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MAXIMUM OF THE MODULUS OF KERNELS IN GAUSS-TURÁN QUADRATURES

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Abstract. We study the kernels $K_{n,s}(z)$ in the remainder terms $R_{n,s}(f)$ of the Gauss-Turán quadrature formulae for analytic functions on elliptical contours with foci at ± 1, when the weight ω is a generalized Chebyshev weight function. For the generalized Chebyshev weight of the first (third) kind, it is shown that the modulus of the kernel $|K_{n,s}(z)|$ attains its maximum on the real axis (positive real semi-axis) for each $n \geq n_0,\ n_0 = n_0(\rho, s)$. It was stated as a conjecture in [*Math. Comp.* **72** (2003), 1855–1872]. For the generalized Chebyshev weight of the second kind, in the case when the number of the nodes n in the corresponding Gauss-Turán quadrature formula is even, it is shown that the modulus of the kernel attains its maximum on the imaginary axis for each $n \geq n_0,\ n_0 = n_0(\rho, s)$. Numerical examples are included.

1. Introduction

We consider the Gauss-Turán quadrature formula with multiple nodes

$$\int_{-1}^{1} f(t)\omega(t)dt = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f) \quad (n \in \mathbb{N};\ s \in \mathbb{N}_0) \tag{1.1}$$

where ω is a nonnegative and integrable function on the interval $(-1, 1)$, which is exact for all algebraic polynomials of degree at most $2(s+1)n-1$. The nodes τ_ν in (1.1) must be zeros of the s-orthogonal polynomials with respect to the weight function $\omega(t)$. The s-orthogonal polynomials $\pi_n = \pi_{n,s}$ with respect to the weight function $\omega(t)$ are polynomials which satisfy the following orthogonality conditions:

$$\int_{-1}^{1} \pi_n(t)^{2s+1} t^k \omega(t)dt = 0\,, \qquad k = 0, 1, \ldots, n-1.$$

Numerically stable methods for constructing nodes τ_ν and coefficients $A_{i,\nu}$ can be found in [3, 10, 13]. For more details on quadrature formulae with multiple nodes see [7] and [9].

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Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and let D be its interior. If the integrand f is analytic on D and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (1.1) admits the contour integral representation (see [14], [11])

$$R_{n,s}(f) = \frac{1}{2\pi i}\oint_\Gamma K_{n,s}(z)f(z)dz. \tag{1.2}$$

The *kernel* is given by $K_{n,s}(z) = \rho_{n,s}(z)/[\pi_{n,s}(z)]^{2s+1}$, $z \notin [-1,1]$, where

$$\rho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t}\,\omega(t)dt.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_{n,s}(\overline{z})| = |K_{n,s}(z)|$. If the weight function in (1.1) is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(-\overline{z})| = |K_{n,s}(z)|$ (see [11, Lemma 2.1]).

A particularly interesting case is the Chebyshev weight $\omega_1(t) = (1-t^2)^{-1/2}$. In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\,dt \qquad (k \ge 0).$$

This means that the Chebyshev polynomials T_n are s-orthogonal on $(-1,1)$ for each $s \ge 0$. Ossicini and Rosati [14] found three other weights $\omega_k(t)$ ($k = 2,3,4$) for which the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third and fourth kind: U_n, V_n, and W_n, which are defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad V_n(\cos\theta) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \quad W_n(\cos\theta) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta},$$

respectively (cf. Gautschi and Notaris [4]). However, these weights depend on s,

$$\omega_2(t) = (1-t^2)^{1/2+s}, \quad \omega_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}}, \quad \omega_4(t) = \frac{(1-t)^{1/2+s}}{(1+t)^{1/2}}.$$

It is easy to see that $W_n(-t) = (-1)^n V_n(t)$, so that in the investigation it is sufficient to study only the first three generalized Chebyshev weights $\omega_k(t)$, $k = 1,2,3$.

The integral representation (1.2) leads directly to the error estimate

$$|R_{n,s}| \le \frac{l(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right),$$

where $l(\Gamma)$ denotes the length of the contour Γ. First maximum depends only on the quadrature rule (i.e., on ω) and not on f. The first unified approach described above was taken by Donaldson and Elliot [2]. They applied it to several kinds of interpolatory and non-interpolatory quadrature rules. Error bounds for Gaussian quadratures of analytic functions were studied by Gautschi and Varga [5] (see also [6]), and later by Schira [15, 16], Hunter and Nikolov [8].

As a contour Γ we take an ellipse $\mathcal{E}_\rho$ with foci at points ± 1 and a sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} :\ z = \frac{1}{2}\left(u + u^{-1}\right), \quad 0 \le \theta \le 2\pi \right\}, \quad u = \rho e^{i\theta}.$$

When $\rho \to 1$, the ellipse shrinks to the interval $[-1,1]$, while with increasing ρ it becomes more and more circle-like.

When ω is the generalized Chebyshev weight of the first (third) kind, it is conjectured, on the basis of numerical experiments (see [11]), that the modulus of the kernel attains its maximum on the real axis (positive real semi-axis) for each $n \geq n_0,\ n_0 = n_0(\rho, s)$.

In this paper we prove those conjectures. Moreover, for the generalized Chebyshev weight of the second kind, in the case when the number of the nodes n in the corresponding Gauss-Turán quadrature formula is even, we show that the modulus of the kernel attains its maximum on the imaginary axis for each $n \geq n_0,\ n_0 = n_0(\rho, s)$. Numerical examples are included.

2. The maximum modulus of the kernel on confocal ellipses

We study the magnitude of $|K_{n,s}(z)|$ on the contour $\mathcal{E}_\rho$ for the generalized Chebyshev weight functions of the first, second and third kind, respectively. The particular case $|K_{n,0}(z)|$ was analyzed in details by Gautschi et al. [5, 6].

2.1. The weight function $\omega_1(t) = (1-t^2)^{-1/2}$.

An explicit representation of the kernel $K_{n,s}^{(1)}(z)$ on the ellipse $\mathcal{E}_\rho$ for the weight function $\omega_1(t)$ was given by Milovanović and Spalević in [11], as well as

$$\left|K_{n,s}^{(1)}(z)\right| = \frac{2^{1-s}\pi}{\rho^n} \frac{\left|Z_{n,s}^{(1)}(\rho e^{i\theta})\right|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{1/2+s}}, \quad z \in \mathcal{E}_\rho, \tag{2.1}$$

where

$$a_j = a_j(\rho) = \frac{1}{2}\left(\rho^j + \rho^{-j}\right), \quad j \in \mathbb{N}, \tag{2.2}$$

and

$$Z_{n,s}^{(1)}(u) = \sum_{k=0}^{s} \binom{2s+1}{s+k+1} u^{-2nk} = \sum_{k=0}^{s} \binom{2s+1}{k} u^{-2n(s-k)}. \tag{2.3}$$

The weight function $\omega_1(t)$ is even, so we can take $\theta \in [0, \pi/2]$.

The following result was conjectured in [11]:

Theorem 2.1. *For each fixed $\rho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\rho, s)$ such that*

$$\max_{z \in \mathcal{E}_\rho} \left|K_{n,s}^{(1)}(z)\right| = K_{n,s}^{(1)}\left(\frac{1}{2}(\rho + \rho^{-1})\right)$$

for each $n \geq n_0$.

Proof. The inequality $|Z_{n,s}^{(1)}(\rho e^{i\theta})| \leq Z_{n,s}^{(1)}(\rho)$ immediately follows from (2.3). Because of that and (2.1), it is sufficient to prove

$$\frac{1}{(a_2 - \cos 2\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{1/2+s}} \leq \frac{1}{(a_2 - 1)^{1/2}(a_{2n} + 1)^{1/2+s}} \tag{2.4}$$

for a sufficiently large n ($n \geq n_0(\rho, s)$) and $\theta \in (0, \pi/2]$, where a_j are given by (2.2).

By squaring (2.4) it is reduced to

$$(a_2 - 1)(a_{2n} + 1)^{2s+1} \leq (a_2 - \cos 2\theta)(a_{2n} + \cos 2n\theta)^{2s+1}. \tag{2.5}$$

The following transformation will be used

$$a_2 - \cos 2\theta = (a_2 - 1) + 2\sin^2\theta. \tag{2.6}$$

Further, we will use

$$\begin{aligned}(a_{2n}+\cos 2n\theta)^{2s+1} &= \left((a_{2n}+1)-2\sin^2 n\theta\right)^{2s+1}\\ &= (a_{2n}+1)^{2s+1}+\sum_{k=1}^{2s+1}(-2)^k\binom{2s+1}{k}(a_{2n}+1)^{2s+1-k}\sin^{2k} n\theta,\end{aligned}$$

i.e.,

$$(a_{2n}+\cos 2n\theta)^{2s+1}=(a_{2n}+1)^{2s+1}-2(\sin^2 n\theta)E_{\rho,s}(n,\theta), \tag{2.7}$$

where

$$E_{\rho,s}(n,\theta)=\sum_{k=1}^{2s+1}(-2)^{k-1}\binom{2s+1}{k}(a_{2n}+1)^{2s+1-k}\sin^{2k-2} n\theta \quad (\geq 0).$$

It is easy to see that $E_{\rho,s}(n,\theta)$ can be represented in the form

$$E_{\rho,s}(n,\theta)=(2s+1)(a_{2n}+1)^{2s}+\sum_{k=2}^{2s+1}(-2)^{k-1}\binom{2s+1}{k}(a_{2n}+1)^{2s+1-k}\sin^{2k-2} n\theta,$$

i.e.,

$$\begin{aligned}E_{\rho,s}(n,\theta) &= (2s+1)(a_{2n}+1)^{2s}\\ &\quad -\sum_{k=1}^{s}2^{2k-1}\binom{2s+1}{2k}(a_{2n}+1)^{2s-2k+1}\sin^{4k-2} n\theta\\ &\quad +\sum_{k=1}^{s}2^{2k}\binom{2s+1}{2k+1}(a_{2n}+1)^{2s-2k}\sin^{4k} n\theta.\end{aligned} \tag{2.8}$$

Using (2.6) and (2.7), the inequality (2.5) is reduced to

$$\begin{aligned}&(a_2-1)\,(a_{2n}+1)^{2s+1}\\ &\quad\leq \left[(a_2-1)+2\sin^2\theta\right]\left[(a_{2n}+1)^{2s+1}-2\,(\sin^2 n\theta)\,E_{\rho,s}(n,\theta)\right],\end{aligned}$$

i.e.,

$$2\sin^2\theta\,(a_{2n}+1)^{2s+1}-2\sin^2 n\theta\left[(a_2-1)+2\sin^2\theta\right]E_{\rho,s}(n,\theta)\geq 0.$$

Dividing this inequality by $2\sin^2\theta$, it becomes

$$(a_{2n}+1)^{2s+1}-\frac{\sin^2 n\theta}{\sin^2\theta}\left[(a_2-1)+2\sin^2\theta\right]E_{\rho,s}(n,\theta)\geq 0. \tag{2.9}$$

By using the well-known fact $|\sin n\theta/\sin\theta|\leq n$, it is easy to see that

$$\frac{\sin^2 n\theta}{\sin^2\theta}\left[(a_2-1)+2\sin^2\theta\right]=(a_2-1)\frac{\sin^2 n\theta}{\sin^2\theta}+2\sin^2 n\theta\leq(a_2-1)n^2+2. \tag{2.10}$$

According to (2.8), we conclude that

$$\begin{aligned}E_{\rho,s}(n,\theta)-(2s+1)(a_{2n}+1)^{2s} = &\sum_{k=1}^{s}\frac{4^k(2s+1)!}{(2k)!(2s-2k)!}(a_{2n}+1)^{2s-2k}\\ &\times\left(\frac{\sin^2 n\theta}{2k+1}-\frac{a_{2n}+1}{2(2s-2k+1)}\right)\sin^{4k-2} n\theta.\end{aligned}$$

Since $\sin^{4k-2} n\theta\leq 1$ and

$$\frac{\sin^2 n\theta}{2k+1}-\frac{a_{2n}+1}{2(2s-2k+1)}\leq\frac{1}{2k+1}-\frac{a_{2n}+1}{2(2s-2k+1)}\,,$$

from the previous equality we obtain

$$E_{\rho,s}(n,\theta)-(2s+1)(a_{2n}+1)^{2s}\le\sum_{k=1}^{s}4^k(a_{2n}+1)^{2s-2k}\left[\binom{2s+1}{2k+1}-\frac{a_{2n}+1}{2}\binom{2s+1}{2k}\right].$$

Therefore,

$$E_{\rho,s}(n,\theta)\le\sum_{k=0}^{s}4^k\binom{2s+1}{2k+1}(a_{2n}+1)^{2s-2k}-\frac{1}{2}\sum_{k=1}^{s}4^k\binom{2s+1}{2k}(a_{2n}+1)^{2s-2k+1}.$$

Using the last inequality and (2.10), we conclude that the left-hand side of (2.9) is greater than or equal to $F(n)\equiv F_{\rho,s}(n)$, where

$$\begin{aligned}F_{\rho,s}(n) &:= (a_{2n}+1)^{2s+1}-\left[(a_2-1)n^2+2\right]\\ &\times\left[\sum_{k=0}^{s}4^k\binom{2s+1}{2k+1}(a_{2n}+1)^{2s-2k}-\frac{1}{2}\sum_{k=1}^{s}4^k\binom{2s+1}{2k}(a_{2n}+1)^{2s-2k+1}\right].\end{aligned}$$

Since $F_{\rho,s}(n)$ (ρ,s – are fixed) is continuous on $\mathbb{R}$ and $\lim_{n\to+\infty}F_{\rho,s}(n)=+\infty$, it follows that $F_{\rho,s}(n)>0$, for each $n>t$, where t is the largest zero of $F_{\rho,s}(n)$. For n_0 we can take $[t]+1$. □

TABLE 1. The smallest possible (s.p.) values of n_0 and their approximations $[t]+1$ (t is the largest zero of F)

	$\rho=1.05$		$\rho=1.3$	
s	$[t]+1$	the s.p. n_0	$[t]+1$	the s.p. n_0
1	41	34	8	7
2	50	46	10	9
3	56	53	11	10
4	59	57	12	11
5	63	61	12	12
6	65	63	13	12
7	67	66	13	13
8	69	68	13	13
9	70	69	14	13

The proof of Theorem 2.1 is not only of theoretical, but also of practical importance. We can use the function $F(n)$ from the proof to estimate n_0. Numerical values of $[t]+1$ (t is the largest zero of F) for some values of ρ and s are presented in Tables 1 and 2. The smallest possible (s.p.) values of n_0 are also presented. We can see that the smallest possible n_0 is estimated by $[t]+1$ very well.

A typical graph illustrating the relationship between n and $F(n)$ is given in Figure 1. Here, $\rho=1.05,\ s=1;\ n\in[1,42]$.

2.2. **The weight function** $\omega_2(t)=(1-t^2)^{s+1/2},\ s\in\mathbb{N}_0$**.** An explicit representation of the kernel $K_{n,s}^{(2)}(z)$ on the ellipse $\mathcal{E}_\rho$ for the weight function $\omega_2(t)$ was given in [11], as well as

$$|K_{n,s}^{(2)}(z)|=\frac{\pi}{4^s\rho^{n+1}}\left[\frac{a_2-\cos 2\theta}{a_{2n+2}-\cos{(2n+2)\theta}}\right]^{s+1/2}|Z_{n,s}^{(2)}(\rho e^{i\theta})|, \tag{2.11}$$

TABLE 2. The smallest possible (s.p.) values of n_0 and their approximations $[t]+1$ (t is the largest zero of F)

	$s=1$		$s=5$	
ρ	$[t]+1$	the s.p. n_0	$[t]+1$	the s.p. n_0
1.01	200	165	305	295
1.02	101	83	153	148
1.03	68	56	103	100
1.04	51	42	78	75
1.05	41	34	63	61
1.06	35	29	52	51
1.07	30	25	45	44
1.08	26	22	40	39
1.09	24	20	36	35
1.1	21	18	32	31
1.2	11	10	17	17
1.3	8	7	12	12
1.4	6	6	10	9
1.5	6	5	8	8
1.6	5	4	7	7
1.7	4	4	6	6
1.8	4	4	6	6
1.9	4	3	5	5
2.0	4	3	5	5

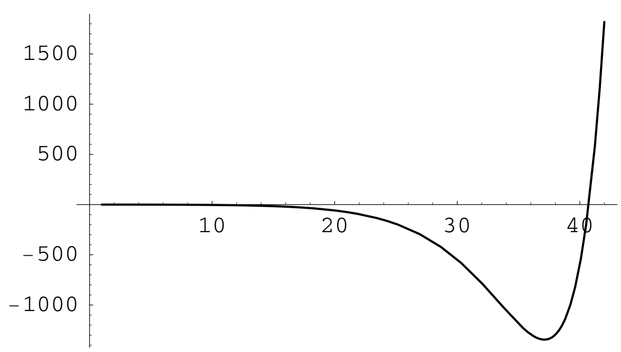

FIGURE 1. The typical graph of $F(n)$.

where

$$Z^{(2)}_{n,s}(\rho e^{i\theta}) = \sum_{k=0}^{s}(-1)^k \binom{2s+1}{s+k+1}(\rho e^{i\theta})^{-2(n+1)k}. \tag{2.12}$$

There we proved the following statement:

Theorem 2.2. *If* $\omega_2(t)=(1-t^2)^{s+1/2}$ *on* $(-1,1)$, $s\in\mathbb{N}_0$, *and* n *is odd, then*

$$\max_{z\in\mathcal{E}_\rho}|K^{(2)}_{n,s}(z)| = \left| K^{(2)}_{n,s}\left(\frac{i}{2}\left(\rho-\rho^{-1}\right)\right)\right|.$$

In this section we consider the case when n is even.

Theorem 2.3. *For each fixed $\rho > 1$ and $s \in \mathbb{N}_0$ there exists even $n_0 = n_0(\rho, s)$ such that*

$$\max_{z \in \varepsilon_\rho} |K_{n,s}^{(2)}(z)| = \left| \, K_{n,s}^{(2)}\Big(\frac{i}{2}\big(\rho - \rho^{-1}\big)\Big) \, \right|$$

for each even $n \geq n_0$.

Proof. First we prove the inequality

$$|Z_{n,s}^{(2)}(\rho e^{i\theta})| \leq Z_{n,s}^{(2)}(i\rho)\,, \quad \theta \in [0, \pi/2),\ n \text{ is even.} \tag{2.13}$$

We note that (see [11, Eq. (3.13)])

$$\begin{aligned} Z_{n,s}^{(2)}(u) &= \sum_{\nu=0}^{[(s-1)/2]} \left(\sum_{k=2\nu}^{2\nu+1} (-1)^k \binom{2s+1}{s+k+1} u^{-2(n+1)k} \right) + \zeta_{n,s}(u) \\ &= \sum_{\nu=0}^{[(s-1)/2]} \binom{2s+1}{s+2\nu+1} u^{-4\nu(n+1)} \left(1 - \alpha u^{-2(n+1)}\right) + \zeta_{n,s}(u), \end{aligned}$$

where $u = \rho e^{i\theta}$, $\alpha = (s - 2\nu)/(s + 2\nu + 2)$, $0 < \alpha < 1$, and

$$\zeta_{n,s}(u) = \zeta_{n,s}(\rho e^{i\theta}) := \begin{cases} 0 & \text{if } s \text{ is odd,} \\ (\rho e^{i\theta})^{-2(n+1)s} & \text{if } s \text{ is even,} \end{cases}$$

as well as $|\zeta_{n,s}(\rho e^{i\theta})| = \zeta_{n,s}(i\rho)$.

Since

$$|Z_{n,s}^{(2)}(u)| \leq \sum_{\nu=0}^{[(s-1)/2]} \binom{2s+1}{s+2\nu+1} \left| u^{-4\nu(n+1)} \left(1 - \alpha u^{-2(n+1)}\right) \right| + |\zeta_{n,s}(u)|\,,$$

introducing $q = \alpha \rho^{-2(n+1)}$, now we get

$$\begin{aligned} |Z_{n,s}^{(2)}(u)| &\leq \sum_{\nu=0}^{[(s-1)/2]} \binom{2s+1}{s+2\nu+1} \rho^{-4\nu(n+1)} \sqrt{1 - 2q\cos(2n+2)\theta + q^2} + \zeta_{n,s}(i\rho) \\ &\leq \sum_{\nu=0}^{[(s-1)/2]} \binom{2s+1}{s+2\nu+1} \rho^{-4\nu(n+1)} (1+q) + \zeta_{n,s}(i\rho) \\ &= \sum_{\nu=0}^{[(s-1)/2]} \binom{2s+1}{s+2\nu+1} (i\rho)^{-4\nu(n+1)} \left(1 - \alpha (i\rho)^{-2(n+1)}\right) + \zeta_{n,s}(i\rho) \\ &= Z_{n,s}^{(2)}(i\rho). \end{aligned}$$

Therefore, in order to prove the statement, on the basis of (2.11) and (2.13), it is sufficient to prove

$$\frac{a_2 - \cos 2\theta}{a_{2n+2} - \cos{(2n+2)\theta}} \leq \frac{a_2 + 1}{a_{2n+2} + 1}\ , \quad \theta \in [0, \pi/2),\ n \text{ is even,}$$

for sufficiently large n ($n \geq n_0;\ n_0 = n_0(\rho)$ – even). This is equivalent to

$$a_{2n+2} + a_{2n+2}\cos 2\theta - a_2 - a_2\cos(2n+2)\theta + \cos 2\theta - \cos(2n+2)\theta \geq 0,$$

and furthermore to

$$a_{2n+2}(1 + \cos 2\theta) - a_2(1 + \cos 2(n+1)\theta) + (1 + \cos 2\theta) - (1 + \cos 2(n+1)\theta) \geq 0,$$

introducing half-angles, to $(a_{2n+2}+1)\cos^2\theta-(a_2+1)\cos^2(n+1)\theta\ge 0$, and to

$$(a_{2n+2}+1)-\frac{\cos^2(n+1)\theta}{\cos^2\theta}(a_2+1)\ge 0. \tag{2.14}$$

Since $|\cos(n+1)\theta/\cos\theta|\le n+1$ for even n, we have

$$(a_{2n+2}+1)-\frac{\cos^2(n+1)\theta}{\cos^2\theta}(a_2+1)\ge(a_{2n+2}+1)-(n+1)^2(a_2+1),$$

which means that (2.14) holds if $(a_{2n+2}+1)-(n+1)^2(a_2+1)\ge 0$.

Since $a_2+1=2a_1^2$ and $a_{2n+2}+1=2a_{n+1}^2$, the last inequality is equivalent to $a_{n+1}^2-[(n+1)a_1]^2\ge 0$ or to $a_{n+1}-(n+1)a_1\ge 0$. Substituting a_1,a_{n+1} by (2.2), this inequality becomes

$$G_\rho(n)\equiv G(n):=\rho^{n+1}-(n+1)\rho-(n+1)\rho^{-1}+\rho^{-(n+1)}\ge 0.$$

Since $G_\rho(n)$ (ρ – is fixed) is continuous on $\mathbb{R}$ and $\lim_{n\to+\infty}G_\rho(n)=+\infty$, it follows that $G_\rho(n)>0$, for each $n>t$, where t is the largest zero of $G_\rho(n)$. For n_0 we can take the smallest even integer which is greater than or equal to t . □

Let $\overline{t}$ be the smallest even integer $\ge t$. If t is an even integer, we have $\overline{t}=t$, otherwise

$$\overline{t}:=\begin{cases}[t]+1 & \text{if } [t] \text{ is odd,}\\ [t]+2 & \text{if } [t] \text{ is even.}\end{cases}$$

We can use the function $G(n)$ from the proof to estimate n_0. Numerical values of $\overline{t}$ for some values of ρ are presented in Table 3. The smallest possible (s.p.) values of n_0, for $s=1,\ldots,10$, are also presented in the same table. We can see that the smallest possible n_0 (which is even) is estimated very well, independently of s.

Finally, observe that the function $G_\rho(n)\equiv G(n)$ in this case has rather simple form. Because of $G(0)=0$, and $G''(n)=2a_{n+1}\log^2\rho>0$, for $n\in[0,+\infty)$, we conclude that $G(n)$ has at most one zero t in the interval $(0,+\infty)$.

2.3. **The weight function** $\omega_3(t)=(1+t)^{1/2+s}(1-t)^{-1/2},\ s\in\mathbb{N}_0$**.** An explicit representation of the kernel $K_{n,s}^{(3)}(z)$ on the ellipse $\mathcal{E}_\rho$ for the generalized Chebyshev weight function of the third kind $\omega_3(t)$ was given in [11], as well as

$$\left|K_{n,s}^{(3)}(z)\right|=\frac{2^{1-s}\pi}{\rho^{n+1/2}}\frac{(a_1+\cos\theta)^{s+1}\left|Z_{n,s}^{(3)}(\rho e^{i\theta})\right|}{(a_2-\cos 2\theta)^{1/2}(a_{2n+1}+\cos(2n+1)\theta)^{1/2+s}}, \tag{2.15}$$

where

$$Z_{n,s}^{(3)}(u)=\sum_{k=0}^{s}\binom{2s+1}{s+k+1}u^{-(2n+1)k}.$$

The following result was conjectured in [11]:

Theorem 2.4. *For each fixed $\rho>1$ and $s\in\mathbb{N}_0$ there exists $n_0=n_0(\rho,s)$ such that*

$$\max_{z\in\mathcal{E}_\rho}\left|K_{n,s}^{(3)}(z)\right|=K_{n,s}^{(3)}\left(\frac{1}{2}(\rho+\rho^{-1})\right)$$

for each $n\ge n_0$.

TABLE 3. The smallest possible (s.p.) values of n_0 and their approximations $\bar{t}$ (t is the largest zero of G)

	the s.p. n_0										
ρ	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	$s=8$	$s=9$	$s=10$	$\bar{t}$
1.01	726	726	728	728	728	730	730	730	730	730	732
1.02	324	324	324	326	326	326	326	326	326	326	328
1.03	200	202	202	202	202	202	202	202	202	202	204
1.04	142	142	144	144	144	144	144	144	144	144	144
1.05	110	110	110	110	110	110	110	110	110	110	110
1.06	88	88	88	88	88	88	88	88	88	88	88
1.07	72	72	74	74	74	74	74	74	74	74	74
1.08	62	62	62	62	62	62	62	62	62	62	62
1.09	54	54	54	54	54	54	54	54	54	54	54
1.1	48	48	48	48	48	48	48	48	48	48	48
1.2	20	20	20	20	20	20	20	20	20	20	20
1.3	12	12	12	12	12	12	12	12	12	12	12
1.4	8	8	8	8	8	8	8	8	8	8	8
1.5	6	6	6	6	6	6	6	6	6	6	6
1.6	4	4	6	6	6	6	6	6	6	6	6
1.7	4	4	4	4	4	4	4	4	4	4	4
1.8	4	4	4	4	4	4	4	4	4	4	4
1.9	2	2	2	2	4	4	4	4	4	4	4
2.	2	2	2	2	2	2	2	2	2	2	2
2.5	2	2	2	2	2	2	2	2	2	2	2

Proof. Because of (2.15), it is sufficient to prove

$$\frac{(a_1+\cos\theta)^{s+1}\left|Z_{n,s}^{(3)}(\rho e^{i\theta})\right|}{(a_2-\cos 2\theta)^{1/2}(a_{2n+1}+\cos(2n+1)\theta)^{1/2+s}} \le \frac{(a_1+1)^{s+1}Z_{n,s}^{(3)}(\rho)}{(a_2-1)^{1/2}(a_{2n+1}+1)^{1/2+s}}$$

for sufficiently large n ($n\ge n_0(\rho,s)$) and $\theta\in(0,\pi]$, where a_j are given by (2.2).

It is obvious that for each $n\ge 1$, we have $(a_1+\cos\theta)^{s+1}\le(a_1+1)^{s+1}$. On the basis of the results from Subsection 2.1, we obtain

$$\begin{aligned}
&\frac{\left|Z_{n,s}^{(3)}(\rho e^{i\theta})\right|}{(a_2-\cos 2\theta)^{1/2}(a_{2n+1}+\cos(2n+1)\theta)^{1/2+s}}\\
&\qquad=\frac{\left|Z_{n+1/2,s}^{(1)}(\rho e^{i\theta})\right|}{(a_2-\cos 2\theta)^{1/2}(a_{2(n+1/2)}+\cos(2(n+1/2))\theta)^{1/2+s}}\\
&\qquad\le\frac{Z_{n+1/2,s}^{(1)}(\rho)}{(a_2-1)^{1/2}(a_{2(n+1/2)}+1)^{1/2+s}}=\frac{Z_{n,s}^{(3)}(\rho)}{(a_2-1)^{1/2}(a_{2n+1}+1)^{1/2+s}},
\end{aligned}$$

for each $n\ge n_0$ ($n_0=n_0(\rho,s)$). Therefore, we conclude that

$$\left|K_{n,s}^{(3)}\left(\frac{1}{2}\left(\rho e^{i\theta}+\rho^{-1}e^{-i\theta}\right)\right)\right|\le K_{n,s}^{(3)}\left(\frac{1}{2}\left(\rho+\rho^{-1}\right)\right),$$

for each $n\ge n_0$ ($n_0=n_0(\rho,s)$).

If t is the largest zero of F, for n_0 we can take $[(2t-1)/2]+1$. $\square$

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