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ORIGINAL PAPER

Internality of generalized averaged Gauss quadrature rules and truncated variants for modified Chebyshev measures of the third and fourth kinds

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Abstract

Gauss quadrature rules are commonly used to approximate integrals determined by a measure with support on a real interval. These rules are known to be internal, i.e., their nodes are in the convex hull of the support of the measure. This allows the application of Gauss rules also when the integrand only is defined on the convex hull of the support of the measure. It is important to be able to estimate the quadrature error that is incurred when using a Gauss rule. Averaged and generalized averaged Gauss quadrature formulas are helpful in this respect. Given an n-node Gauss rule, the associated $(2n + 1)$-node averaged and generalized averaged Gauss rules are easy to compute. However, they are not guaranteed to be internal, and in this situation they cannot be used for integrands that are defined on the convex hull of the support of the measure only. This paper investigates whether averaged and generalized averaged Gauss quadrature formulas for modified Chebyshev measures of the third and fourth kinds are internal. We show that in situations when this is not the case, truncated variants, that use fewer nodes, are internal. Computed examples that illustrate the performance of the quadrature rules discussed are presented.

Keywords Gauss quadrature · Averaged Gauss quadrature rule · Generalized averaged Gauss quadrature rule · Truncated generalized averaged Gauss quadrature rule · Internality of quadrature rules · Modified Chebyshev measure of the third and fourth kinds

Mathematics Subject Classification 65D30 · 65D32

Dedicated to Claude Brezinski on the occasion of his 80th birthday.

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1 Introduction

Let $\mathrm{d}\lambda$ be a nonnegative measure with infinitely many points of support on the interval $[a, b] \subseteq \mathbb{R}$, and assume that all the moments,

$$\nu_k = \int_a^b x^k \,\mathrm{d}\lambda(x), \qquad k = 0, 1, 2, \dots ,$$

are well defined. We let $\{P_k\}_{k=0}^{\infty}$ denote the sequence of monic orthogonal polynomials with respect to the measure $\mathrm{d}\lambda$, where $\deg(P_k) = k$. The polynomials P_k satisfy a three-term recurrence relation of the form

$$P_{k+1}(x) = (x - \alpha_k)P_k(x) - \beta_k P_{k-1}(x), \quad k = 1, 2, \dots , \tag{1}$$

where $P_{-1}(x) \equiv 0$ and $P_0(x) \equiv 1$, $\alpha_k \in \mathbb{R}$, and $\beta_k > 0$ for all $k \geq 1$; see, e.g., [11, 29] for many properties and examples of orthogonal polynomials.

It is well known that among all interpolatory quadrature rules with n nodes for approximating the integral

$$I(f) = \int_a^b f(x) \,\mathrm{d}\lambda(x), \tag{2}$$

the rule with maximum degree of exactness is the n-node Gauss quadrature rule with respect to the measure $\mathrm{d}\lambda$,

$$Q_n^G(f) = \sum_{i=1}^{n} w_i^{(n)} f(x_i^{(n)}).$$

Its nodes $x_i^{(n)}$ $(i = 1, 2, \dots , n)$ are the zeros of the monic orthogonal polynomial P_n and lie in the convex hull of the support of the measure $\mathrm{d}\lambda$, and the weights $w_i^{(n)}$ $(i = 1, 2, \dots , n)$ are positive; see [11, 29] for proofs. The degree of exactness of the Gauss rule Q_n^G is $2n - 1$, that is, $Q_n^G(p) = I(p)$ for all polynomials p of degree not exceeding $2n - 1$.

It is important to be able to estimate the magnitude of the quadrature error

$$\varepsilon_n(f) = |(I - Q_n^G)(f)|, \tag{3}$$

because this helps determine a suitable value of n when applying the rule Q_n^G to approximate the integral (2) to desired accuracy. An unnecessarily large value of n requires the computation of needlessly many nodes and weights, as well as the evaluation of the integrand f at excessively many nodes, while a too small value of n does not yield the required accuracy. The development of methods for estimating the error (3) therefore has received considerable attention over many years.

A popular approach to estimate the error (3) is to use another quadrature rule, A_ℓ, with $\ell > n$ nodes and degree of exactness larger than $2n - 1$. One then can use

$$|(A_\ell - Q_n^G)(f)| \tag{4}$$

as an the estimate of (3).

A classical attractive choice for the rule A_ℓ with $\ell = 2n + 1$ is the Gauss-Kronrod rule with $2n + 1$ nodes, n of which are the nodes of Q_n^G, as its degree of exactness is at least $3n + 1$. However, the $n + 1$ extra nodes are neither guaranteed to be real nor in the convex hull of the support of the measure $d\lambda(x)$; see [21] for a nice recent survey of Gauss-Kronrod rules. Moreover, Gauss-Kronrod rules are somewhat complicated to compute; see [1, 5, 18].

Another approach to define a quadrature rule A_ℓ with $\ell = 2n + 1$ is to construct an $(n + 1)$-node quadrature formula U_{n+1}^θ for approximating the functional $I_\theta(f) = I(f) - \theta Q_n^G(f)$ for some $\theta \in \mathbb{R}\backslash\{0\}$ and use the "stratified" $(2n + 1)$-node quadrature formula

$$A_{2n+1} = \theta Q_n^G + U_{n+1}^\theta \tag{5}$$

to estimate the error (3); see [16, 22] for discussions of this approach. Here the nodes of U_{n+1}^θ are assumed to be distinct from the nodes of Q_n^G.

We will refer to quadrature rules of the form (5) as Q_{2n+1}. The nodes of Q_{2n+1} are the n nodes of Q_n^G and $n + 1$ extra nodes. We will let the latter nodes be the zeros of the polynomial

$$F_{n+1} = P_{n+1} - \bar{\beta}_{n+1} P_{n-1} \tag{6}$$

for some constant $\bar{\beta}_{n+1}$ depending on θ. Polynomials of the form (6) are known as quasi-orthogonal polynomials of order two. Their properties, and in particular, their zeros have been studied by Shohat [25] as well as by Joulak [14].

Two common choices of the coefficient $\bar{\beta}_{n+1}$ are

(i) $\bar{\beta}_{n+1} = \beta_n$. This gives the *averaged rule* Q_{2n+1}^L, which was introduced by Laurie [17]. It has degree of exactness at least $2n + 1$. It corresponds to $\theta = \frac{1}{2}$.
(ii) $\bar{\beta}_{n+1} = \beta_{n+1}$. This leads to the *generalized averaged rule* Q_{2n+1}^S introduced in [26, 27]. Its degree of exactness is at least $2n + 2$.

Numerous computed examples that illustrate the high quality of the error estimate (4) when $\ell = 2n + 1$ and $A_{2n+1} = Q_{2n+1}^L$ or $A_{2n+1} = Q_{2n+1}^L$ for a variety of measures $d\lambda$ have recently been provided in [24].

The quadrature formulas Q_{2n+1}^L and Q_{2n+1}^S have real nodes and positive weights, and are easy to compute; see [17, 23]. However, they are not guaranteed to be internal, i.e., they may have nodes outside the convex hull of the support of the measure $d\lambda$. In fact, it holds for both rules that they may have at most one node to the left of the convex hull of the support $d\lambda$, and at most one node to the right; see [17, 26].

The quadrature rules Q_{2n+1}^L and Q_{2n+1}^S are associated with symmetric tridiagonal matrices of order $2n + 1$ with positive off-diagonal entries. These matrices are determined by the recursion coefficients of the monic orthogonal polynomials (1); see, e.g., [23, 28] for details. The eigenvalues of these matrices are the nodes of the quadrature rules, and the square of suitably normalized first components of the eigenvectors yield the weights; see [10, 11]. This property is used by the Golub-Welsch [12] algorithm for computing the nodes and weights of Gauss-type quadrature rules.

When removing the last row and column of a symmetric tridiagonal matrix with positive off-diagonal entries, the eigenvalues of the reduced matrix so obtained strictly interlace the eigenvalues of the original matrix; see, e.g., [13]. It therefore may be possible to obtain internal quadrature rules with fairly high degree of exactness by truncating the symmetric tridiagonal matrices that are associated with the $(2n+1)$-node average Gauss rule or generalized averaged Gauss rule. Specifically, we consider *truncated generalized averaged quadrature rules* obtained by removing the $r = n-1$ last rows and columns from the symmetric tridiagonal matrix associated with the quadrature rule Q^S_{2n+1}. We refer to the $(2n+2)$-node quadrature rule so obtained as $Q^{(1)}_{n+2}$. Its nodes are the zeros of the polynomial

$$t_{n+2}(x) = (x - \alpha_{n-1})P_{n+1}(x) - \beta_{n+1}P_n(x).$$

This follows from the recursion relations (1).

The present paper is concerned with the internality of generalized averaged Gauss quadrature rules associated with modifications of Chebyshev measures of the third and fourth kinds. The measure for Chebyshev polynomials of the third kind is given by

$$\mathrm{d}\lambda(x) = \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x, \qquad -1 < x < 1, \tag{7}$$

and the measure for Chebyshev polynomials of the fourth kind is

$$\mathrm{d}\lambda(x) = \sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x, \qquad -1 < x < 1. \tag{8}$$

Chebyshev polynomials associated with these measures arise in the approximation of functions on the open interval $-1 < x < 1$, quadrature, and the solution of differential equations; see [9, 11, 19]. We are interested in Gauss quadrature rules associated with modifications of the measures (7) and (8). In Section 2 we modify the measure (7) by a linear divisor, and in Section 3 by a linear divisor and a linear factor. We are interested in studying whether averaged and generalized averaged quadrature rules associated with these modified measures are internal and, if not, whether truncated rules $Q^{(1)}_{n+2}$ associated with generalized averaged quadrature rules are internal.

Orthogonal polynomials associated with modifications of Chebyshev measures of the second kind have been studied by Milovanović et al. [20], and properties of averaged and generalized averaged Gauss quadrature associated with these measures are studied in [8]. Our investigation complements the latter work as well as the study of averaged Gauss and generalized averaged Gauss rules for modified Chebyshev measures of the first kind reported in [6]. A few computed examples are presented in Sections 2-4, and concluding remarks are provided in Section 5.

Claude Brezinski has made numerous profound contributions to orthogonal polynomials, quadrature, function approximation, and extrapolation over many years;

see, e.g., [2–4] as well as his home page[1]. It is a great pleasure to dedicate this work to him.

2 Modification by a linear divisor

Consider the Chebyshev measure of the third kind (7). The recurrence coefficients (1) for monic orthogonal polynomials associated with this measure are

$$\alpha_0 = \frac{1}{2} \quad \text{and} \quad \begin{cases} \alpha_k = 0, \\ \beta_k = \frac{1}{4}, \end{cases} \quad \text{for } k \geqslant 1;$$

see, e.g., [11, 19]. The monic orthogonal polynomials are $\frac{1}{2^n}V_n(x)$, where the $V_n(x)$ are Chebyshev polynomials of the third kind. They are can be written as

$$V_n(\cos t) = \frac{\cos(n+\frac{1}{2})t}{\cos\frac{t}{2}}.$$

We note that $V_n(1) = 1$ and $V_n(-1) = (-1)^n(2n+1)$.

This section discusses quadrature rules with respect to measures obtained by modifying the measure (7) by a linear divisor. Thus, for a constant $c \in \mathbb{R}\backslash\{0\}$, define the modified Chebyshev measure

$$d\widetilde{\lambda}(x) = \sqrt{\frac{1+x}{1-x}} \cdot \frac{dx}{x-\delta} \quad \text{for} \quad -1 < x < 1, \tag{9}$$

where $\delta = -\frac{1}{2}(c + c^{-1})$. We introduce

$$\acute{c} = \begin{cases} c, & |c| < 1, \\ c^{-1}, & |c| \geqslant 1, \end{cases} \quad \text{so that} \quad \delta = -\frac{1}{2}(\acute{c} + \acute{c}^{-1}). \tag{10}$$

Everything in this section will be written solely in terms of $\acute{c}$. For instance, the zeroth moment can be expressed as

$$\mu_0 = 2\pi \cdot \frac{\acute{c}}{1+\acute{c}}$$

for $\acute{c} \neq -1$. This moment is not defined for $\acute{c} = -1$.

We present derivations for the Chebyshev measures of the third kind. Results for Chebyshev measures of fourth kind follow from those for the third kind by replacing x by $-x$ and c by $-c$.

2.1 Monic orthogonal polynomials

Let the measures $d\lambda$ and $d\widetilde{\lambda}$ be defined by (7) and (9), respectively. Let the recurrence coefficients α_k, β_k of the monic orthogonal polynomials P_k associated with

[1] http://math.univ-lille1.fr/~brezinsk

the measure $d\lambda$ be known; cf. (1). Kautsky and Golub [15] and Gautschi [11, eqs. (2.4.24-25)] describe how the recurrence coefficients $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$ for the monic orthogonal polynomials $\widetilde{P}_k$ associated with the measure (9) can be computed from the coefficients α_k and β_k. Our discussion will follow the description by Gautschi [11]. The algorithm described there uses the values

$$r_k(\delta) = \frac{\rho_{k+1}(\delta)}{\rho_k(\delta)}, \quad \rho_k(\delta) = \int_{-1}^{1} \frac{P_k(x)}{\delta - x}\, d\lambda(x) \quad (k = 0, 1, 2, \dots)$$

and $\rho_{-1}(\delta) = 1$. For the measures (7) and (9), we obtain the relations

$$\begin{aligned} r_k &= \delta - \frac{1}{4r_{k-1}} \quad (k \geqslant 1), \\ \widetilde{\alpha}_k &= r_k - r_{k-1}, \quad \text{and} \quad \widetilde{\beta}_k = \frac{r_{k-1}}{4r_{k-2}} \quad (k \geqslant 2), \end{aligned}$$

with the initial values

$$\begin{aligned} r_{-1} &= -\frac{2\pi\acute{c}}{1+\acute{c}}, \quad r_0 = -\frac{1}{2}\acute{c}, \\ \widetilde{\alpha}_0 &= \frac{1-\acute{c}}{2}, \quad \widetilde{\alpha}_1 = 0, \quad \widetilde{\beta}_0 = \frac{2\pi\acute{c}}{1+\acute{c}}, \quad \widetilde{\beta}_1 = \frac{1+\acute{c}}{4}. \end{aligned}$$

An easy induction shows that $r_k = -\frac{1}{2}\acute{c}$ for all $k \geqslant 1$. This leads to the following result.

Theorem 1 *The recurrence coefficients for the monic orthogonal polynomials associated with the measure* $d\widetilde{\lambda}$ *in* (9) *are*

$$\begin{aligned} \widetilde{\alpha}_0 &= \tfrac{1-\acute{c}}{2}, & \widetilde{\alpha}_k &= 0 \quad \text{for} \quad k \geqslant 1, \\ \widetilde{\beta}_0 &= \tfrac{2\pi\acute{c}}{1+\acute{c}}, \quad \widetilde{\beta}_1 = \tfrac{1+\acute{c}}{4}, & \widetilde{\beta}_k &= \tfrac{1}{4} \quad \text{for} \quad k \geqslant 2. \end{aligned}$$

The monic orthogonal polynomials $\widetilde{P}_k$ *with respect to* $d\widetilde{\lambda}$ *are*

$$\widetilde{P}_k(x) = \frac{1}{2^k}\big(V_k(x) + \acute{c}\, V_{k-1}(x)\big) \quad \text{for} \quad k \geqslant 0. \tag{11}$$

2.2 Internality of generalized averaged Gauss rules and truncated variants

The following result is a consequence of [28, Theorem 3.1]; related results can be found in [6].

Theorem 2 *The averaged Gauss formula* Q^L_{2n+1} *and the generalized averaged Gauss formula* Q^S_{2n+1} *associated with the measure* $d\widetilde{\lambda}$, *defined by* (9), *both coincide with the*

Table 1 Example 1: The smallest node of averaged Gauss rules Q_{2n+1} for the measure $d\widetilde{\lambda}$ for several values of c and n

c	n	x_1	c	n	x_1
	5	−9.58144212765193(−1)		5	−9.51567627092748(−1)
	10	−9.88635810821284(−1)		10	−9.87752883027287(−1)
−0.1	15	−9.94808737281532(−1)	−0.9	15	−9.94541066229161(−1)
	20	−9.97039635400724(−1)		20	−9.96925430010067(−1)
	30	−9.98666154723967(−1)		30	−9.98631935917441(−1)
	5	−9.61044001085879(−1)		5	−9.91110418792453(−1)
	10	−9.89061954783418(−1)		10	−9.96163261105782(−1)
0.1	15	−9.94941871960338(−1)	0.9	15	−9.97760026932085(−1)
	20	−9.97097295114571(−1)		20	−9.98509394574705(−1)
	30	−9.98683690097044(−1)		30	−9.99192146304767(−1)

Gauss-Kronrod formulas for $n \geqslant 3$. *Consequently, the polynomials* F_{n+1} *in* (6) *are Stieltjes polynomials.*

We remark that for $n = 1$, the formulas Q^L_{2n+1} and Q^S_{2n+1} do not coincide, whereas for $n = 2$ they do, but differ from the Gauss-Kronrod rule. For all $n \geq 1$, the quadrature rules Q^L_{2n+1} and Q^S_{2n+1} have n nodes that coincide with the nodes of the Gauss rule Q^G_n; this follows from the construction of the rules Q^L_{2n+1} and Q^S_{2n+1}; see (5) as well as [23]. Consequently, these nodes are internal. For $n \geq 2$, the remaining $n + 1$ nodes, $x^F_1 < x^F_2 < \ldots < x^F_{n+1}$, are the zeros of the polynomial

$$F_{n+1}(x) = \widetilde{P}_{n+1}(x) - \frac{1}{4}\widetilde{P}_{n-1}(x);$$

cf. (6). Since the rules Q^L_{2n+1} and Q^S_{2n+1} coincide for $n \geq 2$, we will simply denote them by Q_{2n+1}.

It suffices to investigate the location of the smallest and largest zeros, x^F_1 and x^F_{n+1}, respectively of F_{n+1}. Since $F_{n+1}(1) = 0$, we have by (11) that $x^F_{n+1} = 1$. Moreover, the condition $x^F_1 \geqslant -1$ is equivalent to

$$\frac{V_{n+1}(-1) + \hat{c}V_n(-1)}{V_{n-1}(-1) + \hat{c}V_{n-2}(-1)} = \frac{2n + 1 - (2n - 1)\hat{c}}{2n - 3 - (2n - 5)\hat{c}} \geqslant 1.$$

This inequality holds since $\hat{c} \leqslant 1$. We have established the following result.

Theorem 3 *For* $n \geqslant 2$, *the averaged quadrature rule* Q_{2n+1} *associated with the measure* $d\widetilde{\lambda}$ *defined by* (9) *is internal. The truncated variants of* Q_{2n+1} *have all nodes in the open interval* $(-1, 1)$. *They therefore also are internal.*

Example 1 Table 1 shows the smallest nodes x_1 of averaged Gauss quadrature rules Q_{2n+1} for several values of n and the parameter c for the measure $\mathrm{d}\widetilde{\lambda}$ defined by (9). As predicted by the theory, the nodes x_1^L are inside the open interval $(-1, 1)$. The largest node, x_{n+1}, always is one. All computations reported in this paper were carried out using Matlab and high-precision arithmetic.

3 Modifications by a linear divisor and a linear factor

This section is concerned with the measure

$$\mathrm{d}\widehat{\lambda}(x) = (x-\gamma)\,\mathrm{d}\widetilde{\lambda}(x) = \frac{x-\gamma}{x-\delta}\sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x \quad \text{for} \quad -1 < x < 1, \tag{12}$$

where

$$\gamma = -(\frac{1}{2}c + c^{-1}), \qquad \delta = -\frac{1}{2}(c + c^{-1}), \tag{13}$$

and the constant c is the same as in Section 2; hence $c \in \mathbb{R}\backslash\{-1, 0\}$. Thus,

$$\mathrm{d}\widehat{\lambda}(x) = (x-\gamma)\,\mathrm{d}\widetilde{\lambda}(x),$$

where $\mathrm{d}\widetilde{\lambda}$ is given by (9).

3.1 Monic orthogonal polynomials

Let the $\widehat{P}_k$ denote the monic orthogonal polynomials associated with the measure (12) and let $\widehat{\alpha}_k, \widehat{\beta}_k$ be the recurrence coefficients for these polynomials. The polynomials $\widehat{P}_k$ are related to the polynomials $\widetilde{P}_k$ associated with the measure (9) for $k \geqslant 0$ by the relation

$$\widehat{P}_k(x) = \frac{\widetilde{P}_{k+1}(x) - r_k\widetilde{P}_k(x)}{x-\gamma}, \quad \text{where} \quad r_k = \frac{\widetilde{P}_{k+1}(\gamma)}{\widetilde{P}_k(\gamma)}, \tag{14}$$

under the assumption that $\widetilde{P}_k(\gamma) \neq 0$ for all k; see [11, Theorem 1.55].

Gautschi [11, eqs. (2.4.12-13)] describes an algorithm for computing the recursion coefficients for the measure $\mathrm{d}\widehat{\lambda}$ (12) by using the recursion coefficients for the measure $\mathrm{d}\widetilde{\lambda}$ defined by (9). This algorithm yields

$$r_0 = -\tfrac{c+2}{2c}, \quad r_1 = -\tfrac{c^2+2c+4}{2c(c+2)}, \quad \text{if } |c| < 1, \tag{15}$$

$$r_0 = -\tfrac{c^2+c+1}{2c}, \quad r_1 = -\tfrac{c^4+c^3+2c^2+c+2}{2c(c^2+c+1)}, \quad \text{if } |c| \geqslant 1, \tag{16}$$

and

$$r_k = \gamma - \frac{1}{4r_{k-1}} \quad (k \geqslant 2). \tag{17}$$

The initial recursion coefficients for $|c| \geqslant 1$ then are

$$\begin{aligned} \widehat{\alpha}_0 &= \tfrac{c^3+c^2+c-1}{2c(c^2+c+1)}, \quad \widehat{\alpha}_1 = -\tfrac{c^2+2c+2}{2(c^2+c+1)(c^4+c^3+2c^2+c+2)}, \\ \widehat{\beta}_1 &= \tfrac{(c+1)(c^4+c^3+2c^2+c+2)}{4c(c^2+c+1)^2}, \end{aligned}$$

and for $|c| < 1$, we have

$$\widehat{\alpha}_0 = \tfrac{1}{c+2}, \quad \widehat{\alpha}_1 = -\tfrac{c^2(c^2+2c+2)}{2(c+2)(c^2+2c+4)}, \quad \widehat{\beta}_1 = \tfrac{(c+1)(c^2+2c+4)}{4(c+2)^2}.$$

Moreover,

$$\widehat{\alpha}_k = r_{k+1} - r_k \quad \text{and} \quad \widehat{\beta}_k = \frac{r_k}{4r_{k-1}} \quad (k \geqslant 2). \tag{18}$$

It is known (see, e.g., [6, Theorem 4]) that every sequence $(r_k)_{k=1}^{\infty}$ that satisfies (17) with $r_1 \neq -\frac{1}{2}z^{-1}$ is of the form

$$r_k = -\frac{1}{2z} \cdot \frac{z^{2k-2} + A}{z^{2k-4} + A} \qquad (k \in \mathbb{N}), \tag{19}$$

where

$$z = \frac{c^2+2 + \sqrt{c^4+4}}{2c} \tag{20}$$

and A is a real constant. When r_1 is given by (15) or (16), we obtain

$$A = \begin{cases} z^{-5}\left(\frac{c^2+\sqrt{c^4+4}}{2}\right)^2, & |c| < 1, \\ [1mm] z^{-3}\left(\frac{c^2+\sqrt{c^4+4}}{2}\right)^{-2}, & |c| \geqslant 1. \end{cases} \tag{21}$$

Note that in either case A has the same sign as z and c, and

$$0 < |A| < |z|^{-3}. \tag{22}$$

The relations (18) and (19) can be used to derive explicit expressions for $\widehat{\alpha}_k$ and $\widehat{\beta}_k$. This yields the following result.

Theorem 4 *The recurrence coefficients for the monic orthogonal polynomials associated with the measure* (12) *are given by*

$$\widehat{\alpha}_k = -\frac{Az^{2k-5}(z^2-1)^2}{2(z^{2k-2}+A)(z^{2k-4}+A)} \quad \text{and} \quad \widehat{\beta}_k = \frac{1}{4} + \frac{Az^{2k-6}(z^2-1)^2}{4(z^{2k-4}+A)^2} \tag{23}$$

for $k \geqslant 2$, *where* z *and* A *are defined by* (20) *and* (21). *Hence* $\widehat{\alpha}_k < 0$ *and* $\widehat{\beta}_k - \frac{1}{4}$ *are of the same sign as* c.

3.2 Internality of the averaged and generalized averaged Gauss formulas Q^L_{2n+1} and Q^S_{2n+1}

The non-Gaussian nodes of the formulas Q^L_{2n+1} and Q^S_{2n+1} are the zeros of the polynomial

$$T_{n+1}(x) = \widehat{P}_{n+1}(x) - \widehat{\beta}_N \widehat{P}_{n-1}(x), \tag{24}$$

for $N = n$ and $N = n + 1$, respectively; cf. (6). Let x^L_1 and x^S_1 be the smallest zeros of T_{n+1} for $N = n$ and $N = n + 1$, respectively. Similarly, let x^L_{n+1} and x^S_{n+1} be the largest zeros of T_{n+1} for $N = n$ and $N = n + 1$, respectively. The quadrature rules Q^L_{2n+1} and Q^S_{2n+1} are internal if x^L_1 and x^S_1 are bounded below by -1 and x^L_{n+1} and x^S_{n+1} are bounded above by 1. These conditions are equivalent to $x^{n+1}T_{n+1}(x) \geqslant 0$ for $x = \pm 1$, which is equivalent to

$$\frac{\widehat{P}_{n+1}(x)}{\widehat{P}_{n-1}(x)} \geqslant \widehat{\beta}_N \quad \text{for} \quad x = \pm 1; \tag{25}$$

see, e.g., [17] for an analogous discussion. By (11) and (14), we have

$$4 \cdot \frac{\widehat{P}_{n+1}(x)}{\widehat{P}_{n-1}(x)} = \begin{cases} \dfrac{1 - 2r_{n+1}}{1 - 2r_{n-1}} & \text{if } x = 1, \\[2ex] \dfrac{2(1-\hat{c}) + \big(2 + (2n+1)(1-\hat{c})\big)(1 + 2r_{n+1})}{2(1-\hat{c}) + \big(2 + (2n-3)(1-\hat{c})\big)(1 + 2r_{n-1})} & \text{if } x = -1. \end{cases} \tag{26}$$

The following result on the relative sizes of the recursion coefficients $\widehat{\beta}_n$ and $\widehat{\beta}_{n+1}$ will be used below.

Lemma 1 $\widehat{\beta}_n > \widehat{\beta}_{n+1}$ *if* $c > 0$, *and* $\widehat{\beta}_n < \widehat{\beta}_{n+1}$ *if* $c < 0$.

Proof By (23) we have

$$\widehat{\beta}_n - \widehat{\beta}_{n+1} = \frac{Az^{2n-6}(z^2-1)^3(z^{4n-6}-A^2)}{4(z^{2n-2}+A)^2(z^{2n-4}+A)^2},$$

which is of the same sign as z and c.

Combining Lemma 1 with (24) yields the following result.

Corollary 1

(i) *If* $c<0$, *then* $x^L_{n+1}>1$ *implies* $x^S_{n+1}>1$,
(ii) *if* $c<0$, *then* $x^S_1>-1$ *implies* $x^L_1>-1$, *and*
(iii) *if* $c>0$, *then* $x^L_1>-1$ *implies* $x^S_1>-1$.

We are in a position to show the main result of this section.

Theorem 5 *Assume that* $n\geqslant 2$. *Then the quadrature formula* Q^S_{2n+1} *is internal if and only if* $c>0$. *The quadrature rule* Q^L_{2n+1} *is not internal. More precisely*:

(i) $x^L_1>-1$ *and* $x^L_{n+1}>1$,
(ii) $x^S_1>-1$, *but* $x^S_{n+1}<1$ *for* $c>0$ *and* $x^S_{n+1}>1$ *for* $c<0$.

Proof

(a) We first show that $x^L_{n+1}>1$. By Corollary 1, if $c<0$, then this implies that $x^S_{n+1}>1$. By (18) and (26), the condition (25) for $x=1$ and $n\geqslant 2$ reduces to

$$\frac{1-2r_{n+1}}{1-2r_{n-1}}-\frac{r_n}{r_{n-1}}\geqslant 0,$$

which by (19) can be expressed as

$$\frac{z+\frac{z^{2n}+A}{z^{2n-2}+A}}{z+\frac{z^{2n-4}+A}{z^{2n-6}+A}}-\frac{(z^{2n-2}+A)(z^{2n-6}+A)}{(z^{2n-4}+A)^2}\geqslant 0$$

and expands into

$$Az(z-1)^2(z^2-1)(A+z^{2n-5})(A-z^{2n-3})\geqslant 0.$$

This is false, because (22) implies that $Az>0$, $(A+z^{2n-5})(A-z^{2n-3})<0$, and also $z^2-1>0$.

(b) We next verify that if $c>0$, then $x^S_{n+1}<1$. This is equivalent to

$$\frac{z+\frac{z^{2n}+A}{z^{2n-2}+A}}{z+\frac{z^{2n-4}+A}{z^{2n-6}+A}}-\frac{(z^{2n}+A)(z^{2n-4}+A)}{(z^{2n-2}+A)^2}\geqslant 0.$$

This expression expands into the inequality

$$Az^{2n-6}(z-1)(z^2-1)(z^3-1)(z^{2n-3}-A)\geqslant 0,$$

which is trivially correct.

(c) We proceed to show that $x^L_1, x^S_1>-1$. Assume that $c>0$. By Corollary 1, it suffices to show that $x^L_1>-1$, i.e., (25) for $N=n$ and $x=-1$. This is equivalent to

$$L-\frac{r_n}{r_{n-1}}>0,\quad \text{where}\quad L=\frac{2(1-\hat{c})+\big(2+(2n+1)(1-\hat{c})\big)(1+2r_{n+1})}{2(1-\hat{c})+\big(2+(2n-3)(1-\hat{c})\big)(1+2r_{n-1})}, \tag{27}$$

and expands into

$$\frac{4(1-\hat{c})R_1+(2+(2n-1)(1-\hat{c}))R_2}{r_{n-1}\big(2(1-\hat{c})+(2+(2n-3)(1-\hat{c}))(1+2r_{n-1})\big)}\geqslant 0, \tag{28}$$

where

$$R_1=r_{n-1}(1+r_n+r_{n+1})=\frac{(z-1)\big(2z^{4n-5}+Az^{2n-4}(z-1)(z^2+1)-2A^2\big)}{4z^2(z^{2n-2}+A)(z^{2n-6}+A)}>0,$$

$$R_2=r_{n-1}-r_n+2r_{n-1}(r_{n+1}-r_n)=\frac{Az^{2n-7}(z-1)^2(z+1)^3(z^{2n-3}+A)}{2(z^{2n-2}+A)(z^{2n-4}+A)(z^{2n-6}+A)}>0.$$

Moreover, since $z\geqslant\sqrt{2}+1$ and $A<z^{-3}$, we have

$$-(1+2r_{n-1})=(z-1)\left(1-\frac{A(z+1)}{z(z^{2n-6}+A)}\right)>\frac{\sqrt{2}}{2}\quad \text{for } n\geqslant 2.$$

This implies that

$$2(1-\hat{c})+(2+(2n-3)(1-\hat{c}))(1+2r_{n-1})<(1-\hat{c})(2+3(1+2r_{n-1}))<0.$$

It follows that both the numerator and denominator in the left-hand side of (28) are positive for $n\geqslant 2$. Thus, (28) follows.

(d) Finally, in the case $c<0$, we will show that $x_1^S>-1$. By Corollary 1, this implies that $x_1^L>-1$. We need to prove that

$$L>\frac{r_{n+1}}{r_n}\quad \text{for } n\geqslant 2,$$

where L is defined in (27). Since by (18) and (23), we have

$$\frac{r_{n+1}}{r_n}=4\widehat{\beta}_{n+1}<1,$$

and it suffices to show that $L>1$, i.e., that

$$L-1=\frac{4(1-\hat{c})(1+r_{n-1}+r_{n+1})-2\big(2+(2n-1)(1-\hat{c})\big)(r_{n-1}-r_{n+1})}{2(1-\hat{c})+(2+(2n-3)(1-\hat{c}))(1+2r_{n-1})}>0.$$

Since in this case $r_k>0$ for all k, the denominator is positive. Therefore, it remains to prove that

$$4(1-\hat{c})(1+r_{n-1}+r_{n+1})>2\big(2+(2n-1)(1-\hat{c})\big)(r_{n-1}-r_{n+1}),$$

i.e., that

$$\frac{1+r_{n-1}+r_{n+1}}{r_{n-1}-r_{n+1}} > n - \frac{1}{2} + \frac{1}{1-\acute{c}}.$$

Since $\frac{1}{1-\acute{c}} < 1$ (recall that $\acute{c} < 0$), it suffices to show that

$$\frac{r_{n-1}+r_{n+1}}{r_{n-1}-r_{n+1}} > n + \frac{1}{2},$$

which is equivalent to

$$(n + \frac{3}{2})r_{n+1} > (n - \frac{1}{2})r_{n-1}.$$

Actually, we will prove that the sequence $(k + \frac{1}{2})r_k$ is increasing in k. Specifically, we will show that

$$(n + \frac{1}{2})r_n > (n - \frac{1}{2})r_{n-1},$$

which is equivalent to

$$\frac{r_{n-1} + r_n}{r_{n-1} - r_n} > 2n. \tag{29}$$

We use induction over n. The base case is $n = 2$. Then, by (19), (29) simplifies to the trivial inequality

$$2(A + 1)^2 + 5A(z - \frac{1}{z})^2 > 0.$$

For the inductive step, we verify that

$$\frac{r_n + r_{n+1}}{r_n - r_{n+1}} > \frac{r_{n-1} + r_n}{r_{n-1} - r_n} + 2.$$

This multiplies out into $2r_{n-1}r_{n+1} \geqslant r_n(r_{n-1} + r_{n+1})$, and by (19), when expanded and simplified, reduces to

$$Az^{2n-6}(z^2 - 1)^3(z^{2n-2} - A) < 0.$$

This inequality holds because $z^2 > 1$ and $A < 0$, thus proving (29).

Example 2 Table 2 shows the smallest and largest nodes, x_1^L and x_{n+1}^L, respectively, of averaged Gauss quadrature rules Q_{2n+1}^L for several values of n and the parameter c for the measure $d\hat{\lambda}$ defined by (12). As predicted by the theory developed above, the smallest node x_1^L is inside the interval $[-1, 1]$, while the largest node x_{n+1}^L is not.

Example 3 Table 3 displays the extreme nodes x_1^S and x_{n+1}^S of the generalized averaged Gauss quadrature rules Q_{2n+1}^S for the measure $d\hat{\lambda}(x)$ given by (12) for several values of n and c. In agreement with the theory developed above, the smallest node,

Table 2 Example 2: The smallest and largest nodes of averaged Gauss rules Q^L_{2n+1} for the measure $d\hat{\lambda}$ for several values of c and n

c	n	x^L_1	x^L_{n+1}
−0.9	5	−9.56252677710693(−1)	1+2.4854(−6)
−0.9	10	−9.88375642391720(−1)	1+7.1313(−11)
−0.9	15	−9.94729022774889(−1)	1+2.2002(−15)
−0.9	20	−9.97005429844440(−1)	1+7.0983(−20)
−0.9	30	−9.98655844571235(−1)	1+8.0241(−29)
−0.1	5	−9.58856005129239(−1)	1+4.7571(−15)
−0.1	10	−9.88738301958147(−1)	1+2.3852(−28)
−0.1	15	−9.94840523653745(−1)	1+1.5417(−41)
−0.1	20	−9.97053350474925(−1)	1+1.1112(−54)
−0.1	30	−9.98670309976306(−1)	1+6.7814(−81)
0.9	5	−9.89801057715791(−1)	1+1.4330(−6)
0.9	10	−9.95868016309400(−1)	1+2.8739(−11)
0.9	15	−9.97638433414493(−1)	1+7.5439(−16)
0.9	20	−9.98445859437810(−1)	1+2.2173(−20)
0.9	30	−9.99167643851814(−1)	1+2.2594(−29)
1.1	5	−9.90420650413612(−1)	1+2.8099(−6)
1.1	10	−9.96118166775301(−1)	1+1.3923(−10)
1.1	15	−9.97767423021090(−1)	1+9.0197(−15)
1.1	20	−9.98521880161805(−1)	1+6.5413(−19)
1.1	30	−9.99200752415474(−1)	1+4.0571(−27)

Table 3 Example 3: The smallest and largest nodes of the generalized averaged Gauss rule Q^S_{2n+1} for the measure $d\hat{\lambda}$ for several values of c and n

c	n	x^S_1	x^S_{n+1}
−0.9	5	−9.56266728289617(−1)	1+5.2764(−6)
−0.9	10	−9.88375642682738(−1)	1+1.5134(−10)
−0.9	15	−9.94729022774896(−1)	1+4.6694(−15)
−0.9	20	−9.97005429844440(−1)	1+1.5064(−19)
−0.9	30	−9.98655844571235(−1)	1+1.7029(−28)
−0.1	5	−9.58856005129407(−1)	1+9.0860(−14)
−0.1	10	−9.88738301958147(−1)	1+4.5558(−27)
−0.1	15	−9.94840523653745(−1)	1+2.9446(−40)
−0.1	20	−9.97053350474925(−1)	1+2.1224(−53)
−0.1	30	−9.98670309976306(−1)	1+1.2952(−79)
0.9	5	−9.89791207698374(−1)	1−5.9056(−6)
0.9	10	−9.95868016106703(−1)	1−1.1847(−10)
0.9	15	−9.97638433414487(−1)	1−3.1097(−15)
0.9	20	−9.98445859437810(−1)	1−9.1401(−20)
0.9	30	−9.99167643851814(−1)	1−9.3136(−29)
1.1	5	−9.90401924412408(−1)	1−1.1005(−5)
1.1	10	−9.96118165838869(−1)	1−5.4551(−10)
1.1	15	−9.97767423021027(−1)	1−3.5341(−14)
1.1	20	−9.98521880161805(−1)	1−2.5630(−18)
1.1	30	−9.99200752415474(−1)	1−1.5896(−26)

x_1^S, is inside the interval $[-1, 1]$, while the largest node, x_{n+1}^S, is inside this interval only if $c > 0$.

3.3 Internality of the truncated quadrature formula $Q_{n+2}^{(1)}$

This subsection considers the truncated generalized averaged quadrature rule $Q_{n+2}^{(1)}$ that is obtained by removing the last $n-1$ rows and columns of the symmetric tridiagonal matrix of order $2n+1$, whose eigenvalues are the nodes of Q_{2n+1}^S. The nodes of $Q_{n+2}^{(1)}$ are the zeros of the polynomial

$$t_{n+2}(x) = (x - \widehat{\alpha}_{n-1})\widehat{P}_{n+1}(x) - \widehat{\beta}_{n+1}\widehat{P}_n(x).$$

This formula is internal if its smallest zero, x_1^t, and its largest zero, x_{n+2}^t, both lie in $[-1, 1]$. This is equivalent to

$$\frac{(x - \widehat{\alpha}_{n-1})\widehat{P}_{n+1}(x)}{\widehat{\beta}_{n+1}\widehat{P}_n(x)} \geqslant 1 \quad \text{for} \quad x = \pm 1; \tag{30}$$

see, e.g., [7] for a related discussion. By (11) and (14), we have

$$2 \cdot \frac{\widehat{P}_{n+1}(x)}{\widehat{P}_n(x)} = \begin{cases} \frac{1-2r_{n+1}}{1-2r_n} & \text{if } x = 1, \\ -\frac{2(1-\hat{c})+\left(2+(2n+1)(1-\hat{c})\right)(1+2r_{n+1})}{2(1-\hat{c})+\left(2+(2n-1)(1-\hat{c})\right)(1+2r_n)} & \text{if } x = -1. \end{cases} \tag{31}$$

Theorem 6 *The quadrature formula $Q_{n+2}^{(1)}$ is internal for all c and every $n \geqslant 2$.*

Proof We will first show that $x_{n+2}^t \leqslant 1$, i.e., that (30) holds for $x = 1$. It follows from (18) and (31) that this is equivalent to

$$1 \leqslant \frac{(1-\widehat{\alpha}_{n-1})\widehat{P}_{n+1}(1)}{\widehat{\beta}_{n+1}\widehat{P}_n(1)} = 2(1-\widehat{\alpha}_{n-1}) \cdot \frac{r_n(1-2r_{n+1})}{r_{n+1}(1-2r_n)}. \tag{32}$$

By (19), we have

$$\frac{1-2r_k}{r_k} = \frac{1}{r_k} - 2 = -\frac{(z+1)(z^{2n-3}+A)}{z^{2n-2}+A}$$

and, hence, by using (22), it follows that

$$\begin{aligned} \frac{r_n(1-2r_{n+1})}{r_{n+1}(1-2r_n)} &= \frac{(z^{2n-1}+A)(z^{2n-2}+A)}{(z^{2n}+A)(z^{2n-3}+A)} = 1 - \frac{Az^{2n-3}(z-1)^2(z+1)}{(z^{2n}+A)(z^{2n-3}+A)} \\ &\geqslant 1 - \frac{A(z-1)^2(z+1)}{z^{2n}+A} \geqslant 1 - \frac{|z|^{-3}(|z|+1)^3}{z^{2n}+|z|^{-3}} \geqslant \frac{1}{2}. \end{aligned}$$

Moreover, by (23), we have $1-\widehat{\alpha}_{n-1} > 1$, which together with the above inequality shows (32).

We proceed to show that $x_1^t \geqslant -1$, i.e., that (30) holds for $x = -1$. We will first prove that

$$-\frac{\widehat{P}_{n+1}(-1)}{\widehat{P}_n(-1)} > \frac{1+2r_{n+1}}{1+2r_n}. \tag{33}$$

Introduce the quantities

$$a = 2(1-\acute{c}), \quad b = 2+(2n-1)(1-\acute{c}), \quad x = 1+2r_{n+1}, \quad y = 1+2r_n.$$

Then using (31), the inequality (33) can be expressed as

$$\frac{a+(a+b)x}{a+by} > \frac{x}{y},$$

which is equivalent to

$$\frac{a(xy+y-x)}{y(a+by)} > 0.$$

Since a is positive, y and $a+by$ are of the same sign, and

$$xy+y-x = 1+4r_n+4r_nr_{n+1} = -\frac{2(z-1)^2}{z}r_n > 0,$$

the proof of (33) is complete.

The inequality (30) readily follows for $x = -1$ if we can show that

$$\frac{1+\widehat{\alpha}_{n-1}}{\widehat{\beta}_n} \cdot \frac{1+2r_{n+1}}{1+2r_n} = 2(1+r_n-r_{n-1})\frac{r_n(1+2r_{n+1})}{r_{n+1}(1+2r_n)} \geqslant 1.$$

In this case, it follows from (19) that

$$\frac{r_n(1+2r_{n+1})}{r_{n+1}(1+2r_n)} = \frac{(z^{2n-1}-A)(z^{2n-2}+A)}{(z^{2n-3}-A)(z^{2n}+A)} = 1+\frac{Az^{2n-3}(z+1)(z^2-1)}{(z^{2n-3}-A)(z^{2n}+A)} > 1.$$

Therefore, it suffices to show that

$$1+r_n-r_{n-1} \geqslant \frac{1}{2},$$

i.e., that

$$r_{n-1}-r_n \leqslant \frac{1}{2}.$$

The latter inequality is easy to verify. We have

Table 4 Example 4: The two outermost nodes of truncated generalized averaged quadrature rules $Q^{(1)}_{n+2}$ for the measure $d\widehat{\lambda}$ for several values of c and n

c	n	x^t_1	x^t_{n+2}
	5	−9.12569364812230(−1)	9.78480361103346(−1)
	10	−9.68366438732778(−1)	9.92187753480289(−1)
−0.1	15	−9.83850057500048(−1)	9.96001055730301(−1)
	20	−9.90230550263921(−1)	9.97576672634387(−1)
	30	−9.95319162343563(−1)	9.98836424622862(−1)
	5	−9.56573294152220(−1)	9.76882963334918(−1)
	10	−9.83332991704801(−1)	9.91846036346949(−1)
0.9	15	−9.90812105799422(−1)	9.95877013278884(−1)
	20	−9.94057904178699(−1)	9.97518446981950(−1)
	30	−9.96851060157643(−1)	9.98817155910107(−1)
	5	−9.20005057726076(−1)	9.77811816443556(−1)
	10	−9.70252427064411(−1)	9.92042412994509(−1)
2	15	−9.84575257020554(−1)	9.95947976675102(−1)
	20	−9.90580316621724(−1)	9.97551676433712(−1)
	30	−9.95437815551145(−1)	9.98828124534837(−1)

$$
\begin{aligned}
r_{n-1} - r_n &= \frac{Az^{2n-7}(z^2-1)^2}{2(z^{2n-4}+A)(z^{2n-6}+A)} \leqslant \frac{z^{-3}\cdot z^{2n-7}(z^2-1)^2}{2(z^{2n-4}-z^{-3})(z^{2n-6}-z^{-3})} \\
&= \frac{z^{2n-4}(z^2-1)^2}{2(z^{2n-1}-1)(z^{2n-3}-1)} < \frac{1}{2z^{2n-4}} \leqslant \frac{1}{2}.
\end{aligned}
$$

Example 4 Table 4 displays the outermost nodes x^t_1 and x^t_{n+2} of truncated generalized averaged Gauss quadrature formulas $Q^{(1)}_{n+2}$ for the measure $d\widehat{\lambda}$ given by (12) computed in high-precision arithmetic. In agreement with the results above, these nodes are inside the interval $[-1, 1]$, making these rules internal.

Finally, as noted in Section 1, we can obtain analogous results for the Chebyshev measure of the fourth kind (8) by replacing c by $-c$. In particular, the truncated quadrature rule $Q^{(1)}_{n+2}$ is internal for the measure (8), whereas the formula Q^S_{2n+1} is internal if and only if $c < 0$.

4 The numerical performance of the quadrature rules

This section presents a few computed examples that illustrate the application of the quadrature rules Q^L_{2n+1}, Q^S_{2n+1}, and $Q^{(1)}_{n+2}$ to estimate the magnitude of the quadrature error (3) incurred when applying Gauss rules Q^G_n. Specifically, we will evaluate and compare the error estimates

Table 5 Example 5: The error estimates (34) and the actual error "Error" (3) for some values of c and n

c	n	$E_{AG} \equiv E_{GA}$	E_{TGA}	Error	$I(f)$
	5	3.7614(−2)	8.7288(−2)	3.7678(−2)	
	10	1.2289(−4)	1.2269(−4)	1.2289(−4)	
−2	15	2.2774(−11)	2.2768(−11)	2.2774(−11)	4.2632
	20	1.2708(−19)	1.2707(−19)	1.2708(−19)	
	25	5.5115(−29)	5.5113(−29)	5.5115(−29)	
	30	3.1107(−39)	3.1106(−39)	3.1107(−39)	
	5	2.9851(−1)	2.8318(−1)	2.9851(−1)	
	10	2.3656(−5)	2.3636(−5)	2.3656(−5)	
0.1	15	4.2795(−12)	4.2789(−12)	4.2795(−12)	1.5516(−1)
	20	2.4317(−20)	2.4316(−20)	2.4317(−20)	
	25	1.0766(−29)	1.0766(−29)	1.0766(−29)	
	30	6.1886(−40)	6.1885(−40)	6.1886(−40)	
	5	2.7463	2.7105	2.7461	
	10	1.7011(−4)	1.7014(−4)	1.7011(−4)	
0.9	15	3.2741(−11)	3.2742(−11)	3.2741(−11)	5.8310(−1)
	20	1.9706(−19)	1.9706(−19)	1.9706(−19)	
	25	9.1082(−29)	9.1079(−29)	9.1082(−29)	
	30	5.4075(−39)	5.4074(−39)	5.4075(−39)	

$$\begin{aligned} E_{AG} &= |Q_{2n+1}^{L}(f) - Q_n^G(f)|, \\ E_{GA} &= |Q_{2n+1}^{S}(f) - Q_n^G(f)|, \\ E_{TGA} &= |Q_{n+2}^{(1)}(f) - Q_n^G(f)| \end{aligned} \tag{34}$$

with the actual error magnitude (3) for several values of n and parameter c.

Example 5 Consider the evaluation of the integral

$$\int_{-1}^{1} g(x)\,\mathrm{d}x \tag{35}$$

with

$$g(x) = \frac{\exp(3x)\sin(10x)}{x-\delta} \cdot \sqrt{\frac{1+x}{1-x}}, \qquad -1 < x < 1,$$

and δ defined by (10). This integral can be expressed as the integral

$$I(f) = \int_{-1}^{1} f(x)\,\mathrm{d}\widetilde{\lambda}(x), \quad f(x) = \exp(3x)\sin(10x), \tag{36}$$

with an analytic integrand by using the measure (9). We apply the quadrature rules of Section 2 to approximate the integral (36). Table 5 displays the error (3) in the

Table 6 Example 6: The error estimates (34) and the actual error "Error" (3) for some values of c and n

c	n	E_{AG}	E_{GA}	E_{TGA}	Error	$I(f)$
	5	1.4122	1.4122	1.2883	1.4122	
	10	1.5852(−4)	1.5852(−4)	1.5834(−4)	1.5852(−4)	
−2	15	2.8774(−11)	2.8774(−11)	2.8768(−11)	2.8774(−11)	−2.4628(−1)
	20	1.6213(−19)	1.6213(−19)	1.6211(−19)	1.6213(−19)	
	25	7.1183(−29)	7.1183(−29)	7.1180(−29)	7.1183(−29)	
	30	4.0629(−39)	4.0629(−39)	4.0628(−39)	4.0629(−39)	
	5	2.8613	2.8613	2.6992	2.8613	
	10	2.3900(−4)	2.3900(−4)	2.3879(−4)	2.3900(−4)	
0.1	15	4.3211(−11)	4.3211(−11)	4.3204(−11)	4.3211(−11)	1.5953
	20	2.4495(−19)	2.4495(−19)	2.4494(−19)	2.4495(−19)	
	25	1.0822(−28)	1.0822(−28)	1.0822(−28)	1.0822(−28)	
	30	6.2102(−39)	6.2102(−39)	6.2101(−39)	6.2102(−39)	
	5	3.8160	3.8162	3.7047	3.8159	
	10	2.6207(−4)	2.6207(−4)	2.6199(−4)	2.6207(−4)	
0.9	15	4.8845(−11)	4.8845(−11)	4.8843(−11)	4.8845(−11)	1.1435
	20	2.8578(−19)	2.8578(−19)	2.8577(−19)	2.8578(−19)	
	25	1.2940(−28)	1.2940(−28)	1.2940(−28)	1.2940(−28)	
	30	7.5673(−39)	7.5673(−39)	7.5671(−39)	7.5673(−39)	
	5	3.5076	3.5080	3.4008	3.5076	
	10	2.4347(−4)	2.4347(−4)	2.4339(−4)	2.4347(−4)	
1.1	15	4.5301(−11)	4.5301(−11)	4.5298(−11)	4.5301(−11)	1.0848
	20	2.6458(−19)	2.6458(−19)	2.6457(−19)	2.6458(−19)	
	25	1.1964(−28)	1.1964(−28)	1.1964(−28)	1.1964(−28)	
	30	6.9897(−39)	6.9897(−39)	6.9896(−39)	6.9897(−39)	

column labeled "Error" in the Gauss formula $Q_n^G(f)$, as well as the error estimates (34). All the error estimates can be seen to be very accurate.

Example 6 We illustrate the approximation of the integral (35) with the integrand with

$$g(x) = \exp(3x)\sin(10x) \cdot \frac{x-\gamma}{x-\delta} \cdot \sqrt{\frac{1+x}{1-x}}, \qquad -1 < x < 1,$$

and δ and γ defined by (13). We express this integral as

$$I(f) = \int_{-1}^{1} f(x)\, \mathrm{d}\widehat{\lambda}(x) \tag{37}$$

Table 7 Example 7: The error estimates (34) (when available) and the actual error (3), labeled "Error", for some values of c and n

c	n	E_{AG}	E_{GA}	E_{TGA}	Error	$I(f)$
	5	–	–	8.9438(−2)	1.6648(−1)	
	10	–	–	1.3056(−2)	3.7274(−2)	
−0.9	15	–	–	3.9011(−3)	1.4795(−2)	−2.2311
	20	–	–	1.5916(−3)	7.4788(−3)	
	25	–	–	7.7637(−4)	4.3369(−3)	
	30	–	–	4.2578(−4)	2.7505(−3)	
	5	–	1.6426(−2)	8.8773(−3)	1.3939(−2)	
	10	–	2.1786(−3)	7.7454(−4)	1.8396(−3)	
0.5	15	–	6.6314(−4)	1.7548(−4)	5.5956(−4)	−3.3012(−1)
	20	–	2.8392(−4)	5.9798(−5)	2.3953(−4)	
	25	–	1.4672(−4)	2.5659(−5)	1.2377(−4)	
	30	–	8.5446(−5)	1.2775(−5)	7.2081(−5)	
	5	–	6.7877(−3)	3.7641(−3)	5.8056(−3)	
	10	–	8.2640(−4)	2.9972(−4)	7.0359(−4)	
1.1	15	–	2.4410(−4)	6.5609(−5)	2.0767(−4)	3.4925(−1)
	20	–	1.0291(−4)	2.1952(−5)	8.7527(−5)	
	25	–	5.2680(−5)	9.3125(−6)	4.4801(−5)	
	30	–	3.0486(−5)	4.6003(−6)	2.5925(−5)	

with the integrand given by (36) and the measure defined by (12). The quadrature rules of Section 3 are applied to approximate the integral (37). Table 6 shows the error (3) in the Gauss rules $Q_n^G(f)$, in the columns labeled "Error", as well as the error estimates (34). The error estimates are seen to provide very accurate estimates of the actual error (3).

Example 7 We seek to approximate the integral (35) with the integrand

$$g(x) = \ln(2-x)\ln(1-x) \cdot \frac{x-\gamma}{x-\delta} \cdot \sqrt{\frac{1+x}{1-x}}, \qquad -1 < x < 1.$$

The parameters δ and γ are defined by (13). Similarly as above, we simplify the integrand by using the measure (12). Then our task becomes to approximate the integral

$$I(f) = \int_{-1}^{1} f(x)\, \mathrm{d}\hat{\lambda}(x), \quad f(x) = \ln(2-x)\ln(1-x), \tag{38}$$

by quadrature rules of Section 3. Table 7 displays the error (3) in the Gauss rules $Q_n^G(f)$ in the column labeled "Error", as well as computable error estimates (34). We note that the integrand in (38) is not defined for $x > 1$, and recall that the averaged rules Q^L_{2n+1} have a node larger than one for all values of c. They therefore cannot be evaluated. The generalized averaged rules Q^S_{2n+1} have a node larger than one for $c < 0$, but are interior for $c > 0$. They therefore can be used to approximate (38) when $c > 0$. Finally, the truncated rules $Q^{(1)}_{n+2}(f)$ can be evaluated both for positive and negative values of c.

5 Conclusion

This paper studies quadrature rules associated with two kinds of modifications of Chebyshev measures of the third and fourth kinds. The internality of averaged and generalized averaged Gauss rules is established for some measures, as well as for truncated generalized averaged Gauss rules. Computed examples illustrate the theory and show the quality of the computed error estimates.

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Declarations

Conflict of interest The authors declare no competing interests.

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