Forward and inverse kinematics for vertical 5-axis turning center with angular head of non-intersectional axes, with compensation for table moving caused by base thermal dilatation

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Abstract. The paper presents the solution for forward and inverse kinematics of the vertical 5-axis turning centers with 2 linear and 3 rotational axes (C_y, X, Z, B_t and C_t) which for the 5-axis milling achieves the motion accomplished by 3 linear and 2 rotational axes (X, Y, Z, B_t and C_y). It has been done in such a way to provide for machine motion programming as if machining were performed on a 5-axis gantry milling machine. This has essentially facilitated machine programming, because tool positions and orientations required for programming are determined disregarding the workpiece swiveling during machining and current positions and orientations taken by the tool during machining relative to the workpiece. Turning center has a 2-rotary-axis head with axes C_t and B_t which do not intersect. This type of angular head has increased the possibilities of machining and allowed for performing certain types of machining without machine's taking the singular positions, but it has made the machine control algorithm more complex. A high number of rotating of the table, required for turning, causes heating of the table bearing support and base thermal dilatation. If milling or drilling is done immediately after turning, the table and X axis motion control should be corrected to eliminate the error in machining appeared due to dilatation, as has been done in this paper.
Keywords: Vertical five-axis turning centers; Forward and inverse kinematics; Thermal errors

1. Introduction

Nowadays, the precision and productivity that users demand from 5-axis machining of complex workpiece surfaces is gradually increasing. To satisfy this requirement different structures of the 5-axis machines are developed. The machine denominations, where L is a linear axis and R is a rotation axis, will be given now. LLLRR-The cutting tool is supported by a two-rotary-axis head, one for head rotation and another for tool tilting. This configuration is used in large gantry machine tools. RRLLL-The workpiece is supported by a double turning table, i.e. the work table has two rotational axes. This configuration is commonly used in small compact machines or in machines with auxiliary rotary tables. RLLLR-The workpiece is supported by a rotary turning table and the tool has one rotational degree of freedom (swivelling head).

The present paper deals with the control algorithm development for vertical 5-axis turning centre, where the work table becomes the axis of auxiliary motion (C_y axis), whose swivelling, with cutting tool motion along the X axis according to the corresponding law, produces motion corresponding to the motion along the Y axis, not existing here. The ram carrying turning tools is replaced by the turning, drilling and milling unit for rotating tools. By addition of the replaceable two-axis angular head to this unit, the 5-axis milling and drilling is possible to achieve. This way, a machine with 2 linear and 3 rotational axes was obtained, which for the 5-axis machining achieves the motion accomplished by 3 linear and 2 rotational axes. A machine with C_y, X, Z, B_t and C_t axes, the RLLRR machine type, was thus obtained.

In the control algorithm given in this paper, the compensation for error caused by machine base thermal dilatation has been carried out, because it is the biggest error and because it is very difficult to eliminate it by corresponding structure, cooling and mounting. When workpiece turning is performed with a large number of revolutions there occurs a substantial heating of the work table bearing support. The bearing support temperature is transferred to the machine base, causing its thermal dilatation. This causes the table rotational axis shift by a few tenths of a millimetre. The work table moving along the X and Y axes (δ_{xc} and δ_{yc}) has impact on milling and drilling accuracy. To eliminate the influence of the work table thermal dilatation on machining accuracy, real-time measurements and machine control algorithm correction are required, as has been done in this paper.

Control algorithm of a vertical 5-axis turning centre was integrated into its control system, developed at Lola Institute too. The control system was obtained by extending the Lola-Industrial Robot Language [1] with commands for machine tool, by integration of the new solutions for forward and inverse kinematics of this machine in this control system and by adapting of its trajectory planner to novel commands for tool moving.

In this paper, forward and inverse kinematics has been solved for such machine with an angular head, where axes do not intersect (Fig. 1). Such angular head increases the machining possibilities and helps to avoid singularity positions of the machine work table. Some possible singular positions of the work table have been discussed. Control algorithm for the work table and C_t axis has been given, eliminating their singular positions.

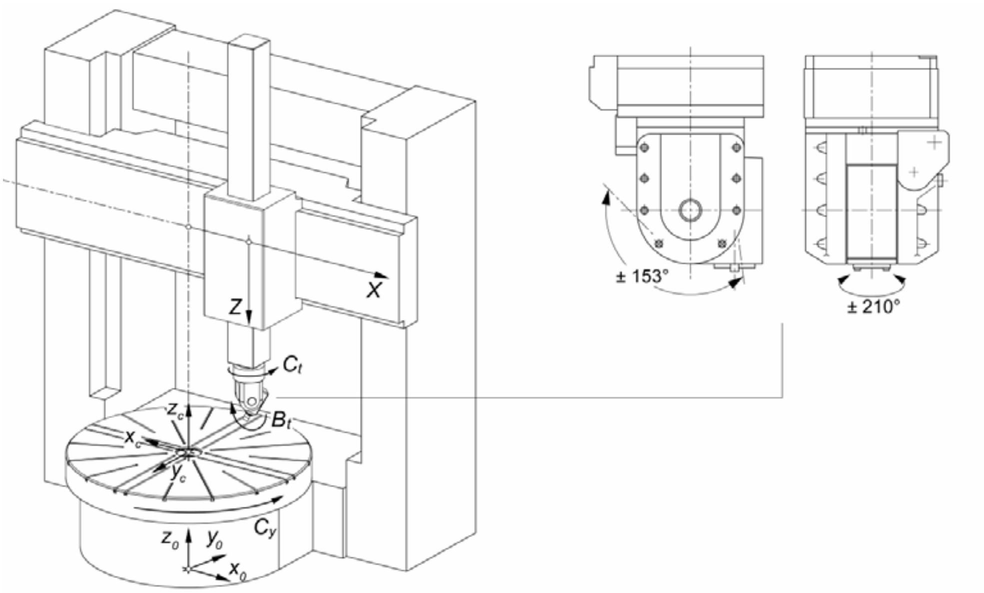

Figure 1. Vertical 5-axis turning center with two axis head.

2. Coordinate frames of machine components and matrices determining their relations

This section defines coordinate frames for the components of vertical 5-axis turning centre and matrices determining their relations. Machine components, their links and coordinate frames are denoted using the Denavit-Hartenberg convention (D-H) [2,3,4]. The machine is viewed as a system consisting of two entities performing cooperative motion; one entity comprises a work table with a workpiece, swivelling according to a certain law, and another is a serial mechanism with 2 translational and 2 rotational axes, carrying the cutting tool. (Fig. 2) shows frames for machine components. The machine base is denoted by *0*, and the last serial component by *4*. The table rotation by angle *c* is denoted with C_y. The first 2 serial links are translational and another 2 rotational, so the corresponding translatory movements d_1 and d_2 and angles θ_3 and θ_4 are variables. It was adopted that the angle θ_3 is positive when the component's 3 rotation is in the negative mathematical direction, and that the angles θ_4 and *c* are positive when the component's 4 rotation and the rotating table is in the positive mathematical direction. D-H parameters of machine components are given in Tab. 1.

Table 1 D-H parameters of vertical 5-axis turning centre components

Link	Variable	a [mm]	a_y [mm]	d [mm]	α [°]	θ [°]
1-Z	d_1	0	0	d_{1c}	$\alpha_1=90, \alpha'_1=90$	$\theta_{1a}=90$
2-X	d_2	0	0	0	0	0
3	$(-)\theta_3$	a_3	0	0	$\alpha_3=90$	$\theta_{3a}=90$
4	θ_4	0	0	0	$\alpha_4=-90$	0
5-C_y	C	δ_{xc}	δ_{yc}	0	0	0

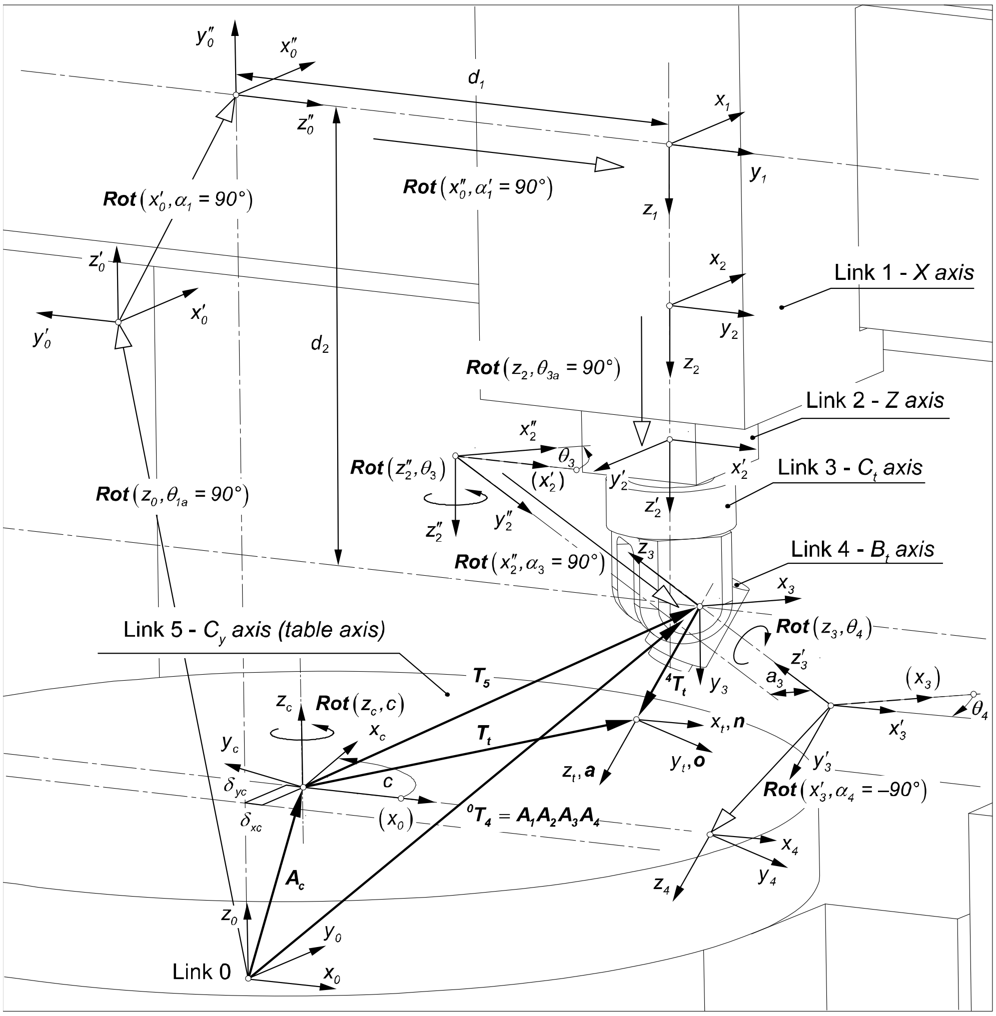

Figure 2. Coordinate frames of vertical 5-axis turning centre components.

The homogenous matrix that transforms the coordinates of a point from frame $x_n y_n z_n$ to frame $x_m y_m z_m$ is denoted by $^n\boldsymbol{T}_m$. The homogenous transformation describing the relation between one link and the next is called $\boldsymbol{A}_i$=$\boldsymbol{A}(i\text{-}1,i)$ matrix [4]. The following homogenous matrices for the coordinate frames of the machine links are defined to derive the kinematic equations for the machine:

$$\boldsymbol{A_1} = \boldsymbol{A}(0,1) = \boldsymbol{Trans}(z_0, d_{1c})\boldsymbol{Rot}(z_0, \theta_{1a})\boldsymbol{Rot}(x_0', \alpha_1)\boldsymbol{Trans}(z_0'', d_1)\boldsymbol{Rot}(x_0'', \alpha_1')$$

$$\boldsymbol{A}_2 = \boldsymbol{A}(1,2) = \boldsymbol{Trans}(z_1, d_2)$$

$$\boldsymbol{A}_3 = \boldsymbol{A}(2,3) = \boldsymbol{Rot}(z_2, \theta_{3a})\boldsymbol{Rot}(z_2'', \theta_3)\boldsymbol{Trans}(x_2'', a_3)\boldsymbol{Rot}(x_2'', \alpha_3)$$

$$\boldsymbol{A}_4 = \boldsymbol{A}(3,4) = \boldsymbol{Rot}(z_3, \theta_4)\boldsymbol{Rot}(x_3', \alpha_4)$$

By using the convenient shorthand notation $sin(\varphi)=s_\varphi$ and $cos(\varphi)=c_\varphi$ the transformation matrices defined above are written as follows:

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0 & d_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & d_{1c} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} s_3 & 0 & c_3 & s_3 a_3 \\ c_3 & 0 & -s_3 & c_3 a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{A}_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{A}_c = \begin{bmatrix} c_c & -s_c & 0 & \delta_{xc} \\ s_c & c_c & 0 & \delta_{yc} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (1)-(5)$$

3. Programming of the machine

5-axis programs are generated by CAD/CAM systems or, manually, by G codes. Programming of the milling operations on the vertical 5-axis turning centre is performed in the movable working table coordinate system $x_c y_c z_c$ and not in the machine basic coordinate system $x_0 y_0 z_0$ (Fig. 2).

In programming by G codes the tool orientation is given in Euler angles or RPY angles via A_t, B_t, and C_t (in degrees) or by the tool direction vector which points from the tool tip towards the toolholder. If we define the approach vector $\boldsymbol{a}=a_{x5}\boldsymbol{i}+a_{y5}\boldsymbol{j}+a_{z5}\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors along the x_c, y_c and z_c coordinate axes of the rotating table, which lies in the z_t direction from which the tool approaches the workpiece (Fig. 2), the components of the tool direction vector would be: $-a_{x5}$, $-a_{y5}$ and $-a_{z5}$.

The output of the CAM systems is cutter locations, X_t, Y_t, Z_t, $-a_{x5}$, $-a_{y5}$ and $-a_{z5}$, which define the tool positions and the tool direction vectors with respect to the workpiece coordinate system given in the CL data file [5,6,3]. The tool path between two CL points is a straight line relative to the workpiece. CL motion commands from the CL data file are further converted in motion commands of the NC program (in G code).

Afterward, the tool path, given in NC program, is converted in the sequence of consecutive positions of all machine axes that will produce the desired tool location (inverse kinematics). The calculation of inverse kinematics can be performed either by the CAD/CAM system, by the post-processor or by the NC unit. The control system developed in Lola Institute for vertical 5-axis turning centre calculates complete path interpolation and inverse kinematics in real time.

4. Forward kinematics

The forward kinematics is used to calculate the tool position and orientation X_t, Y_t, Z_t, B_t and C_t from the machine axis variables: c, d_1, d_2, θ_3 and θ_4. In vertical 5-axis turning centre we will determine the position and orientation of the component 4 and the tool relative to the rotating table. It is obvious from (Fig. 2) that the component 4 position and orientation relative to the machine base is given by the equation ${}^0\boldsymbol{T}_4=\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\boldsymbol{A}_4$ and relative to the rotating table by the equation

$$\boldsymbol{T}_5=\begin{bmatrix} n_{x5} & o_{x5} & a_{x5} & X_5 \\ n_{y5} & o_{y5} & a_{y5} & Y_5 \\ n_{z5} & o_{z5} & a_{z5} & Z_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\boldsymbol{A}_c^{-1}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\boldsymbol{A}_4=$$
$$=\begin{bmatrix} c_c c_{34}+s_c s_3 c_4 & c_c s_3 - s_c c_3 & -c_c c_3 s_4 - s_c s_{34} & X_5 \\ -s_c c_{34}+c_c s_3 c_4 & -s_c s_3 - c_c c_3 & s_c c_3 s_4 - c_c s_{34} & Y_5 \\ -s_4 & 0 & -c_4 & Z_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

where is: $X_5=c_c(d_1+c_3a_3-\delta_{xc})+s_c(s_3a_3-\delta_{yc})$,
$Y_5=-s_c(d_1+c_3a_3-\delta_{xc})+c_c(s_3a_3-\delta_{yc})$, $Z_5=d_{1c}-d_2$
Tool position relative to the rotating table is determined by matrix:

$$\boldsymbol{T}_t=\boldsymbol{T}_5\,{}^4\boldsymbol{T}_t=\boldsymbol{T}_5\begin{bmatrix} 1 & 0 & 0 & -r_t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Here l_t is length, r_t is radius and ${}^4\boldsymbol{T}_t$ is tool position matrix relative to the component 4 (Fig. 2). In an initial position it is: $c=\theta_3=\theta_4=C_t=B_t=0°$,

$$\boldsymbol{T}_t=\begin{bmatrix} 1 & 0 & 0 & d_1-r_t \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & d_{1c}-d_2-l_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Now, using the tool orientation matrix terms [$\boldsymbol{n}$ $\boldsymbol{o}$ $\boldsymbol{a}$], obtained by Eq. (6), tool orientation angles will be determined.

4.1. Calculations of the RPY tool orientation angles

There follows the analysis and discussion of the solutions used to calculate angles $-90° \le B_t \le 90°$ and $-180° \le C_t \le 180°$ defined by RPY orientation angles. Tool orientation matrix for

RPY angles and for the case when $A_t=180°$ reads:
ORIZYX(C_t,B_t,A_t) = $\mathbf{Rot}(z,C_t)\,\mathbf{Rot}(y,B_t)\,\mathbf{Rot}(x,A_t)$, i.e.

$$\begin{bmatrix} n_{x5} & o_{x5} & a_{x5} \\ n_{y5} & o_{y5} & a_{y5} \\ n_{z5} & o_{z5} & a_{z5} \end{bmatrix} = \begin{bmatrix} c_{C_t}c_{Bt} & s_{C_t} & -c_{C_t}s_{Bt} \\ s_{C_t}c_{Bt} & -c_{C_t} & -s_{C_t}s_{Bt} \\ -s_{Bt} & 0 & -c_{Bt} \end{bmatrix} \tag{9}$$

From here it is obtained $\mathbf{Rot}(z,C_t)^{-1}\,[\mathbf{n}\ \mathbf{o}\ \mathbf{a}] = \mathbf{Rot}(y,B_t)\,\mathbf{Rot}(x,A_t)$, i.e.

$$\begin{bmatrix} c_{C_t}n_{x5}+s_{C_t}n_{y5} & c_{C_t}o_{x5}+s_{C_t}o_{y5} & c_{C_t}a_{x5}+s_{C_t}a_{y5} \\ -s_{C_t}n_{x5}+c_{C_t}n_{y5} & -s_{C_t}o_{x5}+c_{C_t}o_{y5} & -s_{C_t}a_{x5}+c_{C_t}a_{y5} \\ n_{z5} & o_{z5} & a_{z5} \end{bmatrix} = \begin{bmatrix} c_{B_t} & 0 & -s_{B_t} \\ 0 & -1 & 0 \\ -s_{B_t} & 0 & -c_{B_t} \end{bmatrix} \tag{10}$$

and $[\mathbf{n}\ \mathbf{o}\ \mathbf{a}]\,\mathbf{Rot}(x,A_t)^{-1}\,\mathbf{Rot}(y,B_t)^{-1} = \mathbf{Rot}(z,C_t)$,

i.e. $$\begin{bmatrix} c_{B_t}n_{x5}-s_{B_t}a_{x5} & -o_{x5} & -s_{B_t}n_{x5}-c_{B_t}a_{x5} \\ c_{B_t}n_{y5}-s_{B_t}a_{y5} & -o_{y5} & -s_{B_t}n_{y5}-c_{B_t}a_{y5} \\ c_{B_t}n_{z5}-s_{B_t}a_{z5} & -o_{z5} & -s_{B_t}n_{z5}-c_{B_t}a_{z5} \end{bmatrix} = \begin{bmatrix} c_{C_t} & -s_{C_t} & 0 \\ s_{C_t} & c_{C_t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Using terms of the matrix Eqs. (9), (10) and (11), it is possible to determine the angles B_t and C_t in a few ways. These solutions will be analyzed now. Some solutions differ mutually by $\pm 180°$, however, in some cases, when the argument of a function *atan2* reads *0,0* inaccurate results are obtained, which are inapplicable.

The solutions of equations that yield values of the angles B_t and C_t will be analyzed using Eq. (9) to obtain tool orientation angle, by virtue of known values for these angles. Afterward, using the terms of this matrix, orientation angles will be calculated. Only solutions that for each tool orientation position yield solutions equal to starting tool orientation angles will be adopted.

4.1.1. Calculations of the angle B_t

The terms (3,1) and (3,3) of matrix Eqs. (9) and (10) yield:

$$B_t = a\tan 2(-n_{z5}, -a_{z5}) \tag{12}$$

This solution is independent of the angle C_t, while the components n_{z5} and a_{z5} (Fig. 3) of the argument of a function *atan2* are not simultaneously equal to zero in any position of the tool, which would result in ambiguity or inaccurate solution. Consequently, this solution gives an accurate result in any tool position.

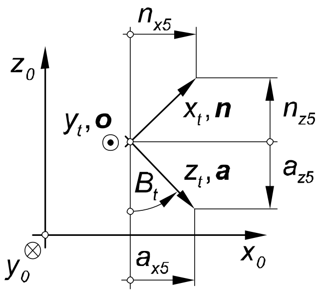

Figure 3. Tool orientation vectors.

If calculations for the angle B_t are done after those for the angle C_t, then other expressions can be also used. Using terms (1,3) and (3,3) or (1,3) and (1,1) or (3,1) and (1,1), respectively, of Eq. (10) the expressions (13), (14) and (15) can be obtained. Like the expression (12), the expressions (13), (14) and (15) always yield accurate results.

$$B_t = atan2(-c_{C_t}a_{x5} - s_{C_t}a_{y5}, -a_{z5}) \tag{13}$$

$$B_t = atan2(-c_{C_t}a_{x5} - s_{C_t}a_{y5}, c_{C_t}n_{x5} + s_{C_t}n_{y5}) \tag{14}$$

$$B_t = atan2(-n_{z5}, c_{C_t}n_{x5} + s_{C_t}n_{y5}) \tag{15}$$

The term (1,3) of Eq. (11) (Fig. 3) yields the solutions

$$B_t = atan\,2(a_{x5}, -n_{x5}) \text{ and } B_t = atan\,2(-a_{x5}, n_{x5}) \qquad (16), (17)$$

The values for these two solutions differ mutually by $\pm 180°$. If $C_t \in (90°, 180°]$ or if $C_t \in (-90°, -180°]$ an accurate result is obtained by Eq. (16), and if $C_t \in (-90°, 90°)$ an accurate result is obtained by Eq. (17). However, for the values of $C_t = -90°$ and $C_t = -90°$, the components n_{x5} and a_{x5} are always equal to zero. Therefore, these two equations are inapplicable in these cases.

The term (2,3) of Eq. (11) gives the solutions

$$B_t = atan2(a_{y5}, -n_{y5}) \text{ and } B_t = atan2(-a_{y5}, n_{y5}) \qquad (18), (19)$$

The values for these solutions also differ mutually by $\pm 180°$. If $C_t \in (0°, 180°)$, an accurate result is obtained by Eq. (19), and if $C_t \in (0°, -180°)$ an accurate result is obtained by Eq. (18). However, for the values of $C_t = 0°$ and $C_t = 180°$, the components n_{y5} and a_{y5} are always equal to zero. Therefore, these two equations in these two cases are inapplicable.
Sine of the angle B_t can be calculated, for the known value of the angle C_t, using the terms (1,3) or (2,3) of Eq. (9), therefore $sin(B_t) = -a_{x5}/c_{Ct}$ or $sin(B_t) = -a_{y5}/s_{Ct}$. Cosine of the angle B_t can be calculated, in this case, using the terms (1,1) or (2,1) of mentioned equation, therefore $cos(B_t) = n_{x5}/c_{Ct}$ or $cos(B_t) = n_{y5}/s_{Ct}$. However, using these terms does not always yield accurate results, and in some cases there occurs division by zero, so these terms are not suitable for calculations of the angle B_t.

4.1.2. Calculations of the angle C_t

The terms (1,2) and (2,2) of Eq. (11) give

$$C_t = atan2(o_{x5}, -o_{y5}) \tag{20}$$

The components o_{x5} and o_{y5} of the argument of a function *atan2* are independent of the angle B_t, they always lie in the x_0y_0 plane ($o_{z5}=0$) and are never simultaneously equal to zero. Therefore, the expression (20) can be used to calculate the angle C_t.

When the value of angle B_t is known, the angle C_t can be calculated using the expressions obtained by the terms (2,1) and (1,1) or (1,2) and (1,1) or (2,2) and (1,1) of Eq. (11), respectively.

$$C_t = atan2(c_{B_t} n_{y5} - s_{B_t} a_{y5}, c_{B_t} n_{x5} - s_{B_t} a_{x5}) \tag{21}$$

$$C_t = atan2(o_{x5}, c_{B_t} n_{x5} - s_{B_t} a_{x5}) \tag{22}$$

$$C_t = atan2(c_{B_t} n_{y5} - s_{B_t} a_{y5}, -o_{y5}) \tag{23}$$

Like Eq.(20), the Eqs. (21, 22 and 23) always yield accurate results.
The term (2,3) of Eq. (10) gives the solutions

$C_t = atan2(a_{y5}, a_{x5})$ and $C_t = atan2(-a_{y5}, -a_{x5})$ (24), (25)

The values of these two solutions differ mutually by $\pm 180°$. If $B_t \in [-90°, 0°)$ accurate results are obtained by Eq. (24), and if $B_t \in [0°, 90°)$ accurate results are obtained by Eq. (25). However, for $B_t = 0°$ the components a_{x5} and a_{y5} are always equal to zero. Hence, these two equations in this case are inapplicable.

The term (1,2) of Eq. (10) gives the solutions

$C_t = atan2(n_{y5}, n_{x5})$ and $C_t = atan2(-n_{y5}, -n_{x5})$ (26), (27)

If $B_t \in (-90°, 90°)$ Eq. (26) yields an accurate result, while Eq. (27) yields the result differing by $\pm 180°$. However, for $B_t = \pm 90°$, the components n_{x5} and n_{y5} are always equal to zero. Consequently, these two equations in this case are inapplicable.

5. Inverse kinematics

The inverse kinematics is used to determine the set of axis variables c, d_1, d_2, θ_3 and θ_4 that will produce the desired cutter location (X_t, Y_t, Z_t, $-a_{x5}$, $-a_{y5}$ and $-a_{z5}$) given in the CL data file, or X_t, Y_t, Z_t, C_t and B_t given in G codes. Control unit primarily utilizes the inverse kinematics. In this case, during machining in each interpolation period, i.e. for each interpolated point of motion, it determines the cutting tool position and orientation relative to the swivelling table, respectively movable work table coordinate system. This section discusses some possible machine singular positions. Control algorithms are given for the work table and the axis C_t that eliminate their singular positions.

Here we will determine machine component positions in virtue of its tool position relative to the rotating table (C_y axis) given by the matrix $\boldsymbol{T}_t$, Eq. (7). The component 4 position relative to the rotating table is defined by the expression:

$$\boldsymbol{T}_5 = \boldsymbol{T}_t\,{}^4\boldsymbol{T}_t^{-1} \tag{28}$$

This way, for tool orientation defined by RPY and for $A_t = 180^\circ$ the preceding equation reads:

$$\boldsymbol{T}_5 = \begin{bmatrix} c_{C_t}c_{B_t} & s_{C_t} & -c_{C_t}s_{B_t} & c_{C_t}c_{B_t}r_t + c_{C_t}s_{B_t}l_t + X_t \\ s_{C_t}c_{B_t} & -c_{C_t} & -s_{C_t}s_{B_t} & s_{C_t}c_{B_t}r_t + s_{C_t}s_{B_t}l_t + Y_t \\ -s_{B_t} & 0 & -c_{B_t} & -s_{B_t}r_t + c_{B_t}l_t + Z_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

Multiplying both sides of Eq. (6) by the matrix $\boldsymbol{A}_c$ on the left side we obtain Eq. $\boldsymbol{A}_c\boldsymbol{T}_5 = {}^0\boldsymbol{T}_4 = \boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\boldsymbol{A}_4$, i.e.

$$\begin{bmatrix} c_c n_{x5} - s_c n_{y5} & c_c o_{x5} - s_c o_{y5} & c_c a_{x5} - s_c a_{y5} & c_c X_5 - s_c Y_5 + \delta_{xc} \\ s_c n_{x5} + c_c n_{y5} & s_c o_{x5} + c_c o_{y5} & s_c a_{x5} + c_c a_{y5} & s_c X_5 + c_c Y_5 + \delta_{yc} \\ n_{z5} & o_{z5} & a_{z5} & Z_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} c_{34} & s_3 & -c_3 s_4 & d_1 + c_3 a_3 \\ s_3 c_4 & -c_3 & -s_{34} & s_3 a_3 \\ -s_4 & 0 & -c_4 & d_{1c} - d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{30}$$

Multiplying Eq. (30) consecutively by $\boldsymbol{A}_1^{-1}$, then by $\boldsymbol{A}_2^{-1}$ and lastly by $\boldsymbol{A}_3^{-1}$ on the left side we obtain Eq. $\boldsymbol{A}_3^{-1}\boldsymbol{A}_2^{-1}\boldsymbol{A}_1^{-1}\boldsymbol{A}_c\boldsymbol{T}_5 = {}^3\boldsymbol{T}_4 = \boldsymbol{A}_4$, i.e.

$$\begin{bmatrix} .... & .... & s_3(s_c a_{x5} + c_c a_{y5}) + c_3(c_c a_{x5} - s_c a_{y5}) & .... \\ .... & .... & -a_{z5} & .... \\ .... & .... & c_3(s_c a_{x5} + c_c a_{y5}) - s_3(c_c a_{x5} - s_c a_{y5}) & .... \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{31}$$

5.1. Calculation of the angle c

The term (2,4) of Eq. (30) reads:

$$s_c X_5 + c_c Y_5 + \delta_{yc} = s_3 a_3 \tag{32}$$

The term (1,2) of Eq. (30) yields $s_3 = c_c o_{x5} - s_c o_{y5}$ (33)

The terms (2,3) and (3,1) of Eq. (30) yield $s_3 = (s_c a_{x5} + c_c a_{y5}) / n_{z5}$ (34)

The terms (2,1) and (3,3) of Eq. (30) yield $s_3 = (-s_c n_{x5} - c_c n_{y5}) / a_{z5}$ (35)

It is possible to calculate s_3 only by the help of the components of vector $\boldsymbol{a}$, obtained from CL data file, but in this case it is needed to have the information on the sign of the angle θ_4. Namely, for $\theta_4 \neq 0°$, the terms (2,3) and (3,3) of Eq. (30) yield

$$s_3 = (-s_c a_{x5} - c_c a_{y5}) / (sign(\theta_4)\sqrt{1 - a_{z5}^2}) \quad (36)$$

and for $\theta_4 = 0°$ is $s_3 = 0$.

Now, Eq. (32) can be written in the following form:

$$\frac{tan(c)}{\sqrt{1 + tan^2(c)}} p_x + \frac{p_y}{\sqrt{1 + tan^2(c)}} = -\delta_{yc} \quad (37)$$

Eqs. (33) and (37) yield $p_x = X_5 + o_{y5}\, a_3$ and $p_y = Y_5 - o_{x5}\, a_3$ (38)

Eqs. (34) and (37) yield $p_x = X_5 - a_{x5}\, a_3 / n_{z5}$ and $p_y = Y_5 - a_{y5}\, a_3 / n_{z5}$ (39)

Eqs. (35) and (37) yield $p_x = X_5 + n_{x5}\, a_3 / a_{z5}$ and $p_y = Y_5 + n_{y5}\, a_3 / a_{z5}$ (40)

Eqs. (36) and (37) yield $p_x = X_5 + a_{x5}\, a_3 / (sign(\theta_4)\sqrt{1 - a_{z5}^2})$ and

$$p_y = Y_5 + a_{y5}\, a_3 / (sign(\theta_4)\sqrt{1 - a_{z5}^2}) \text{ for } \theta_4 \neq 0° \quad (41)$$

and $p_x = X_5$ and $p_y = Y_5$ for $\theta_4 = 0°$.

Eq. (37) can be written in the form $p_x\, tan(c) + p_y = -\delta_{yc}\sqrt{1 + tan^2(c)}$ and

$p_x^2\, tan^2(c) + p_y^2 + 2 p_x\, p_y\, tan(c) = \delta_{yc}^2\,(1 + tan^2(c))$ i.e.

$(p_x^2 - \delta_{yc}^2)\, tan^2(c) + 2\, p_x\; p_y\; tan(c) + p_y^2 - \delta_{yc}^2 = 0$.

The solution for the preceding equation is

$: c = arctan((-p_x\, p_y \pm \sqrt{p_x^2\, p_y^2 - (p_x^2 - \delta_{yc}^2)(p_y^2 - \delta_{yc}^2)})/(p_x^2 - \delta_{yc}^2)$.

As the table swivelling angle decreases with the table moving in the direction of the y_o axis, caused by thermal dilatations ($\delta_{yc}>0$), the sign – will be adopted in the preceding expression, such that:

$$c_{90} = a\tan 2(-p_x\, p_y - \delta_{yc}\sqrt{p_x^2 + p_y^2 - \delta_{yc}^2},\; p_x^2 - \delta_{yc}^2) \quad (42)$$

The table swivelling angle c in the expression (42) is denoted by c_{90} for the reason that this expression always gives angle c in the range [$-90°,90°$], except when $p_x^2 < \delta_{yc}^2$, which is a special case. For the case when the angular head axes intersect, $a_3=0$, and there is no thermal dilatation, $\delta_{yc}=0$, the solution for Eq. (32) would read: $c=atan2(-Y_5,X_5)$.

5.1.1. Algorithm for calculations of the angle c

The X axis feed will be limited so that the cutting tool moves from the table axis to the maximum positive value. To achieve this, and given that the value of X_5 can be negative too, the table swivelling angle c should be in the minimum range of [$-180°,180°$]. In order to reduce the additional positioning of the workpiece and angular head during milling, the range of [$-360°+c_{90},360°+c_{90}$] will be adopted for the table swivelling angle c. Hence, it is necessary to extend the range of the angle c, obtained by the expression (42), from [$-90°,90°$] to [$-360°+c_{90},360°+c_{90}$]. To achieve this, but also to avoid uncontrolled work table swivelling in singular positions by approximately $\pm 180°$ or $\pm 360°$, this paper proposes a novel algorithm for calculating the angle c. It consists of three steps presented below.

Step 1. Step 1 involves the calculations of the angle c for the case without thermal dilatation. The range of the angle c obtained here is [$-180°,180°$]. For $\delta_{yc}=0$, the expression (42) reads:

$$c = atan2(-p_y, p_x) \tag{43}$$

Let angle c be denoted by c_{prev} for the previous interpolation period. The angle c increment for the next interpolation period will be

$$\Delta c = c - c_{prev} \tag{44}$$

Step 2. In Step 2 it is checked if the value of the angle c, calculated in Step 1 for a single interpolation period, changes by approximately $\Delta c \pm 180°$ or $\Delta c \pm 360°$. The procedure is presented, which makes this impossible and which, if necessary, extends the range of the angle c from [$-180°,180°$] to [$-360°,360°$]. For calculations of the angle c in the next interpolation step, the value will be assigned to $c_{prev}=c$.

Step 3. In Step 3 the angle c is determined for the case if $\delta_{yc}\neq 0$. First, in virtue of the expression (42) the value of the angle c_{90} is calculated. Using this and the value of the angle c obtained in Step 2, the range of the angle c is extended from [$-90°,90°$] to [$-360°+c_{90},360°+c_{90}$].

The difference between the angle c_{90} obtained by the expression (42) and the angle c calculated in Step 2 will be denoted by

$$\Delta c_{90} = c_{90} - c \tag{45}$$

Now, the value of the table swivelling angle (c_{90}) will be corrected by $\pm 180°$ or by $\pm 360°$ as follows:
If $\Delta c<-7\pi/4$, then $c=c_{90}+2\pi$;

If $\Delta c \in [-\pi/4, -5\pi/4]$, then $c=c_{90}+\pi$;
If $\Delta c \in [\pi/4, 5\pi/4]$, then $c=c_{90}-\pi$;
If $\Delta c > 7\pi/4$, then $c=c_{90}-2\pi$;
Else $c=c_{90}$;
Note: If $p_x^2 + p_y^2 < \delta_{yc}^2$ in the expression (42), the discriminate of the argument of a function in this expression (42) being smaller than zero, therefore its argument is not a real number. In this case, the solution does not exist for angle c that will lead the surface of machining to the machine X axis. It should be waited for the table base to cool and for the table axis to return to the zero position.

5.2. Work table singular positions

A special issue associated with rotary axes is the problem of singular configuration. Sometimes the singular configuration represents the borderline between two possible solutions for the inverse kinematics, but most often the singular configuration is at the end position of one of the rotary axes. The problem with the singularity is that the C axis sometimes has to make a quick turn, often $\pm 180°$, in order to produce the desired tool motion. In the subsection 5.1.1. the algorithm given prevents the work table to swivel by $\pm 180°$ or $\pm 360°$, in a single interpolation period.
This algorithm has also extended the table swivelling range to $[-360°+c_{90}, 360°+c_{90}]$, whereby many singular positions were avoided as well.

Some possible singular positions of the work table have still remained. They will be discussed now. Analyzing the expression (42), it is noticeable that in some specific situations, in order to correct motion caused by thermal dilatation δ_{yc}, the work table swivelling occurs by $\pm 180°$ or $\pm 90°$ in short time intervals. These singular positions will be presented below.

1. If $p_x=0$, then $c = atan2(-\sqrt{p_y^2 - \delta_{yc}^2}, -\delta_{yc})$. This entails that for the case when $p_y \to \delta_{yc}$, and for $\delta_{yc}>0$, there holds $c \to 180°$, and for $\delta_{yc}<0$, it holds $c \to 0°$. If $p_y \le \delta_{yc}$, the discriminate $D = p_y^2 - \delta_{yc}^2$ is smaller than or equal to zero, therefore the assigned motion is not achievable.
2. If $p_y=0$, then $c = atan2(-\delta_{yc}, \sqrt{p_x^2 - \delta_{yc}^2})$. This entails that for the case when $p_x \to \delta_{yc}$ and for $\delta_{yc}>0$, there holds $c \to 90°$, and for $\delta_{yc}<0$, it holds $c \to 90°$. If $p_x \le \delta_{yc}$, the discriminate $D = p_x^2 - \delta_{yc}^2$ is smaller than zero, therefore the assigned motion is not achievable.
3. A special case of machining is when $p_y=0$ and $\delta_{yc}=0$. Then $c=atan2(0,p_x)$ holds, so in changing the sign of the parameter p_x, there occurs the table swivelling by the angle of $\pm 180°$. Table swivelling is performed here at programmed velocity and programmed acceleration, while the remaining 4 axes do not rotate. The control algorithm itself provides for further continuation of motion along the X axis in a positive direction.

During previous 3 motions the machine passes through singular points, so abrupt table swivelling occurs. Owing to the two-axis angular head with the axis that do not intersect it is possible in singular position to change its orientation in the way to change the angle C_t for $\pm 180^\circ$ and to change the sign of the angle B_t. This will change the parameters p_x and p_y in Eq. (42), and thereby avoid work table singular positions.

5.3. Calculations of the angle θ_3

The angle θ_3, similar to the angle C_t, as described in subsection 4.2.2, can be calculated in a number of ways. In case that programming is done in G code and if tool orientation is assigned by the angles B_t and C_t, using Eq. (9), we can calculate tool orientation matrix coefficients. Afterward, using terms (1,2) and (2,2) of Eq. (30), we can calculate the angle θ_3 by the equation:

$$\theta_3 = atan2(c_c o_{x5} - s_c o_{y5}, -s_c o_{x5} - c_c o_{y5}) \tag{46}$$

In 4.1.2. it has been shown that components o_{x5} and o_{y5} of the argument of a function *atan2* do not depend on the angle B_t, that they always lie in the x_0y_0 plane and that they are never simultaneously equal to zero. Hence, for this case, the expression (46) can be used for calculations of the angle θ_3. When the table swivelling angle c equals zero, the expression (20) used to calculate the angle C_t is identical to the expression (46), which now reads: $\theta_3=atan2(o_{x5},-o_{y5})$.

As mentioned in section 3, when CAD/CAM system is used in programming, its output are the assigned tool position and direction vector ***a*** given in the CL data file. This means that in this case in order to determine the machine axis position all 9 tool orientation matrix terms cannot be used, but only the components a_{x5}, a_{y5} and a_{z5}. The term (3,3) of Eq. (31) yields

$$\theta_3 = atan2(s_c a_{x5} + c_c a_{y5}, c_c a_{x5} - s_c a_{y5}) \tag{47}$$

And:

$$\theta_3 = atan2(s_c a_{x5} + c_c a_{y5}, c_c a_{x5} - s_c a_{y5}) \tag{48}$$

Previous 2 equations, for the case when the work table swivelling angle c equals zero, are identical to Eqs. (24) and (25). As mentioned in 4.1.2. the values of these two solutions mutually differ by $\pm 180^\circ$. For $B_t=0^\circ$ the components a_{x5} and a_{y5} equal zero, so Eq. (47) always yields $\theta_3=0^\circ$, while Eq. (48) always gives here $\theta_3=180^\circ$, irrespective of the current value of the angle θ_3. Now, we will present the algorithm that will in the application of either of the two previous equations prevent abrupt angular head swivelling around the vertical axis and that will yield solution for the angle θ_3 for $B_t=0^\circ$. Let θ_{3prev} denote the angle θ_3 in the previous interpolation period.

If θ_3-$\theta_{3prev}\in[\pi/2,3\pi/2]$, then $\theta_3=\theta_3-\pi$;
If θ_3-$\theta_{3prev}\in[-\pi/2,3-\pi/2]$, then $\theta_3=\theta_3+\pi$;
If $a_{x5}=0$ and $a_{y5}=0$, then $\theta_3=\theta_{3prev}$;

5.4. Calculations of the angle θ_4

When programming is performed in CAD/CAM system, by the help of the CL data file, the angle θ_4 can be calculated using the terms (1,2) and (2,2) of Eq. (31), such that:

$$\theta_4 = a\tan 2(c_3(s_c a_{y5} - c_c a_{x5}) - s_3(s_c a_{x5} + c_c a_{y5}), -a_{z5}) \tag{49}$$

In case that tool orientation is given by the angles B_t and C_t, the angle θ_4 can be calculated in other manner using different terms of Eqs. (30) and (31), such as, e.g.

$$\theta_4 = a\tan 2(-n_{z5}, -a_{z5}) \tag{50}$$

5.5. Calculations of the parameters d_1 and d_2 (the X and Z axes positions)

The term (1,4), Eq. (30), determining motion along the *X* axis yields:

$$X = d_1 = c_c X_5 - s_c Y_5 - c_3 a_3 + \delta_{xc} \tag{51}$$

The term (3,4), Eq. (30), determining motion along the *Z* axis yields:

$$Z = d_2 = d_{1c} - Z_5 \tag{52}$$

This way, using the cutting tool position relative to the work table, matrix $\boldsymbol{T}_t$, by the help of the expressions (42), (46) or (47) or (48), (49) or (50), (51) and (52), respectively, we have calculated the positions c, θ_3, θ_4, d_1 and d_2 of machine

6. Conclusion

It has been shown that on the 5-axis turning centre with 2 linear and 3 rotational axes, besides the turning, it is possible to achieve 5-axis milling, drilling and boring identical to that on the milling machine with 3 linear and 2 rotational axes. Control algorithm allowing for this was presented. Thanks to the proposed control algorithm, machine programming is possible in identical way as done for the 5-axis milling machine, which essentially simplifies writing the machining program or taking over the CL data from CAD/CAM systems developed for the milling machines.
Forward and inverse kinematics has been solved for the case of utilizing the 2-axis angular head with non-intersecting axes, which increases the achievability of machining in some cases. For the case of control of the angular head with intersecting axes, the mentioned algorithm is simplified.

The proposed algorithm fully eliminates the inaccuracy of machining caused due to base thermal extension. Here the solving of the table swivelling angle was a specific problem. The algorithm was given that performs compensation of the base thermal extension error by correcting mentioned angle. Also, this algorithm extends the angle range, so it is greater than ±*360*°. Thus, additional positioning of the work table and angular head during machining is decreased.

If necessary, it is possible to incorporate into the control algorithm the compensations for some other errors of the machine, such as, for example, the error caused due to cross-bar deflection carrying the ram with angular head, whose weight is significant.

Acknowledgement. This research is supported by the Ministry of Science and Technological Development of Serbia under the project: Research and development of the new generation of vertical 5-axis turning centre (2008-2010), no. 14026.

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