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On the remainder term of Gauss–Radau quadratures for analytic functions

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Abstract

For analytic functions the remainder term of Gauss–Radau quadrature formulae can be represented as a contour integral with a complex kernel. We study the kernel on elliptic contours with foci at the points ± 1 and a sum of semi-axes $\varrho > 1$ for the Chebyshev weight function of the second kind. Starting from explicit expressions of the corresponding kernels the location of their maximum modulus on ellipses is determined. The corresponding Gautschi's conjecture from [On the remainder term for analytic functions of Gauss–Lobatto and Gauss–Radau quadratures, Rocky Mountain J. Math. 21 (1991), 209–226] is proved.
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1. Introduction

In this paper we prove a conjecture of Gautschi [1] for the Gauss–Radau quadrature formula

$$\int_{-1}^{1} w(t) f(t)\,\mathrm{d}t = \sum_{\nu=1}^{N} \lambda_\nu f(\tau_\nu) + R_N(f)_w, \tag{1.1}$$

with respect to the Chebyshev weight function of the second kind $w(t) = w_2(t) = \sqrt{1-t^2}$ and with a fixed node at -1 (or 1).

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and $\mathscr{D} = \operatorname{int}\Gamma$ its interior. If the integrand f is analytic in $\mathscr{D}$ and continuous on $\overline{\mathscr{D}}$, then the remainder term $R_N(f)_w$ in (1.1) admits the contour integral

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representation

$$R_N(f)_w = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma K_N(z; w) f(z)\, \mathrm{d}z, \tag{1.2}$$

where the kernel is given by

$$K_N(z; w) = \frac{1}{\omega_N(z; w)} \int_{-1}^{1} \frac{w(t)\omega_N(t; w)}{z - t}\, \mathrm{d}t, \quad z \notin [-1, 1],$$

and $\omega_N(z; w) = \prod_{\nu=1}^{N} (z - \tau_\nu)$. It is clear that

$$K_N(\bar{z}; w) = \overline{K_N(z; w)}. \tag{1.3}$$

The integral representation (1.2) leads to the error estimate

$$|R_N(f)_w| \leqslant \frac{\ell(\Gamma)}{2\pi} \left(\max_{z\in\Gamma} |K_N(z; w)| \right) \left(\max_{z\in\Gamma} |f(z)| \right), \tag{1.4}$$

where $\ell(\Gamma)$ is the length of the contour Γ. In order to get estimate (1.4), one has to study the magnitude of $|K_N(z; w)|$ on Γ. Note that the previous formulae hold for every interpolatory quadrature rule with mutually different nodes on $[-1, 1]$.

Many authors have used (1.4) to derive bounds of $|R_N(f)_w|$. Two choices of the contour Γ have been widely used: (1) a circle C_r with a center at the origin and a radius $r\ (> 1)$, i.e., $C_r = \{z :\ |z| = r\}$, $r > 1$, and (2) an ellipse $\mathscr{E}_\varrho$ with foci at the points ± 1 and a sum of semi-axes $\varrho > 1$,

$$\mathscr{E}_\varrho = \{z \in \mathbb{C} | z = \tfrac{1}{2}(\varrho e^{\mathrm{i}\theta} + \varrho^{-1} e^{-\mathrm{i}\theta}),\ 0 \leqslant \theta < 2\pi\}. \tag{1.5}$$

When $\varrho \to 1$ the ellipse shrinks to the interval $[-1, 1]$, while with increasing ϱ it becomes more and more circle-like. The advantage of the elliptical contours, compared to the circular ones, is that such a choice needs the analyticity of f in a smaller region of the complex plane, especially when ϱ is near 1.

Since the ellipse $\mathscr{E}_\varrho$ has length $\ell(\mathscr{E}_\varrho) = 4\varepsilon^{-1} E(\varepsilon)$, where ε is the eccentricity of $\mathscr{E}_\varrho$, i.e., $\varepsilon = 2/(\varrho + \varrho^{-1})$, and

$$E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2\theta}\, \mathrm{d}\theta$$

is the complete elliptic integral of the second kind, estimate (1.4) reduces to

$$|R_N(f)_w| \leqslant \frac{2E(\varepsilon)}{\pi\varepsilon} \left(\max_{z\in\mathscr{E}_\varrho} |K_N(z; w)| \right) \|f\|_\varrho, \quad \varepsilon = \frac{2}{\varrho + \varrho^{-1}}, \tag{1.6}$$

where $\|f\|_\varrho = \max_{z\in\mathscr{E}_\varrho} |f(z)|$. As we can see, the bound on the right-hand side in (1.6) is a function of ϱ, so it can be optimized with respect to $\varrho > 1$.

This approach was discussed first for Gaussian quadrature rules, in particular with respect to the Chebyshev weight functions (cf. [4,5])

$$w_1(t) = \frac{1}{\sqrt{1 - t^2}}, \quad w_2(t) = \sqrt{1 - t^2}, \quad w_3(t) = \sqrt{\frac{1 + t}{1 - t}}, \quad w_4(t) = \sqrt{\frac{1 - t}{1 + t}},$$

and later has been extended to Bernstein–Szegő weight functions [8] and some symmetric weight functions including especially the Gegenbauer weight functions [10], as well as to Gauss–Lobatto and Gauss–Radau (cf. [1–3,6,9]), and to Gauss–Turán (cf. [7]) quadrature rules.

In [1] Gautschi considered Gauss–Radau and Gauss–Lobatto quadrature rules relative to the four Chebyshev weight functions w_i, $i = 1, 2, 3, 4$, and derived explicit expressions of the corresponding kernels $K_N(z; w_i)$, $i = 1, 2, 3, 4$, in terms of the variable $u = \varrho e^{\mathrm{i}\theta}$; they are the key for determining the maximum point of $|K_N(z; w_i)|$, $i = 1, 2, 3, 4$, on $\Gamma = \mathscr{E}_\varrho$ given by (1.5). Note that $z = (u + u^{-1})/2$. For Gauss–Lobatto quadratures it was proved that $|K_N(z; w_1)|$ attains its maximum on $\mathscr{E}_\varrho$ on the real axis (cf. [1, Theorem 4.1]). For w_2, w_3 and w_4 only empirical results and conjectures on the location of the maximum point on $\mathscr{E}_\varrho$ were presented. These conjectures have been proved by Schira (see [9]).

For Gauss–Radau quadratures with a fixed node at -1, Gautschi proved that the corresponding kernel $K_N(z; w)$ for Chebyshev weight functions $w = w_1$ and $w = w_4$ attains its maximum modulus on $\mathscr{E}_\varrho$ on the negative real axis (cf. [1, Theorems 4.4 and 4.5]). For the remaining cases $w = w_2$ and $w = w_3$ only empirical results and conjectures on the location of the maximum point on $\mathscr{E}_\varrho$ for the corresponding kernels are presented in [1]. They are also mentioned in [9], but without the proof. In this paper we derive an analytic proof of the conjecture for $w = w_2$.

Thus, we are concerned with the Gauss–Radau quadrature rule (1.1) with respect to the weight function $w = w_2$ with $N = n + 1$ nodes and a fixed node at -1 or 1. Because of symmetry, it is enough to consider only one case, e.g., when $\tau_1 = -1$. It is well known that the node polynomial in this case can be expressed as

$$\omega_{n+1}(z; w) = (1 + z)\pi_n(z; w^R),$$

where $\pi_n(\cdot; w^R)$ denotes the monic polynomial of degree n orthogonal with respect to the weight function $w^R(t) = (1 + t)w_2(t) = (1 + t)^{3/2}(1 - t)^{1/2}$.

2. The maximum of the kernel for Chebyshev weight function of the second kind

Gautschi [1, Eq. (3.16)] derived the explicit representation of the kernel on $\mathscr{E}_\varrho$

$$K_{n+1}(z; w_2) = \frac{\pi}{u^{n+1}} \frac{(u - u^{-1})(u^{-1} + (n + 2)/(n + 1))}{u^{n+2} - u^{-(n+2)} + ((n + 2)/(n + 1))(u^{n+1} - u^{-(n+1)})}, \tag{2.1}$$

where $z = (u + u^{-1})/2$ and $u = \varrho e^{i\theta}$. Using (2.1) we can determine the modulus of the kernel on $\mathscr{E}_\varrho$. It is easy to prove

$$|u - u^{-1}| = \sqrt{2}\sqrt{a_2 - \cos 2\theta}, \tag{2.2}$$

$$\left|u^{-1} + \frac{n+2}{n+1}\right| = \frac{2}{\sqrt{\varrho}}\sqrt{\frac{n+2}{n+1}}\sqrt{d - \sin^2\frac{\theta}{2}}, \tag{2.3}$$

where

$$a_j = a_j(\varrho) = \tfrac{1}{2}(\varrho^j + \varrho^{-j}), \quad j \in \mathbb{N}, \quad \varrho > 1, \tag{2.4}$$

and

$$d = d(\varrho) = \frac{1}{4}\left(\frac{n+2}{n+1}\varrho + 2 + \frac{n+1}{n+2}\varrho^{-1}\right). \tag{2.5}$$

Further,

$$\begin{aligned}
&\left|u^{n+2} - u^{-(n+2)} + \frac{n+2}{n+1}\left(u^{n+1} - u^{-(n+1)}\right)\right|^2 \\
&\quad = \left[\left(\varrho^{n+2} - \varrho^{-(n+2)}\right)\cos(n+2)\theta + \frac{n+2}{n+1}\left(\varrho^{n+1} - \varrho^{-(n+1)}\right)\cos(n+1)\theta\right]^2 \\
&\qquad + \left[\left(\varrho^{n+2} + \varrho^{-(n+2)}\right)\sin(n+2)\theta + \frac{n+2}{n+1}\left(\varrho^{n+1} + \varrho^{-(n+1)}\right)\sin(n+1)\theta\right]^2 \\
&\quad = [\varrho^{2(n+2)} + \varrho^{-2(n+2)} + 2 - 4\cos^2(n+2)\theta] \\
&\qquad + \left(\frac{n+2}{n+1}\right)^2\left[\varrho^{2(n+1)} + \varrho^{-2(n+1)} + 2 - 4\cos^2(n+1)\theta\right] \\
&\qquad + 2\left(\frac{n+2}{n+1}\right)\left[\left(\varrho^{2n+3} + \varrho^{-(2n+3)}\right)\cos\theta - \left(\varrho + \varrho^{-1}\right)\cos(2n+3)\theta\right] \\
&\quad = 4\,\frac{n+2}{n+1}\,B(\theta),
\end{aligned}$$

where

$$B(\theta) = B(\varrho, \theta) = \frac{n+1}{n+2}(a_{n+2}^2 - \cos^2(n+2)\theta) + \frac{n+2}{n+1}(a_{n+1}^2 - \cos^2(n+1)\theta) + a_{2n+3}\cos\theta - a_1\cos(2n+3)\theta. \tag{2.6}$$

In this way we get

$$|K_{n+1}(z; w_2)| = \frac{\pi\sqrt{2}}{\varrho^{n+3/2}}\left[\frac{(a_2 - \cos 2\theta)(d - \sin^2(\theta/2))}{B(\theta)}\right]^{1/2}. \tag{2.7}$$

For the location of the maximum point of $|K_{n+1}(z; w_2)|$ on $\mathscr{E}_\varrho$ the following conjectures are presented in [1, p. 224]: if $n \leqslant 3$, the maximum is attained at $z = -\frac{1}{2}(\varrho + \varrho^{-1})$ on every ellipse $\mathscr{E}_\varrho$, $\varrho > 1$; if $n \geqslant 4$ there exist parameters $\varrho_0 = \varrho_0(n)$ and $\varrho_0' = \varrho_0'(n)$, $\varrho_0 > \varrho_0'$, such that the maximum is attained at $z = -\frac{1}{2}(\varrho + \varrho^{-1})$ when $\varrho \in (1, \varrho_0') \cup (\varrho_0, +\infty)$. Otherwise, the maximum point moves on the ellipse $\mathscr{E}_\varrho$ from somewhere close to the imaginary axis to the negative real axis as ϱ increases.

With increasing n the parameters ϱ_0' converge to one rather rapidly (cf. [1, Table 4.2]). Hence, the part of the conjecture when $n \geqslant 4$ and $\varrho \in (1, \varrho_0')$ is less important for practical use because the corresponding maximum tends to infinity for increasing n and $\varrho \in (1, \varrho_0')$. Therefore, the error bound (1.4) is rather poor in this case.

Theorem 2.1. *For each integer $n \geqslant 4$ there exists $\varrho_0 = \varrho_0(n)$ such that the kernel of the $(n+1)$-point Gauss–Radau formula (with fixed node at -1) for the Chebyshev weight function of the second kind attains its maximum modulus on $\mathscr{E}_\varrho$ on the negative real axis when $\varrho > \varrho_0$. For $n = 1, 2, 3$, the maximum of the kernel $K_{n+1}(z; w_2)$ attains its maximum on the negative real axis on every ellipse $\mathscr{E}_\varrho$ $(\varrho > 1)$. The maximum is given by*

$$\max_{z\in\mathscr{E}_\varrho} |K_{n+1}(z; w_2)| = \left|K_{n+1}\left(-\frac{1}{2}(\varrho + \varrho^{-1}); w_2\right)\right| = \frac{\pi}{\varrho^{n+1}} \frac{(\varrho - \varrho^{-1})(((n+2)/(n+1)) - \varrho^{-1})}{\varrho^{n+2} - \varrho^{-(n+2)} - ((n+2)/(n+1))(\varrho^{n+1} - \varrho^{-(n+1)})}.$$

Proof. First we show that the denominator of the previous fraction, i.e., the function

$$V(\varrho, n) = \varrho^{n+2} - \varrho^{-(n+2)} - \frac{n+2}{n+1}(\varrho^{n+1} - \varrho^{-(n+1)})$$

is always positive $(\varrho > 1, n \in \mathbb{N})$. If n is fixed $(V(\varrho) := V(\varrho, n))$, we get $V'(\rho) = (n+2)(\varrho - 1)(\varrho^n - \varrho^{-(n+3)}) > 0$ and $V(1) = 0$.

We consider four separate cases: $n = 1$, $n = 2$, $n = 3$, and $n \geqslant 4$.

Case $n = 1$: Using (2.1) we get

$$|K_2(z; w_2)| = \left|\frac{\pi(u^2 - 1)(3u + 2)}{u(2u^6 + 3u^5 - 3u - 2)}\right| = \frac{\pi\varrho^{-1}}{|(u+1)/(3u+2)||2u^3 + u^2 + u + 2|}.$$

Now, we prove that both of the expressions in the denominator for fixed ϱ and $\theta \in [0, \pi]$ attain their minimum modulus when $\theta = \pi$.

The first expression can be written as

$$\frac{u+1}{3u+2} = \frac{1}{3}\left(1 + \frac{1}{3u+2}\right),$$

from which it is evident that it attains its minimum modulus when $\theta = \pi$ $(|1/(3u+2)| < 1$ and attains maximum when $\theta = \pi)$.

For the second expression we can get the following equality:

$$|2u^3 + u^2 + u + 2|^2 = 32x^3 + (8\varrho^2 + 1)x^2 + (4\varrho^4 - 22\varrho^2 + 4)x + (4\varrho^6 - 3\varrho^4 - 3\varrho^2 + 4),$$

where $x = \Re(u) = \varrho \cos\theta$, $x \in [-\varrho, \varrho]$. We denote the right-hand side of the last equality as $F_\varrho(x) = F(x)$. Now it suffices to prove $F(x) - F(-\varrho) \geqslant 0$, for $x \in [-\varrho, \varrho]$ and $\varrho > 1$:

$$\begin{aligned} F(x) - F(-\varrho) &= 32(x^3 + \varrho^3) + (8\varrho^2 + 1)(x^2 - \varrho^2) + (4\varrho^4 - 22\varrho^2 + 4)(x + \varrho) \\ &= 2(x + \varrho)[16x^2 + (4\varrho^2 - 16\varrho + 4)x + (2\varrho^4 - 4\varrho^3 + 5\varrho^2 - 4\varrho + 2)]. \end{aligned}$$

For the discriminant of the polynomial in brackets D there holds

$$D = -16(\varrho - 1)^2(7\varrho^2 + 6\varrho + 7) \leqslant 0,$$

which completes the proof of this case.

Case $n = 2$: Using (2.1) we get

$$\begin{aligned} K_3(z; w_2) &= \frac{\pi}{u^3} \frac{(u - u^{-1})\left(u^{-1} + \frac{4}{3}\right)}{u^4 - u^{-4} + \frac{4}{3}\left(u^3 - u^{-3}\right)} = \frac{\pi(u^2 - 1)(4u + 3)}{u(3u^8 + 4u^7 - 4u - 3)} \\ &= \frac{\pi}{u} \frac{4u + 3}{3u^6 + 4u^5 + 3u^4 + 4u^3 + 3u^2 + 4u + 3}, \end{aligned}$$

i.e.,

$$|K_3(z; w_2)| = \frac{\pi\varrho^{-1}}{|g(u)|}, \tag{2.8}$$

where

$$g(u) = \frac{3u^6 + 4u^5 + 3u^4 + 4u^3 + 3u^2 + 4u + 3}{4u + 3}.$$

By (2.8) and (1.3) it suffices to prove that $|g(u)|$ attains its minimum when $\theta = \pi$ $(u = -\varrho)$, i.e.,

$$g(u)g(\overline{u}) - g(-\varrho)g(-\varrho) \geqslant 0, \quad \theta \in [0, \pi].$$

After some calculation we get

$$g(u)g(\overline{u}) - g(-\varrho)g(-\varrho) = \frac{4\varrho^2\,(y + 1)}{(3 - 4\varrho)^2|4u + 3|^2} P(\varrho, y),$$

where $y = \cos\theta$ and

$$\begin{aligned} P(\varrho, y) = {}& 144\varrho^4(3 - 4\varrho)^2 y^5 \\ & + \varrho^3(1536\varrho^4 - 4608\varrho^3 + 5856\varrho^2 - 3600\varrho + 864)y^4 \\ & + \varrho^2(576\varrho^6 - 2400\varrho^5 + 2500\varrho^4 - 2208\varrho^3 + 2808\varrho^2 - 1728\varrho + 324)y^3 \\ & + \varrho(384\varrho^8 - 1152\varrho^7 + 1080\varrho^6 - 196\varrho^5 - 192\varrho^4 - 504\varrho^3 + 1248\varrho^2 - 900\varrho + 216)y^2 \\ & + (144\varrho^{10} - 600\varrho^9 + 913\varrho^8 - 600\varrho^7 + 432\varrho^6 - 432\varrho^5 + 418\varrho^4 - 768\varrho^3 \\ & + 864\varrho^2 - 432\varrho + 81)y \\ & + (42\varrho^{11} - 144\varrho^{10} + 258\varrho^9 - 337\varrho^8 + 426\varrho^7 - 432\varrho^6 + 378\varrho^5 - 418\varrho^4 \\ & + 384\varrho^3 - 288\varrho^2 + 216\varrho - 81). \end{aligned}$$

Now it suffices to prove $P(\varrho, y) \geqslant 0$. If we take $\varrho = x + 1$ and $y = t - 1$, $x > 0$, $t \in [0, 2]$, we have

$$P(x+1, t-1) = 42x^{11} + \sum_{i=0}^{10} b_i(t)x^i,$$

where

$$\begin{aligned}
b_{10}(t) &= 6(24t + 29),\\
b_9(t) &= 24(16t^2 + 3t + 28),\\
b_8(t) &= 576t^3 + 576t^2 - 887t + 2170,\\
b_7(t) &= 1536t^4 - 3936t^3 + 8280t^2 - 8512t + 6230,\\
b_6(t) &= 2(1152t^5 - 2688t^4 + 146t^3 + 7852t^2 - 10332t + 5845),\\
b_5(t) &= 4(2592t^5 - 10344t^4 + 14118t^3 - 4914t^2 - 3766t + 3185),\\
b_4(t) &= 4(4644t^5 - 20640t^4 + 32517t^3 - 19866t^2 + 2093t + 1862),\\
b_3(t) &= 4(4176t^5 - 19224t^4 + 31556t^3 - 21350t^2 + 4606t + 539),\\
b_2(t) &= 4(1944t^5 - 9048t^4 + 15036t^3 - 10458t^2 + 2548t + 49),\\
b_1(t) &= 48t(6t^2 - 14t + 7)^2,\\
b_0(t) &= 4t(6t^2 - 14t + 7)^2.
\end{aligned}$$

By computing their zeros, it can be seen that all functions $b_i(t)$, $i = 0, 1, \ldots, 10$, are nonnegative when $t \in [0, 2]$.

Case $n = 3$: We prove this case in the same way as the previous one. From (2.1) we get

$$\begin{aligned}
K_4(z; w_2) &= \frac{\pi}{u^4}\,\frac{(u - u^{-1})\left(u^{-1} + \frac{5}{4}\right)}{u^5 - u^{-5} + \frac{5}{4}\left(u^4 - u^{-4}\right)} = \frac{\pi(u^2 - 1)(5u + 4)}{u(4u^{10} + 5u^9 - 5u - 4)}\\
&= \frac{\pi}{u}\,\frac{5u + 4}{4u^8 + 5u^7 + 4u^6 + 5u^5 + 4u^4 + 5u^3 + 4u^2 + 5u + 4},
\end{aligned}$$

i.e.,

$$|K_4(z; w_2)| = \frac{\pi\varrho^{-1}}{|h(u)|}, \tag{2.9}$$

where

$$h(u) = \frac{4u^8 + 5u^7 + 4u^6 + 5u^5 + 4u^4 + 5u^3 + 4u^2 + 5u + 4}{5u + 4}.$$

By (2.9) and (1.3) it suffices to prove that $|h(u)|$ attains its minimum when $\theta = \pi$ $(u = -\varrho)$, i.e.,

$$h(u)h(\overline{u}) - h(-\varrho)h(-\varrho) \geqslant 0, \quad \theta \in [0, \pi].$$

After some calculation we get

$$h(u)h(\overline{u}) - h(-\varrho)h(-\varrho) = \frac{4\varrho^2(y + 1)}{(4 - 5\varrho)^2|5u + 4|^2}R(\varrho, y).$$

After denoting $\varrho = x + 1$ and $y = t - 1$, $x > 0$, $t \in [0, 2]$, we have

$$R(x+1, t-1) = 90x^{15} + \sum_{i=0}^{14} c_i(t)x^i,$$

where

$$
\begin{aligned}
c_{14}(t) &= 50(8t+11),\\
c_{13}(t) &= 20(50t^2+98t+81),\\
c_{12}(t) &= 1600t^3+5000t^2+2761t+3798,\\
c_{11}(t) &= 4(1000t^4-840t^3+7220t^2-4467t+3264),\\
c_{10}(t) &= 2(3200t^5-400t^4-10318t^3+56192t^2-56697t+23670),\\
c_9(t) &= 20(800t^6-2912t^5+7400t^4-13230t^3+21642t^2-18807t+6462),\\
c_8(t) &= 25600t^7-86400t^6+46736t^5+317824t^4-845684t^3+1139736t^2-830739t+250710,\\
c_7(t) &= 2(81920t^7-456640t^6+986944t^5-895744t^4+15024t^3+685464t^2-580020t+170073),\\
c_6(t) &= 446464t^7-2786432t^6+6890432t^5-8288384t^4+4522092t^3-184992t^2-896592t+313902,\\
c_5(t) &= 4(167936t^7-1095968t^6+2844320t^5-3685848t^4+2414562t^3-638574t^2-49554t+47295),\\
c_4(t) &= 4(151040t^7-1005920t^6+2666584t^5-3558984t^4+2473269t^3-803946t^2+63414t+17280),\\
c_3(t) &= 20(16384t^7-110112t^6+294624t^5-398136t^4+283140t^3-98334t^2+11970t+675),\\
c_2(t) &= 4(25600t^7-172640t^6+463536t^5-629256t^4+451212t^3-160218t^2+21690t+225),\\
c_1(t) &= 64t(-16t^3+54t^2-54t+15)^2,\\
c_0(t) &= 4t(-16t^3+54t^2-54t+15)^2.
\end{aligned}
$$

By computing their zeros, it can be seen that all functions $c_i(t)$, $i=0,1,\ldots,14$, are nonnegative when $t\in[0,2]$, which completes the proof of this case.

Case $n \geqslant 4$: By (2.7) it is enough to prove

$$\frac{(a_2-\cos 2\theta)(d-\sin^2(\theta/2))}{B(\theta)} \leqslant \frac{(a_2-1)(d-1)}{B(\pi)},$$

for sufficiently large ϱ ($\varrho>\varrho_0$) and $\theta\in[0,\pi)$.

The last inequality is reduced to

$$
\begin{aligned}
&[(a_2-1)+2\sin^2\theta]\left[(d-1)+\cos^2\frac{\theta}{2}\right]B(\pi)\\
&\quad\leqslant (a_2-1)(d-1)\left\{\frac{n+1}{n+2}[(a_{n+2}^2-1)+\sin^2(n+2)\theta]\right.\\
&\qquad+\frac{n+2}{n+1}[(a_{n+1}^2-1)+\sin^2(n+1)\theta]\\
&\qquad\left.+a_{2n+3}\left(2\cos^2\frac{\theta}{2}-1\right)-a_1\left(2\cos^2(2n+3)\frac{\theta}{2}-1\right)\right\},
\end{aligned}
$$

and further, using the value of $B(\pi)$ from (2.6) on the right-hand side,

$$
\begin{aligned}
&(a_2-1)(d-1)B(\pi)+\left[(a_2-1)\cos^2\frac{\theta}{2}+\left(d-\sin^2\frac{\theta}{2}\right)2\sin^2\theta\right]B(\pi)\\
&\quad\leqslant (a_2-1)(d-1)B(\pi)+(a_2-1)(d-1)\left[2a_{2n+3}\cos^2\frac{\theta}{2}\right.\\
&\qquad\left.-2a_1\cos^2(2n+3)\frac{\theta}{2}+\frac{n+1}{n+2}\sin^2(n+2)\theta+\frac{n+2}{n+1}\sin^2(n+1)\theta\right].
\end{aligned}
$$

Table 1

n	ϱ_0	n	ϱ_0
4	4.7394	9	16.4698
5	7.7651	10	18.5332
6	10.0870	11	20.5829
7	12.2672	12	22.6229
8	14.3854	13	24.6557

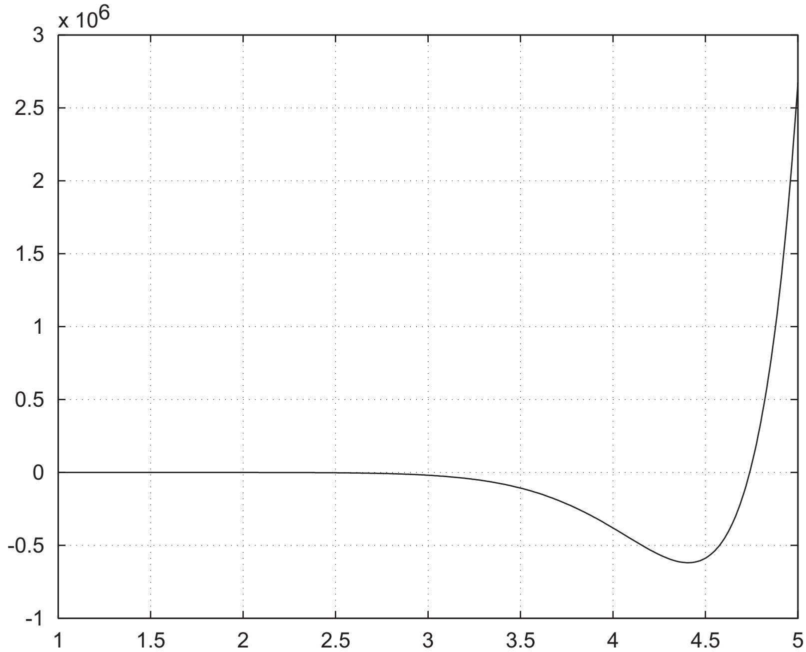

Fig. 1. The typical graph of $G(\varrho)$.

After dividing this by $\cos^2(\theta/2)$ we have

$$2\,(a_2-1)(d-1)\,a_{2n+3}-2(a_2-1)(d-1)\,a_1\frac{\cos^2(2n+3)(\theta/2)}{\cos^2(\theta/2)}$$
$$-\,[(a_2-1)+8\,(d-\sin^2(\theta/2))\sin^2(\theta/2)]B(\pi)$$
$$+\,(a_2-1)(d-1)\left[\frac{n+1}{n+2}\sin^2(n+2)\theta+\frac{n+2}{n+1}\sin^2(n+1)\theta\right]\geqslant 0.$$

Since

$$\max_{\theta\in[0,\pi]}\frac{\cos^2(2n+3)(\theta/2)}{\cos^2(\theta/2)}=(2n+3)^2,$$

$$C=C(\varrho):=\max_{\theta\in[0,\pi]}8\left(d-\sin^2\frac{\theta}{2}\right)\sin^2\frac{\theta}{2}=\begin{cases}8(d-1) & \text{if } d\geqslant 2,\\ 2d^2 & \text{if } d<2,\end{cases} \tag{2.10}$$

$$\min_{\theta\in[0,\pi]}\left\{\frac{n+1}{n+2}\sin^2(n+2)\theta+\frac{n+2}{n+1}\sin^2(n+1)\theta\right\}=0,$$

we conclude that the left-hand side of the last inequality is greater than or equal to $G(\varrho) = G_n(\varrho)$, where

$$G(\varrho) = 2(a_2 - 1)(d - 1)a_{2n+3} - 2(a_2 - 1)(d - 1)a_1(2n + 3)^2 - (a_2 - 1 + C)B(\pi).$$

Using (2.4), (2.5) and (2.10) we get

$$G(\varrho) = \frac{1}{8}\left(\frac{n+2}{n+1} - \frac{n+1}{n+2}\right)\varrho^{2n+6} + \sum_{i=-(2n+6)}^{2n+5} l_i(n)\varrho^i, \quad \varrho \geqslant \varrho_d, \tag{2.11}$$

where ϱ_d is the unique zero greater than 1 of the equation $d(\varrho)=2$. When $\varrho \in (1, \varrho_d)$, $G(\varrho)$ is less than the function from right-hand side of (2.11). Since $G(\varrho)$ is continuous when $\varrho > 1$ and $\lim_{\varrho\to+\infty} G(\varrho) = +\infty$, it follows that $G(\varrho) > 0$, for each $\varrho > \varrho_0$, where ϱ_0 is the largest zero of $G(\varrho)$. □

The values of ϱ_0 for some values of n are displayed in Table 1. All values of ϱ_0, except when $n = 4$, are optimal in a sense that for $\varrho < \varrho_0$ (and $\varrho > \varrho_0'$) the point $z = -\frac{1}{2}(\varrho + \varrho^{-1})$ is not a maximum point. The optimal value of ϱ_0 when $n = 4$ is 4.7385. The values for $n = 5, \ldots, 10$ coincide with the values ϱ_n given in [1, Table 4.2].

A typical graph of $G(\varrho)$ is displayed in Fig. 1. Here $n = 4$.

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