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Original scientific paper

PULL-DOWN INSTABILITY OF THE QUADRATIC NONLINEAR OSCILLATORS

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Abstract. *A nonlinear vibration system, over a span of convincing periodic motion, might break out abruptly a catastrophic instability, but the lack of a theoretical tool has obscured the prediction of the outbreak. This paper deploys the amplitude-frequency formulation for nonlinear oscillators to reveal the critically important mechanism of the pseudo-periodic motion, and finds the quadratic nonlinear force contributes to the pull-down phenomenon in each cycle of the periodic motion, when the force reaches a threshold value, the pull-down instability occurs. A criterion for prediction of the pull-down instability is proposed and verified numerically.*

Key words: *Micro-electromechanical system, Eardrum oscillator, Pull-in instability, Asymmetrical oscillation, Vibration attenuation, Energy harvesting*

1. INTRODUCTION

The nonlinear oscillators with even nonlinearities, especially the quadratic nonlinear oscillators [1-5], became a hot topic recently in both mathematics and engineering, because scientists and engineers cannot exactly differentiate between the periodic motion and the

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fake periodic motion with ease. Scientists have now found that a convincing periodic motion might be a fake one, which is called as the pseudo-periodic motion [6,7], after some cycles of seeming periodic motion, its motion property will be suddenly changed, and the instability occurs, and scientists are often unable to spot the pseudo-periodic motion. Though mathematicians can reveal easily the bifurcation property of such a nonlinear oscillator [8], the pseudo-periodic motion of the quadratic nonlinear oscillator has never been reported, and the critically important mechanism of the pseudo-periodic motion needs to be revealed, so that its instability can be completely avoided.

We consider an important nonlinear oscillator which involves even nonlinearities, the Toda oscillator [9,10,11], which plays an important role in physics:

$$\frac{d^2u}{dt^2}+k(e^u-1)=0 \tag{1}$$

where k is constant. Eq. (1) can be written equivalently in the following form

$$\frac{d^2u}{dt^2}+u+\frac{k}{2}u^2+\frac{k}{6}u^3+\frac{k}{24}u^4+\cdots=0 \tag{2}$$

The Toda oscillator behaves periodically when k is small, however the amplitude might lead to either infinity or zero when the absolute value of k reaches a threshold value.

As another example, we consider a microelectromechanical system, which can be modelled by the following differential equation [12]:

$$\frac{d^2u}{dt^2}+u-\frac{K}{1-u}=0 \tag{3}$$

where u is dimensionless displacement and K is a voltage-related parameter. Eq. (3) can be written as

$$\frac{d^2u}{dt^2}+u-K(1+u^2+u^3+u^4+\cdots)=0 \tag{4}$$

The even nonlinearities in Eq. (4) are the main contributor to the pull-in instability, which should be avoided in the practical applications. The pull-in instability has been widely studied in the micro electromechanical systems (MEMS) [13-17], when the voltage is larger than its threshold value, the pull-in instability occurs, otherwise the systems move periodically. In this paper we focus on a similar phenomenon in quadratic nonlinear oscillators [18,19], and we call it as the pull-down instability.

2. THE QUADRATIC NONLINEAR OSCILLATOR

We consider the following quadratic nonlinear oscillator

$$\frac{d^2u}{dt^2}+u+\beta u^2=0 \tag{5}$$

with the following initial conditions

$$u(0) = A, u'(0) = 0 \tag{6}$$

where u is the displacement, β is a positive constant, A is the initial amplitude.

Eq. (5) is called as the eardrum oscillator [19]. For small values of β, a periodic motion is predicted; however, when it increases to a threshold value, the kinetic energy will be gradually consumed, and finally the periodic motion is forbidden, this phenomenon is called as the pull-down instability.

Fig. 1 shows the dynamical properties of Eq. (5) when A=0.1. When β<<1, a periodic solution is predicted as shown in Fig. 1(b). When β is larger than a threshold value, the pull-down instability occurs as shown in Fig. 1(d). For β near to the threshold value, the system will have a periodic motion first, and after some cycles, the pull-down instability happens suddenly. This phenomenon is called as the pseudo-periodic motion [6,7] as shown in Fig. 1(c).

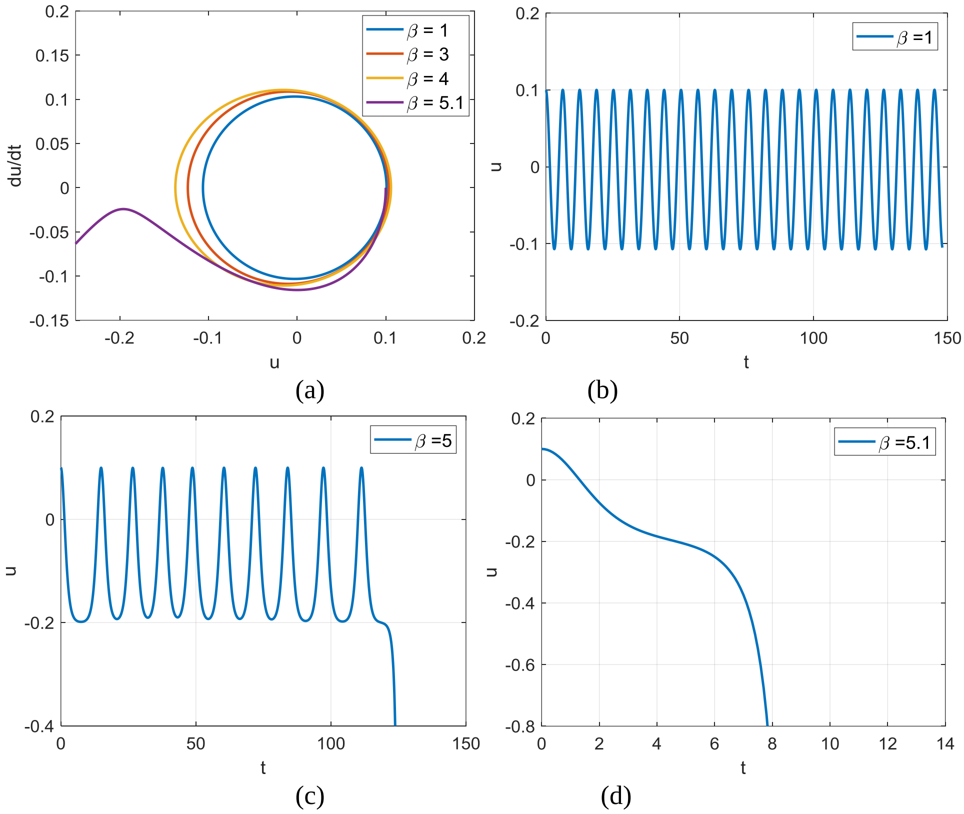

Fig. 1 Periodic motion and pull-down instability of the quadratic nonlinear oscillator when A=0.1. (a) phase diagram; (b) periodic motion; (c) pseudo-periodic motion; and (d) pull-down instability

A periodic motion can be generally expressed by a sinusoidal function $u(t)=B\sin(\omega t+\theta)$, where B and θ are determined by the initial conditions. The displacement of the harmonic motion varies from B to negative B periodically.

In Fig. 1, we find that during the periodic motion (Fig. 1(b)) and the pseudo-periodic motion (Fig. 1(c)), the quadratic force pulls down the system in each cycle. The amplitude changes from 0.1 to negative 0.11 in Fig. 1(b) and negative 0.2 in Fig. 1(c), respectively. This unsymmetrical oscillation phenomenon is interesting and amazing because it cannot be seen in any a conservative nonlinear oscillator [20-22], and it can be concluded that the quadratic nonlinear oscillator cannot be always used for vibration attenuation and an energy harvesting system [23,24]. The unsymmetrical amplitude in each cycle certainly frustrates scientists and engineers, because even the van der Pol oscillator [25] leads to a sinusoidal motion when time tends to infinity.

When β increases to a threshold value, the pull-down instability occurs, and a lack of a theoretical model to predict the time and the condition of the pull-down outbreak has greatly reduced the operation reliability, and hindered its applications.

3. PERIODIC MOTION WITH ASYMMETRIC AMPLITUDES

This section introduces the frequency formulation for nonlinear oscillators [26] and extends to the nonlinear oscillators with even nonlinearities.

Considering a nonlinear equation in the form

$$\frac{d^2u}{dt^2}+\lambda(u)=0\,,\ \ x(0)=A, x'(0)=0 \tag{7}$$

where λ is a nonlinear function of u, generally it can be expressed as

$$\lambda(u)=a_1u+a_3u^3+\cdots+a_{2n+1}u^{2n+1} \tag{8}$$

For a nonlinear oscillator, it requires

$$\frac{\lambda(u)}{u}>0 \tag{9}$$

The frequency formulation is [26]

$$\omega^2=\left.\frac{\lambda(u)}{u}\right|_{u=\frac{\sqrt{3}}{2}A} \tag{10}$$

The frequency formulation is simple but it gives a relatively high accuracy of the frequency-amplitude relationship, and it becomes a universal tool to various vibration systems [27-29].

For $\lambda(u)=a_1u+a_3u^3$, Eq. (7) is the Duffing equation, and Eq. (10) leads to the following result

$$\omega^2=a_1+\frac{3}{4}a_3A^2 \tag{11}$$

Eq. (11) is same as that obtained by the homotopy perturbation method [30]. Using the frequency formulation of Eq. (10), for Eqs. (5) and (6), we have

$$\omega=\sqrt{1+\frac{\sqrt{3}}{2}\beta A} \tag{12}$$

Table 1 shows Eq. (12) has good accuracy of the approximate period, $T=2\pi/\omega$, for $\beta A<<1$, and its accuracy deteriorates greatly when the value of βA becomes large. The errors arise in the quadratic force, $f=\beta u^2$. When the quadratic force is small, Eq. (5) can be solved by the perturbation method [19], but in practical applications, we do not know the criterion for the small value.

Table 1 Comparison the approximate periods with the exact one

A	β	Exact period	Approximate Period Eq.(12)	Error	Approximate Period Eq.(20)	Error (%)
0.1	1	6.266	6.0276	3.8%	6.300	0.55%
0.3	1	6.627	5.5979	15.5%	6.450	2.66%
0.5	0.3	6.308	5.9109	6.2%	6.323	0.24%
0.7	0.3	6.399	5.7795	9.6%	6.363	0.56%

Eq. (10) is valid for nonlinear oscillators with only odd nonlinear terms, our problem has a quadratic nonlinear term, so the frequency formulation has to be modified.

We re-write Eq. (5) in the form

$$\frac{d^2u}{dt^2}+u(1-\beta u)=0, u<0 \tag{13}$$

$$\frac{d^2u}{dt^2}+u(1+\beta u)=0, u>0 \tag{14}$$

Eqs. (13) and (14) give, respectively, the following frequency formulations:

$$\omega_1^2=1-\frac{\sqrt{3}}{2}\beta A \tag{15}$$

$$\omega_2^2=1+\frac{\sqrt{3}}{2}\beta A \tag{16}$$

The solution can be approximately expressed as

$$u(t)=\begin{cases} A\cos\omega_2 t, u>0 \\ (A+\Delta A)\cos\omega_1 t, u<0 \end{cases} \tag{17}$$

where ΔA in the increment of the amplitude when $u<0$.

The periods of Eqs. (13) and (14) are, respectively, given as

$$T_1=\frac{2\pi}{\omega_1}=\frac{2\pi}{\sqrt{1-\frac{\sqrt{3}}{2}\beta A}} \tag{18}$$

$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\sqrt{1+\frac{\sqrt{3}}{2}\beta A}} \tag{19}$$

So the period of Eq. (5) can be calculated as

$$T = \frac{T_1+T_2}{2} = \frac{\pi}{\sqrt{1-\frac{\sqrt{3}}{2}\beta A}} + \frac{\pi}{\sqrt{1+\frac{\sqrt{3}}{2}\beta A}} \tag{20}$$

Eq. (20) shows much better accuracy for $\beta A<0.5$, see Table 1. When $\beta A=1.1547$, the period by Eq. (20) becomes infinite large.

4. Pull-Down Instability

When βA tends to the threshold value, the pull-down instability occurs. The motion of Eq. (5) can be decomposed into two parts as given in Eqs. (13) and (14). It is obvious that Eq. (13) has a tendance of a pull-down motion, and Eq. (14) gives an opposite motion. We consider an extreme case when $u_1=-2A$ for Eq. (13), and $u_2=A$ for Eq. (14), the combined displacement is $u_1+u_2=-2A+A=-A$, so Eq. (5) is still periodic. However if $u_1<-2A$, we have $|u_1+u_2|>A$, under such case, the motion is pulled down, and its periodic motion is forbidden.

We consider the extreme condition of $u_1=-2A$, Eq. (13) can be written approximately as

$$\frac{d^2u}{dt^2} + (1-2\beta A)u = 0 \tag{21}$$

This predicts

$$\omega_1^2 = 1-2\beta A \tag{22}$$

The pull-down instability occurs when $\omega_1^2 < 0$, that is

$$\beta A > 0.5 \tag{23}$$

So $\beta A=0.5$ is the threshold value.

We give another approach to determination of the threshold value. Eq. (5) can be written in the form

$$\frac{1}{2}(\frac{du}{dt})^2 + \frac{1}{2}u^2 + \frac{1}{3}\beta u^3 = H \tag{24}$$

where H is the Hamilton constant. By the initial conditions given in Eq. (6), Eq. (24) becomes

$$\frac{1}{2}(\frac{du}{dt})^2 = \frac{1}{2}(A^2-u^2) + \frac{1}{3}\beta(A^3-u^3) \tag{25}$$

When $u = -2A$, the kinetic energy tends to zero, so the pull-down instability occurs when

$$\frac{1}{2}\left(\frac{du}{dt}\right)^2(u=-2A)=\frac{1}{2}(A^2-(-2A)^2)+\frac{1}{3}\beta(A^3-(-2A)^3)<0 \tag{26}$$

Eq. (26) results in $\beta A>0.5$ as shown in Eq. (23). When $\beta A<0.5$, a periodic motion or a pseudo-periodic motion is predicted. When $\beta A>0.5$, the pull-down instability occurs. To verify this criterion, we consider various values of A and β, and the results are illustrated in Figs. 2-4.

When the value of βA is less than the threshold value, 0.5, but it is near to the value, the pseudo-periodic motion is observed, for example, βA=0.495 in Fig. 2(c); while it is far from the value, for example, βA=0.3 in Fig. 2(c), a periodic motion is seen. So far, we have no criterion for exactly predicting the value of βA for a pseudo-periodic motion, where an unsymmetrical oscillation deteriorates gradually in each cycle, and when u reaches negative 2A, the convincing periodic motion stops suddenly.

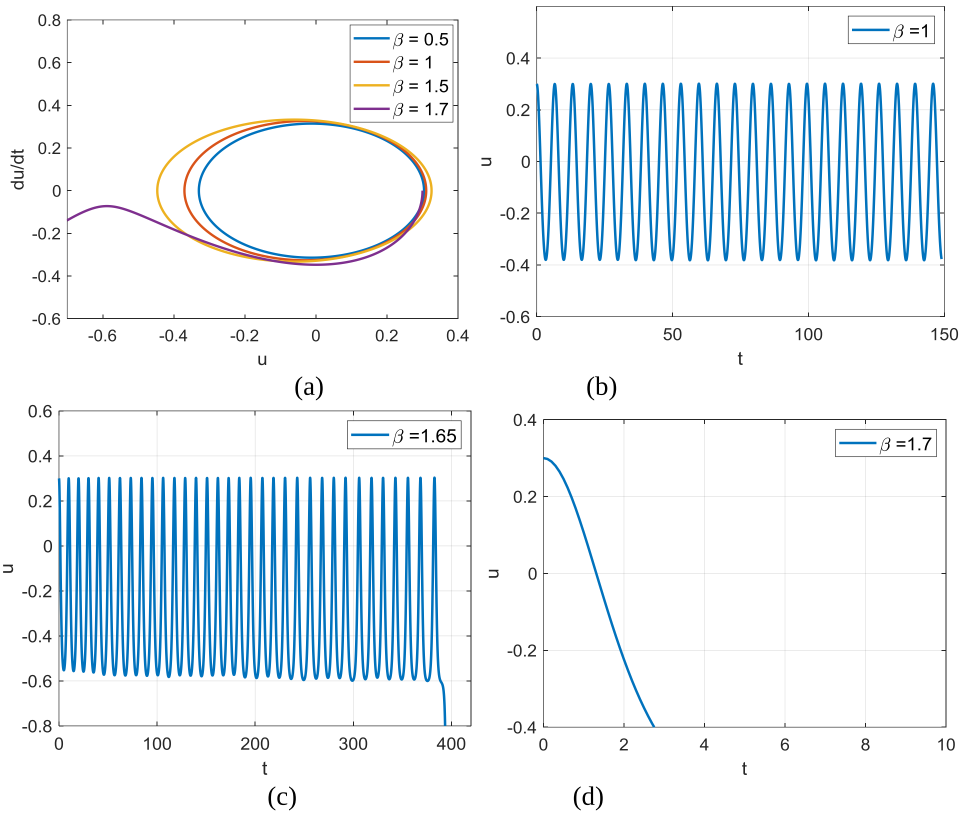

Fig. 2 The criterion of the pull-down instability when A=0.3. (a) phase diagram; (b) periodic motion βA=0.3; (c) pseudo-periodic motion βA=0.495 and (d) pull-down instability βA=0.51

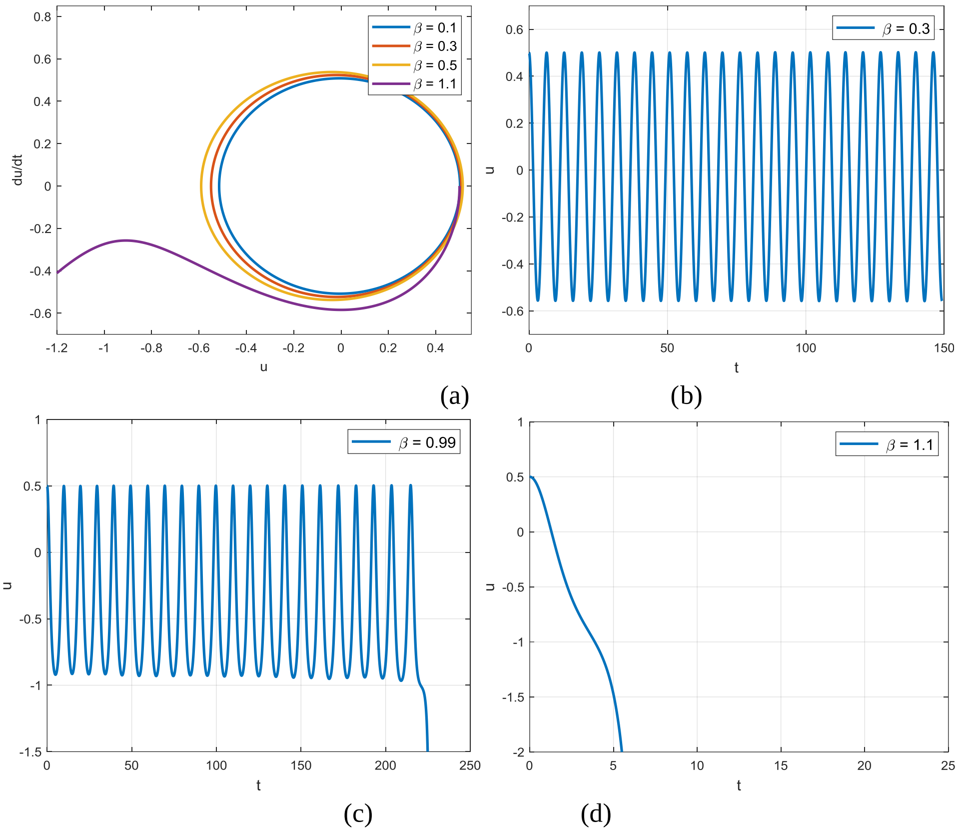

Fig. 3 The criterion of the pull-down instability when A=0.5. (a) phase diagram; (b) periodic motion βA=0.15; (c) pseudo-periodic motion βA=0.495 and (d) pull-down instability βA=0.55

In Figs. 2-4, we can see that when βA is small, a periodic motion is predicted. With gradually increasing βA, a pseudo-periodic motion might occur. Finally, when βA is larger than 0.5, the pull-in instability is observed.

5. Conclusion

This paper proposes an extremely important concept of the pull-down breakout for nonlinear oscillators with even nonlinearities, and it will become a powerful research tool in nonlinear vibration theory. A sophisticated modification of the frequency formulation is suggested, and a criterion is built for judging the pull-down instability, the unsymmetrical amplitude motion, and the pseudo-periodic motion. The amplitude increment, ΔA, in Eq. (17) will be discussed in a forthcoming paper, and when the absolute value of ΔA is larger than A, the pull-down instability occurs abruptly. Hence, the amplitude change is the best parameter to predict the pseudo-periodic motion and the time when the pull-down instability occurs.

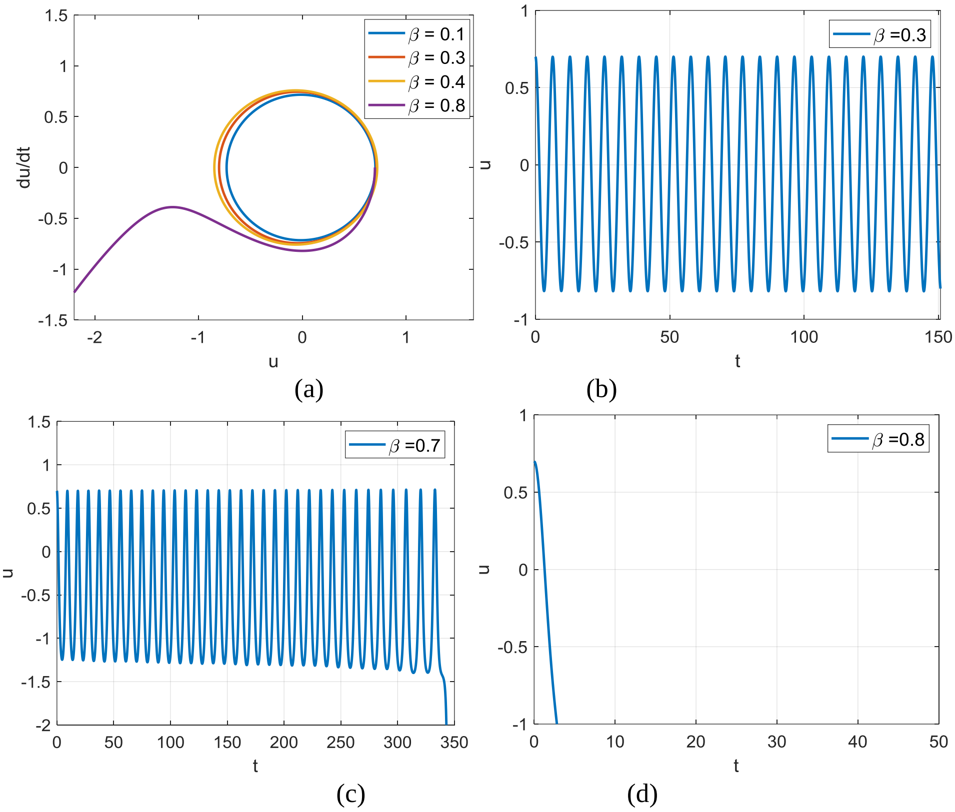

Fig. 4 The criterion of the pull-down instability when A=0.7: (a) phase diagram; (b) periodic motion βA=0.21; (c) pseudo-periodic motion βA=0.49 and (d) pull-down instability βA=0.56

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