IMA Journal of Numerical Analysis (2009) **29**, 486–507
doi:10.1093/imanum/drm040
Advance Access publication on May 16, 2008

Error estimates for Gauss–Turán quadratures and their Kronrod extensions

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[Received on 18 March 2007; revised on 29 October 2007]

We study the kernel $K_{n,s}(z)$ of the remainder term $R_{n,s}(f)$ of Gauss–Turán–Kronrod quadrature rules with respect to one of the generalized Chebyshev weight functions for analytic functions. The location on the elliptic contours where the modulus of the kernel attains its maximum value is investigated. This leads to effective L^∞-error bounds of Gauss–Turán–Kronrod quadratures. Following Kronrod, using the modulus of the difference of Gauss–Turán quadratures and their Kronrod extensions, we derive new error estimates for Gauss–Turán quadratures and compare them with the effective L^1-error bounds derived in Milovanović & Spalević (2005, *BIT*, **45**, 117–136).

Keywords: Gauss–Turán quadratures; Kronrod extensions; s-orthogonal polynomials; Stieltjes polynomials; remainder term; error estimate; analytic function.

1. Introduction

Let w be an integrable weight function on the interval $(-1, 1)$.

Gauss–Turán quadrature formulae of the form

$$\int_{-1}^{1} w(t) f(t) \mathrm{d}t = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} \lambda_{i,\nu} f^{(i)}(\tau_\nu) + E_{n,s}(f),$$

or quadrature formulae with the highest degree of algebraic precision with multiple nodes, has extensively been studied in the last decades from both an algebraic and a numerical point of view. Numerically stable methods for constructing nodes τ_ν and coefficients $\lambda_{i,\nu}$ can be found in Milovanović *et al.* (2004) and Shi & Xu (2007). Some interesting theoretical results concerning this theory have recently been obtained (see Shi, 2005; Kroó & Peherstorfer, 2007; and references therein). For more details on

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quadratures with multiple nodes and corresponding orthogonal polynomials, see the books of Ghizzetti & Ossicini (1970) and Engels (1980) and the survey paper of Milovanović (2001).

We consider the error term $R_{n,s}(f)$ of a Gauss–Turán–Kronrod quadrature formula, first introduced in Li (1994),

$$\int_{-1}^{1} w(t) f(t)\mathrm{d}t = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} \sigma_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{\mu=1}^{n+1} K_\mu f(\hat{\tau}_\mu) + R_{n,s}(f),$$

which is exact for all algebraic polynomials of degree less than or equal to $2(s+1)n+n+1$. The nodes τ_ν are the zeros of the corresponding s-orthogonal polynomial $\pi_n(t) \equiv \pi_{n,s}(t)$ of degree n satisfying the condition

$$\int_{-1}^{1} w(t)t^m[\pi_n(t)]^{2s+1}\,\mathrm{d}t = 0, \quad m = 0, 1, \ldots, n-1. \tag{1.1}$$

The nodes $\hat{\tau}_\mu$ are the zeros of the generalized Stieltjes polynomial $\hat{\pi}_{n+1}$ satisfying the condition

$$\int_{-1}^{1} w(t)t^m[\pi_n(t)]^{2s+1}\hat{\pi}_{n+1}(t)\mathrm{d}t = 0, \quad m = 0, 1, \ldots, n.$$

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and $\mathcal{D}$ its interior. If the integrand f is an analytic function in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$, then the remainder term admits the well-known contour integral representation

$$R_{n,s}(f) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma K_{n,s}(z; w) f(z)\mathrm{d}z. \tag{1.2}$$

The kernel is given by

$$K_{n,s}(z; w) = \frac{\rho_{n,s}(z; w)}{[\pi_{n,s}(z; w)]^{2s+1}\hat{\pi}_{n+1}(z; w)}, \quad z \notin [-1, 1], \tag{1.3}$$

where

$$\rho_{n,s}(z; w) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}\hat{\pi}_{n+1}(t)}{z-t} w(t)\mathrm{d}t. \tag{1.4}$$

In view of $K_{n,s}(\bar{z}) = \overline{K_{n,s}(z)}$, the modulus of $K_{n,s}$ is symmetric with respect to the real axis: $|K_{n,s}(\bar{z})| = |K_{n,s}(z)|$.

The integral representation (1.2) leads to a general error estimate, by using Hölder's inequality,

$$|R_{n,s}(f)| = \frac{1}{2\pi}\left|\oint_\Gamma K_{n,s}(z; w) f(z)\mathrm{d}z\right| \leqslant \frac{1}{2\pi}\left(\oint_\Gamma |K_{n,s}(z; w)|^r |\mathrm{d}z|\right)^{1/r}\left(\oint_\Gamma |f(z)|^{r'} |\mathrm{d}z|\right)^{1/r'},$$

i.e.

$$|R_{n,s}(f)| \leqslant \frac{1}{2\pi}\|K_{n,s}\|_r \|f\|_{r'}, \tag{1.5}$$

where $1 \leqslant r \leqslant +\infty$, $1/r + 1/r' = 1$ and

$$\|f\|_r := \begin{cases} \left(\oint_\Gamma |f(z)|^r |\mathrm{d}z|\right)^{1/r}, & 1 \leqslant r < +\infty, \\ \max_{z\in\Gamma} |f(z)|, & r = +\infty. \end{cases}$$

The case $r = +\infty$ ($r' = 1$) gives

$$|R_{n,s}(f)| \leqslant \frac{1}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z;\omega)|\right)\|f\|_1. \tag{1.6}$$

On the other hand, for $r = 1$ ($r' = +\infty$), the estimate (1.5) is reduced to

$$|R_{n,s}(f)| \leqslant \frac{1}{2\pi}\left(\oint_\Gamma |K_{n,s}(z)||\mathrm{d}z|\right)\|f\|_\infty. \tag{1.7}$$

L^1-error bounds of the type (1.7) were considered in Hunter (1995) for Gauss quadrature formulae and in Milovanović & Spalević (2005) for Gauss–Turán formulae.

L^1-error bounds of the type (1.7) for Gauss–Turán–Kronrod quadrature formulae with generalized Chebyshev weight functions were considered in detail in Milovanović & Spalević (2006). In this paper, we consider L^∞-error bounds of the type (1.6) for those quadratures. Also, using the modulus of the difference of Gauss–Turán quadratures and their Kronrod extensions, we derive new error estimates for Gauss–Turán quadratures and compare them with the effective L^1-error bounds for Gauss–Turán quadratures derived in Milovanović & Spalević (2005).

2. L^∞-error bounds for Gauss–Turán–Kronrod quadratures

As a contour Γ, we take an ellipse $\mathcal{E}_\rho$ with foci at the points ± 1 and a sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{z \in \mathbb{C}\colon z = \frac{1}{2}(u + u^{-1}), 0 \leqslant \theta \leqslant 2\pi\right\}, \quad u = \rho\,\mathrm{e}^{i\theta}.$$

In this section, we study the magnitude of $|K_{n,s}(z;\omega)|$ on the contour $\mathcal{E}_\rho$ for generalized Chebyshev weight functions of the first, second and third kind, respectively. The consideration may be restricted to the upper half of $\mathcal{E}_\rho$, i.e. to the interval $\theta \in [0,\pi]$. The case of the generalized Chebyshev weight function of the fourth kind is analogous to the one for the generalized Chebyshev weight function of the third kind, and it will be therefore omitted. We investigate on the elliptic contours the location where the modulus of the kernel attains its maximum value. This leads to effective L^∞-error bounds for Gauss–Turán–Kronrod quadratures. Similar results are hard to obtain for any other weight functions as no simple explicit formulae exist for both the corresponding s-orthogonal polynomials and the associated Stieltjes polynomials.

Classical Gauss–Kronrod quadrature formulae received particular attention in the past 40 years, especially because of their practical use in packages for automatic integration. Concerning this theory, see, for instance, the recent papers by De La Calle Ysern & Peherstorfer (2007) and Spalević (2007). Of particular importance are those Gauss–Kronrod quadrature formulae all of whose nodes are real and whose coefficients are positive. Efficient methods for their computing have been proposed recently by Laurie (1997) and Calvetti *et al.* (2000). The corresponding MATLAB routines are downloadable from the website http://www.cs.purdue.edu/archives/2002/wxg/codes/ which contains a suite of many other useful routines, in part assembled as a companion piece to the book of Gautschi (2004). However, there are some cases when the classical real and positive Gauss–Kronrod quadrature formulae do not exist. This is proved in Peherstorfer & Petras (2000) for the Gauss–Kronrod quadrature formula with respect to the Gegenbauer weight function $(1-t^2)^{\lambda-1/2}$ for $\lambda > 3$, i.e. ω_2 for $s > 2$, and sufficiently large n. Numerical tests (using routines from Gautschi, 2004) suggest the same for Gauss–Kronrod quadratures with respect to w_3 when $s = 2$, $n \geqslant 6$, and when $s = 3, 4, \ldots, 10$, $n \geqslant 2$. In all these cases, Gauss–Turán–Kronrod quadratures can be seen as a real alternative.

The particular case of the modulus of the kernel in Gauss quadrature formulae was analysed in detail in Gautschi & Varga (1983) and Gautschi *et al.* (1990). Concerning this approach, see also Hunter & Nikolov (2000) and Schira (1996, 1997), for Gauss quadrature formulae and Milovanović & Spalević (2003) and Milovanović *et al.* (2007) for Gauss–Turán quadrature formulae. In all those papers, in fact, the error estimate of Gautschi's type, easily derivable from (1.6),

$$|R_{n,s}(f)| \leqslant \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z;w)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{2.1}$$

where $\ell(\Gamma)$ is the length of the contour Γ, was analysed. The error estimate (1.7) is evidently stronger than (2.1) because of the inequality

$$\oint_{\Gamma}|K_{n,s}(z)||\mathrm{d}z| \leqslant \ell(\Gamma)\left(\max_{z\in\Gamma}|K_{n,s}(z;w)|\right).$$

However, error estimates of the type (1.6) or (2.1) remain very effective and simple to use, especially in cases when the maximum of the modulus of the kernel is attained on the real or imaginary axis.

2.1 *The weight function* $w_1(t) = (1-t^2)^{-1/2}$

It is well known that $\pi_{n,s}(t;w_1) = T_n(t)$, for all $s \in \mathbb{N}$. $T_n(t)$ is the Chebyshev polynomial of the first kind of degree n. In Li (1994) it is shown that $\hat{\pi}_{n+1}(t;w_1) = (1-t^2)U_{n-1}(t)$, where $U_{n-1}(t)$ is the Chebyshev polynomial of the second kind of degree $n-1$. We use the following equalities (see Ossicini & Rosati, 1975, equation 4.1):

$$[T_n(t)]^{2s+1} = 2^{-2s}\sum_{k=0}^{s}\binom{2s+1}{s-k}T_{n(2k+1)}(t),$$

$$\int_0^{\pi}\frac{\cos n\theta}{z-\cos\theta}\mathrm{d}\theta = \frac{2\pi}{u-u^{-1}}u^{-n}, \quad n \in \mathbb{N}_0,$$

when $j > 0$,

$$\begin{aligned}
I_j^{(1)} &= \int_0^{\pi}\frac{\sin\theta\sin n\theta\cos(2j+1)n\theta}{z-\cos\theta}\mathrm{d}\theta \\
&= \frac{1}{4}\left[\int_0^{\pi}\frac{\cos[(2j+1)n+(n-1)]\theta}{z-\cos\theta}\mathrm{d}\theta + \int_0^{\pi}\frac{\cos[(2j+1)n-(n-1)]\theta}{z-\cos\theta}\mathrm{d}\theta \right.\\
&\qquad \left. - \int_0^{\pi}\frac{\cos[(2j+1)n+(n+1)]\theta}{z-\cos\theta}\mathrm{d}\theta - \int_0^{\pi}\frac{\cos[(2j+1)n-(n+1)]\theta}{z-\cos\theta}\mathrm{d}\theta\right] \\
&= \frac{1}{4}\frac{2\pi}{u-u^{-1}}\left[\frac{1}{u^{(2j+1)n+n-1}} + \frac{1}{u^{(2j+1)n-n+1}} - \frac{1}{u^{(2j+1)n+n+1}} - \frac{1}{u^{(2j+1)n-n-1}}\right] \\
&= \frac{\pi}{2(u-u^{-1})}\left[\frac{u^2-1}{u^{(2j+1)n+n+1}} - \frac{u^2-1}{u^{(2j+1)n-n+1}}\right] = \frac{\pi}{2}\left[\frac{1}{u^{2(j+1)n}} - \frac{1}{u^{2jn}}\right],
\end{aligned}$$

and $I_0^{(1)} = \dfrac{\pi}{2}\dfrac{1}{u^{2n}}$.

Now we get

$$\rho_{n,s}(z;\omega_1) = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \frac{(1-t^2)U_{n-1}(t)[T_n(t)]^{2s+1}}{z-t}\mathrm{d}t$$
$$= \int_{-1}^{1} \frac{U_{n-1}(t)\sqrt{1-t^2}}{z-t}\left[\frac{1}{2^{2s}}\sum_{j=0}^{s}\binom{2s+1}{s-j}T_{n(2j+1)}(t)\right]\mathrm{d}t.$$

By substituting $t=\cos\theta$, we have, in view of $T_n(\cos\theta)=\cos n\theta$ and $U_{n-1}(\cos\theta)=\sin n\theta/\sin\theta$,

$$\rho_{n,s}(z;\omega_1) = \frac{1}{2^{2s}}\int_0^{\pi}\frac{\sin n\theta}{z-\cos\theta}\left[\sum_{j=0}^{s}\binom{2s+1}{s-j}\cos(2j+1)n\theta\right]\sin\theta\,\mathrm{d}\theta$$
$$= \frac{1}{2^{2s}}\sum_{j=0}^{s}\binom{2s+1}{s-j}I_j^{(1)}$$
$$= \frac{\pi}{2^{2s+1}}\left[\sum_{j=0}^{s}\binom{2s+1}{s-j}\frac{1}{u^{2(j+1)n}} - \sum_{j=1}^{s}\binom{2s+1}{s-j}\frac{1}{u^{2jn}}\right]$$
$$= \frac{\pi}{2^{2s+1}}\left\{\frac{1}{u^{2(s+1)n}} + \sum_{j=0}^{s-1}\left[\binom{2s+1}{s-j} - \binom{2s+1}{s-j-1}\right]\frac{1}{u^{2(j+1)n}}\right\}$$
$$= \frac{\pi}{2^{2s+1}}\frac{1}{u^{2n}}B_{n,s}^{(1)}(u),$$

where

$$B_{n,s}^{(1)}(u) = \sum_{j=0}^{s}k(j)\frac{1}{u^{2jn}},$$

$$k(j) = \begin{cases} \binom{2s+1}{s-j} - \binom{2s+1}{s-j-1}, & j=0,1,\ldots,s-1, \\ 1, & j=s. \end{cases}$$

According to (1.3) and well-known facts $T_n(z) = (u^n+u^{-n})/2$ and $U_{n-1}(z) = (u^n-u^{-n})/(u-u^{-1})$, we get

$$K_{n,s}(z;\omega_1) = \frac{-4\pi}{u^{2n}}\frac{B_{n,s}^{(1)}(u)}{(u-u^{-1})(u^n-u^{-n})(u^n+u^{-n})^{2s+1}}. \tag{2.2}$$

Further, using the equalities

$$|u^n+u^{-n}| = [2(a_{2n}+\cos 2n\theta)]^{1/2}, \quad |u^n-u^{-n}| = [2(a_{2n}-\cos 2n\theta)]^{1/2},$$

where

$$a_j = a_j(\rho) = \frac{1}{2}(\rho^j+\rho^{-j}), \quad j\in\mathbb{N}, \tag{2.3}$$

we get an explicit representation of $|K_{n,s}(z;\, w_1)|$ in the form

$$|K_{n,s}(z;\, w_1)| = \frac{\pi}{2^{s-1/2}\rho^{2n}} \frac{z|B^{(1)}_{n,s}(\rho\, e^{i\theta})|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n} - \cos 2n\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{s+1/2}}, \tag{2.4}$$

where (see Milovanović & Spalević, 2005, Lemma 4.1)

$$|B^{(1)}_{n,s}(\rho\, e^{i\theta})| = \left[\rho^{-2ns}\sum_{j=0}^{s} A_j \cos 2jn\theta\right]^{1/2},$$

with

$$A_0 = \frac{1}{x^{s/2}}\sum_{\nu=0}^{s}[k(s-\nu)]^2 x^\nu, \quad x = \rho^{4n},$$

$$A_j = \frac{2}{x^{(s-j)/2}}\sum_{\nu=0}^{s-j} k(s-\nu)k(s-\nu-j)x^\nu, \quad j = 1, \ldots, s. \tag{2.5}$$

The graphs $\theta \mapsto |K_{n,s}(z;\, w_1)|$ $(z \in \mathcal{E}_\rho)$ for certain values of n, s and ρ are displayed in Fig. 1.

THEOREM 2.1 For every fixed $s \in \mathbb{N}$ and $\rho > 1$, there exists $n_0 = n_0(\rho, s) \in \mathbb{N}$, such that

$$\max_{z\in\mathcal{E}_\rho} |K_{n,s}(z;\, w_1)| = \left|K_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1});\, w_1\right)\right| = \frac{4\pi\, |B^{(1)}_{n,s}(\rho)|}{\rho^{2n}(\rho-\rho^{-1})(\rho^n-\rho^{-n})(\rho^n+\rho^{-n})^{2s+1}}, \tag{2.6}$$

for every $n \geqslant n_0$. When $s = 0$, $K_{n,s}(z;\, w_1)$ attains its maximum modulus on the real axis for every $n \in \mathbb{N}$.

Proof. By (2.4) and (2.5), consideration may be restricted to the first quarter of $\mathcal{E}_\rho$.

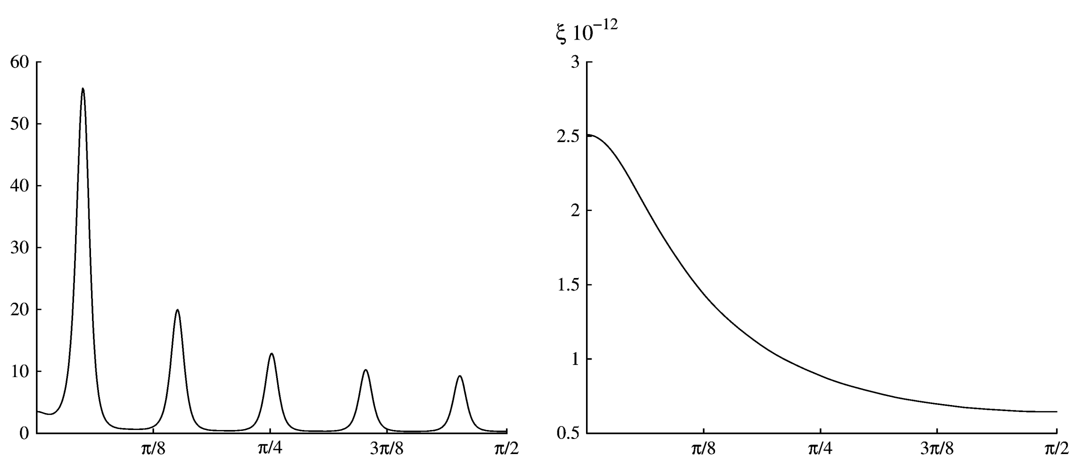

FIG. 1. The functions $\theta \mapsto |K_{10,3}(z;\, w_1)|$ $(z \in \mathcal{E}_{1.05})$ (left) and $\theta \mapsto |K_{15,2}(z;\, w_1)|$ $(z \in \mathcal{E}_{1.3})$ (right).

Because of (2.4), it suffices to prove that

$$\frac{\sum_{j=0}^{s} A_j \cos 2jn\theta}{(a_2 - \cos 2\theta)(a_{2n} - \cos 2n\theta)(a_{2n} + \cos 2n\theta)^{2s+1}} \leqslant \frac{\sum_{j=0}^{s} A_j}{(a_2 - 1)(a_{2n} - 1)(a_{2n} + 1)^{2s+1}},$$

for sufficiently large n, $\theta \in (0, \pi/2]$ and $s \in \mathbb{N}$, where a_j are given by (2.3) and A_k by (2.5). When $s = 0$, the last inequality obviously holds for each $n \in \mathbb{N}$.

The last inequality is reduced to

$$\left(\sum_{j=0}^{s} A_j - 2\sum_{j=1}^{s} A_j \sin^2 jn\theta\right)(a_2 - 1)(a_{2n} - 1)(a_{2n} + 1)^{2s+1}$$
$$\leqslant \left(\sum_{j=0}^{s} A_j\right)[(a_2 - 1) + 2\sin^2\theta][(a_{2n} - 1) + 2\sin^2 n\theta][(a_{2n} + 1)^{2s+1} - 2E_{n,s}(\rho, \theta)\sin^2 n\theta],$$

where (see Milovanović *et al.*, 2007, Thoerem 2.1)

$$E_{\rho,s}(n, \theta) = \sum_{j=1}^{2s+1} (-2)^{j-1}\binom{2s+1}{j}(a_{2n} + 1)^{2s+1-j}\sin^{2j-2} n\theta \quad (\geqslant 0).$$

Further, we have

$$(a_2 - 1)(a_{2n} - 1)(a_{2n} + 1)^{2s+1}\sum_{j=0}^{s} A_j - 2(a_2 - 1)(a_{2n} - 1)(a_{2n} + 1)^{2s+1}\sum_{j=1}^{s} A_j \sin^2 jn\theta$$
$$\leqslant \{(a_2 - 1)(a_{2n} - 1)(a_{2n} + 1)^{2s+1} - 2(a_{2n} - 1)(a_2 - 1)E_{n,s}(\rho, \theta)\sin^2 n\theta$$
$$+ [2(a_{2n} - 1)\sin^2\theta + 2(a_2 - 1)\sin^2 n\theta + 4\sin^2\theta\sin^2 n\theta](a_{2n} + \cos 2n\theta)^{2s+1}\}\sum_{j=0}^{s} A_j.$$

If $\sin n\theta = 0$, the last inequality obviously holds. After dividing this by $2\sin^2 n\theta \sum_{j=0}^{s} A_j$ and denoting $Q = \sum_{j=0}^{s} A_j$, it is reduced to

$$(a_{2n} + \cos 2n\theta)^{2s+1}\left[\frac{\sin^2\theta}{\sin^2 n\theta}(a_{2n} - 1) + (a_2 - 1) + 2\sin^2\theta\right] - (a_2 - 1)(a_{2n} - 1)E_{n,s}(\rho, \theta)$$
$$+ \frac{A_1}{Q}(a_2 - 1)(a_{2n} - 1)(a_{2n} + 1)^{2s+1} + \frac{1}{Q}\left(\sum_{j=2}^{s} A_j \frac{\sin^2 jn\theta}{\sin^2 n\theta}\right)$$
$$\times (a_2 - 1)(a_{2n} - 1)(a_{2n} + 1)^{2s+1} \geqslant 0. \tag{2.7}$$

In Milovanović *et al.* (2007, Theorem 2.1) it is shown that

$$E_{\rho,s}(n, \theta) \leqslant \sum_{j=0}^{s} 4^j\binom{2s+1}{2j+1}(a_{2n} + 1)^{2s-2j} - \frac{1}{2}\sum_{j=1}^{s} 4^j\binom{2s+1}{2j}(a_{2n} + 1)^{2s-2j+1}. \tag{2.8}$$

Using (2.8) and the well-known fact $|\sin n\theta/\sin\theta| \leqslant n$, we conclude that the left-hand side of (2.7) is greater than or equal to $(a_{2n}-1)G_{\rho,s}(n)$, where

$$G_{\rho,s}(n) = \frac{A_1}{Q}(a_2-1)(a_{2n}+1)^{2s+1} + \frac{1}{n^2}(a_{2n}-1)^{2s+1} + (a_2-1)\left\{(a_{2n}-1)^{2s} - \left[\sum_{j=0}^{s} 4^j \binom{2s+1}{2j+1}(a_{2n}+1)^{2s-2j} - \frac{1}{2}\sum_{j=1}^{s} 4^j \binom{2s+1}{2j}(a_{2n}+1)^{2s-2j+1}\right]\right\}.$$

Since $G_{\rho,s}(n)$ (ρ and s are fixed) is continuous when $n \geqslant 1$ and $\lim_{n\to+\infty} G_{\rho,s}(n) = +\infty$, it follows that $G_{\rho,s}(n) > 0$ for each $n > r$, where r is the largest zero of $G_{\rho,s}(n)$. For n_0 we can take $[r]+1$. □

The proof of Theorem 2.1 is not only of theoretical but also of practical importance. We can use the function $G_{\rho,s}(n)$ from the proof to estimate n_0. Numerical values of $[r]+1$ (r is the largest zero of $G_{\rho,s}$) for certain values of ρ and s are presented in Table 1. The smallest possible (s.p.) values of n_0 are also presented. We can see that the s.p. n_0 is very well estimated by $[r]+1$.

A typical graph illustrating the relationship between n and $G_{\rho,s}(n)$ is displayed in Fig. 2 (right).

THEOREM 2.2 For every fixed $s \in \mathbb{N}$ and $n \in \mathbb{N}$ there exists $\rho_0 = \rho_0(n,s) > 1$, such that

$$\max_{z\in\mathcal{E}_\rho} |K_{n,s}(z;\omega_1)| = \left|K_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1});\omega_1\right)\right|,$$

for every $\rho > \rho_0$.

Proof. We can repeat the same computation which led to (2.7), where we can fix n and let ρ be a variable. Since $G_{n,s}(\rho)$ is continuous when $\rho > 1$, and $\lim_{\rho\to+\infty} G_{n,s}(\rho) = +\infty$ (n and s are fixed), it follows that $G_{n,s}(\rho) > 0$ for each $\rho > t$, where t is the largest zero of $G_{n,s}(\rho)$. For ρ_0, we can take t. □

We can use the function $G_{n,s}(\rho)$ from the proof to estimate ρ_0. Numerical values of t (t is the largest zero of $G_{n,s}$) for some values of n and s are presented in Table 2. The s.p. values of ρ_0 are also presented. We can see that the s.p. ρ_0 is estimated by t very well.

A typical graph illustrating the relationship between ρ and $G_{n,s}(\rho)$ is displayed in Fig. 2 (left).

TABLE 1 *Numerical values of* $[r]+1$ *and the s.p. values of* n_0

	$s=3$		$s=6$	
ρ	$[r]+1$	s.p. n_0	$[r]+1$	s.p. n_0
1.04	63	60	77	76
1.07	37	35	45	44
1.1	26	25	32	32
1.2	14	13	17	17
1.5	7	6	8	8
1.9	4	4	5	5

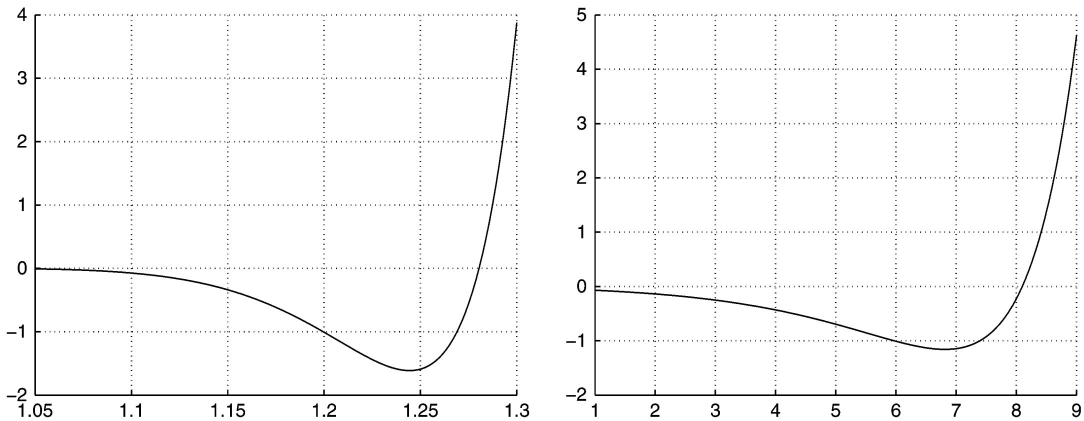

FIG. 2. The functions $G_{6,1}(\rho)$ (left) and $G_{1.2,1}(n)$ (right).

TABLE 2 *The s.p. values of ρ_0 and numerical values of t*

	$s=2$		$s=5$	
n	s.p. ρ_0	t	s.p. ρ_0	t
3	1.9611	2.1003	2.7746	2.8150
4	1.6226	1.7179	2.0861	2.1131
5	1.4619	1.5335	1.7824	1.8021
7	1.3057	1.3529	1.5018	1.5144
10	1.2033	1.2342	1.3264	1.3344
14	1.1407	1.1617	1.2227	1.2280

Using the fact $|B_{n,s}^{(1)}(\rho)| \leqslant |B_{n,s}^{(1)}(1)| = \binom{2s+1}{s}$, we can simplify (2.6) and obtain

$$\left|K_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1}); w_1\right)\right| \leqslant 4\pi\binom{2s+1}{s}\frac{1}{\rho^{2n}(\rho-\rho^{-1})(\rho^n-\rho^{-n})(\rho^n+\rho^{-n})^{2s+1}}. \tag{2.9}$$

From (1.6) and (2.9), we conclude that $|R_{n,s}(f)| = \mathrm{O}(\rho^{-(2n(s+2)+1)})$, when $n \to +\infty$.

When $\rho < \rho_0(n,s)$ or $n < n_0(\rho,s)$, using the inequalities

$$(a_2 - \cos 2\theta)(a_{2n} + \cos 2n\theta) \geqslant (a_2-1)(a_{2n}+1), \quad 0 \leqslant \theta \leqslant \pi/2,$$

and $|B_{n,s}^{(1)}(\rho\,\mathrm{e}^{\mathrm{i}\theta})| \leqslant |B_{n,s}^{(1)}(\rho)|$, we get the following crude estimate:

$$|K_{n,s}(z; w_1)| \leqslant \left|K_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1}); w_1\right)\right|\left(\frac{\rho^n+\rho^{-n}}{\rho^n-\rho^{-n}}\right)^{2s}, \quad z \in \mathcal{E}_\rho.$$

2.2 *The weight function* $w_2(t) = (1-t^2)^{s+1/2}$

It is well known that $\pi_{n,s}(t; w_2) = U_n(t)$. In Milovanović & Spalević (2006) it is shown that $\hat{\pi}_{n+1}(t; w_2) = T_{n+1}(t)$. We use the following equalities (see Ossicini & Rosati, 1975, equation 4.2):

$$(1-t^2)^s[U_n(t)]^{2s+1} = 2^{-2s}\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s-k}U_{n(2k+1)+2k}(t),$$

$$\int_0^\pi \frac{\sin\theta\sin(n+1)\theta}{z-\cos\theta}\mathrm{d}\theta = \frac{\pi}{u^{n+1}}, \quad n\in\mathbb{N}_0, \tag{2.10}$$

when $j>0$,

$$\begin{aligned} I_j^{(2)} &= \int_0^\pi \frac{\sin\theta\cos(n+1)\theta\sin(2j+1)(n+1)\theta}{z-\cos\theta}\mathrm{d}\theta \\ &= \frac{1}{2}\left[\int_0^\pi \frac{\sin\theta\sin[2(j+1)(n+1)]\theta}{z-\cos\theta}\mathrm{d}\theta + \int_0^\pi \frac{\sin\theta\sin[2j(n+1)]\theta}{z-\cos\theta}\mathrm{d}\theta\right] \\ &= \frac{\pi}{2}\left[\frac{1}{u^{2(j+1)(n+1)}} + \frac{1}{u^{2j(n+1)}}\right], \end{aligned}$$

and $I_0^{(2)} = \dfrac{\pi}{2}\dfrac{1}{u^{2(n+1)}}$.

Now we get

$$\begin{aligned} \rho_{n,s}(z; w_2) &= \int_{-1}^{1}(1-t^2)^{s+1/2}\frac{[U_n(t)]^{2s+1}T_{n+1}(t)}{z-t}\mathrm{d}t \\ &= \int_{-1}^{1}\sqrt{1-t^2}\{(1-t^2)^s[U_n(t)]^{2s+1}\}\frac{T_{n+1}(t)}{z-t}\mathrm{d}t \\ &= \frac{1}{2^{2s}}\int_{-1}^{1}\sqrt{1-t^2}\frac{T_{n+1}(t)}{z-t}\left[\sum_{j=0}^{s}(-1)^j\binom{2s+1}{s-j}U_{n(2j+1)+2j}(t)\right]\mathrm{d}t. \end{aligned}$$

By substituting $t=\cos\theta$, we have, in view of $T_n(\cos\theta)=\cos n\theta$ and $U_{n-1}(\cos\theta) = \sin n\theta/\sin\theta$,

$$\begin{aligned} \rho_{n,s}(z; w_2) &= \frac{1}{2^{2s}}\int_0^\pi \frac{\cos(n+1)\theta}{z-\cos\theta}\left[\sum_{j=0}^{s}(-1)^j\binom{2s+1}{s-j}\sin(2j+1)(n+1)\theta\right]\sin\theta\,\mathrm{d}\theta \\ &= \frac{1}{2^{2s}}\sum_{j=0}^{s}(-1)^j\binom{2s+1}{s-j}I_j^{(2)} \\ &= \frac{\pi}{2^{2s+1}}\left[\sum_{j=0}^{s}(-1)^j\binom{2s+1}{s-j}\frac{1}{u^{2(j+1)(n+1)}} + \sum_{j=1}^{s}(-1)^j\binom{2s+1}{s-j}\frac{1}{u^{2j(n+1)}}\right] \\ &= \frac{\pi}{2^{2s+1}}\left\{\frac{(-1)^s}{u^{2(s+1)(n+1)}} + \sum_{j=0}^{s-1}(-1)^j\left[\binom{2s+1}{s-j} - \binom{2s+1}{s-j-1}\right]\frac{1}{u^{2(j+1)(n+1)}}\right\} \\ &= \frac{\pi}{2^{2s+1}}\frac{1}{u^{2(n+1)}}B_{n,s}^{(2)}(u), \end{aligned}$$

where

$$B_{n,s}^{(2)}(u) = \sum_{j=0}^{s} (-1)^j k(j) \frac{1}{u^{2j(n+1)}}.$$

According to (1.3), we get

$$K_{n,s}(z; w_2) = \frac{\pi}{2^{2s}} \left[\frac{u - u^{-1}}{u^{n+1} - u^{-(n+1)}} \right]^{2s+1} \frac{B_{n,s}^{(2)}(u)}{u^{2(n+1)}(u^{n+1} + u^{-(n+1)})}. \tag{2.11}$$

In the same way as in the previous case, we get the explicit representation of $|K_{n,s}(z; w_2)|$,

$$|K_{n,s}(z; w_2)| = \frac{\pi}{2^{2s+1/2}\rho^{2(n+1)}} \left[\frac{a_2 - \cos 2\theta}{a_{2n+2} - \cos(2n+2)\theta} \right]^{s+1/2} \frac{|B_{n,s}^{(2)}(\rho\, e^{i\theta})|}{[a_{2n+2} + \cos(2n+2)\theta]^{1/2}}. \tag{2.12}$$

In the proof of the next theorem, we use the following two lemmas and the function

$$\overline{B}_{n,s,\ell}(u) = \sum_{j=0}^{s-\ell} (-1)^j k(j+\ell) u^{-2(n+1)j}, \quad s \in \mathbb{N}_0, \ \ \ell = 0, 1, \ldots, s. \tag{2.13}$$

LEMMA 2.1 For $s \in \mathbb{N}_0$, $\ell = 0, 1, \ldots, s$, n odd and $r > 1$, we have that $\overline{B}_{n,s,\ell}(r) = \overline{B}_{n,s,\ell}(ri) > 0$.

Proof. We prove this by induction. For $s = 0$, we have $\overline{B}_{n,0,0}(u) = 1$. To pass from $s - 1$ to s, we use the following equalities:

$$\begin{aligned}
\overline{B}_{n,s,0}(u) &= (2 - u^{-2(n+1)})\overline{B}_{n,s-1,0}(u) + \overline{B}_{n,s-1,1}(u), \\
\overline{B}_{n,s,\ell}(u) &= \overline{B}_{n,s-1,\ell-1}(u) + 2\overline{B}_{n,s-1,\ell}(u) + \overline{B}_{n,s-1,\ell+1}(u), \quad \ell = 1, \ldots, s-2, \\
\overline{B}_{n,s,s-1}(u) &= \overline{B}_{n,s-1,s-2}(u) + 2, \\
\overline{B}_{n,s,s}(u) &= 1,
\end{aligned} \tag{2.14}$$

which follow from the well-known identity

$$\binom{2s+1}{k} = \binom{2s-1}{k} + 2\binom{2s-1}{k-1} + \binom{2s-1}{k-2}.$$

□

LEMMA 2.2 For $s \in \mathbb{N}_0$, n odd, $\rho > 1$ and $\ell = 0, 1, \ldots, s$, we have that

$$\frac{|\overline{B}_{n,s,\ell}(\rho\, e^{\theta i})|}{(a_{2n+2} - \cos(2n+2)\theta)^{s/2}} \leqslant \frac{\overline{B}_{n,s,\ell}(\rho i)}{(a_{2n+2} - 1)^{s/2}}$$

.

Proof. This lemma can also be proved by induction. The statement for $s = 0$ is obvious because of $\overline{B}_{n,0,0}(u) = 1$. We demonstrate the induction step for the case $\ell = 0$. The other cases ($\ell = 1, 2, \ldots, s-2$ and $\ell = s-1$) are similar.

First, we prove

$$\frac{|2(\rho\,\mathrm{e}^{\theta i})^{2n+2}-1|}{(a_{2n+2}-\cos(2n+2)\theta)^{1/2}} \leqslant \frac{2(\rho i)^{2n+2}-1}{(a_{2n+2}-1)^{1/2}}. \tag{2.15}$$

Using the notations $r=\rho^{2n+2}$ and $\phi=(2n+2)\theta$ we get

$$\frac{|2(\rho\,\mathrm{e}^{\theta i})^{2n+2}-1|}{(a_{2n+2}-\cos(2n+2)\theta)^{1/2}} = \sqrt{\frac{4r^2-4r\cos\phi+1}{1/2(r+r^{-1})-\cos\phi}} = \sqrt{4r+\frac{2r^2-1}{1/2(r+r^{-1})-\cos\phi}},$$

from which we see that the left-hand side of (2.15) attains its maximum when $\theta=\pi/2$.

Using (2.14) and (2.15) we get

$$\begin{aligned}\frac{|\overline{B}_{n,s,0}(\rho\,\mathrm{e}^{\theta i})|}{(a_{2n+2}-\cos(2n+2)\theta)^{s/2}} &\leqslant \frac{|\overline{B}_{n,s-1,0}(\rho\,\mathrm{e}^{\theta i})|}{(a_{2n+2}-\cos(2n+2)\theta)^{(s-1)/2}}\cdot\frac{|2(\rho\,\mathrm{e}^{\theta i})^{2n+2}-1|}{\rho^{2n+2}(a_{2n+2}-\cos(2n+2)\theta)^{1/2}}\\ &\quad+\frac{|\overline{B}_{n,s-1,1}(\rho\,\mathrm{e}^{\theta i})|}{(a_{2n+2}-\cos(2n+2)\theta)^{(s-1)/2}}\cdot\frac{1}{(a_{2n+2}-\cos(2n+2)\theta)^{1/2}}\\ &\leqslant \frac{\overline{B}_{n,s-1,0}(\rho i)}{(a_{2n+2}-1)^{(s-1)/2}}\cdot\frac{2(\rho i)^{2n+2}-1}{(\rho i)^{2n+2}(a_{2n+2}-1)^{1/2}}\\ &\quad+\frac{\overline{B}_{n,s-1,1}(\rho i)}{(a_{2n+2}-1)^{(s-1)/2}}\cdot\frac{1}{(a_{2n+2}-1)^{1/2}}=\frac{\overline{B}_{n,s,0}(\rho i)}{(a_{2n+2}-1)^{s/2}}.\end{aligned}$$

□

THEOREM 2.3 $|K_{n,s}(z;w_2)|$ attains its maximum on the imaginary axis when n is odd, i.e.

$$\max_{z\in\mathcal{E}_\rho}|K_{n,s}(z;w_2)| = \left|K_{n,s}\left(\frac{i}{2}(\rho-\rho^{-1});w_2\right)\right| = \frac{\pi|B_{n,s}^{(2)}(\rho i)|(\rho+\rho^{-1})^{2s+1}}{4^s\rho^{2(n+1)}(\rho^{n+1}+\rho^{-(n+1)})(\rho^{n+1}-\rho^{-(n+1)})^{2s+1}}. \tag{2.16}$$

When n is even, for all $s\in\mathbb{N}_0$ and $\rho>1$, there exists an even $n_0=n_0(\rho,s)\in\mathbb{N}$, such that $|K_{n,s}(z;w_2)|$ attains its maximum on the imaginary axis for each $n\geqslant n_0$. The maximum is given by

$$\max_{z\in\mathcal{E}_\rho}|K_{n,s}(z;w_2)| = \frac{\pi|B_{n,s}^{(2)}(\rho i)|(\rho+\rho^{-1})^{2s+1}}{4^s\rho^{2(n+1)}(\rho^{n+1}-\rho^{-(n+1)})(\rho^{n+1}+\rho^{-(n+1)})^{2s+1}}. \tag{2.17}$$

Proof. For n odd, note that $B_{n,s}^{(2)}=\overline{B}_{n,s,0}$. By (2.12) and Lemma 2.2, it suffices to prove that the function

$$H(\theta)=\frac{(a_2-\cos 2\theta)^{s+1/2}}{[a_{2n+2}-\cos(2n+2)\theta]^{s/2}[a_{2n+2}^2-\cos^2(2n+2)\theta]}$$

attains its maximum at $\theta=\pi/2$, which is obvious.

The case when n is even can be proved in the same way as Theorem 2.4 from Milovanović *et al.* (2007). All we have to change is the parameter α. The new values of α are

$$\alpha=\frac{k(2\nu+1)}{k(2\nu)},\quad \alpha>0.$$

□

In the same way as Theorem 2.2 we can prove the next theorem.

THEOREM 2.4 For every even n and every $s \in \mathbb{N}_0$, there exists $\rho_0 = \rho_0(n,s) > 1$, such that $|K_{n,s}(z;w_2)|$ attains its maximum on the imaginary axis for each $\rho > \rho_0$.

Similarly as in the Section 2.1, we can simplify (2.16) and (2.17). We obtain

$$\left|K_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1});w_2\right)\right| \leqslant \frac{\pi}{4^s}\binom{2s+1}{s}\frac{1}{\rho^{2(n+1)}(\rho^{n+1}+\rho^{-(n+1)})}\left[\frac{\rho+\rho^{-1}}{\rho^{n+1}-\rho^{-(n+1)}}\right]^{2s+1} \tag{2.18}$$

and conclude that $|R_{n,s}(f)| = \mathrm{O}(\rho^{-(2n(s+2)+3)})$, when $n \to +\infty$.

When the maximum point is not on the imaginary axis (n is even, $\rho < \rho_0(n,s)$ or $n < n_0(\rho,s)$), using the inequalities

$$(a_{2n+2}-\cos(2n+2)\theta)(a_{2n+2}+\cos(2n+2)\theta) \geqslant (a_{2n+2}-1)(a_{2n+2}+1),$$

and $|B_{n,s}^{(2)}(\rho\,\mathrm{e}^{\mathrm{i}\theta})| \leqslant |B_{n,s}^{(2)}(\rho\mathrm{i})|$, we get the following crude estimate:

$$|K_{n,s}(z;w_2)| \leqslant \left|K_{n,s}\left(\frac{i}{2}(\rho-\rho^{-1});w_2\right)\right|\left(\frac{\rho^{n+1}+\rho^{-(n+1)}}{\rho^{n+1}-\rho^{-(n+1)}}\right)^{2s}, \quad z \in \mathcal{E}_\rho.$$

2.3 *The weight function* $w_3(t) = (1-t)^{-1/2}(1+t)^{s+1/2}$

Let $V_n(t)$ and $W_n(t)$ represent Chebyshev polynomials of degree n of the third and fourth kind, respectively.

It is well known that $\pi_{n,s}(t;w_3) = V_n(t)$. In Milovanović & Spalević (2006) it is shown that $\hat{\pi}_{n+1}(t;w_3) = (1-t)W_n(t)$. We use the following equality (see Ossicini & Rosati, 1975, equation 4.6),

$$(1+t)^s[V_n(t)]^{2s+1} = \frac{1}{2^s}\sum_{j=0}^{s}\binom{2s+1}{s-j}V_{n(2j+1)+j}(t),$$

and (2.10), when $j > 0$,

$$\begin{aligned} I_j^{(3)} &= \int_0^\pi \frac{\sin\theta\cos[(2j+1)n+j+1/2]\theta\sin(n+1/2)\theta}{z-\cos\theta}\mathrm{d}\theta \\ &= \frac{1}{2}\left[\int_0^\pi \frac{\sin\theta\sin[(2j+1)n+j+n+1]\theta}{z-\cos\theta}\mathrm{d}\theta - \int_0^\pi \frac{\sin\theta\sin[(2j+1)n+j-n]\theta}{z-\cos\theta}\mathrm{d}\theta\right] \\ &= \frac{\pi}{2}\left[\frac{1}{u^{(2j+1)n+j+n+1}} - \frac{1}{u^{(2j+1)n+j-n}}\right], \end{aligned}$$

and $I_0^{(3)} = \dfrac{\pi}{2}\left[\dfrac{1}{u^{(2j+1)n+j+n+1}}\right]$.

Now we get

$$\begin{aligned} \rho_{n,s}(z;w_3) &= \int_{-1}^{1}\frac{(1+t)^{s+1/2}}{(1-t)^{1/2}}\frac{(1-t)[V_n(t)]^{2s+1}W_n(t)}{z-t}\mathrm{d}t \\ &= \int_{-1}^{1}\sqrt{1-t^2}\frac{W_n(t)}{z-t}\left[\frac{1}{2^s}\sum_{j=0}^{s}\binom{2s+1}{s-j}V_{n(2j+1)+j}(t)\right]\mathrm{d}t. \end{aligned}$$

By substituting $t = \cos\theta$, in view of

$$V_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos\theta/2}, \quad W_n(\cos\theta) = \frac{\sin(n+1/2)\theta}{\sin\theta/2},$$

we have

$$\begin{aligned}
\rho_{n,s}(z; \omega_3) &= \frac{1}{2^s}\int_0^\pi \frac{\sin^2\theta}{z-\cos\theta}\frac{\sin(n+1/2)\theta}{\sin\theta/2}\left[\sum_{j=0}^{s}\binom{2s+1}{s-j}\frac{\cos[(2j+1)n+j+1/2]\theta}{\cos\theta/2}\right]\mathrm{d}\theta \\
&= \frac{1}{2^{s-1}}\sum_{j=0}^{s}\binom{2s+1}{s-j}I_j^{(3)} \\
&= \frac{\pi}{2^s}\left[\sum_{j=0}^{s}\binom{2s+1}{s-j}\frac{1}{u^{(2j+1)n+j+n+1}} - \sum_{j=1}^{s}\binom{2s+1}{s-j}\frac{1}{u^{(2j+1)n+j-n}}\right] \\
&= \frac{\pi}{2^s}\left\{\frac{1}{u^{(2s+1)n+s+n+1}} + \sum_{j=0}^{s-1}\left[\binom{2s+1}{s-j} - \binom{2s+1}{s-j-1}\right]\frac{1}{u^{(2j+1)n+j+n+1}}\right\} \\
&= \frac{\pi}{2^s}\frac{1}{u^{2n+1}}B_{n,s}^{(3)}(u),
\end{aligned}$$

where

$$B_{n,s}^{(3)}(u) = \sum_{j=0}^{s}k(j)\frac{1}{u^{(2n+1)j}}.$$

According to (1.3) and the well-known facts

$$V_n(z) = \frac{u^{n+1}+u^{-n}}{u+1}, \quad W_n(z) = \frac{u^{n+1}-u^{-n}}{u-1},$$

we get

$$K_{n,s}(z; \omega_3) = \frac{-\pi}{2^{s-1}}\left(\frac{u+1}{u^{n+1}+u^{-n}}\right)^{2s+1}\frac{B_{n,s}^{(3)}(u)}{u^{2n}(u-1)(u^{n+1}-u^{-n})}. \tag{2.19}$$

As in the two previous cases, we get an explicit representation of $|K_{n,s}(z; \omega_3)|$:

$$|K_{n,s}(z; \omega_3)| = \frac{\sqrt{2}\pi}{2^s\rho^{2n+1}}\frac{(a_1+\cos\theta)^{s+1}}{(a_2-\cos 2\theta)^{1/2}}\frac{|B_{n,s}^{(3)}(\rho\,\mathrm{e}^{\mathrm{i}\theta})|}{(a_{2n+1}-\cos(2n+1)\theta)^{1/2}(a_{2n+1}+\cos(2n+1)\theta)^{s+1/2}}. \tag{2.20}$$

THEOREM 2.5 For every fixed $\rho > 1$ and $s \in \mathbb{N}$, there exists $n_0 = n_0(\rho, s) \in \mathbb{N}$, such that

$$\begin{aligned}
\max_{z\in\mathcal{E}_\rho}|K_{n,s}(z; \omega_3)| &= \left|K_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1}); \omega_3\right)\right| \\
&= \frac{\pi}{2^{s-1}}\frac{|B_{n,s}^{(3)}(\rho)|}{\rho^{2n}(\rho-1)(\rho^{n+1}-\rho^{-n})}\left(\frac{\rho+1}{\rho^{n+1}+\rho^{-n}}\right)^{2s+1},
\end{aligned} \tag{2.21}$$

for every $n \geqslant n_0$. When $s = 0$, $K_{n,s}(z; \omega_3)$ attains its maximum modulus on the positive real axis for every $n \in \mathbb{N}$.

Proof. Because of (2.20), it suffices to prove that

$$\frac{(a_1 + \cos\theta)^{s+1}|B_{n,s}^{(3)}(\rho\, e^{i\theta})|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} - \cos(2n+1)\theta)^{1/2}(a_{2n+1} + \cos(2n+1)\theta)^{1/2+s}}$$

$$\leqslant \frac{(a_1 + 1)^{s+1}|B_{n,s}^{(3)}(\rho)|}{(a_2 - 1)^{1/2}(a_{2n+1} - 1)^{1/2}(a_{2n+1} + 1)^{1/2+s}},$$

for sufficiently large n ($n \geqslant n_0(n, s)$) and $\theta \in (0, \pi]$ (for each $n \in \mathbb{N}$ when $s = 0$).

It is obvious that

$$(a_1 + \cos\theta)^{s+1} \leqslant (a_1 + 1)^{s+1}. \tag{2.22}$$

On the basis of the results from the proof of Theorem 2.1, we have

$$\frac{|B_{n,s}^{(3)}(\rho\, e^{i\theta})|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} - \cos(2n+1)\theta)^{1/2}(a_{2n+1} + \cos(2n+1)\theta)^{1/2+s}}$$

$$= \frac{|B_{n+1/2,s}^{(1)}(\rho\, e^{i\theta})|}{(a_2 - \cos 2\theta)^{1/2}[a_{2(n+1/2)} - \cos(2(n+1/2))\theta]^{1/2}[a_{2(n+1/2)} + \cos(2(n+1/2))\theta]^{1/2+s}}$$

$$\leqslant \frac{B_{n+1/2,s}^{(1)}(\rho)}{(a_2 - 1)^{1/2}(a_{2(n+1/2)} - 1)^{1/2}(a_{2(n+1/2)} + 1)^{1/2+s}}$$

$$= \frac{B_{n,s}^{(3)}(\rho)}{(a_2 - 1)^{1/2}(a_{2n+1} - 1)^{1/2}(a_{2n+1} + 1)^{1/2+s}}.$$

The last inequality holds for sufficiently large n when $s > 0$ and each n ($\in \mathbb{N}$) when $s = 0$, and with (2.22) that completes the proof. For n_0, we can take $[(2r - 1)/2] + 1$, where r is the largest zero of $G_{\rho,s}(n)$. □

In the same way as Theorem 2.2, we can prove the following result.

THEOREM 2.6 For every fixed $s \in \mathbb{N}$ and $n \in \mathbb{N}$, there exists $\rho_0 = \rho_0(n, s) > 1$, such that $|K_{n,s}(z; \omega_3)|$ attains its maximum on the positive real axis for all $\rho > \rho_0$.

In the same way as in the Sections 2.1 and 2.2, we can show that $|R_{n,s}(f)| = O(\rho^{-(2n(s+2)+2)})$, when $n \to +\infty$.

In the cases when the maximum point is not on the positive real axis ($\rho < \rho_0(n, s)$ or $n < n_0(\rho, s)$), using the inequalities

$$\frac{a_1 + \cos\theta}{\sqrt{a_2 - \cos 2\theta}\sqrt{a_{2n+1} + \cos(2n+1)\theta}} < \frac{a_1 + 1}{\sqrt{a_2 - 1}\sqrt{a_{2n+1} + 1}}, \quad 0 < \theta \leqslant \pi,$$

and $|B_{n,s}^{(3)}(\rho\, e^{i\theta})| \leqslant |B_{n,s}^{(3)}(\rho)|$, we get the following crude estimate:

$$|K_{n,s}(z; \omega_3)| \leqslant \left|K_{n,s}\left(\frac{1}{2}(\rho + \rho^{-1}); \omega_3\right)\right| \left(\frac{\rho^{n+1/2} + \rho^{-(n+1/2)}}{\rho^{n+1/2} - \rho^{-(n+1/2)}}\right)^{2s}, \quad z \in \mathcal{E}_\rho.$$

3. Error estimates for Gauss–Turán quadratures

A very popular method for obtaining a practical error estimate in numerical integration is to use two quadrature formulae A and B, where the nodes used by formula B form a proper subset of those used by formula A, and where rule A is also of higher degree of precision. Kronrod (1964a,b) originated this method, which has been used many times to date. For more details, see, e.g. Monegato (1982, 2001), Laurie (1996) and Spalević (2007). The difference $|A(f) - B(f)|$, i.e. $|R^{(A)}(f) - R^{(B)}(f)|$ where f is the integrand, is usually quite a good estimate of the error for the rule B. Following this idea, taking Kronrod extensions of Gauss–Turán quadratures (K) as rule A and Gauss–Turán quadratures (GT) as rule B, we derive new error estimates for Gauss–Turán quadratures which have the same form (const $\cdot \|f\|_\infty$) as the effective L^1-error bounds from Milovanović & Spalević (2005).

The following integrals for $k \in \mathbb{N}_0$ and $a > 1$ will be useful in this section:

$$\tilde{J}_k(a) = \int_0^\pi \frac{\cos k\theta}{(a+\cos\theta)^{2s+1}(a-\cos\theta)}\mathrm{d}\theta, \quad M_k(\rho) = \frac{1}{\pi}\int_0^\pi (a_1 \pm \cos\theta)^k\,\mathrm{d}\theta.$$

In Milovanović & Spalević (2006, Lemma 3.1) it is shown that

$$\tilde{J}_k(a) = \frac{\pi x^{s+1}}{x^2-1}\left[\frac{x^{k/2}+x^{-k/2}}{(x+1)^{2s}} - \frac{2^{2s}\sqrt{x}}{(x-1)^{2s}}\sum_{\nu=0}^{2s} x^{\nu/2}\frac{h_\nu^{(1)}+h_\nu^{(2)}}{2^\nu(2s-\nu)!}\right.$$
$$\left.\times\left(\sum_{\ell=0}^{\nu}(-2)^\ell\binom{2s+\ell}{\ell}(x-1)^{-\ell}\sum_{p=0}^{\nu-\ell}2^p(x+1)^{-p}\right)\right],$$

where

$$a = \frac{x+1}{2\sqrt{x}}, \quad x > 1, \quad h_\nu^{(1)} = (-1)^{\nu+k+1}\frac{(2s+1+k)!}{(\nu+k+1)!}x^{-(\nu+k+1)/2}$$

and

$$h_\nu^{(2)} = \begin{cases} (-1)^{\nu-k+1}\frac{(2s+1-k)!}{(\nu-k+1)!}x^{-(\nu-k+1)/2}, & k \leqslant \nu+1, \\ 0, & k > \nu+1. \end{cases}$$

In the rest of the paper, we use

$$\tilde{J}_0(a) = \frac{2\pi x^{s+1}}{(x-1)^2}\left[\frac{x-1}{(x+1)^{2s+1}} + \left(\frac{2}{x-1}\right)^{2s}\cdot\psi(s,x)\right],$$

where

$$\psi(s,x) = \sum_{\nu=1}^{s}\frac{(-1)^\nu}{2^\nu}\binom{2s+1}{\nu+1}\sum_{\ell=0}^{\nu}(-1)^\ell\binom{2s+\ell}{\ell}\left(\frac{2}{x-1}\right)^\ell\left(1-\left(\frac{2}{x+1}\right)^{\nu-\ell+1}\right).$$

From Gradshteyn & Ryzhik (2000, equations 3.661.3 and 3.616.1), we have that

$$M_k(\rho) = \left(\frac{\rho-\rho^{-1}}{2}\right)^k P_k\left(\frac{\rho+\rho^{-1}}{\rho-\rho^{-1}}\right) = (2\rho)^{-k}\sum_{\nu=0}^{k}\binom{k}{\nu}^2\rho^{2\nu},$$

where P_k is the Legendre polynomial of degree k.

3.1 *The weight function* $w_1(t) = (1 - t^2)^{-1/2}$

First, we derive a connection between the kernel $K_{n,s}^{(\mathrm{GT})}(z; w_1)$ from the Gauss–Turán quadrature formula and the kernel $K_{n,s}^{(\mathrm{K})}(z; w_1)$ from the Kronrod extension of the Gauss–Turán quadrature formula. The term $\rho_{n,s}(z; w_1) = \rho_{n,s}^{(\mathrm{K})}(z; w_1)$ can be written in another form as

$$\rho_{n,s}(z; w_1) = \frac{\pi}{2^{2s+1}} \left[\frac{u^{-n} - u^n}{u^n} Z_{n,s}^{(1)}(u) + \binom{2s+1}{s} \right],$$

where (see Milovanović & Spalević, 2003)

$$Z_{n,s}^{(1)}(u) = \sum_{j=0}^{s} \binom{2s+1}{s-j} \frac{1}{u^{2jn}}.$$

According to (1.3), we get

$$K_{n,s}^{(\mathrm{K})}(z; w_1) = K_{n,s}^{(\mathrm{GT})}(z; w_1) + \frac{\frac{\pi}{2^{2s+1}}\binom{2s+1}{s}}{(1 - z^2)[T_n(z)]^{2s+1} U_{n-1}(z)}. \tag{3.1}$$

THEOREM 3.1 For the remainder term of the Gauss–Turán quadrature formula $R_{n,s}^{(\mathrm{GT})}(f; w_1) = R_{n,s}(f; w_1)$ and the remainder term of the Kronrod extension of the Gauss–Turán quadrature formula $R_{n,s}^{(\mathrm{K})}(f; w_1) = E_{n,s}(f; w_1)$, we have that

$$|R_{n,s}^{(\mathrm{K})}(f; w_1) - R_{n,s}^{(\mathrm{GT})}(f; w_1)| \leqslant V_{n,s,\rho}^{(1)} \|f\|_\infty, \quad |R_{n,s}^{(\mathrm{K})}(f; w_1) - R_{n,s}^{(\mathrm{GT})}(f; w_1)| \leqslant W_{n,s,\rho}^{(1)} \|f\|_\infty,$$

where

$$V_{n,s,\rho}^{(1)} = \frac{\sqrt{\pi}}{2^s} \binom{2s+1}{s} \sqrt{\tilde{J}_0(a_{2n})}, \quad W_{n,s,\rho}^{(1)} = 2\pi \binom{2s+1}{s} \frac{1}{(\rho^n + \rho^{-n})(\rho^n - \rho^{-n})^{2s+1}}.$$

Proof. According to (1.2), we obtain

$$\begin{aligned} |R_{n,s}^{(\mathrm{K})}(f; w_1) - R_{n,s}^{(\mathrm{GT})}(f; w_1)| &= \frac{1}{2\pi} \left| \oint_\Gamma [K_{n,s}^{(\mathrm{K})}(z; w_1) - K_{n,s}^{(\mathrm{GT})}(z; w_1)] f(z) \mathrm{d}z \right| \\ &\leqslant \frac{1}{2\pi} \|K_{n,s}^{(\mathrm{K})}(z; w_1) - K_{n,s}^{(\mathrm{GT})}(z; w_1)\|_1 \|f\|_\infty. \end{aligned}$$

Further, we get

$$\begin{aligned} &\oint_{\mathcal{E}_\rho} |K_{n,s}^{(\mathrm{K})}(z; w_1) - K_{n,s}^{(\mathrm{GT})}(z; w_1)||\mathrm{d}z| \\ &= \frac{\pi}{2^{2s+1}} \binom{2s+1}{s} \oint_{\mathcal{E}_\rho} \frac{1}{|1 - z^2||T_n(z)|^{2s+1}|U_{n-1}(z)|} |\mathrm{d}z| \\ &= \frac{\pi}{2^s} \binom{2s+1}{s} \int_0^{2\pi} \frac{1}{(a_{2n} - \cos 2n\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{s+1/2}} \mathrm{d}\theta \\ &= \frac{\pi}{2^{s-1}} \binom{2s+1}{s} \int_0^{\pi} \sqrt{\frac{1}{(a_{2n} - \cos\theta)(a_{2n} + \cos\theta)^{2s+1}}} \mathrm{d}\theta. \end{aligned}$$

We can continue applying Hölder's inequality with different choices of the parameters r and r', such that $1/r + 1/r' = 1$. The case $r = r' = 2$ gives the bound with the factor $V^{(1)}_{n,s,\rho}$, whereas the case $r = \infty$ and $r' = 1$ gives the bound with the factor $W^{(1)}_{n,s,\rho}$. □

The bound with $V^{(1)}_{n,s,\rho}$ is sharper than the bound with $W^{(1)}_{n,s,\rho}$, whereas the factor $W^{(1)}_{n,s,\rho}$ is simpler and exhibits more clearly its behaviour with respect to ρ and n. As is seen from Fig. 3, the factors $V^{(1)}_{n,s,\rho}$ and $W^{(1)}_{n,s,\rho}$ (as functions of ρ) very quickly become the same as the upper bound on $L_{n,s}(\mathcal{E}_\rho)$ in (4.18) from Theorem 4.3 in Milovanović & Spalević (2005). Numerical tests also suggest that $V^{(1)}_{n,s,\rho}$ is greater than or equal to $L_{n,s}(\mathcal{E}_\rho)$, but less than or equal to the upper bound on $L_{n,s}(\mathcal{E}_\rho)$ mentioned above, i.e. $V^{(1)}_{n,s,\rho}$ can be seen as a new upper bound on $L_{n,s}(\mathcal{E}_\rho)$ which is sharper than the bound from Milovanović & Spalević (2005). The same can be said in the Sections 3.2 and 3.3.

3.2 *The weight function* $w_2(t) = (1 - t^2)^{s+1/2}$

We derive a connection between the kernel $K^{(GT)}_{n,s}(z; w_2)$ from the Gauss–Turán quadrature formula and the kernel $K^{(K)}_{n,s}(z; w_2)$ from the Kronrod extension of the Gauss–Turán quadrature formula. The term $\rho_{n,s}(z; w_2)$ can be written in another form as

$$\rho_{n,s}(z; w_2) = \frac{\pi}{2^{2s+1}}\left[\frac{u^{n+1} + u^{-(n+1)}}{u^{n+1}} Z^{(2)}_{n,s}(u) - \binom{2s+1}{s}\right],$$

where (see Milovanović & Spalević, 2003)

$$Z^{(2)}_{n,s}(u) = \sum_{j=0}^{s} (-1)^j \binom{2s+1}{s-j} \frac{1}{u^{2j(n+1)}}.$$

According to (1.3), we get

$$K^{(K)}_{n,s}(z; w_2) = K^{(GT)}_{n,s}(z; w_2) - \frac{\frac{\pi}{2^{2s+1}}\binom{2s+1}{s}}{[U_n(z)]^{2s+1} T_{n+1}(z)}. \tag{3.2}$$

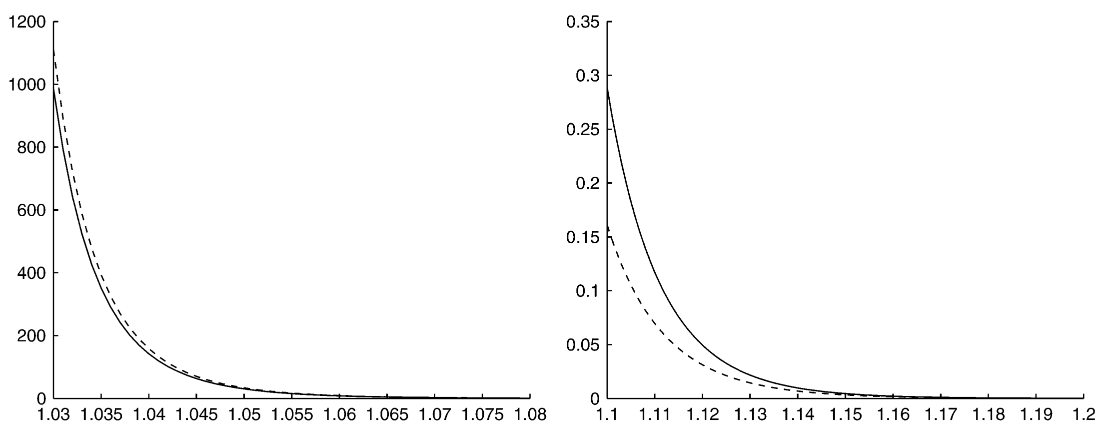

FIG. 3. The upper bound on $L_{n,s}(\mathcal{E}_\rho)$ (dashed line) and $V^{(1)}_{n,s,\rho}$ (solid line, left) ($W^{(1)}_{n,s,\rho}$ right) as functions of ρ when $n = 10$ and $s = 3$.

THEOREM 3.2 For the remainder term of the Gauss–Turán quadrature formula $R_{n,s}^{(GT)}(f;\omega_2)$ and the remainder term of the Kronrod extension of the Gauss–Turán quadrature formula $R_{n,s}^{(K)}(f;\omega_2)$, we have that

$$|R_{n,s}^{(K)}(f;\omega_2)-R_{n,s}^{(GT)}(f;\omega_2)|\leqslant V_{n,s,\rho}^{(2)}\|f\|_\infty,\quad |R_{n,s}^{(K)}(f;\omega_2)-R_{n,s}^{(GT)}(f;\omega_2)|\leqslant W_{n,s,\rho}^{(2)}\|f\|_\infty,$$

where

$$V_{n,s,\rho}^{(2)}=\frac{\sqrt{\pi}}{2^{2s+1}}\binom{2s+1}{s}\sqrt{M_{2s+2}(\rho^2)}\sqrt{\tilde{J}_0(a_{2n+2})},$$
$$W_{n,s,\rho}^{(2)}=\frac{\pi}{2^s}\binom{2s+1}{s}\frac{M_{s+1}(\rho^2)}{(\rho^{n+1}+\rho^{-(n+1)})(\rho^{n+1}-\rho^{-(n+1)})^{2s+1}}.$$

Proof. According to (1.2), we get

$$\begin{aligned}|R_{n,s}^{(K)}(f;\omega_2)-R_{n,s}^{(GT)}(f;\omega_2)|&=\frac{1}{2\pi}\left|\oint_\Gamma[K_{n,s}^{(K)}(z;\omega_2)-K_{n,s}^{(GT)}(z;\omega_2)]f(z)\mathrm{d}z\right|\\&\leqslant\frac{1}{2\pi}\|K_{n,s}^{(K)}(z;\omega_2)-K_{n,s}^{(GT)}(z;\omega_2)\|_1\|f\|_\infty.\end{aligned}$$

Further, we get

$$\begin{aligned}&\oint_{\mathcal{E}_\rho}|K_{n,s}^{(K)}(z;\omega_2)-K_{n,s}^{(GT)}(z;\omega_2)||\mathrm{d}z|\\
&=\frac{\pi}{2^{2s+1}}\binom{2s+1}{s}\oint_{\mathcal{E}_\rho}\frac{1}{|U_n(z)|^{2s+1}|T_{n+1}(z)|}|\mathrm{d}z|\\
&=\frac{\pi}{2^{2s+1}}\binom{2s+1}{s}\int_0^{2\pi}\frac{(a_2-\cos 2\theta)^{s+1}}{(a_{2n+2}-\cos(2n+2)\theta)^{s+1/2}(a_{2n+2}+\cos(2n+2)\theta)^{1/2}}\mathrm{d}\theta\\
&\leqslant\frac{\pi}{2^{2s+1}}\binom{2s+1}{s}\left(\int_0^{2\pi}(a_2-\cos 2\theta)^{2s+2}\,\mathrm{d}\theta\right)^{1/2}\\
&\quad\times\left(\int_0^{2\pi}\frac{1}{(a_{2n+2}-\cos(2n+2)\theta)^{2s+1}(a_{2n+2}+\cos(2n+2)\theta)}\mathrm{d}\theta\right)^{1/2}\\
&=\frac{\pi}{2^{2s}}\binom{2s+1}{s}\left(\int_0^{\pi}(a_2-\cos\theta)^{2s+2}\,\mathrm{d}\theta\right)^{1/2}\left(\int_0^{\pi}\frac{1}{(a_{2n+2}-\cos\theta)^{2s+1}(a_{2n+2}+\cos\theta)}\mathrm{d}\theta\right)^{1/2}\\
&=\frac{\pi^{3/2}}{2^{2s}}\binom{2s+1}{s}\sqrt{M_{2s+2}(\rho^2)}\sqrt{\tilde{J}_0(a_{2n+2})}.\end{aligned}$$

Similarly as in the proof of the previous theorem, applying Hölder's inequality with $r=1$ and $r'=\infty$, we obtain the bound with the factor $W_{n,s,\rho}^{(2)}$. □

3.3 *The weight function* $w_3(t) = (1-t)^{-1/2}(1+t)^{s+1/2}$

We derive a connection between the kernel $K_{n,s}^{(\mathrm{GT})}(z; w_3)$ from the Gauss–Turán quadrature formula and the kernel $K_{n,s}^{(\mathrm{K})}(z; w_3)$ from the Kronrod extension of the Gauss–Turán quadrature formula. The term $\rho_{n,s}(z; w_3)$ can be written in another form as

$$\rho_{n,s}(z; w_3) = \frac{\pi}{2^s}\left[\frac{1-u^{2n+1}}{u^{2n+1}} Z_{n,s}^{(3)}(u) + \binom{2s+1}{s}\right],$$

where (see Milovanović & Spalević, 2003)

$$Z_{n,s}^{(3)}(u) = \sum_{j=0}^{s} \binom{2s+1}{s-j} \frac{1}{u^{j(2n+1)}}.$$

According to (1.3), we get

$$K_{n,s}^{(\mathrm{K})}(z; w_3) = K_{n,s}^{(\mathrm{GT})}(z; w_3) + \frac{\frac{\pi}{2^s}\binom{2s+1}{s}}{(1-z)[V_n(z)]^{2s+1} W_{n+1}(z)}. \tag{3.3}$$

THEOREM 3.3 For the remainder term of the Gauss–Turán quadrature formula $R_{n,s}^{(\mathrm{GT})}(f; w_3)$ and the remainder term of the Kronrod extension of the Gauss–Turán quadrature formula $R_{n,s}^{(\mathrm{K})}(f; w_3)$, we have that

$$|R_{n,s}^{(\mathrm{K})}(f; w_3) - R_{n,s}^{(\mathrm{GT})}(f; w_3)| \leqslant V_{n,s,\rho}^{(3)} \|f\|_\infty, \quad |R_{n,s}^{(\mathrm{K})}(f; w_3) - R_{n,s}^{(\mathrm{GT})}(f; w_3)| \leqslant W_{n,s,\rho}^{(3)} \|f\|_\infty,$$

where

$$V_{n,s,\rho}^{(3)} = \frac{\sqrt{\pi}}{2^s}\binom{2s+1}{s}\sqrt{M_{2s+2}(\rho)}\sqrt{\tilde{J}_0(a_{2n+1})},$$

$$W_{n,s,\rho}^{(3)} = 2\pi\binom{2s+1}{s}\frac{M_{s+1}(\rho)}{(\rho^{n+1/2}+\rho^{-(n+1/2)})(\rho^{n+1/2}-\rho^{-(n+1/2)})^{2s+1}}.$$

Proof. According to (1.2), we get

$$\begin{aligned}|R_{n,s}^{(\mathrm{K})}(f; w_3) - R_{n,s}^{(\mathrm{GT})}(f; w_3)| &= \frac{1}{2\pi}\left|\oint_\Gamma [K_{n,s}^{(\mathrm{K})}(z; w_3) - K_{n,s}^{(\mathrm{GT})}(z; w_3)] f(z)\mathrm{d}z\right| \\ &\leqslant \frac{1}{2\pi}\|K_{n,s}^{(\mathrm{K})}(z; w_3) - K_{n,s}^{(\mathrm{GT})}(z; w_3)\|_1 \|f\|_\infty.\end{aligned}$$

Further, we get

$$\begin{aligned}&\oint_{\mathcal{E}_\rho} |K_{n,s}^{(\mathrm{K})}(z; w_3) - K_{n,s}^{(\mathrm{GT})}(z; w_3)||\mathrm{d}z| \\ &= \frac{\pi}{2^s}\binom{2s+1}{s}\oint_{\mathcal{E}_\rho}\frac{1}{|1-z||V_n(z)|^{2s+1}|W_n(z)|}|\mathrm{d}z|\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{2^s}\binom{2s+1}{s}\int_0^{2\pi}\frac{(a_1+\cos\theta)^{s+1}}{(a_{2n+1}+\cos(2n+1)\theta)^{s+1/2}(a_{2n+1}-\cos(2n+1)\theta)^{1/2}}\,\mathrm{d}\theta\\
&\leqslant \frac{\pi}{2^s}\binom{2s+1}{s}\left(\int_0^{2\pi}(a_1+\cos\theta)^{2s+2}\,\mathrm{d}\theta\right)^{1/2}\\
&\quad\times\left(\int_0^{2\pi}\frac{1}{(a_{2n+1}+\cos(2n+1)\theta)^{2s+1}(a_{2n+1}-\cos(2n+1)\theta)}\,\mathrm{d}\theta\right)^{1/2}\\
&= \frac{\pi}{2^{s-1}}\binom{2s+1}{s}\left(\int_0^{\pi}(a_1+\cos\theta)^{2s+2}\,\mathrm{d}\theta\right)^{1/2}\left(\int_0^{\pi}\frac{1}{(a_{2n+1}+\cos\theta)^{2s+1}(a_{2n+1}-\cos\theta)}\,\mathrm{d}\theta\right)^{1/2}\\
&= \frac{\pi^{3/2}}{2^{s-1}}\binom{2s+1}{s}\sqrt{M_{2s+2}(\rho)}\sqrt{\tilde{J}_0(a_{2n+1})}.
\end{aligned}
$$

Similarly as in the proofs of the previous two theorems, applying Hölder's inequality with $r = 1$ and $r' = \infty$, we obtain the bound with the factor $W^{(2)}_{n,s,\rho}$. □

REMARK 3.1 Concerning some interesting historical details on Gauss–Kronrod quadrature rules, see Gautschi (2005).

Acknowledgments

We are grateful to the referees for suggestions, which have improved the first version of this paper.

Funding

Swiss National Science Foundation (SCOPES Joint Research Project No. IB7320-111079 'New Methods for Quadrature'); Serbian Ministry of Science (Research Projects: 'Approximation of linear operators' (No. #144005) & 'Orthogonal systems and applications' (No. #144004C)).

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