.17. *Supposed (X, x_0) is a based path connected space. Then X is n-semilocally simply connected if and only if X has an open cover $\mathcal{U}$ so that $\pi_n^{\text{Sp}}(\mathcal{U}, x_0)$ is trivial.*

Proof. Suppose X is n-semi-locally simply connected. Then by definition every point of X has an assocaited neighborhood U where every n-loop η can be contracted to a single point. Let $\mathcal{U}$ be an open cover of X and consider $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ with any generator $[p * \gamma]$. A generator of the Spanier group is just an n-loop contained in some neighborhood U of $\mathcal{U}$. Since all elements of the Spanier group are formed by a product of generators and X is n-semi-locally simply connected we have that $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ is trivial. Conversely, suppose that X exhibits an open cover $\mathcal{U}$ such that $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ is trivial. Then any element is null-homotopic and it follows at once that X is n-semi-locally simply connected. ♣

This means, that whenever X is n-semilocally simply connected it follows that it's n-th Spanier group must be trivial. Note; however, the converse is not always true. Let us give an example

Example 3.18. Let,

$$S_k = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \middle| \left(x_1 - \frac{1}{k}\right)^2 + \sum_{j=2}^{n+1} x_j^2 = \frac{1}{k^2} \right\}$$

We define the n-dimensional earring by $\mathbb{E}_n = \bigcup_{k \in \mathbb{N}} S_k$.

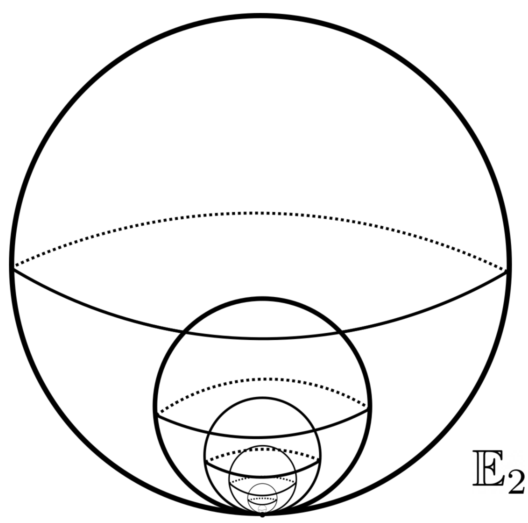

Figure 3.3: The 2-dimensional Earring Space .

It has been shown by Eda-Kawamura in [8] that $\mathbb{E}_n$ is $(n-1)$-connected and additionally, the canonical homomorphism

$$\pi_n(\mathbb{E}_n) \to \prod_{k=1}^{\infty} \pi_n(S_k) \cong \mathbb{Z}^{\omega}$$

is an isomorphism. Further, using this fact, the authors of [**?**] show that $\pi_n^{\mathrm{Sp}}(\mathbb{E}_n) \cong 0$.

Example 3.19. Let us given another non-trival example. Let (n, m) be a pair of integers; we will call this pair **dyadic unital** when the dyadic rational $\frac{2m-1}{2^n}$ is containted in $(0, 1)$. For each dyadic unital pair (n, m) consider the upper semi-circle which is given by $\mathbb{D}(n, m) = \{(x, y) \in \mathbb{R}^2 \mid (x - \frac{2m-1}{2^n})^2 + y^2 = (\frac{1}{2^n})^2; x \geq 0\}$. In addition, define $B = [0, 1] \times \{0\}$; this will be the "base" of the dyadic arc space. In total, we define the Dyadic Arc space as follows:

$$\mathbb{D} = B \cup \bigcup_{(n,m)} \mathbb{D}(n, m)$$

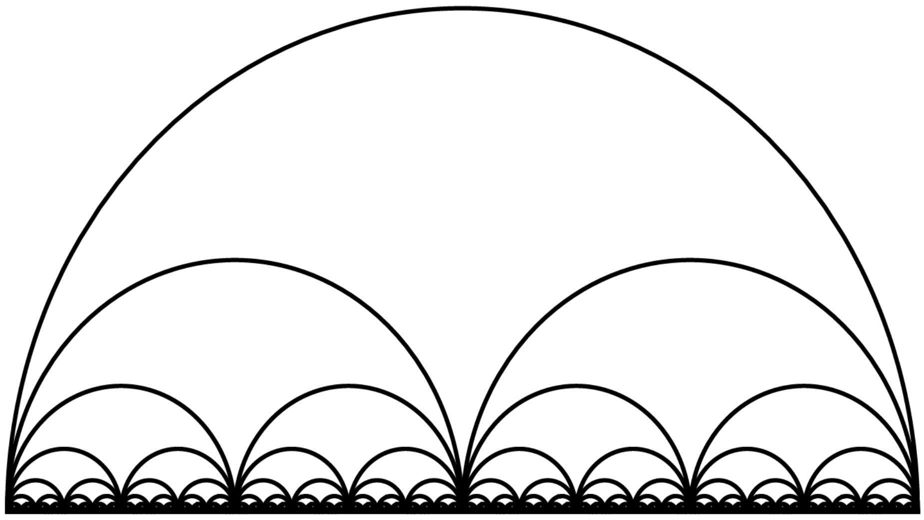

Figure 3.4: The Dyadic Arc Space $\mathbb{D}$.

where the base point is $x_0 = (0, 0)$. It is of particular interest that the base B of the Dyadic Arc is nowhere semi-locally simply connected. It has been shown that the $\pi_1(\mathbb{D})$ embeds as a subgroup in the inverse limit $\varprojlim F_{2^n-1}$ (which is also equivalent to its first shape group $\check{\pi}_1(\mathbb{D})$.

Example 3.20. It should be noted in general by (insert ref) that all one-dimensional spaces have trivial Spanier groups. Perhaps the most extereme examples this would apply to are those spaces which are nowhere semi-locally simply connected (i.e they are everywhere "wild"). This is because the Spanier group is a subgroup of the shape kernel for every space and the shape kernel is trivial for one-dimensional spaces; It thus follows that the Spanier group is trivial.

Proposition 3.21. *For every based space* (X, x_0) *we have that*

$$\pi_n^{\mathrm{Sp}}(X, x_0) \subseteq \Pi_n^{\mathrm{Sp}}(X, x_0)$$

In addition, if every open cover $\mathcal{U}$ *admits a refinement* $\mathcal{V}$ *so that* $\Pi_n^{\mathrm{Sp}}(\mathcal{V}, x_0) \subseteq \pi_n^{\mathrm{Sp}}(\mathcal{V}, x_0)$*, then we have the equality:* $\Pi_n^{\mathrm{Sp}}(X, x_0) = \pi_n^{\mathrm{Sp}}(X, x_0)$.

This follows from the observations we have made above. Let us now turn our attention to two properties that will determine when the n-Spanier and thick n-Spanier agree

Corollary 3.22. *If* $\mathcal{U}$ *is an open cover of* X *where all finte intersections of elements of* $\mathcal{U}$ *are* $(n-1)$*-connected, then the Thick n-th Spanier group of* $\mathcal{U}$ *is equal to the n-th Spanier group with respect to* $\mathcal{U}$.

It has been shown in [1] that whenever a space is T_1 and paracompact it must be that $\Pi_1^{\mathrm{Sp}}(X, x_0) = \pi_1^{\mathrm{Sp}}(X, x_0)$ (our goal will be to generalize this for $n \geq 2$). Let us recall that if $\mathcal{U}$ is some open cover of a space X and $x \in X$, then the star of x with respect to $\mathcal{U}$ is defined to be the union: $\mathrm{St}(x, \mathcal{U}) = \bigcup \{U \in \mathcal{U} \mid x \in U\}$. Let us now introduce the notion of barycenteric refinement

Definition 3.23. A **barycentric refinement** of a cover $\mathcal{U}$ is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that, for each $x \in X$, there is a $U \in \mathcal{U}$ where $\mathrm{St}(x, \mathcal{V}) \subseteq U$.

In particular, it's known that any T_1 space is paracompact if and only if every open cover has an open barycentric refinement. We know extend this result to the case of $n \geq 2$.

Remark 3.24. Spanier groups define a natural group topology on $\pi_n(X, x_0)$ generated by the basis of normal subgroups $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ at the identity element $[c_x]$. Hence, a general open set in $\pi_n(X, x_0)$ is a coset $[\gamma]\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ for $\mathcal{U} \in \Lambda$. To see that this is indeed the case let $[\alpha], [\beta] \in \pi_n(X, x_0)$ and consider $W = [\alpha][\beta^-]\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$; it's clear that $U = [\alpha]\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$, $V = [\beta^-]\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$ are both neighborhoods of $[\alpha], [\beta^-]$ respectivly and it's clear that $U \cdot V^{-1} = [\alpha]\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)[\beta^-]\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) \subseteq W$. This estbalishes that Spanier groups indeed define a topology on homotopy groups.

In order to see this topologically enriched version of $\pi_n(X, x_0)$ is functorial, it will suffice to show that a map induces a continuous homomorphism on homotopy groups. Observe that since the Spanier topology is a group topology, it suffices to check continuity at the identity element. That is given a based map $f(X, x_0) \to (Y, y_0)$ and an open cover $\mathcal{V}$ of Y, we define $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$. It's easy to see that the induced homomorphism $f_\# : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ will statisy $f_\#(\pi_n(\mathcal{U}, x_0)) \subseteq \pi_n(\mathcal{V}, x_0)$. So then we see that $f_\#$ is continuous with respect to the topology.

Chapter 4

Comparison with the Shape Kernel

We now want to establish a certain collection of results regarding the relationship between the shape kernel Ψ_X^n and the nth-Thick Spanier group. Up to this point we've observed that $\pi_n^{\mathrm{Sp}}(X, x_0) \subseteq \ker \Psi_X^n$ for any space X and that if X is T_1 and paracompact then $\Pi_n^{\mathrm{Sp}}(X, x_0) = \pi_n^{\mathrm{Sp}}(X, x_0)$. We now will establish the following result that yields an equality between the shape kernel and the nth Spanier group.

It has been established in [6,Proposition 4.8] that $\pi_1^{\mathrm{Sp}}(X, x_0) \subseteq \ker \Psi_X$. In addition, it has been shown in [1] that $\Pi_1^{\mathrm{Sp}}(X, x_0) \subseteq \ker \Psi_X$. We will now show that both of these results generalize to the case of $n > 1$.

Theorem 4.1. $\pi_n^{\mathrm{Sp}}(X, x_0) \subseteq \ker \Psi_X^n$ *for any space X.*

Proof. Let $[\alpha] \in \pi_n^{\mathrm{Sp}}(X, x_0)$ and let $\mathcal{U}$ be an open cover of X. Define $V_U = p^{-1}[\mathrm{St}(U, N(\mathcal{U}))]$, and $\mathcal{V} = \{V_U \mid U \in \mathcal{U}\}$ (which is also an open cover of X). Now, since $[\alpha] \in \pi_n^{\mathrm{Sp}}(\mathcal{V}, x_0)$ we can write

$$[\alpha] = \prod_{i=1}^{m} [\beta_i * \gamma_i]$$

where $\gamma_i : S^n \to V_{U_i}$ for $U_i \in \mathcal{U}$. Now observe that $V_{U_i} = p_{\mathcal{U}}^{-1}[\mathrm{St}(U_j, N(\mathcal{U}))]$. Since, $p(V_{U_i}) \subseteq \mathrm{St}(U, N(\mathcal{U}))$ although note that $\mathrm{St}(U, N(\mathcal{U}))$ is contractible and consequently $p_{\mathcal{U}} \circ \gamma_i$ is null-homotopic in $N(\mathcal{U})$ for all $i \in I$. This then implies that the induced projection can be written

$$P_{\mathcal{U}\#}([\alpha]) = \prod_{i=1}^{m} [p_{\mathcal{U}} \circ \beta_i * p_{\mathcal{U}} \circ \gamma_i] = 0$$

this then implies that $[\alpha] \in \ker(p_{\mathcal{U}})_\#$.

♣

Theorem 4.2. *If X is* T_1 *and paracompact, then* $\Pi_n^{\mathrm{Sp}}(X, x_0) = \pi_n^{\mathrm{Sp}}(X, x_0)$.

Proof. Let X be T_1 and paracompact and $\mathcal{U}$ an arbitrary open cover of X. Further, let $\mathcal{V}$ be a barycentric refinement of $\mathcal{U}$. Assume $g \in \Pi_n^{\mathrm{Sp}}(\mathcal{V}, x_0)$ is a generator so that $g = [p * \gamma]$ where $p : I \to X$ is a path and $\gamma : S^n \to \bigcup_{j=1}^{m} V_j$ for $V_j \in \mathcal{V}$; It will suffice to show that g is a generator of $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$. Indeed, observe that there are $2^m - 1$ possible open sets of $\mathcal{V}$ for which the terminal point $p(1)$ can lie. So let k be such that $1 \leq k \leq m$ where $p(1) \in \bigcap_{i=1}^{k} V_i$; then $p(1) \in V_i$ for every $i = 1, \dots, k$. It follows that

$$\bigcup_{i=1}^{k} V_i \subseteq \mathrm{St}(p(1), \mathcal{V}) \subseteq U$$

for some $U \in \mathcal{U}$. Recall, $[p * \gamma]$ is a loop in U and conseuqently a generator of $\pi_n^{\text{Sp}}(\mathcal{U}, x_0)$. The inclusion $\Pi_n(\mathcal{V}, x_0) \subseteq \pi_n^{\text{Sp}}(\mathcal{U}, x_0)$ follows.

♣

Corollary 4.3. *If X is metrizable, then $\Pi_n^{\text{Sp}}(X, x_0) = \pi_n^{\text{Sp}}(X, x_0)$.*

We next aim to establish that in the case where the space X is T_1 and paracompact we have that $\ker \Psi_X^n = \Pi_n^{\text{Sp}}(X, x_0)$. We need to first make a few definitions.

Definition 4.4. Let $\mathcal{U}, \mathcal{V} \in \Lambda$. We say that $\mathcal{V}$ is a **barycentric-star refinement** of $\mathcal{U}$ if for every $x \in X$, we have that $\text{St}(x, \mathcal{V}) \subseteq U$ for $U \in \mathcal{U}$. We use the notation $\mathcal{U} \leq_* \mathcal{V}$ to denote this.

Lemma 4.5. *If X is $T1$ and paracompact, then for every open cover $\mathcal{U}$ there is an open cover $\mathcal{V}$ such that $\mathcal{U} \leq_* \mathcal{V}$.*

Proof. Suppose X is paracompact, $\mathcal{U}$ is an open cover of X and let $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ an open locally finite refinement of $\mathcal{U}$. By (lemma I need to put in here) since X is normal, let

$$\mathcal{W} = \{W_\alpha \mid \alpha \in A\}$$

be a shrinking of $\mathcal{V}$. Now, $\mathcal{W}$ must be a locally finite also. Pick $x \in X$ and let $A_x = \bigcap\{V_\alpha \mid x \in \bar{W}_\alpha\}$ ♣

Definition 4.6. We will call a map $\sigma : |\partial\Delta_n| \to X$ $\mathcal{U}$ **-admissable** if there exists a $k \in \mathbb{N}$ where the extension map $\rho : \text{sd}^k(|\partial\Delta_n|)_{n-1} \to X$ such that for any n-simplex τ of $\text{sd}^k |\partial\Delta_n|$ we have that $\rho(\tau) \subset U$ for some $U \in \mathcal{U}$.

To show the opposite inclusion $\ker \Psi_n \subseteq \pi_n^{\text{Sp}}(X, x_0)$ takes considerably more effort and we need introduce some additional concepts and notation before our presentation of its proof.

Lemma 4.7. *Suppose X is paracompact, Hausdorff, and UV^n. For every $\mathcal{U} \in O(X)$, there exists $\mathcal{V} \in O(X)$ such that $\mathcal{U} \leq_*^n \mathcal{V}$.*

Lemma 4.8. *If $m, n \in \mathbb{N}$, $\mathcal{U}$ is an open cover of X, and $f : ((\text{sd}^m \Delta_{n+1})_n, d_0) \to (X, x_0)$ is a map such that for every $(n+1)$-simplex σ of $\text{sd}^m \Delta_{n+1}$, we have that $f(\partial\sigma) \subseteq U$ for some $U \in \mathcal{U}$, then $f_\#(\pi_n((\text{sd}^m \Delta_{n+1})_n, d_0)) \subseteq \pi_n^{\text{Sp}}(\mathcal{U}, x_0)$.*

Proof. The case of $n = 1$ is proved in [2]. Indeed, suppose $n \geq 2$ and let $K = \text{sd}^m \Delta_{n+1}$. The set $\mathcal{W} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of K_n such that $f_\#(\pi_n^{\text{Sp}}(\mathcal{W}, d_0)) \subseteq \pi_n^{\text{Sp}}(\mathcal{U}, x_0)$ and for every $(n+1)$-simplex σ in K, we have that $\partial\sigma \subseteq f^{-1}(U)$ for some $U \in \mathcal{U}$. Then it will suffice to prove that $\pi_n^{\text{Sp}}(\mathcal{W}, d_0) = \pi_n(K_n, d_0)$. Let S be the set of n-simplices of K. Observe that since $n \geq 2$, K_n is simple connected. By standard simplical homology we have that the reduced singular homology groups of K_n must be trivial in dimensions $< n$ and $H_n(K_n)$ is finitely generated free abelian. A set of free generators for $H_n(K_n)$ can be selected by fixing the homology class of a simplicial map $g_\sigma : \partial\Delta_{n+1} \to K_n$ that sends $\partial\Delta_{n+1}$ homeomorphically onto the boundary of an $(n+1)$-simplex of $\sigma \in S$. Hence, K_n is $(n-1)$-connected and the Hurewicz homomorphism $h : \pi_k(K_n, d_0) \to H_k(K_n)$ is an isomorphism for all $1 \leq k \leq n$. In particular, let $p_\sigma : I \to K_n$ be any path from d_0 to $g_\sigma(d_0)$. Then $\pi_n(K_n, d_0)$ is freely generated by the path-conjudates $[p_\sigma * g_\sigma], \sigma \in S$. By assumption, for every $\sigma \in S$, $[p_\sigma * g_\sigma]$ is a generator of $\pi_n^{\text{Sp}}(\mathcal{W}, d_0)$. Since $\pi_n^{\text{Sp}}(\mathcal{W}, d_0)$ contains all generators of $\pi_n(K_n, d_0)$, the equality $\pi_n^{\text{Sp}}(\mathcal{W}, d_0) = \pi_n(K_n, d_0)$ must follow.

♣

To characterize the triviality of relative Spanier groups, we establish the following:

Definition 4.9. Suppose $n \geq 0$. We say a space X is

1. semilocally π_n-trivial at $x \in X$ if there exists an open neighborhood U of X such that every map $S^n \to U$ is null-homotpic in X.

2. semilocally n-connected at $x \in X$ if there exists an open neighborhood U of X such that every map $S^k \to X, 0 \leq k \leq n$ is null-homotpic in X.

It should be noted that we will say X is semilocally π_n-trivial (resp. semilocally n-connected) if it has this property at all of its points.

Proof. Let $\mathcal{U} \in O(X)$. Since X is UV^n, for every $U \in \mathcal{U}$ and $x \in U$, there exists an open neighborhood $W(U, x)$ such that $W(U, x) \subseteq U$ and such that for all $1 \leq k \leq n$, each map $f : \partial\Delta_{k+1} \to W(U, x)$ extends to a map $g : \Delta_{k+1} \to U$. Let $\mathcal{W} = \{W(U, x) \mid U \in \mathcal{U}, x \in U\}$ and note that $\mathcal{U} \leq^n \mathcal{W}$. Since X is paracompact Hausdorff, by **Lemma 3.30** we have that there exists $\mathcal{V} \in O(X)$ such that $\mathcal{W} \leq_* \mathcal{V}$. Fix $x' \in X$. Then $\mathrm{St}(x', \mathcal{V}) \subseteq W(U, x)$ for some $x \in U \in \mathcal{U}$. Then $\mathrm{St}(x', \mathcal{U}) \subseteq U$. Moreover, if $1 \leq k \leq n$ and $f : \partial\Delta_{k+1} \to \mathrm{St}(x', \mathcal{V})$ is a map, then since f has image in $W(U, x)$ there exists an extension $g : \Delta_{k+1} \to U$. This proves that $\mathcal{U} \leq^n_* \mathcal{V}$.

♣

We will next present two lemmas, for these we introduce some common notation we will appeal to. Let $n \in \mathbb{N}$ be fixed, a geometric simplicial complex K which consists of $(n+1)$-simplices and their faces, and a subcomplex $L \subseteq K$ with $\dim(L) \leq n$. Let $M[k] = L \cup K_k$ denote the union of L and the k-skeleton of K. Since, $L \subseteq K_n, M[n] = K_n$ is the union of the boundaries of the $(n+1)$-simplicies of K. At a later time we shall consider the cases where (1) $K = \mathrm{sd}^m \Delta_{n+1}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ and $L = \{d_0\}$. We now introduce our Lemmas:

Lemma 4.10. *(Recursive Extensions). Suppose that* $1 \leq k \leq n$, $\mathcal{U} \leq_* \mathcal{V} \leq^{k-1}_* \mathcal{W}, m \in \mathbb{N}$ *and* $f : M[k-1] \to X$ *is a map such that for every* $n+1$*-simplex* σ *of* K*, we have* $f(\sigma \cap M[k-1]) \subseteq W_\sigma$ *for some* $W_\sigma \in \mathcal{W}$*. Then there exists a continuous extension* $g : M[k] \to X$ *of* f *such that for every* $(n+1)$*-simplex* σ *of* K*, we have* $g(\sigma \cap M[k]) \subseteq U_\sigma$ *for some* $U_\sigma \in \mathcal{U}$*.*

Proof. Supposing the hypothesis, we need to extend f to the k-simplicies of $M[k]$ that don't lie in L. Let τ be a k-simplex of $M[k]$ that does not lie in L and let S_τ be the set of $(n+1)$-simplices in K that contain τ. By assumption, S_τ is non-empty. We make some general observations to begin. Note that since f maps the $(k-1)$-skeleton of each $(n+1)$-simplex $\sigma \in S_\tau$ into W_σ and $\partial\tau$ lies in this $(k-1)$-skeleton, we have $f(\partial\tau) \subseteq \bigcap_{\sigma \in S_\tau} W_\sigma$. Thus, for all τ, we have that:

$$f(\partial\tau) \subseteq \bigcap_{\sigma \in S_\tau} \mathrm{St}(W_\sigma, \mathcal{V})$$

.

Fix a vertex v_τ of τ and let $x_\tau = f(v_\tau)$. Then $x_\tau \in W_\sigma \subseteq \mathrm{St}(x_\tau, \mathcal{W})$ whenver $\sigma \in S_\tau$. Now, obsreve that since $\mathcal{V} \leq^{k-1}_* \mathcal{W}$, we may find $V_\tau \in \mathcal{V}$ such that $\mathrm{St}(x_\tau, \mathcal{W}) \subseteq V_\tau$ and such that every map $\partial\Delta_k \to \mathrm{St}(x_\tau, \mathcal{W})$ extends to a map $\Delta_k \to V_\tau$. In particular, $f|_{\partial\tau} : \partial\tau \to W_\sigma$ extends to a map $\tau \to V_\tau$. We define $g : M[k] \to X$ so that it will agree with f on $M[k-1]$ and so that the restriction of g to τ is a choice of continuous extensions $\tau \to V_\tau$ of $f|_{\partial_\tau}$.

We now select sets U_σ. Fix an $(n+1)$-simplex σ of K. If the k-skeleton of σ lies entirely in L, we choose any $U_\sigma \in \mathcal{U}$ which satisfies $W_\sigma \subseteq U_\sigma$. Now, suppose that there exists at least one k-simplex in σ which is not in L. Then whenever τ is a k-simplex of σ not in L, we have that $W_\sigma \subseteq \mathrm{St}(x_\tau, \mathcal{W}) \subseteq V_\tau$. Fix a point $y_\sigma \in W_\sigma$. The assumption that $\mathcal{U} \leq_* \mathcal{V}$ implies that there exists a $U_\sigma \in \mathcal{U}$ such that $\mathrm{St}(y_\sigma, \mathcal{V}) \subseteq U_\sigma$. In this case, we have $W_\sigma \subseteq V_\tau \subseteq U_\sigma$ whenever τ is a k-simplex of σ not in L.

Lastly, we will checek that g satisfies the desired property. Once again, fix an $(n+1)$-simplex σ of K. If τ is a k-simplex of σ not in L, our definition of g gives $g(\tau) \subseteq V_\tau \subseteq U_\sigma$. If τ' is a k-simplex in $\sigma \cap L$, then $g(\tau') = f(\tau') \subseteq W_\sigma \subset U_\sigma$. This shows that $g(\sigma \cap M[k]) \subseteq U_\sigma$ for each $(n+1)$-simplex σ of K. ♣

The next lemma is a direct, recursive application of the previous lemma.

Lemma 4.11. *Assume there exists a sequence of open covers as follows:*

$$\mathcal{U} = \mathcal{W}_n \leq_* \mathcal{V}_n \leq_*^{n-1} \mathcal{W}_{n-1} \leq_* \cdots \leq_*^2 \mathcal{W}_2 \leq_* \mathcal{V}_2 \leq_*^1 \mathcal{W}_1 \leq_* \mathcal{V}_1 \leq_*^0 \mathcal{W}_0 = \mathcal{W}$$

and a map $f_0 : M[0] \to X$ such that for every $(n+1)$-simplex σ of K, we have $f_0(\sigma \cap M[0]) \subseteq W$ for some $W \in \mathcal{W}$. Then there exists an extension $f_n : M[n] \to X$ of f_0 such that for every $(n+1)$-simplex σ of K, we have $f_n(\partial\sigma) \subseteq U$ for some $U \in \mathcal{U}$.

We are now in the appropriate situation to give the proof. We shall apply the extension results of the previous two lemmas, that is $K = \mathrm{sd}^m \Delta_{n+1}$ for some $m \in \mathbb{N}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ so that $M[k] = L \cup K_k$ consists of the boundary of Δ_{n+1} and the k-simpices of $\mathrm{sd}^m \Delta_{n+1}$ not in the boundary. Observe that $M[n]$ is the union of boundaries of the $(n+1)$-simplices of $\mathrm{sd}^m \Delta_{n+1}$. We now proceed with the proof;

Lemma 4.12. *Let $n \geq 1$. Suppose X is paracompact, Hausdorff, and UV^{n-1}. Then for every open cover $\mathcal{U}$ of X, there exists $(\mathcal{V}, V_0) \in \Lambda$ such that* $\ker(p_{\mathcal{V}\#}) \subseteq \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$.

Proof. Suppose $\mathcal{U} \in O(X)$. Since X is paracompact, Hausdorff, and UV^{n-1} we can apply **Lemma 3.33** and **Lemma 3.34** to find a sequence of refinements:

$$\mathcal{U} = \mathcal{U}_n \leq_* \mathcal{V}_n \leq_*^{n-1} \mathcal{U}_{n-1} \leq_* \cdots \leq_*^2 \mathcal{U}_2 \leq_* \mathcal{V}_2 \leq_*^1 \mathcal{U}_1 \leq_* \mathcal{V}_1 \leq_*^0 \mathcal{U}_0$$

and then one last refinement $\mathcal{U}_0 \leq_* \mathcal{V}_0 = \mathcal{V}$. Let $V_0 \in \mathcal{V}$ be any set containing x_0 and recall that since X is paracompact Hausdorff $(\mathcal{V}, V_0) \in \Lambda$. Now, we will show that $\ker(p_{\mathcal{V}\#}) \subseteq \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$. Note that $p_{\mathcal{V}}^{-1}(\mathrm{St}(V, N(\mathcal{V}))) \subseteq V$ for some choice of canonical map $p_{\mathcal{V}}$.

Suppose $[f] \in \ker(p_{\mathcal{V}\#})$ is represented by a map $f : (|\partial\Delta_{n+1}|, d_0) \to (X, x_0)$. Our objective is to show that $[f] \in \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$. Then, $p_{\mathcal{V}} \circ f : |\partial\Delta_{n+1}| \to |N(\mathcal{V})|$ is null-homotopic and extends to a map $h : |\Delta_{n+1}| \to |N(\mathcal{V})|$. Indeed, set $Y_V = h^{-1}(\mathrm{St}(V, N(\mathcal{V})))$ so that $\mathcal{Y} = \{Y_V \mid V \in \mathcal{V}\}$ is an open cover of $|\Delta_n n + 1|$. We will find a particular simplicial approximation for h using this cover $\mathcal{Y}$ [17]: Let λ be a Lebesgue number for the cover $\mathcal{Y}$ so that any subset of Δ_{n+1} of diameter less than λ lies in some element of $\mathcal{Y}$. Find an $m \in \mathbb{N}$ such that each simplex in $\mathrm{sd}^m \Delta_{n+1}$ lies in a set $Y_{V_a} \in \mathcal{Y}$ for some $V_a \in \mathcal{V}$. Note that the assignment $a \mapsto V_a$ on vertices extends to a simplicial approximation $h' : \mathrm{sd}^m \Delta_{n+1} \to N(\mathcal{V})$ of h; that is a map h' such that

$$h(\mathrm{St}(a, \mathrm{sd}^m \Delta_{n+1})) \subseteq \mathrm{St}(h'(a), N(\mathcal{V})) = \mathrm{St}(V_a, N(\mathcal{V}))$$

for every vertx a. Now, let $K = \mathrm{sd}^m \Delta_{n+1}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ so that $M[k] = L \cup K_k$. First, we will extend $f : L \to X$ to a map $f_0 : M[0] \to X$. For every vertex a in K, pick a point $f_0(a) \in V_a$. In particular, if $a \in L$, take $f_0(a) = f(a)$. This choice is well defined since on boundary vertices $a \in L$. This is because we have that $p_{\mathcal{V}} \circ f(a) = h(a) \in \mathrm{St}(V_a \mid |N(\mathcal{V})|)$ and thus $f(a) \in p_{\mathcal{V}}^{-1}(\mathrm{St}(V_a, |N(\mathcal{V})|)) \subseteq V_a$.

Note that h' maps every simplex $\sigma = [a_0, a_1, \ldots, a_k]$ of K to the simplex of the nerver $N(\mathcal{V})$ spanned by $\{h'(a_i) \mid 0 \leq i \leq k\}$. By definition of the nerve, we have that $\bigcap\{V_{a_i} \mid 0 \leq i \leq k\} \neq \emptyset$. Now, select a point $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \leq i \leq k\}$. Note that by our initial selection of refinements we have $\mathcal{U}_0 let_* \mathcal{V}$. If $\sigma = [a_0, a_1, \ldots, a_{n+1}]$ is an $(n+1)$-simplex of K, then $\mathrm{St}(x_\sigma, \mathcal{V}) \subseteq U_\sigma$ for some $U_\sigma \in \mathcal{U}$. In particular, $\{f_0(a_i) \mid 0 \leq i \leq n+1\} \subseteq \bigcup\{V_{a_i} \mid 0 \leq i \leq n+1\} \subseteq U_\sigma$. Thus, f_0 map the 0-skeleton of σ into U_σ. If $1 \leq k \leq n$, then τ is a k-face of $\sigma \cap L$ with $a_i \in \tau$, then we have that $p_{\mathcal{V}} \circ f_0(int(\tau)) = p_{\mathcal{V}} \circ f(int(\tau)) = h(int(\tau)) \subseteq h(\mathrm{St}(a_i, K)) \subseteq \mathrm{St}(V_{a_i}, |N(\mathcal{V})|)$. It then follows

$$f_0(\tau) \subseteq p_{\mathcal{V}}^{-1}(\mathrm{St}(V_{a_i}, |N(\mathcal{V})|)) \subseteq V_{a_i} \subseteq U_\sigma.$$

This implies that for every n-simplex in $\sigma \cap L$, we have that $f_0(\tau) \subseteq U_\sigma$. So we conclude that for every $(n+1)$-simplex σ of K, we have that $f_0(\sigma \cap M[0]) \subseteq U_\sigma$.

Observe that by our choice of sequence of refinements, we are precisely in the situation to apply **Lemma 3.34**. In doing so, we obtain an extension $f_n : M[n] \to X$ of f such that for every $(n+1)$-simplex σ of K, we have that $f_n(\partial\sigma) \subseteq U_\sigma$ for some $U_\sigma \in \mathcal{U}_n = \mathcal{U}$. Further, by (check that I have lemma) we have that

$$[f] = [f_n|_{\partial\Delta_{n+1}}] \in \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0).$$

This proves the opposite inclusion.

♣

One of the properties we have made us of thus far is the notion of π_n-shape injectivity. Of particular use, is a weaker property, which we can introduce now:

Definition 4.13. We will say that a space X is n-homotopically Hausdorff at $x \in X$ if no non-trivial element of $\pi_n(X, x)$ has a representing map in every neighborhood of x. We say that X is n-homotpically Hausdorff it it is n-homotopically Hausdorff at all of its points.

It should be observed at once that if a space is π_n-shape injectieve then it's automatically n-homotopically Hausdorff. Let us now give an example which highlights the use of the equality of the n-th Spanier group and the Shape Kernel.

Example 4.14. Fix an integer $n \geq 2$ and let $\ell_j : S^n \to \mathbb{E}$ be the inclusion of the j-th sphere defined $f : \mathbb{E}_n \to \mathbb{E}_n$ to be the shift map which is given by $f \circ \ell_j = \ell_{j+1}$. Let $M_f = \mathbb{E}_n \times [0,1]/\sim, (x, 0) \sim (f(x), 1)$ which is the mapping torus of f. We identify $\mathbb{E}_n$ with the image of $\mathbb{E}_n \times \{0\}$ in M_f and take b_0 to be the basepoint of M_f. Now, let $\alpha : I \to M_f$ be the loop where $\alpha(t)$ is the image of (b_0, t). Then M_f is locally contractible at all points other than those in the image of α. Also, every point $\alpha(t)$ has a neighborhood that deformation retracts onto a homeomorphic copy of $\mathbb{E}_n$. So, observe that since $\mathbb{E}_n$ is UV^{n-1}, so is X. Notice that $\pi_n^{\mathrm{Sp}}(M_f, b_0) = \ker(\pi_n(M_f, b_0) \to \check{\pi}_n(M_f, b_0))$ (and this follows from (Insert Theorm)).

In particular, the Spanier group of M_f contains all elements $[\alpha^k * g]$ where $g : S^n \to \mathbb{E}_n$ is a based map with $k \in \mathbb{Z}$. Using the universal covering map $E \to M_f$ that "unwinds" α and the relation $[g] = [\alpha * (f \circ g)]$ in $\pi_n(M_f, b_0)$, it is not hard to show that these are in fact the only elements of the n-th Spanier group. We see that

$$\ker(\pi_n(M_f, b_0) \to \check{\pi}_n(M_f, b_0)) = \{[\alpha^k * g] \mid [g] \in \pi_n(\mathbb{E}_n, b_0)\}$$

this is an uncountable subgroup.

It follows from this description that, even though M_f is not π_n-shape injective, M_f is n-homotopically Hausdorff. Indeed, it suffices to check this at the points $\alpha(t), t \in I$. Let's give the argument for $\alpha(0) = b_0$, the other points are similar. If $0 \neq h \in \pi_n(M_f, b_0)$ has a representative in every neighborhood of b_0 in M_f, then clearly $h \in \ker(\Psi_n)$. Hence, $h = [\alpha^k * g]$ for $[g] \in \pi_n(\mathbb{E}_n, b_0)$. Since M_f retracts onto the circle parameterized by α^k, the hypothesis on h can only hold if $k = 0$. However, there exists a basis of neighborhoods of b_0 in M_f that deformation retracts onto an open neighborhood of b_0 in $\mathbb{E}_n$. Thus, we see $[g]$ has a representative in every neighborhood of b_0 in $\pi_n(\mathbb{E}_n, b_0)$, giving $h = [g] \in \ker(\pi_n(\mathbb{E}_n, b_0) \to \check{\pi}_n(\mathbb{E}_n, b_0)) = 0$.

4.1 When is Ψ_n an Isomorphism?

Kozlowski-Segal [**?**] that if X is paracompact Hausdorff and UV^n, then $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$ is an isomorphism. In fact, this result was first proved for compact metric spaces in [15]. The assumption that a space is UV^n assumes that small maps $S^n \to X$ may be contracted by small null-homotopies. For instance, the cone $C\mathbb{E}_n$ is UV^{n-1} but not UV^n. Note; however, that $C\mathbb{E}_n$ is contractible and so Ψ_n is clearly an isomorphism of trivial groups. Our Spanier group based approach allows us to generalize Kozlowski-Segal's theorem in a way that includes this example by removing the need for "small" homotopies in dimension n. For the sake of brevity, we will cometimes supress the pointedness of open covers and just write $\mathcal{U}$ for elements of Λ.

Lemma 4.15. *Let $n \geq 1$. Assume that X is paracompact, Hausdoff, and UV^{n-1}. If $([f_{\mathcal{U}}])_{\mathcal{U} \in \Lambda} \in \check{\pi}_1(X, x_0)$, then for every $\mathcal{U} \in \Lambda$, there exists $[g] \in \pi_n(X, x_0)$ such that $(p_{\mathcal{U}})_{\#}([g]) = [f_{\mathcal{U}}]$.*

Proof. Let $(\mathcal{U}, U_0) \in \Lambda$ and $p_{\mathcal{U}}$ be fixed, consider representing a map $f_{\mathcal{U}} : (|\partial\Delta_{n+1}|, d_0) \to (|N(\mathcal{U})|, U_0)$. Let $\mathcal{U}' = \{p_{\mathcal{U}}^{-1}(\mathrm{St}(U, |N(\mathcal{U})|)) \mid U \in \mathcal{U}\}$. Since $p_{\mathcal{U}}^{-1}(\mathrm{St}(U, |N(\mathcal{U})|)) \subseteq U$ for all $U \in \mathcal{U}$, we have $\mathcal{U} \leq \mathcal{U}'$. Applying **Lemma 3.30** and **LemmaX** we select the following sequence of refinements of $\mathcal{U}'$. First, we choose a star refinement $\mathcal{U}' \leq_{**} \mathcal{W}$ so that for every $W \in \mathcal{W}$, there exists a $U' \in \mathcal{U}$ such that $\mathrm{St}(W, \mathcal{W}) \subseteq U'$. In this case, we can choose the projection map $p_{\mathcal{U}'\mathcal{W}} : |N(\mathcal{W})| \to |N(\mathcal{U}')|$ so that if $p_{\mathcal{U}'\mathcal{W}}(W) = U'$ on vertices, then $\mathrm{St}(W, \mathcal{W}) \subseteq U'$ in X. This choice will be important near the end of the proof.

To constuct g, we must take further refinements. First, we choose a sequence of refinements

$$\mathcal{W} = \mathcal{W}_n \leq_* \mathcal{V}_n \leq_*^{n-1} \mathcal{W}_{n-1} \leq_* \cdots \leq_*^2 \mathcal{W}_2 \leq_* \mathcal{V}_2 \leq_*^1 \mathcal{W}_1 \leq_* \mathcal{V}_1 \leq_*^0 \mathcal{W}_0$$

followed by one last refinement $\mathcal{W}_0 \leq_* \mathcal{V}_0 = \mathcal{V}$. Let $V_0 \in \mathcal{V}$ be any set which contains x_0 and recall that since X is paracompact Hausdorff $(\mathcal{V}, V_0) \in \Lambda$. For some choice of canconical map $p_{\mathcal{V}}$, we have that $p_{\mathcal{V}}^{-1}(\mathrm{St}(V, N(\mathcal{V}))) \subseteq V$ for all $V \in \mathcal{V}$.

Recall that we have assumed the existence of a map $f_{\mathcal{V}} : (\partial\Delta_{n+1}, d_0) \to (|N(\mathcal{V})|, V_0)$ such that $p_{\mathcal{U}\mathcal{V}\#}([f_{\mathcal{V}}]) = [f_{\mathcal{U}}]$. Set $Y_V = f_{\mathcal{V}}^{-1}(st(V, N(\mathcal{V})))$ so that $\mathcal{Y} = \{Y_V \mid V \in \mathcal{V}\}$ is an open cover of $\partial\Delta_{n+1}$. As before, we find a simplicial approximation for $f_{\mathcal{V}}$. Find $m \in \mathbb{N}$ such that the star $\mathrm{St}(a, sd^m\partial\Delta_{n+1})$ of each vertex a in $\mathrm{sd}^m\,\partial\Delta_{n+1}$ lies in a set $Y_{V_a} \in \mathcal{Y}$ for some $V_a \in \mathcal{V}$. Since $f_{\mathcal{V}}(d_0) = V_0$, we can take $V_{d_0} = V_0$. Then the assignment $a \mapsto V_a$ on vertices extends to a simplicial approximation $f' : \mathrm{sd}^m\,\partial\Delta_{n+1} \to |N(\mathcal{V})|$ of $f_{\mathcal{V}}$, i.e a simplicial map f' such that

$$f_{\mathcal{V}}(\mathrm{St}(a, \mathrm{sd}^m\,\partial\Delta_{n+1})) \subseteq \mathrm{St}(f'(a), |N(\mathcal{V})|) = \mathrm{St}(V_a, |N(\mathcal{V})|)$$

for every vertex a.

We can begin to define g with the constant map $\{d_0\} \to X$ which sends d_0 to x_0. In preparation for applications of **Lemma 3.35**, let $K = \mathrm{sd}^m \partial\Delta_{n+1}$ and $L = \{d_0\}$ so that $K[k] = K_k$. First, we define a map $g_0 : M[0] \to X$ by selecting, for each vertex $a \in K_0$, a point $g_0(a) \in V_a$. In particular, set $g_0(d_0) = x_0$. This choice is well defined since we have that $p_{\mathcal{V}}(x_0) = V_0 \in \mathrm{St}(V_{d_0}, N(\mathcal{V}))$ and thus $g_0(d_0) = x_0 \in p_{\mathcal{V}}^{-1}(\mathrm{St}(V_{d_0}, N(\mathcal{V}))) \subseteq V_{d_0}$. Indeed, note that f' maps every simplex $\sigma = [a_0, a_1, \ldots, a_k]$ of K to the simplex of $|N(\mathcal{V})|$ which is spanned by $\{V_{a_i} \mid 0 \le i \le k\}$. Now, by definition of the nerve, it must be that $\bigcap\{V_{a_i} \mid 0 \le i \le k\} \neq \emptyset$. Pick a pint $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \le i \le k\}$. By our initial choice of refinements, we have that $\mathcal{U}_0 \le_* \mathcal{V}$. If $\sigma = [a_0, a_1, \cdots, a_n]$ is an n-simplex of K, then $\mathrm{St}(x_\sigma, \mathcal{V}) \subseteq U_{0,\sigma}$. Thus, g_0 maps the 0-skeleton of σ into $U_{0,\sigma}$. If $d_0 \in \sigma$, then $g_0(d_0) \in p_{\mathcal{V}}^{-1}(\mathrm{St}(V_{d_0}, N(\mathcal{V}))) \subseteq V_{d_0} \subseteq U_{0,\sigma}$. This then implies that for every n-simplex σ of K, we have that $g_0(\sigma \cap M[0]) \subseteq U_{0,\sigma}$. From the above, we are now ready to recursivley apply Lemma 3.35. Indeed, it will be observed that this is similar to the proof of Lemma 3.37 where the dimension $n+1$ is shifted down by one so we omit the details here. We obtain an extension $g : K = M[n] \to X$ of g_0 so that for every n-simplex σ of K, we have $g(\sigma) \subseteq W_\sigma$ for some $W_\sigma \in \mathcal{W} = \mathcal{U}_n$. Now that we have g defined, we want to show that $f_{\mathcal{U}} \simeq p_{\mathcal{U}} \circ g$. Since $f' \simeq f_{\mathcal{V}}$ (by simplicial approximation), $p_{\mathcal{U}\mathcal{V}} \simeq p_{\mathcal{U}\mathcal{U}'} \circ p_{\mathcal{U}'\mathcal{W}} \circ p_{\mathcal{W}\mathcal{V}}$ (for any choice of projection maps), and $p_{\mathcal{U}\mathcal{V}} \circ f_{\mathcal{V}} \simeq f_{\mathcal{U}}$ (for any choice of projection $p_{\mathcal{U}\mathcal{V}}$), it will suffice to show that

$$p_{\mathcal{U}\mathcal{U}'} \circ p_{\mathcal{U}'\mathcal{W}} \circ p_{\mathcal{W}\mathcal{V}} \circ f' \simeq p_{\mathcal{U}} \circ g.$$

We will do this by proving that the simplicial map $F = p_{\mathcal{U}\mathcal{U}'} \circ p_{\mathcal{U}'\mathcal{W}} \circ p_{\mathcal{W}\mathcal{V}} \circ f' : K \to |N(\mathcal{U})|$ is a simplicial approximation for $p_{\mathcal{U}} \circ g$. Recall that this can be done by verifying the "star-condition" $p_{\mathcal{U}} \circ g(\mathrm{St}(a, K)) \subseteq \mathrm{St}(F(a) : |N(\mathcal{U})|)$ for any vertex $a \in K$ [17]. Since, $n \ge 1$, we have that $\mathcal{W} \le_{**} \mathcal{V}$. Hence, just like our choice of $p_{\mathcal{U}'\mathcal{W}}$ we may choose $p_{\mathcal{W}\mathcal{V}}$ so that whenever $p_{\mathcal{W}\mathcal{V}}(V) = W$, then $\mathrm{St}(V, \mathcal{V}) \subseteq W$. Also, we choose $p_{\mathcal{U}'\mathcal{U}}$ to map $p_{\mathcal{U}}^{-1}(\mathrm{St}(U, |N(\mathcal{U})|)) \mapsto U$ on vertices. Fix a vertex $a_0 \in K$. To check the star-conditon, we'll check that $p_{\mathcal{U}} \circ g(\sigma) \subseteq \mathrm{St}(F(a_0), N(\mathcal{U})|)$ for each n-simplex σ having a_0 as a vertex. Pick an n-simplex $\sigma = [a_0, a_1, \ldots, a_n] \subseteq K$ having a_0 as a vertex. Recall that $f'(a_i) = V_{a_i}$ for every i. Set $p_{\mathcal{W}\mathcal{V}}(V_{a_i}) = W_i$ and $p_{\mathcal{U}'\mathcal{W}}(W_i) = p_{\mathcal{U}}^{-1}(\mathrm{St}(U_i, |N(\mathcal{U})|)) \in \mathcal{U}'$ for some $U_i \in \mathcal{U}$. Then $F(a_i) = U_i$ for every i. it now suffices to check that $p_{\mathcal{U}} \circ g(\sigma) \subseteq \mathrm{St}(U_0, |N(\mathcal{U})|)$. Recall that by our selection of $p_{\mathcal{U}'\mathcal{W}}$, we have that $\mathrm{St}(W_0, \mathcal{W}) \subseteq p_{\mathcal{U}}^{-1}(\mathrm{St}(U_0, |N(\mathcal{U})|)$. Then it's enough to check that $g(\sigma) \subseteq \mathrm{St}(W_0, \mathcal{W})$. By construction of g, we have that $g(\sigma) \subseteq W_\sigma$ for some $W_\sigma \in \mathcal{W}$. Since $g(a_0) \in W_0 \cap W_\sigma$. We have that $g(\sigma) \subseteq W_\sigma \subseteq \mathrm{St}(W_0, \mathcal{W})$, this completes the proof.

♣

We now state and prove our second theorem,

Theorem 4.16. *Let $n \ge 1$ and $x_0 \in X$. If X is paracompact, Hausdorff, UV^{n-1}, and semilocally π_n – trivial, then $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$ is an isomorphism.*

Proof. By hypothesis X is paracompact, Hausdorff, and UV^{n-1}, we hae that $\pi_n^{\mathrm{Sp}}(X, x_0) = \ker(\Psi_n)$ by (TM). Since X is semilocally π_n-trivial, we have that $\pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0) = 1$ for some $\mathcal{U} \in \Lambda$ with $\ker(p_{\mathcal{V}\#} \subseteq \pi_n^{\mathrm{Sp}}(\mathcal{U}, x_0)$. So then

$$p_{\mathcal{V}\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathcal{V})|, V_0)$$

is an injection. Let $([f_{\mathcal{U}}])_{\mathcal{U} \in \Lambda} \in \check{\pi}_n(X, x_0)$. By Lemma 3.40, for each $\mathcal{U} \in \Lambda$, there exists $[g_{\mathcal{U}}] \in \pi_n(X, x_0)$ such that $p_{\mathcal{U}}([g_{\mathcal{U}}]) = [f_{\mathcal{U}}]$. If $\mathcal{V} \le \mathcal{W}$, then

$$p_{\mathcal{V}\#}([g_{\mathcal{V}}]) = [f_{\mathcal{V}}] = p_{\mathcal{V}\mathcal{W}\#}([f_{\mathcal{W}}]) = p_{\mathcal{V}\mathcal{W}\#} \circ p_{\mathcal{W}\#}([g_{\mathcal{W}}] = p_{\mathcal{V}\#}([g_{\mathcal{W}}])$$

Now, since $p_{\mathcal{V}\#}$ is injective, it follows that $[g_{\mathcal{W}}] = [g_{\mathcal{V}}]$ whenever $\mathcal{V} \leq \mathcal{W}$. Setting $[g] = [g_{\mathcal{V}}]$ gives $\Psi_n([g]) = ([f_{\mathcal{U}}])_{\mathcal{U}\in\Lambda}$. Hence, Ψ_n is surjective.

♣

4.2 Examples

Let us give a few examples that illustrate the use of the above theorems. These will aid us to understand more clearly what their utility is.

Example 4.17. Fix $n \geq 2$. When X is a metrizable, UV^{n-1} space, the cone CX and unreduced suspension SX are also UV^{n-1} and semilocally π_n-trivial but need not be UV^n. In particular, this occues when $X = \mathbb{E}_n$ or if $X = Y \vee \mathbb{E}_n$ where Y is a CW-complex. In such cases, it can be seen that $\Psi_n : \pi_n(SX) \to \check{\pi}_n(SX)$ is an isomorphism. In addition, one point unions of such cones and suspensions (e.g $CX \vee CY$ or $CX \vee SY$) will also meet the hypotheses of Theorem 3.41 (checking this; however, may be fairly technical [3]) but need not be UV^n.

Example 4.18. The converse of Theorem 3.41 does not hold. For $n \geq 2$, $\mathbb{E}_n$ is UV^{n-1} but not semilocally π_n-trivial at the wedgepoint x_0. However, $\Psi_n : \pi_n(\mathbb{E}_n, x_0) \to \check{\pi}_n(X, x_0)$ is an isomorphism where both groups are canonically isomorphic to $\mathbb{Z}^{\omega}$ [8]. In addition, the infinite direct product $\prod_{\mathbb{N}} S^n$, $\Psi_k : \pi_k\Big(\prod_{\mathbb{N}} S^n, x_0\Big) \to \check{\pi}_k\Big(\prod_{\mathbb{N}} S^n, x_0\Big)$ is an isomorphism for all $k \geq 1$ even though $\prod_{\mathbb{N}} S^n$ is not UV^{k-1} when $k - 1 \geq n$.

Example 4.19. Let $n \geq 2$ and $X = \mathbb{E}_1 \vee S^n$ (See Figure 3.5). Note that since $\mathbb{E}_1$ is aspherical [5] and [6] and X is semilocally π_n-trivial. Note; however, that X is not UV^1 since it has $\mathbb{E}_1$ as a retract. It has been shown in [3] that

$$\pi_n(X) \cong \bigoplus_{\pi_1(\mathbb{E}_1)} \pi_n(S^n) \cong \bigoplus_{\pi_1(\mathbb{E}_1)} \mathbb{Z}$$

and that Ψ_n is injective. In particular, we may represent elements of $\pi_n(X)$ as finite-support sums $\sum_{\beta\in\pi_1(\mathbb{E}_1)} m_\beta$ with $m_\beta \in \mathbb{Z}$. We will show that Ψ_n is not surjective.

First, identify $\pi_1(X)$ with $\pi_1(\mathbb{E}_1)$ and recall from [7] that we may represent the elements of $\pi_1(\mathbb{E}_1)$ as countably infinite reduced words indexed by countable linear order (with a countable alphabet $\beta_1, \beta_2, \beta_3, \dots$). We see here that β_j is represented by a loop $S^1 \to \mathbb{E}_1$ going around the j-th circle. Let X_j be the union of S^n and the largest j circles of $\mathbb{E}_1$ such that $X = \operatorname{proj}\lim_j X_j$. Next, identify $\pi_1(X_j)$ with the free group F_j on generators $\beta_1, \beta_2, \dots, \beta_j$ and note that $\pi_n(X_j) \cong \bigoplus_{F_j} \mathbb{Z}$. In this sense, we may view an element of $\pi_n(X_j)$ as finite-support sums $\sum_{w\in F_j} m_w$ of integers indexed over reduced words in F_j. Let $d_{j+1,j} : F_j + 1 \to F_j$ be the homomorphism that deletes the letter β_{j+1}. Consider the inverse limit $\check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$. Now, the map $X \to X_j$ collapses all but the first $j - circles$ of $\mathbb{E}_1$ and this induces a homomorphism $d_j : \pi_1(X) \to F_j$. There exists a canonical homomorphism $\phi : \pi_1(X) \to \check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$ given by $\phi(\beta) = (d_1(\beta), d_2(\beta), \dots)$, which is known to be an injection [16] but not a surjection. As an example, if $x_k = \prod_{j=1}^{k} [\beta_1, \beta_j]$, then $(x_1, x_2, x_3, \dots)$ is an element of $\check{\pi}(X)$ not in the image of ϕ.

The bonding map $b_{j+1,j} : \pi_n(X_{j+1}) \to \pi_n(X_j)$ sends a sum $\sum_{w\in F_{j+1}} m_w$ to $\sum_{v\in F_j} p_v$ where $p_v = \sum_{d_{j+1,j}(w)=v} m_w$. In a similar manner, the projection $b_j : \pi_n(X) \to \pi_n(X_j)$ will send the sum $\sum_{\beta\in\pi_1(X)} n_\beta$ to $\sum_{v\in F_j} m_v$ where we have $m_v = \sum_{d_j(\beta)=v} m_\beta$. Next, let $y_j \in \pi_n(X)$ be the sum

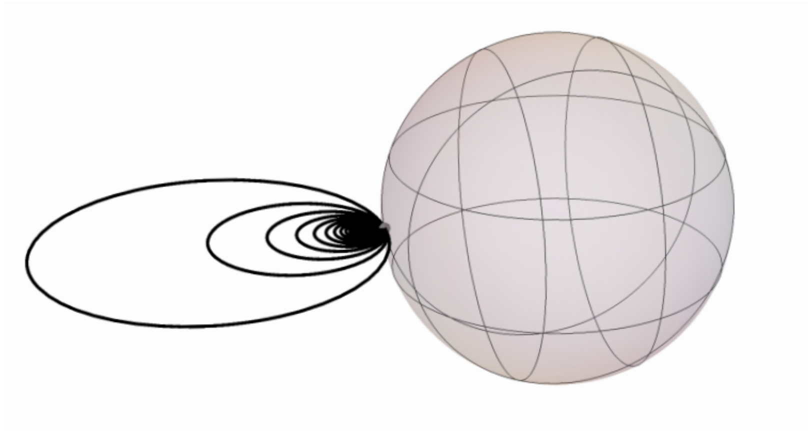

Figure 4.1: The space $X = \mathbb{E}_1 \vee S^2$

whose only non-zero coefficient is the x_j-coefficient, which is 1. Now, since $d_{j+1,j}(x_{j+1}) = x_j$, it's clear that $(y_1, y_2, y_3, \cdots) \in \check{\pi}_n(X)$. Suppose that $\Psi_n(\sum_\beta m_\beta) = (y_1, y_2, y_3, \ldots)$. We can write $\sum_\beta m_\beta$ as a finite sum $\sum_{i=1}^r m_{\beta_i}$ for non-zero m_{β_i}, then we need have $\sum_{d_j(\beta_i)=x_j} m_{\beta_i} = 1$ for all $j \in \mathbb{N}$. Since there are only finitely many β_i involved, there must exist at least one i for which $d_j(\beta_i) = x_j$ for infinitely many j. For such i, we have that $\phi(\beta_i) = (x_1, x_2, x_3, \ldots)$, which as previously mentioned is impossible. This shows that Ψ is not a surjection.

Example 4.20. Let $n \geq 2$ and $\ell_j : S^n \to \mathbb{E}_n$ be the inclusion of the j-th n-sphere in $\mathbb{E}_n$. Let X be the space obtained by attaching $(n+1)$-cells to $\mathbb{E}_n$ using the attaching maps ℓ_j. Recall that $\mathbb{E}_n$ is UV^{n-1} and so it follows easily that X is UV^{n-1}. Note; however, that X is not semilocally π_n-trivial at the wedgepoint x_0 of $\mathbb{E}_n$. Observe that the infinite concatenation of maps $\prod_{j\geq k} \ell_j = \ell_k \cdot \ell_{k+1} \cdots$ are not null-homotopic (utilizing the standard argument for the harmonic archipelago) but are all homotopic to each other. So then we have that $\pi_n(X, x_0) \neq 0$; however, for sufficiently fine open covers $\mathcal{U} \in O(X)$, $|N(\mathcal{U})|$ is homotopy equivalent to a wedge of $(n+1)$-spheres and is thus n-connected. Then we have that $\check{\pi}_n(X, x_0) = 0$: so then despite X being UV^{n-1} it need not be that Ψ is an isomorphism. In fact, $\pi_n(X, x_0) = \pi_n^{\mathrm{Sp}}(X, x_0) = \ker(\Psi_n)$.

Additionally, notice that since $\mathbb{E}_{n-1}$ is $(n-1)$-connected and $\pi_n(\mathbb{E}_n) \cong H_n(\mathbb{E}_n) \cong \mathbb{Z}^{\mathbb{N}}$, X will itelf will be $(n-1)$-connected. A Meyer-Vietoris Sequence arguement similar to that in [13] can be used to show that $\pi_n(X, x_0) \cong H_n(X) \cong \mathbb{Z}^{\mathbb{N}} / \bigoplus_{\mathbb{N}} \mathbb{Z}$.

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