ISOLATED SUBORDERS AND THEIR APPLICATION TO COUNTING CLOSURE OPERATORS

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Abstract. In this paper we investigate the interplay between isolated suborders and closures. Isolated suborders are a special kind of suborders and can be used to diminish the number of elements of an ordered set by means of a quotient construction. The decisive point is that there are simple formulae establishing relationships between the number of closures in the original ordered set and the quotient thereof induced by isolated suborders. We show how these connections can be used to derive a recursive algorithm for counting closures, provided the ordered set under consideration contains suitable isolated suborders.

1. Introduction

A widespread and common concept in various areas of mathematics and computer science are hull or closure operators, i.e., idempotent, isotone and extensive endofunctions on some ordered set. The best-known examples include the topological closure in traditional analysis, the (reflexive) transitive closure of a relation or a graph and the Kleene closure in language theory. There are also more complicated and sophisticated appearances as for example in automated reasoning (see e.g. [EBJ$^+$14]), data base theory (as in [DHLM92]) or the algebraic analysis of connected components (as done in [Glü17]). Most of this work uses hulls or closures mainly as a tool for specific purposes but does not investigate the properties of these operators.

If we take a look at the work dealing with actual properties of closures we see that most of this work is concerned with closures on powerset lattices which falls also under the term ”Moore family” (here [CM03] gives a survey; for more recent results see e.g. [BD18]). Some other work deals with closures on lattices although closures can also be defined on general ordered sets.

In the recent year, counting structures of interest has become a rising area of research in order theory and related topics. For example, [AJS21] gives numbers of so-called $d\ell$-structures of various kinds, [QRRV20] counts join-endomorphisms in lattices, [BPBV18] generates and counts a certain kind of bisemilattices whereas the topic of [BBW21] are topological spaces, and different kinds of posets are counted in [FJST20]. However, to our best knowledge, there is no work dealing with the number of closures on general lattices or orders. The only results we are aware of concern the power set lattice $(\mathcal{P}(S), \subseteq)$. For this

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special structure, the exact number of closure operators is known only up to $|S| = 7$ (for curiosity, there are 14.087.648.235.707.352.472 of them, as shown in [CIR10] only in 2010).

The present work introduces a heuristic method for structuring ordered sets in a way that eases under certain circumstances the computation of the number of closure operators. The key idea are so called isolated suborders which are intuitively speaking suborders which have contact with the rest of the ordered set only via their least and greatest element. By means of quotient orders we can reduce the number of elements of the ordered set under consideration and obtain an ordered set with a certain structure for which there are closed formulae for the number of closure operators available. Similar ideas were used already in the predecessor work of [Glü21]. However, there the area of application was restricted to lattices whereas the present work generalizes the results to general ordered sets. Additionally to this generalization, the present work contains also thoughts about the computation of isolated suborders.

The remainder is organized as follows: in Section 2 we introduce some notation we will use in the sequel (on other places, we introduce notation ad hoc to spare the reader annoying look-ups). Section 3 introduces the main topic of the present work, closures, and shows important relations between different characterizations of closures. The main tool of the present work, isolated suborders, are the topic of Section 4 whereas Section 5 studies the interplay between closures and isolated suborders. Because we look at isolated suborders not only as an object of study but will use them as tool in an algorithm we investigate in Section 6 how to compute isolated suborders. In Section 7 we put all our results together to obtain an algorithm for counting closures which can make use of favorable structures of an ordered set under consideration. Finally, the concluding Section 8 gives a short retrospect of the work, raises some open issues and sketches ways of further research.

2. Basic Notions and Properties

In this paper we presuppose general knowledge of order and lattice theory and refer e.g. to [DP02, Grä11, Rom08] for the basics and e.g. to [Bir67, JR92] for more advanced topics. We will, however, recapitulate some topics in order to clarify matters and to obtain a concise notion.

The symbol $\leq$ and derived variants thereof (by indexes or primes) denote always an order. Given an order $\leq$ we use the symbol $\not\leq$ for the relation $\not\leq \Leftrightarrow_{def} \not(x \leq y)$. The associated strict, reverse order, and strict reverse order are denoted by $<$, $\geq$ and $>$, resp. In the case of existence, $\bot$ and $\top$ (also possibly indexed) stay for the least and greatest element of an order. The sets of maximal, minimal and least elements of a subset $S' \subseteq S$ are denoted by $\max(S')$, $\min(S')$ and $\mathsf{lst}(S')$, resp An element x is said to *majorize* an element y if $x \geq y$ holds, and we extend this concept to sets by $\mathsf{maj}(x, S') =_{def} \{y \in S' \mid y \geq x\}$. Two elements x and y are called *comparable*, written $x \lesseqgtr y$, if $x \leq y$ or $y \leq x$ holds. Consequently, we call x and y *incomparable* if they are not comparable and denote this by $x \not\lesseqgtr y$. As usual, a *chain* is a set of elements which are pairwise comparable. A subset $S' \subseteq S$ of an ordered set $(S, \leq)$ is called *convex* if for all $x, y \in S'$ and all $z \in S$ the implication $x \leq z \leq y \Rightarrow z \in S'$ holds. For intervals we use the common notations $[a, b] =_{def} \{x \mid a \leq x \wedge x \leq b\}$ and $]a, b] =_{def} [a, b] \setminus \{a\}$. For a relation $R \subseteq S \times S$ and a subset $S' \subseteq S$ we define the *restriction* of R to S' routinely by $R|_{S'} =_{def} \{(s', t') \in R \mid (s', t') \in S' \times S'\}$.

Given an equivalence relation $E \subseteq S \times S$ we denote the equivalence class of an element $s \in S$ under E by $[x]_E$. For the set of equivalence classes of such an equivalence relation

$E \subseteq S \times S$ we use the notation S/E. For an arbitrary relation $R \subseteq S \times S$ and an equivalence relation $E \subseteq S \times S$ we define the *quotient* $R/E \subseteq S/E \times S/E$ by $([x]_E, [y]_E) \in S/E \Leftrightarrow_{def} \exists x' \in [x]_E \exists y' \in [y]_E : (x', y') \in R$. If $(S, \leq)$ is an ordered set and $E \subseteq S \times S$ is an equivalence relation such that $(S/E, \leq /E)$ is an ordered set we say that E is *order generating*. In this case, we may write also $\leq_E$ instead of $\leq /E$. Clearly, if under these circumstances both $[x]_E$ and $[y]_E$ are singleton sets we have the equivalence $x \leq y \iff [x]_E \leq_E [y]_E$ to which we will often refer as homomorphism properties. Examples for order generating equivalences are the identity and the universal relation. However, the relation $\sim$ on the ordered set $(\mathbb{Z}, \leq)$ (where $\leq$ denotes the usual ordering of the integers), defined by $x \sim y \Leftrightarrow_{def} x \cdot y \neq 0 \vee x = y = 0$, is obviously not order generating.

Since we will have to unite the sets of a set system $\mathbf{C}$ (i.e., $\mathbf{C}$ is a set of sets) we use the abbreviation $\bigcup \mathbf{C} =_{def} \bigcup_{C \in \mathbf{C}} C$ to make the text easier readable. Conversely, dealing with a set C, we use $C^{\{\}}$ to denote the set of singleton sets $\{\{c\} \mid c \in C\}$.

3. Closures

The main topic of this work, closures, can be characterized in two different ways, namely as endofunctions on ordered sets or as subsets of ordered sets. At the end of this section we will see that these characterizations are cryptomorphic; we start with the functional definition:

Definition 3.1. Given an ordered set $(S, \leq)$ an endofunction c on S is called a *closure operator* if it fulfills the following properties for all $x, y \in S$:

(1) $x \leq c(x)$ (extensitivity)
(2) $x \leq y \Rightarrow c(x) \leq c(y)$ (isotony)
(3) $c(c(x)) = c(x)$ (idempotence)

A useful easy consequence of this definition is the following corollary:

Corollary 3.1. *Let c be a closure function and $x, y \in S$ elements such that $y \geq c(x)$ ($y = c(x)$) holds. Then $x \leq y$ holds.*

Proof. This follows simply from $x \leq c(x)$ and transitivity of $\leq$. □

The next definition is a characterization of closures as subsets of ordered sets:

Definition 3.2. Given an ordered set $(S, \leq)$ a subset $C \subseteq S$ is called a *closure system* if for every $s \in S$ the set $\mathsf{maj}(s, C)$ has a least element.

The set of all closure systems of an ordered set $(S, \leq)$ is denoted by $\mathcal{C}(S)$ (here we assume that the order on S is clear from context).

Remark 3.2. In the literature, e.g. [Grä11] one finds a more complicated version of Definition 3.2 which imposes also the requirement that a closure system is closed under binary infima. However, a reviewer of [Glü21] pointed out that this requirement is redundant. This observation lead in the consequence to the generalization of [Glü21] from lattices to the present form handling general ordered sets. This is a rare case where generalization lead also to simplification due to the more concise formulation of Definition 3.2. □

It turns out that closure functions generate closure systems:

Lemma 3.3. *For every closure function c the set $\mathsf{fix}(c)$ of fixpoints of c is a closure system.*

Proof. Let $s \in S$ be an arbitrary element. Clearly, $c(s) \in \mathsf{fix}(c)$ holds, and due to extensitivity of c we have $c(s) \in \mathsf{maj}(s, \mathsf{fix}(c))$. Let us now pick an arbitrary $f \in \mathsf{fix}(c)$ with the property $s \leq f$. Now isotony of c and $f \in \mathsf{fix}(c)$ imply $c(s) \leq f$ which completes the proof. □

Also, closure systems determine closure functions in a unique way:

Lemma 3.4. *Let C be a closure system. Then there exists exactly one closure function c with* $\mathsf{fix}(c) = C$.

Proof. We define the function c by $c(s) =_{def} \mathsf{lst}(\mathsf{maj}(s, C))$ (note that this is well-defined due to the properties of a closure system according to Definition 3.2). Let us first check that c is indeed a closure function:

- Extensivity: this is obvious since every element is mapped to a majorizing one.
- Isotony: under the assumption $s \leq t$ we have $\mathsf{maj}(s, C) \supseteq \mathsf{maj}(t, C)$ and hence $c(s) = \mathsf{lst}(\mathsf{maj}(s, C)) \leq \mathsf{lst}(\mathsf{maj}(t, C)) = c(t)$.
- Idempotence: by already shown extensivity we have $s \leq c(s)$ and hence $\mathsf{maj}(s, C) \supseteq \mathsf{maj}(c(s), C)$. Let us now pick an arbitrary $s' \in \mathsf{maj}(s, C)$. Because $c(s)$ is the least element of $\mathsf{maj}(s, C)$ we have $c(s) \leq s'$, hence $s' \in \mathsf{maj}(c(s), C)$, implying $\mathsf{maj}(s, C) = \mathsf{maj}(c(s), C)$. Now $c(s) = c(c(s))$ follows from this set equality and definition of c.

Let us now assume that there is another closure function c' with $\mathsf{fix}(c') = C$, and pick an arbitrary $s \in S$. Because c' is idempotent we have $c'(s) \in C$ and hence $c(c'(s)) = c(s)$ from where we conclude that $c'(s) \leq c(s)$ holds (this is due to Corollary 3.1). Symmetrically we obtain $c(s) \leq c'(s)$ and hence $c(s) = c'(s)$ which shows uniqueness of c. □

Lemmata 3.3 and 3.4 establish a cryptomorphic one-to-one correspondence between closure functions and closure systems on an ordered set. Since the main contribution of this work deals with counting of closures (both functions and systems) it is sufficient to use the more convenient characterization. In this case, closure systems are much easier to handle than closure functions.

4. Isolated Suborders

The main tool for structuring ordered sets we will use is the subject of the following definition:

Definition 4.1. Let $(S, \leq)$ be an ordered set. A subset $S' \subseteq S$ is called an *isolated suborder* if it fulfills the following properties:

(1) S' has a greatest element $\top_{S'}$ and least element $\bot_{S'}$.
(2) $\forall x \notin S' \forall y' \in S' : y' \leq x \Rightarrow \top_{S'} \leq x$
(3) $\forall x \notin S' \forall y' \in S' : x \leq y' \Rightarrow x \leq \bot_{S'}$

Intuitively, an isolated suborder S' can be "entered from below" only via $\bot_{S'}$ and "left upwards" only via $\top_{S'}$. We call an isolated suborder *nontrivial* if S' does not equal S. A *summit isolated suborder* is a suborder S' such that $\top_{S'} \in \max(S)$ holds. If $|S'| > 1$ holds we call an isolated suborder *nonsingleton*, and a *useful* isolated suborder is a nontrivial non-singleton isolated suborder.

Another property we will need to make our ideas work is that an order does not "branch upwards" at an element under consideration (in our case at its top element):

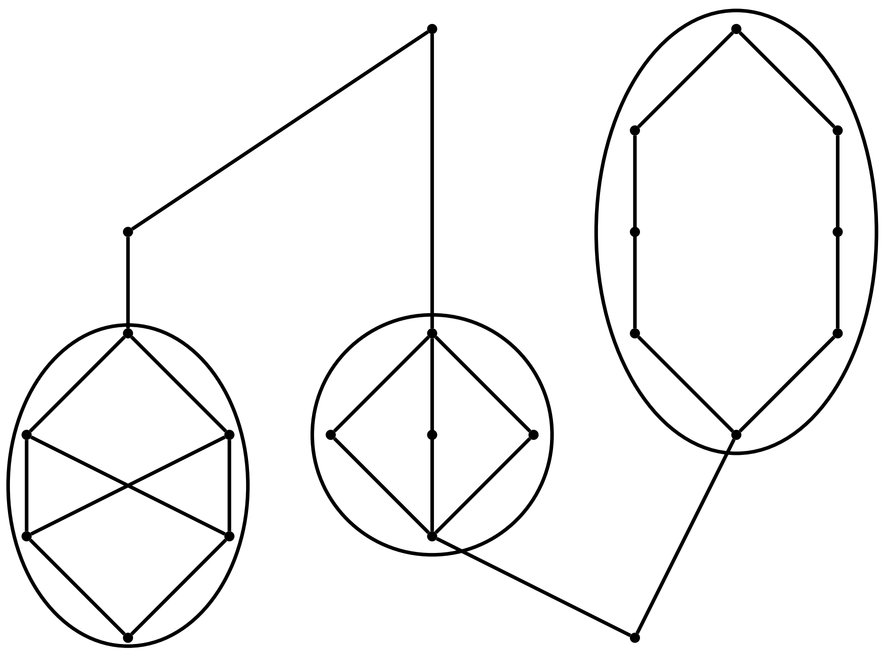

Figure 1: Various Kinds of Isolated Suborders

Definition 4.2. Given an ordered set $(S, \leq)$ we call an element $b \in S$ a *bottleneck* of an element $x \in S$ if the following conditions are fulfilled:

(1) $b > x$,
(2) $[x, b]$ is a chain, and
(3) $y > x \Rightarrow (y \in [x, b] \vee y > b)$ holds for all $y \in S$.

We note that this definition is equivalent to meet-irreducibility of x. However, because later proofs make use of the properties from Definition 4.2 (in particular, the element b will be referenced) we will stick with the definition given above.

Consequently, an *isolated suborder with bottleneck* is an isolated suborder S' such that $\top_{S'}$ has a bottleneck. Obviously, given an element x with a bottleneck b, every element in $]x, b]$ is also a bottleneck of x.

Figure 1 illustrates these definitions. At the left, in isolated suborder with bottleneck is shown. In the middle, we find an isolated suborder without bottleneck, and at the right a summit isolated suborder is given. Note that neither the overall order nor the isolated suborder with bottleneck are lattices (in contrast, the other two isolated suborders are indeed lattices).

We will use isolated suborders to derive quotients of an order. To this end, we define for an isolated suborder $S' \subseteq S$ an equivalence relation $\sim_{S'}$ by $x \sim_{S'} y \Leftrightarrow_{def} (x \in S' \;\Leftrightarrow\; y \in S')$. Clearly, for an element $x \notin S'$ the equivalence class $[x]_{\sim_{S'}}$ is the singleton set $\{x\}$, and for all elements $x \in S'$ the equivalence class $[x]_{\sim_{S'}}$ coincides with S'. A crucial point is that $\sim_{S'}$ is even order generating:

Lemma 4.1. *Let S' be an isolated suborder of an ordered set $(S, \leq)$. Then $\sim_{S'}$ is order generating.*

Proof. It is easy to see that $\leq / \sim_{S'}$ (we use this notation since we do not know yet whether $\sim_{S'}$ is order generating) is both reflexive and transitive so it remains to show that it is also antisymmetric. To this end, we pick two arbitrary $[s]_{\sim_{S'}}, [t]_{\sim_{S'}} \in S/\sim_{S'}$ such that both $[s]_{\sim_{S'}} \leq / \sim_{S'} [t]_{\sim_{S'}}$ and $[t]_{\sim_{S'}} \leq / \sim_{S'} [s]_{\sim_{S'}}$ hold. In the case $s, t \notin S'$ we have $[s]_{\sim_{S'}} = \{s\}$ and $[t]_{\sim_{S'}} = \{t\}$ and hence both $s \leq t$ and $t \leq s$ by homomorphism properties. Now $[s]_{\sim_{S'}} = [t]_{\sim_{S'}}$ is an easy consequence of the antisymmetry of $\leq$. If $s, t \in S'$ holds we have $[s]_{\sim_{S'}} = [t]_{\sim_{S'}}$ by construction of $\sim_{S'}$. For the last case we assume w.l.o.g. that $s \in S'$ and $t \notin S'$ hold. From $[s]_{\sim_{S'}} \leq / \sim_{S'} [t]_{\sim_{S'}}$ we conclude that there is an $s_1 \in S'$ with $s_1 \leq t$ (note that $[t]_{\sim_{S'}}$ is the singleton set $\{t\}$). By definition of an isolated suborder this implies $\top_{S'} \leq t$, and symmetrically we obtain $t \leq \bot_{S'}$. This leads to the chain $t \leq \bot_{S'} \leq \top_{S'} \leq t$, implying among other things $t = \bot_{S'}$ and hence $t \in S'$, contradicting the choice of t. This finishes the proof since it shows that the last case can not occur. □

This lemma justifies the writing $\leq_{\sim_{S'}}$ which we will use from now on.

An important property of isolated suborders is their convexitivity:

Lemma 4.2. *Let S' be an isolated suborder of an ordered set $(S, \leq)$. Then S' is convex.*

Proof. Let $s, t \in S'$ be arbitrary elements of S' with $s \leq t$ and assume that there is an $u \notin S'$ such that $s \leq u \leq t$ holds. By $\bot_{S'} \leq s \leq u$ we have $\bot_{S'} \leq u$. On the other hand, from $u \leq t$ and $u \notin S'$ we obtain $u \leq \bot_{S'}$ by definition of an isolated suborder; so altogether we have $u = \bot_{S'}$. However, this is a contradiction to the assumption $u \notin S'$. □

In particular, this means that an isolated suborder S' is the same as the interval $[\bot_{S'}, \top_{S'}]$. However, not every interval is an isolated suborder: consider as counterexample the interval $[\emptyset, \{1\}]$ in the ordered set $(\mathcal{P}(\{1, 2\}), \subseteq)$. In this setting, we have $\emptyset \subseteq \{2\}$ but not $\{1\} \subseteq \{2\}$ so the second part of Definition 4.1 is not fulfilled.

The next lemma states intuitively spoken that isolated suborders with common elements can not lie side by side:

Lemma 4.3. *Let S_1 and S_2 be two isolated suborders with $S_1 \cap S_2 \neq \emptyset$. Then the set $\{\bot_{S_1}, \bot_{S_2}, \top_{S_1}, \top_{S_2}\}$ is a chain.*

Proof. Let us pick an arbitrary $s_{12} \in S_1 \cap S_2$. In the case $\top_{S_1} \in S_2$ the inequality $\top_{S_1} \leq \top_{S_2}$ is easy to see. If $\top_{S_1}$ is not an element of S_2 we can conclude $\top_{S_2} \leq \top_{S_1}$ from $s_{12} \in S_2$, $s_{12} \leq \top_{S_1}$ and the properties of an isolated suborder. Symmetrically, we can show that $\bot_{S_1}$ and $\bot_{S_2}$ are comparable. Now the rest is an easy consequence of $\bot_{S_1}, \bot_{S_2} \leq s_{12} \leq \top_{S_1}, \top_{S_2}$. □

Now we see that two isolated suborders with a common element can be merged into one isolated suborder:

Lemma 4.4. *Let S_1 and S_2 be two isolated suborders with $S_1 \cap S_2 \neq \emptyset$. Then $S_{12} =_{def} S_1 \cup S_2$ is an isolated suborder, too.*

Proof. Because $\{\bot_{S_1}, \bot_{S_2}, \top_{S_1}, \top_{S_2}\}$ is a chain due to Lemma 4.3 we can assume w.l.o.g. that $\bot_{S_1} \leq \bot_{S_2}$ holds. If $\top_{S_2} \leq \top_{S_1}$ holds we have $[\bot_{S_2}, \top_{S_2}] \subseteq [\bot_{S_1}, \top_{S_1}]$ and we obtain the claim immediately because isolated suborders are intervals. We do not have to consider the case

$\top_{S_1} < \bot_{S_2}$ because of $S_1 \cap S_2 \neq \emptyset$ so we only have to look only at $\bot_{S_1} \leq \bot_{S_2} \leq \top_{S_1} \leq \top_{S_2}$ as last case.

To show that S_{12} is indeed an isolated suborder we pick arbitrary $s_{12} \in S_{12}$ and $x \notin S_{12}$ such that $s_{12} \leq x$ holds. In the case $s_{12} \in S_2$ we obtain $x \geq \top_{S_2} = \top_{S_{12}}$ because S_2 is an isolated suborder so let us now assume that $s_{12} \in S_1$ holds. Due to the properties of S_1 we have here $x \geq \top_{S_1}$. Knowing this, we can deduce $x \geq \top_{S_2}$ from $\top_{S_1} \in S_2$ and $x \notin S_2$. The case $x \leq s_{12}$ can be treated by a symmetric argumentation. □

An easy observation is now that $S_1 \cup S_2$ is an isolated suborder with bottleneck provided that S_1 and S_2 are isolated suborders with bottlenecks. Moreover, if S_1 and S_2 are nontrivial summit isolated suborders with $\top_{S_1} = \top_{S_2}$ then $S_1 \cup S_2$ is also a nontrivial summit isolated suborder with greatest element $\top_{S_1}$ (or, equivalently, greatest element $\top_{S_2}$). This shows the following theorem together with Lemma 4.4:

Theorem 4.5. *Let $(S, \leq)$ be an ordered set.*

1. *Two different inclusion-maximal suborders with bottleneck of $(S, \leq)$ are disjoint.*
2. *For every $s \in \max(S)$ there is at most one nontrivial inclusion-maximal summit isolated suborder with s as greatest element.*

The next lemma establishes a connection between isolated suborders in an ordered set and a quotient of this order, induced by an isolated suborder.

Lemma 4.6. *Let S' be an isolated suborder of an ordered set $(S, \leq)$ and let $S_{S'}$ be an isolated suborder of $S/{\sim_{S'}}$. Then $S'' =_{def} \bigcup S_{S'}$ is an isolated suborder of S.*

Proof. Clearly, $\bot_{S''} = \bot_{S'}$ holds provided $\bot_{S_{S'}} = S'$, and also $\bot_{S''} = s$ provided $\bot_{S_{S'}} = \{s\}$. Analogous equalities hold also for $\top_{S''}$ instead of $\bot_{S_{S'}}$.

To show the remaining properties of Definition 4.1 we pick arbitrary $s \in S''$ and $t \notin S''$ with $s \leq t$. The construction of S'' yields both $[s]_{\sim_{S'}} \in S_{S'}$ and $[t]_{\sim_{S'}} \notin S_{S'}$, and homomorphism properties lead to $[s]_{\sim_{S'}} \leq_{\sim_{S'}} [t]_{\sim_{S'}}$. Because $S_{S'}$ is an isolated suborder we can therefrom deduce that $\top_{S_{S'}} \leq_{\sim_{S'}} [t]_{\sim_{S'}}$ has to hold. We note that $[t]_{\sim_{S'}}$ and $\top_{S_{S'}}$ are disjoint and consider first the case that $\top_{S_{S'}}$ is a singleton set. In this case, the equality $\top_{S_{S'}} = \{\top_{S''}\}$ holds which implies $\top_{S''} \leq t$. Otherwise, i.e., if $\top_{S_{S'}}$ has more than one element, we have $\top_{S_{S'}} = S'$, implying $\top_{S''} = \top_{S'}$. Here too we have $\top_{S''} \leq t$ by homomorphism and because of disjointness of $[t]_{\sim_{S'}}$ and $\top_{S_{S'}}$. A symmetric argumentation applies to the case $s \geq t$, hence S'' is an isolated suborder according to Definition 4.1. □

Next we extend this claim to isolated suborders with bottleneck:

Lemma 4.7. *Let S' be an isolated suborder of an ordered set $(S, \leq)$ and let $S_{S'}$ be an isolated suborder with bottleneck of $S/{\sim_{S'}}$. Then $S'' =_{def} \bigcup S_{S'}$ is an isolated suborder with bottleneck of S.*

Proof. S'' is an isolated suborder by Lemma 4.6, and by assumption we can choose an arbitrary bottleneck $B \in S/{\sim_{S'}}$ of $S_{S'}$. Now we have the following three possibilities:

1. $\top_{S_{S'}} = S'$: In this case we have $\top_{S''} = \top_{S'}$ and $B = \{b\}$ for some $b \in S$. Hence b is a bottleneck according to Definition 4.2 by homomorphism properties because all elements of the (nonempty) interval $]S', B]$ are singleton sets.
2. $B = S'$: Here $\top_{S_{S'}} = \{\top_{S''}\}$ holds, and our goal is to show that $\bot_{S'}$ is a bottleneck of S''. By assumption, $[\{\top_{S''}\}, S']$ is a chain in $S/{\sim_{S'}}$, hence $[\top_{S''}, \bot_{S'}]$ is a chain in S due to the fact that $S/{\sim_{S'}}$ consists only of singleton sets except possibly S'. Now it is easy to check the remaining properties of Definition 4.2.

3. $\top_{S_{S'}} \neq S' \wedge B \neq S'$: Here, we first investigate the case $S' \in [\top_{S_{S'}}, B]$. Under this condition, B is also a bottleneck of $S_{S'}$ and the argumentation can be carried out analogously to the previous case. If $S' \notin [\top_{S_{S'}}, B]$ holds, $S/{\sim_{S'}}$ consists of singleton sets only, and the properties of Definition 4.2 follow easily from homomorphism properties.

$\square$

An analogous lemma holds also for summit isolated suborders:

Lemma 4.8. *Let S' be an isolated suborder of an ordered set $(S, \leq)$ and let $S_{S'}$ be a summit isolated suborder of $S/{\sim_{S'}}$. Then $S'' =_{def} \bigcup S_{S'}$ is a summit isolated suborder of S.*

Proof. By Lemma 4.6 we know that S'' is an isolated suborder. If $\top_{S_{S'}} = \{s\}$ holds for some $s \in S$ then s is a maximal element in $(S, \leq)$ by homomorphism properties. Moreover, also by homomorphism properties, s is the greatest element of S'' so S'' is a summit isolated suborder in this case. Let us now assume that $\top_{S_{S'}} = S'$ holds. In this case, S' is a maximal element of $(S/{\sim_{S'}}, \leq_{\sim_{S'}})$, hence $\top_{S'}$ is a maximal element of $(S, \leq)$. By construction and homomorphism properties, $\top_{S'}$ is the greatest element of S''. $\square$

In our algorithm we may make use of consecutive quotients induced by various isolated suborders. A (possibly infinite) sequence $S_0, S_1, S_2, \dots$ of ordered sets is called a *quotient sequence* if for all i the ordered set S_{i+1} can be written as the quotient $S_{i+1} = S_i/{\sim_{S'_i}}$ for some isolated suborder S'_i of S_i. Consecutive quotient formation leads to ordered sets whose carrier sets have an increasing depth of set nesting. To be type correct, we introduce the notation $\bigcup^n \mathbf{C}$ inductively by $\bigcup^0 \mathbf{C} =_{def} \mathbf{C}$ and $\bigcup^{n+1} \mathbf{C} =_{def} \bigcup(\bigcup^n \mathbf{C})$. Intuitively spoken, this operation removes n set brackets from the elements of $\mathbf{C}$ and joins them all.

In a quotient sequence, an inclusion-maximal summit isolated suborder containing a fixed maximal element can appear as most once as a factor (if we abstract from set parentheses):

Lemma 4.9. *Let $S_0, S_1, S_2, \dots$ be a quotient sequence such that $S_{i+1} = S_i/{\sim_{S'_i}}$ holds for an inclusion-maximal useful summit isolated suborder S'_i with greatest element $\top_{S'_i}$. Then no S_j with $j > i$ contains a useful summit isolated suborder S'_j with $\top_{S'_i} \in \bigcup^{j-i} S'_j$.*

Proof. Assume that some S_j with $j > i$ contains a useful summit isolated suborder S'_j. Then we could construct an inclusion-maximal summit isolated suborder $S''_i \supsetneq S'_i$ of S_i from S'_j backwards along the lines of Lemmata 4.6 and 4.8, contradicting the inclusion-maximality of S'_i. $\square$

In the next lemma, we show that if we ignore set brackets, an element can appear in at most one inclusion-maximal isolated suborders in a quotient sequence.

Lemma 4.10. *Let $S_0, S_1, S_2, \dots$ be a quotient sequence such that $S_{i+1} = S_i/{\sim_{S'_i}}$ holds for an inclusion-maximal useful isolated suborder with bottleneck S'_i for all $i \geq 0$. Then S'_i and $\bigcup^{j-i} S'_j$ are disjoint for all i, j with $j > i$.*

Proof. By the first part of Theorem 4.5 and Lemma 4.7 it is obvious that S'_i and $\bigcup S'_{i+1}$ are disjoint for all i. Now the claim follows by straightforward induction. $\square$

5. Isolated Suborders and Closure Systems

After investigating isolated suborders and their properties we now turn our attention to the interplay between isolated suborders and closure systems. First we show how closure systems on an ordered set give rise to closure systems of a quotient:

Lemma 5.1. *Let $(S, \leq)$ be an ordered set, S' an isolated suborder of $(S, \leq)$ and consider a closure system C of $(S, \leq)$.*

(1) *If $C \cap S' = \emptyset$ then $C^{\{\}}$ is a closure system of $S/\sim_{S'}$.*
(2) *If $C \cap S' \neq \emptyset$ then $(C \backslash S')^{\{\}} \cup \{S'\}$ is a closure system of $S/\sim_{S'}$.*

Proof. Because $\sim_{S'}$ is order inducing, $S/\sim_{S'}$ is a homomorphic image of $(S, \leq)$. Deploying this fact, Definition 3.2 can now be verified easily on $S/\sim_{S'}$ for the two cases of the Lemma. □

By the nature of things, homomorphisms work in general only in one direction so we expect to have a harder task to show how closure systems of a quotient can induce closure systems of the original ordered set. First we introduce a notion for "almost" closure systems:

Definition 5.1. Let $(S, \leq)$ be an ordered set with greatest element $\top$. A subset $C \subseteq S$ is called a *preclosure system* of $(S, \leq)$ if $C \cup \{\top\}$ is a closure system of $(S, \leq)$. The set of all preclosure systems of $(S, \leq)$ is denoted by $\mathcal{PC}(S)$.

Clearly, every closure system on an ordered set with greatest element is also a preclosure system, and the empty set is a preclosure system on ordered sets with a greatest element. An important observation for the algorithm we will develop is that $|\mathcal{PC}(S)| = 2 \cdot |\mathcal{C}(S)|$ holds if $\mathcal{C}(S)$ is finite (note that this presupposes that S has a greatest element). Another crucial fact in the further course is that a nonempty preclosure system contains a least element majorizing $\perp$ (if the order under consideration has a least element at all):

Lemma 5.2. *Let C be a nonempty preclosure system of an ordered set $(S, \leq)$ with least element $\perp_S$ and greatest element $\top_S$. Then there is a least element $c \in C$ majorizing $\perp_S$.*

Proof. In the case $\top_S \in C$, C is even a closure system, and Definition 3.2 entails the claim obviously. Otherwise, we define the closure system C' by $C' =_{def} C \dot{\cup} \{\top_S\}$, and by definition of a closure system there is a least $c' \in C'$ majorizing $\perp_S$. This element c' can not be $\top_S$ since C' contains at least one element except $\top_S$ (recall the C was supposed to be nonempty), so we conclude $c' \in C$. □

Now we can use preclosure systems to describe the intersection of an isolated suborder and a closure system:

Lemma 5.3. *Let C be a closure system on an ordered set $(S, \leq)$ and let S' be an isolated suborder of S with greatest element $\top_{S'}$ and least element $\perp_{S'}$. Then $C' =_{def} C \cap S'$ is a preclosure system of S'. Moreover, if S' is a summit isolated suborder then C' is a closure system of S'.*

Proof. Let S' be an arbitrary isolated suborder and consider an arbitrary $s' \in S'$. Because C is a closure system it contains a least c which majorizes s'. If $c \in S'$ holds then c is by construction also an element of $C'_{\top_{S'}}$. In the other case $c \notin S'$ we have $\top_{S'} \leq c$ by definition of an isolated suborder. Now it is obvious that $\top_{S'}$ is a smallest element of $C'_{\top_{S'}}$ majorizing s'. This shows the first claim so let us now assume that S' is even a summit isolated suborder. Then $\top_{S'} \in \max(S)$ holds by definition, hence C has to contain also $\top_{S'}$ wherefrom the second claim follows immediately. □

The next lemma is in some sense a "reverse" of Lemma 5.1 in the case of isolated suborders with bottleneck:

Lemma 5.4. *Let $(S, \leq)$ be an ordered set and S' an isolated suborder of S such that $\top_{S'}$ has a least bottleneck b. Assume that $C_{S'}$ is a preclosure system of S' and let C' be a closure system of $S/\sim_{S'}$ with $S' \in C'$. Then $C =_{def} \bigcup(C' \backslash \{S'\}) \cup C_{S'}$ is a closure system of $(S, \leq)$.*

Proof. Let us pick an arbitrary $s \in S$ with the goal to show that there is a least $c \in C$ with $s \leq c$ in order to fulfill Definition 3.2. To this end, we have several cases to consider:

1. $s \notin S'$: In this case we have $[s]_{\sim_{S'}} = \{s\}$, and by assumption there is a least $c' \in C'$ with $\{s\} \leq_{\sim_{S'}} c'$. Now we have the following possibilities:
   a) $c' \neq S'$: then we have $c' = \{c''\}$ for some $c'' \in S$. By homomorphism properties, c'' majorizes s in $(S, \leq)$, so let us pick an arbitrary $\hat{c} \in C$ with $s \leq \hat{c}$. If $[\hat{c}]_{\sim_{S'}} = \{\hat{c}\}$ holds we have $c' \leq_{\sim_{S'}} \{hatc\}$ since C' is a closure system (note that by construction of C, $\{\hat{c}\}$ has to be an element of C'). Otherwise, we have $[\hat{c}]_{\sim_{S'}} = S'$, and because C' is a closure system, we have $c' \leq_{\sim_{S'}} S'$. In both cases we have $c'' \leq \hat{c}$ by homomorphism properties.
   b) $c' = S'$: here we distinguish the following cases:
      i) $C_{S'} \neq \emptyset$: by Lemma 5.2 there is a least $c'' \in C_{S'}$ majorizing $\perp_{S'}$. By homomorphism and the assumption $\{s\} \leq_{\sim_{S'}}$ we have $s \leq \perp_{S'}$, and by transitivity we get $s \leq c''$. Let us now consider an arbitrary $\hat{c} \in C$ with $s \leq \hat{c}$. If $\hat{c} \in S'$ we get $\hat{c} \in C_{S'}$ by construction of C. Because c'' is the least element of $C_{S'}$ we obtain $c'' \leq \hat{c}$ immediately. In the case $\hat{c} \notin S'$ we have $\{\hat{c}\} \in C'$ by construction of C, and as above we get $\{s\} \leq_{\sim_{S'}} \{\hat{c}\}$. Because C' is a closure system we obtain $S' \leq_{\sim_{S'}} \{\hat{c}\}$ and hence $c'' \leq \hat{c}$.
      ii) $C_{S'} = \emptyset$: because of $b \notin S'$ we have $[b]_{\sim_{S'}} = \{b\}$, and because C' is a closure system there is a least $b' \in C'$ majorizing $\{b\}$. Moreover, $b' = \{c''\}$ has to hold for some $c'' \in S$ (note that we have $S' <_{\sim_{S'}} \{b\} \leq_{\sim_{S'}} b'$), and by homomorphism and transitivity we obtain $s \leq c''$. As usual we pick an arbitrary $\hat{c} \in C$ with $s \leq \hat{c}$ and observe that $[\hat{c}]_{\sim_{S'}} = \{\hat{c}\}$ holds (note that we assume here $C_{S'} = \emptyset$). From $s \leq \hat{c}$ we derive $\{s\} \leq_{\sim_{S'}} \{\hat{c}\}$ and hence $S' \leq_{\sim_{S'}} \{\hat{c}\}$ (recall $c' = S'$ and the properties of c'). However, due to $C_{S'} = \emptyset$ and construction of C we obtain $\hat{c} \notin S'$ and by order theory and homomorphism this leads to $\{b\} \leq_{\sim_{S'}} \{\hat{c}\}$ from where we conclude that $b' \leq_{\sim_{S'}} \{\hat{c}\}$ and eventually $c'' \leq \hat{c}$ hold.
2. $s \in S'$: clearly, S' is the least element of C' majorizing $[s]_{\sim_{S'}}$ since $[s]_{\sim_{S'}} = S'$ and $S' \in C'$ hold. We have two cases:
   a) $C_{S'}$ contains an element s' majorizing s: then $C_{S'}$ contains also a minimal element c'' majorizing s, and let $\hat{c}$ be an arbitrary element of C such that $s \leq \hat{c}$ holds. In the case $\hat{c} \in C_{S'}$ we get $c'' \leq \hat{c}$ immediately by the choice of c'' so let us assume that $\hat{c} \notin S'$ holds. However, here we have $\top_{S'} \leq \hat{c}$ by properties of an isolated suborder, implying $c'' \leq \hat{c}$.
   b) $C_{S'}$ contains no element majorizing s: as above there is a least $b' = \{c''\} \in C'$ majorizing $\{b\}$. Clearly, c'' majorizes s in $(S, \leq)$ so we pick an arbitrary $\hat{c} \in C$ with $s \leq \hat{c}$. By properties of an isolated suborder we have $\top_{S'} \leq \hat{c}$, and because $C_{S'}$ does not contain $\top_{S'}$ (otherwise $\top_{S'}$ would be an element of $C_{S'}$ majorizing s) we can deduce even $b \leq \hat{c}$. Now it is clear that $c'' \leq \hat{c}$ holds due to $\{b\} \leq_{\sim_{S'}} \{\hat{c}\}$ and closure properties of C'.

In all cases, we constructed in the form of c'' a least element of C majorizing s. □

Remark 5.5. It is necessary for the correctness of Lemma 5.4 to require that $\top_{S'}$ has a bottleneck. This can be seen in Figure 2. At the left, a preclosure system, indicated by encircled elements, on an isolated suborder without bottleneck of its top element, indicated by an ellipse, is shown. In the middle of the figure, we see a closure system on the associated

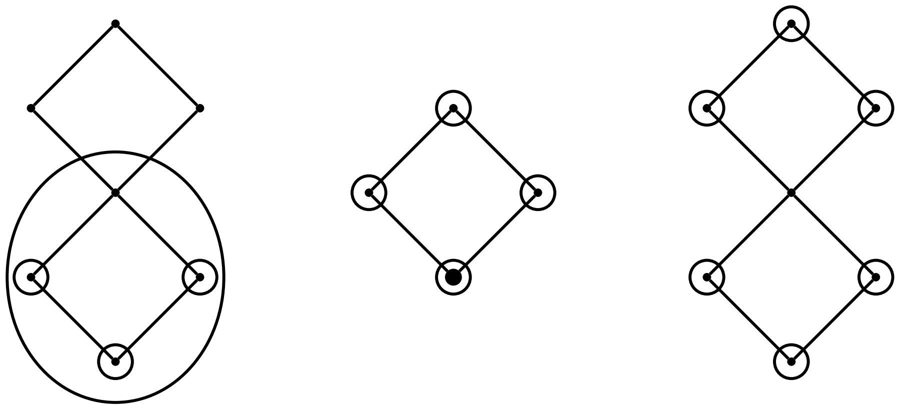

Figure 2: A preclosure system on an isolated suborder (left), a closure system on a quotient (middle) and no closure system on the original order (right)

quotient order, indicated by circles. However, executing the construction from Lemma 5.4 leads to the set of encircled elements in the right picture which does not contain a least element majorizing the middle element (the only unencircled element) and hence is no closure system. This shows one effect of a bottleneck: it releases the top element of an isolated suborder from the responsibility of being the least element majorizing of two elements above it. Moreover, it it necessary to require that $\top_{S'}$ has even a least bottleneck: consider the ordered set $S = ([0,1], \leq)$ (where $\leq$ denotes the usual order on the reals) and the isolated suborder $S' = (\{0\}, \leq)$. We choose the empty set as preclosure $C_{S'}$ of S' and $[0,1]^{\{\}}$ as closure system C'. Then the construction from above yields for C the set $]0,1]$ which is no closure system since it contains no least element majorizing 0. □

The next lemma is a variant of the previous lemma for summit isolated suborders:

Lemma 5.6. *Let $(S, \leq)$ be an ordered set and let S' be a summit isolated suborder of $(S, \leq)$. Assume that $C_{S'}$ is a closure system of S' and let C' be a closure system of $S/\sim_{S'}$. Then $C =_{def} \bigcup(C' \backslash \{S'\}) \cup C_{S'}$ is a closure system of $(S, \leq)$.*

Proof. As in the proof of Lemma 5.4 we pick an arbitrary $s \in S$ and show the existence of a least element of C majorizing s. We distinguish the following cases:

1. $s \notin S'$: analogously to Lemma 5.4 we have $[s]_{\sim_{S'}} = \{s\}$, and by assumption there is a least $c' \in C'$ with $\{s\} \leq_{\sim_{S'}} c'$. Now we have the following possibilities:
   a) $c' \neq S'$: this case can be handled exactly like the same case in the proof of Lemma 5.4.
   b) $c' = S'$: here we have the following cases:
      i) $C_{S'} \neq \emptyset$: also this case can be handled along the lines of Lemma 5.4 (note that $C_{S'}$ as closure system is also a preclosure system).
      ii) $C_{S'} = \emptyset$: in this case we can not resort to the proof of Lemma 5.4 because the argument there uses properties of a smallest bottleneck. However, $C_{S'}$ can not be empty because it has to contain the greatest element of S'.
2. $s \in S'$: by properties of a closure system, $C_{S'}$ contains a least element c' majorizing s. However, By construction of C and because S' is a summit isolated suborder, all elements of $C \backslash C_{S'}$ majorizing s are contained in $C_{S'}$

In all cases we deduced the existence of a least element of C majorizing s. □

Finally, we consider the case that an isolated suborder with bottleneck does not appear in a closure system of the quotient:

Lemma 5.7. *Let $(S, \leq)$ be an ordered set and let S' be an isolated suborder with bottleneck of $(S, \leq)$. Assume that C' is a closure system on $S/\sim_{S'}$ such that $S' \notin C'$. Then $C =_{def} \bigcup C'$ is a closure system on S.*

Proof. Routinely, we pick an arbitrary $s \in S$ with the obligation to show the existence of a least element of C majorizing s. Then the case $s \notin S'$ is analogous to case 1.a) of the proof of Lemma 5.4 (note that $S' \notin C'$ was assumed). Similarly, the case $s \in S'$ can be handled along the lines of case 2.b) of the proof of Lemma 5.4. □

Now we are ready to state our main results about the relationships between closure systems on ordered sets and quotients thereof in the following two theorems:

Theorem 5.8. *Let S' be an isolated suborder with bottleneck of an ordered set $(S, \leq)$, and consider a subset $C \subseteq S$.*

(1) *Assume that $C' =_{def} C \cap S' \neq \emptyset$ holds. Then C is a closure system of S iff C' is a nonempty preclosure system of S' and $(C \backslash S')^{\{\}} \cup \{S'\}$ is a closure system of $S/\sim_{S'}$.*
(2) *Assume that $C \cap S' = \emptyset$ holds. Then C is a closure system of S iff $C^{\{\}}$ is a closure system of $S/\sim_{S'}$.*

Proof. This follows now easily from Lemmata 5.1, 5.3, 5.4 and 5.7 □

Theorem 5.9. *Let S' be a summit isolated suborder of an ordered set $(S, \leq)$, and consider a subset $C \subseteq S$. Then C is a closure system of S iff $C \cap S'$ is a closure system of S' and $(C \backslash S')^{\{\}} \cup \{S'\}$ is a closure system of $S/\sim_{S'}$.*

Proof. This follows simply from Lemmata 5.1, 5.3 and 5.6. □

6. Computing Isolated Suborders

The previous results concerned isolated suborders and their relation closure systems. In this subsection we will become more concrete and deal with the computation of isolated suborders. As representation for an ordered set, we assume that an ordered set is given by its Hasse diagram as a directed graph $G = (S, E)$ where the edges point "upwards". i.e., $(s, t) \in E$ implies $s < t$. For the reverse graph of $G = (S, E)$ we use the notation $G^{\leftarrow} = (S, E^{\leftarrow})$ and for its undirected version we write $G^{\leftrightarrow}$. Given a path $p = s_1 s_2 \ldots s_n$ we write $s \in p$ if $s = s_i$ for some $i \in [1, n]$ holds and say that *p contains s*. We call a node u an (s, t)-separator if every path from s to t contains u. Routinely, we add an element $\perp$ to S and consider the graph $G_\perp =_{def} (S_\perp, E_\perp)$ with $S_\perp = S \dot\cup \{\perp\}$ and $E_\perp = E \dot\cup \{(\perp, m) \mid m \in \min(S)\}$. Intuitively, this adds a least element to the ordered set. Additionally, we may add yet another element $\top$ to $S_\perp$ to obtain the set $S_{\perp,\top} =_{def} S_\perp \dot\cup \{\top\}$ and define the graph $G_{\perp,\top} = (S_{\perp,\top}, E_{\perp,\top})$ by $E_{\perp,\top} =_{def} E_\perp \dot\cup \{(\top, m) \mid m \in \max(S)\}$. Analogously to the construction of $G_\perp$, this adds a greatest element to the ordered set under consideration. All algorithms considered in this section have polynomial running time so these constructions do not affect their asymptotic complexity.

6.1. **Summit Isolated Suborders and Separators.** As we already know, isolated suborders are intervals. Assume now that $[s, m]$ is a summit isolated suborder. In this case, m is a maximal element of S, hence the second condition of Definition 4.1 can be omitted because for every $y' \in [s, m]$ there is no $x \notin [s, m]$ with $y' \leq x$. Clearly, $[s, m]$ has s as least and m as greatest element and we have shown the following lemma:

Lemma 6.1. *An interval* $[s, m]$ *with* $m \in \max(S)$ *is a summit isolated suborder iff the following implication holds:*

$$\forall t \notin [s, m] \forall u \in [s, m] : t \leq u \Rightarrow t \leq s.$$

From this characterization we can deduce the following one:

Lemma 6.2. *An interval* $[s, m]$ *with* $m \in \max(S)$ *is a summit isolated suborder iff for all* $t \notin [s, m]$ *with* $t \lesseqgtr m$ *the inequality* $t \leq s$ *holds.*

Proof. "$\Rightarrow$": Let us pick an arbitrary $t \notin [s, m]$ with $t \lesseqgtr m$. By maximality of m and comparability of t and m we have $t \leq m$, and the claim follows from the substitution $y' := m$ in Lemma 6.1.
"$\Leftarrow$": Similarly as above, maximality of m, comparability of t and m together with $t \notin [s, m]$ imply even $t \leq s$. Now the consequent from Lemma 6.2 is always true. □

This characterization which is valid in arbitrary ordered sets entails the following first characterization for finite ordered sets:

Lemma 6.3. *Let* $(S, \leq)$ *be a finite ordered set with Hasse diagram* $G = (S, E)$*. Then every summit isolated suborder of* S *has the form* $[s, m]$ *with* $m \in \max(S)$ *where* $s \in S$ *is an* $(m, \perp)$*-separator in* $G_\perp^\leftarrow$*.*

Proof. We know already that isolated suborders are intervals so it remains to show the separator property. We omit the trivial and uninteresting cases $s \in \min(S)$ (clearly, every minimal element of $(S, \leq)$ reachable from m in $G^\leftarrow$ is an $(m, \perp)$-separator in $G_\perp^\leftarrow$) and $s = m$ and consider a path $p = s_1 s_2 \dots s_n$ in $G_\perp^\leftarrow$ with $s_1 = m$ and $s_n = \perp$. Then there is an index $i \in [1, n-1]$ with $s_i \in [s, m]$ and $s_{i+1} \notin [s, m]$ and let us assume for the sake of contradiction that $s_i \neq s$ holds. Due to $s_i \in [s, m]$ this implies $s < s_i$ and the property $(s_i, s_{i+1}) \in E^\leftarrow$ implies $s_{i+1} < s_i$ (this follows from maximality of m and isolated suborder properties). On the other hand, Lemma 6.2 implies $s_{i+1} < s$. Altogether, we have $s < s_i$ and $s_{i+1} < s$. But then $s_{i+1} < s_i$ follows already by transitivity of $<$, hence (s_{i+1}, s_i) can not be an edge in the Hasse diagram of S and now $(s_i, s_{i+1}) \notin E^\leftarrow$ contradicts the path property of p. □

Problems like separators in directed graphs can be tackled by modified max-flow algorithms which are in general somehow cumbersome to implement. Luckily, there is a characterization of summit isolated suborders using undirected graphs:

Lemma 6.4. *Let* $(S, \leq)$ *be a finite ordered set with Hasse diagram* $G = (S, E)$*. Then every summit isolated suborder of* S *has the form* $[s, m]$ *with* $m \in \max(S)$ *where* $s \in S$ *is an* $(m, \perp)$*-separator in* $G_\perp^\leftrightarrow$*.*

Proof. The claim is obvious for the case $s \in \min(S) \cup \{\perp, m\}$ so let us assume that $s \notin \min(S) \cup \{\perp, m\}$ holds and let us fix an arbitrary path $p = s_1 s_2 \dots s_n$ in $G_\perp^\leftrightarrow$ with $s_1 = m$ and $s_n = \perp$. Analogously to the proof of Lemma 6.3 there are vertices s_i and s_{i+1} with $s_i \in [s, m]$ and $s_{i+1} \notin [s, m]$, and here, too, we claim that $s_i = s$ holds. Due to $s_i \in [s, m]$ and $s_{i+1} \notin [s, m]$ we have $(s_i, s_{i+1}) \in E^\leftarrow$ (the crucial point here is that m is a maximal element). On the other hand, from $s_i \in [s, m]$ we conclude that there is a path $p' = s'_1 s'_2 \ldots s'_{n'}$ in $G^{\leftarrow}_{\perp}$ with $s'_1 = m$, $s'_{n'} = s_i$ and $s'_{s'} \neq s$ for all $i' \in [1, n']$. Analogously, there is a path $p'' = s''_1 s''_2 \ldots s''_{n''}$ in $G^{\leftarrow}_{\perp}$ with $s''_1 = s_{i+1}$, $s''_{s''} = \perp$ and $s''_{i''} \neq s$ for all $i'' \in [1, n'']$. This means that $s'_1 s'_2 \ldots s'_{n'} s''_1 s''_2 \ldots s''_{n''}$ is a path in $G^{\leftarrow}_{\perp}$ from m to $\perp$, and the claim follows from Lemma 6.3. □

Lemma 6.4 shows that summit isolated suborders are intervals with certain properties. Now we will show that all those intervals are indeed also summit isolated suborders. We start with the following lemma:

Lemma 6.5. *Consider a finite ordered set $(S, \leq)$ with Hasse diagram $G = (S, E)$, a maximal element $m \in \max(S)$ and an $(m, \perp)$-separator s in $G^{\leftarrow}_{\perp}$ with $s \neq \perp$. Then $[s, m]$ is a summit isolated suborder in $(S, \leq)$.*

Proof. Let us pick an arbitrary $t \notin [s, m]$ with $t \lesseqgtr m$ and note that due to maximality of m the last condition is equivalent to $t < m$. To make use of Lemma 6.2 by means of contradiction we assume now that $\not(t \leq s)$ holds. The case $t > s$ is ruled out by maximality of m and the assumption $t \notin [s, m]$ so we have $s \not\gtrless t$. By $t < m$ there is a path p_1 from m to t in $G^{\leftarrow}$ which $s \not\in p_1$ due to $s \not\gtrless t$. Moreover, by construction, there is a path p_2 from t to $\perp$ in $G_{\perp^{\leftarrow}}$ with $s \notin p_2$ by the same argument as above. Now the concatenation of p_1 and p_2 yields a path p from m to $\perp$ in $G^{\leftarrow}_{\perp}$, contradicting the assumption that s is an $(m, \perp)$-separator. □

With an argument analogous to the one from Lemma 6.4 we can show that the claim from Lemma 6.5 holds even for $G^{\leftrightarrow}_{\perp}$ so we obtain the following theorem together with Lemma 6.4:

Theorem 6.6. *Let $(S, \leq)$ be an ordered set and consider an arbitrary element $s \in S$ and an arbitrary maximal element $m \in \max(S)$. Then $[m, s]$ is a summit isolated suborder of $(S, \leq)$ iff s is an $(m, \perp)$-separator in $G^{\leftrightarrow}_{\perp}$.*

Using a result from [Tar72] this leads to the following corollary:

Corollary 6.7. *Given the Hasse diagram (S, E) of a finite order $(S, \leq)$, it can be determined in $\mathcal{O}(|E|)$ time whether S has a useful summit isolated suborder. In the case of existence, a useful summit isolated suborder can be determined also in $\mathcal{O}(|E|)$ time.*

Clearly, this time bound is asymptotically optimal. Note that [Tar72] uses only a simple DFS and does not rely on some sophisticated network flow algorithms. Moreover, we can determine (in the case of existence) an inclusion-maximal useful summit isolated suborder in time linear in $|E|$.

6.2. **General Isolated Suborders and Separators.**

Most of the arguments from the previous subsection can easily be generalized to arbitrary isolated suborders. However, the proof of Lemma 6.4 can not be transferred to general isolated suborders since it exploited the maximality of the greatest element of an isolated suborder. So we can give only a characterization in terms of directed graphs:

Theorem 6.8. *Let $(S, \leq)$ be a finite ordered set with Hasse diagram $G = (S, E)$. Then the isolated suborders of S are exactly the intervals $[s, t]$ where s is a $(t, \perp)$-separator in $G^{\leftarrow}_{\perp,\top}$ and t is an $(s, \top)$-separator in $G_{\perp,\top}$.*

Compared to the computation of summit isolated suborders we face a much more adversary situation if we want to compute general isolated suborders. First, we do not know anything about the top element of such an isolated suborder (in a summit isolated suborder the top element is always a maximal element of the ordered set itself). Second, there may be a superlinear number of general isolated suborders: A chain of length n has $\frac{n(n-1)}{2} - 1$ useful isolated suborders (a linear amount of them, namely $n - 1$, are summit isolated suborders). This seems to make it impossible to come up with a linear time algorithm as in the case of summit isolated suborders. However, we are not interested in all isolated suborders but only in inclusion-maximal ones. Since distinct inclusion-maximal isolated suborders are disjoint (this follows from Lemma 4.4) there at most $|S|$ of them. Moreover, we even do not need to know all incÄºusion-maximal isolated suborders but are satisfied with some inclusion-maximal useful isolated suborder (as we will see in Section 7 we are actually interested only in inclusion-maximal useful isolated suborders with bottlenecks; however, this does not ease our task). At least, separators are computable in polynomial time (consult e.g. [CCMP20] and its references for an extensive survey on this topic) which makes the algorithm from Section 7 reasonable under circumstances discussed there. Possibly, ideas from [ILS12] can lead even to a linear algorithm for this problem.

7. Counting Closure Operators

In this section we will apply our previous results to counting closure systems. First, we consider come special cases in Subsection 7.1 and give afterward a recursive algorithm based on isolated suborders in Subsection 7.2.

7.1. Special Cases.

General orders may consist of several distinct connected components; however, we will see that for counting purposes it suffices to concentrate on orders consisting of one connected component. The following lemma shows a splitting property for closure systems on ordered sets with two or more connected components.

Lemma 7.1. *Let $(S_1, \leq_1)$ and $(S_2, \leq_2)$ be ordered sets with $S_1 \cap S_2 = \emptyset$, and let $C_1 \subseteq S_1$ and $C_2 \subseteq S_2$ be closures systems of S_1 and S_2, resp. Then $C_1 \cup C_2$ is a closure system of $(S_{12}, \leq_{12}) =_{def} (S_1 \cup S_2, \leq_1 \cup \leq_2)$.*

Proof. Let $s \in S_{12}$ be arbitrary, and assume w.l.o.g. that $s \in S_1$ holds. Then there is a least (in C_1) element $c \in C_1$ majorizing s (wrt. $\leq_1$). However, elements from S_1 and S_2 (and hence from C_1 and C_2) are incomparable wrt. $\leq_{12}$ by construction, so c is also a least element (wrt. $\leq_{12}$) majorizing s (wrt. $\leq_{12}$). □

This observation entails the following corollary:

Corollary 7.2. *Let $(S, \leq)$ be an ordered set such that there is a partition of $S = S_1 \dot\cup S_2$ into disjoint nonempty finite subsets S_1 and S_2 such that for all $s_1 \in S_1$ and $s_2 \in S_2$ the elements s_1 and s_2 are incomparable. Then the equality $|\mathcal{C}(S, \leq)| = |\mathcal{C}(S_1, \leq|_{S_1})| \cdot |\mathcal{C}(S_2, \leq|_{S_2})|$ holds.*

Of course, the results of Lemma 7.1 and Corollary 7.2 can be extended to ordered sets with more than two connected components by means of induction.

Next, we will derive closed formulae for the number of closure system on some special kinds of ordered sets which may serve as termination case in the algorithm from the following Subsection 7.2. However, we are not only interested in the overall number of closure systems but also in the number of closure systems containing a given subset of the ordered set under consideration. To deal with this situation we introduce the notations $\mathcal{C}(S)_T =_{def} \{C \in \mathcal{C}(S) \mid T \subseteq C\}$ and $\mathcal{C}(S)_{-s,T} =_{df} \{C \in \mathcal{C}(S)_T \mid s \notin C\}$ for a subsets $T \subseteq S$ and elements $s \in S$ of an ordered set $(S, \leq)$. With this notation, we have the trivial equalities $\mathcal{C}(S) = \mathcal{C}(S)_\emptyset$ (the empty set imposes no constraints) and $\mathcal{C}(S)_T = \mathcal{C}(S)_{T\setminus\{m\}} = \mathcal{C}(S)_{T\cup\{m\}}$ for each $m \in \max(S)$ (every closure system has to contain every maximal element). Moreover, since $\mathcal{C}(S)_T$ is the disjoint union of $\mathcal{C}(S)_{T\cup\{s\}}$ and $\mathcal{C}(S)_{-s,T}$, we have the equality $|\mathcal{C}(S)_T| = |\mathcal{C}(S)_{T\cup\{s\}}| + |\mathcal{C}(S)_{-s,T}|$. In particular, this means that we do not need explicit formulae for $\mathcal{C}(S)_{-s,T}$.

The first special case is that of a chain:

Lemma 7.3. *Let $(S, \leq)$ be a chain with n elements and consider an arbitrary $T \subseteq S$. Then we have $|\mathcal{C}(S)_T| = 2^{n-1-|T\setminus\{\top_S\}|}$.*

Proof. It is straightforward to see that for a finite chain $(S, \leq)$ a set $C \subseteq S$ is a closure system according to Definition 3.2 iff it contains $\top_S$. The claim follows now from the formula for the cardinality of power sets. □

Next, we consider orders with only one layer of elements between the bottom and top element:

Definition 7.1. An ordered set $(S, \leq)$ is called a *diamond of width n* if its carrier set $S = \{\bot_S, \top_S, b_1, \ldots, b_n\}$ consists of $n+2$ pairwise different elements and $b_i \ngeq b_j$ holds for all $i \neq j$. The elements $(b_i)_{1\leq i\leq n}$ are called the *belt elements* of $(S, \leq)$.

Lemma 7.4. *Let $(S, \leq)$ be a diamond of width n and let B be the set of its belt elements. Then the following holds:*

1. $\bot_S \in T \Rightarrow |\mathcal{C}(S)_T| = 2^{n-|T\setminus\{\top_S\}|+1}$
2. $\bot_S \notin T \wedge |T \cap B| > 1 \Rightarrow |\mathcal{C}(S)_T| = 2^{n-|T\setminus\{\top\}|}$
3. $\bot_S \notin T \wedge T \cap B = \{b_i\} \Rightarrow |\mathcal{C}(S)_T| = 2^{n-1} + 1$
4. $\bot_S \notin T \wedge T \cap B = \emptyset \Rightarrow |\mathcal{C}(S)_T| = 2^n + n + 1$

Proof. 1. Here, all elements of $\mathcal{C}(S)_T$ have the form $\{\bot_S, \top_S\} \cup T \cup B$. However, T occupies already $|T\setminus\{\top_S\}| - 1$ places in B so the claim follows again from the cardinality formula for power sets.

2. Because closures have to contain a least element majorizing every element, $|T \cap B| > 1$ implies $\bot \in C$ for every $C \in \mathcal{C}(S)_T$ which reduces this case to the previous one.

3. Consider a closures system $C \in \mathcal{C}(S)_T$. If $\bot \notin C$ holds then $b_j \notin C$ has to hold for all $b_i \neq b_j \in B$ since C has to contain a least element majorizing any element. Hence, the only possibility in this case is $C = \{\bot_S, b_i, \top_S\}$. The case $\bot \in C$ can be treated analogously to the first case and the result follows from summing up.

4. We have 2^n closure systems of the form $\{\bot_S, \top_S\} \cup B'$ with $B' \subseteq B$, n of the form $\{b_i, \top_S\}$ and the trivial closure system $\{\top_S\}$. □

A concept similar to diamonds are bottomless diamonds:

Definition 7.2. An ordered set $(S, \leq)$ is called a *bottomless diamond of width n* if its carrier set $S = \{\top_S, b_1, \ldots, b_n\}$ consists of $n+1$ pairwise different elements and $b_i \ngeq b_j$ holds for all $i \neq j$. The elements $(b_i)_{1\leq i\leq n}$ are called the *belt elements* of $(S, \leq)$.

Lemma 7.5. *Let $(S, \leq)$ be a bottomless diamond of width n and let B be the set of its belt elements. Then $|\mathcal{C}(S)_T| = 2^{n-|T\setminus\{\top_S\}|+1}$.*

Proof. Clearly, every $C \subseteq S$ is a closure system iff $\top_S \in C$ holds so the claim follows by elementary combinatorics. □

7.2. **Counting Closures using Isolated Suborders.** From now we assume on that every ordered set under consideration is finite because we are interested in developing an algorithm for counting the number of closure systems. The algorithm we are going to introduce in subsection 7.2 will make recursive calls computing the number of (pre)closure systems containing already processed elements.

Let $(S, \leq)$ be an ordered set and consider an isolated suborder with bottleneck $S' \subseteq S$ of S and a subset $T \subseteq S$ with $T \cap S' = \emptyset$. Then we can partition the set $\mathcal{C}(S)_T$ into two disjoint sets $\mathcal{C}(S)_T^{S'}$ and $\mathcal{C}(S)_T^{-S'}$ where the first one consists of all elements from $\mathcal{C}(S)_T$ containing an element from S' and the second one consists of all elements from $\mathcal{C}(S)_T$ containing no element from S'. By the first part of Theorem 5.8 we obtain the following equation (we have to subtract 1 in order not to count the empty preclosure twice):

$$|\mathcal{C}(S)_T^{S'}| = |\mathcal{C}(S/{\sim_{S'}})_{T^{\{\}}\cup\{\{S'\}\}}| \cdot (|\mathcal{PC}(S')| - 1). \tag{7.1}$$

By an analogous argumentation we obtain the following equation by means of the second part of Theorem 5.8:

$$|\mathcal{C}(S)_T^{-S'}| = |\mathcal{C}(S/{\sim_{S'}})_{-\{S'\},T^{\{\}}}|. \tag{7.2}$$

Now the equalities $|\mathcal{C}(S/{\sim_{S'}})_{T^{\{\}}}| = |\mathcal{C}(S/{\sim_{S'}})_{T^{\{\}}\cup\{S'\}}| + |\mathcal{C}(S/{\sim_{S'}})_{-\{S'\},T^{\{\}}}|$ and $|\mathcal{PC}(S')| = 2 \cdot |\mathcal{C}(S')|$, together with $|\mathcal{C}(S)_T| = |\mathcal{C}(S)_T^{S'}| + |\mathcal{C}(S)_T^{-S'}|$ lead to

$$|\mathcal{C}(S)_T| = |\mathcal{C}(S/{\sim_{S'}})_{T^{\{\}}\cup\{\{S'\}\}}| \cdot 2(|\mathcal{C}(S')| - 1) + |\mathcal{C}(S/{\sim_{S'}})_{T^{\{\}}}| \tag{7.3}$$

Analogous considerations for a summit isolated suborder S' guide us to the following formula by means of Theorem 5.9:

$$|\mathcal{C}(S)_T| = |\mathcal{C}(S/{\sim_{S'}})_{T^{\{\}}}| \cdot |\mathcal{C}(S')| \tag{7.4}$$

Clearly, $|\mathcal{C}(S)| = |\mathcal{C}(S)_\emptyset|$ holds, so we can use the relationships given in Equations (7.3) and (7.4) for a recursive algorithm if the ordered set under consideration contains a useful summit isolated suborder or a useful isolated suborder with bottleneck. However, this is only feasible if the isolated suborder S' and the set T are disjoint. Fortunately, we can ensure this due to Lemmata 4.9 and 4.10 by choosing first - if possible - an inclusion-maximal nontrivial suborder followed by the choice of inclusion maximal isolated suborders with bottleneck. A formal description of this idea is given in Algorithm 7.2. Of course, if the ordered set under consideration does not contain some kind of useful isolated suborder or does not have a special structure for which a closed formula is available we have to resort to some kind of brute force.

Let us now take a look at the complexity of this algorithm. In every recursive call of `#CLOSURES` the cardinality of the ordered sets in the first arguments is strictly smaller than the cardinality of the ordered set from the first argument of the function call. The first reason for this is that every isolated suborder with bottleneck or every nontrivial summit isolated suborder S' is a strict subset of S. Moreover, we consider only useful isolated suborders S', hence $S/{\sim_{S'}}$ contains strictly less element than S. This enforces termination in the sense that either an ordered set with a special structure (for which a closed formula is known) is obtained or some brute force method is applicated.

Algorithm 1 Counting Closure Operators

```
function #CLOSURES(ordered set S, set T)
    if some special case from Subsection 7.1 is applicable then
        return the respective number
    end if
    if S has a useful summit isolated suborder then
        S′ ← an inclusion maximal useful summit suborder
        return #CLOSURES(S/∼_S′, T^{}) · #CLOSURES(S′, ∅)
    end if
    if S has a useful isolated suborder with bottleneck then
        S′ ← an inclusion maximal useful isolated suborder with bottleneck
        return #CLOSURES(S/∼_S′, T^{} ∪ {S′^{}}) · 2(#CLOSURES(S′, ∅)-1) + #CLOSURES(S/∼_S′, T^{})
    end if
    compute and return |C(S)_T| by some brute force algorithm
end function
```

As described in Section 6 isolated suborders of interest can be computed in polynomial time. Let us now assume that a brute force algorithm takes $c^{|S|}$ time for some $c > 1$ to compute the number of all closure systems. Furthermore, we consider a family of ordered sets which have a useful summit isolated suborder of cardinality $\frac{|S|}{2}$. Then Algorithm 7.2 will make two recursive calls on instances of sizes $\frac{|S|}{2}$ and $\frac{|S|}{2} + 1$. In the worst case, i.e., if no useful isolated suborder is found in these two instances, the algorithm has to resort to a brute force solution. By the results from Subsection 6.1, summit isolated suborders can be determined in $O(|S|^2)$ time so the the overall running time is bounded by $|S|^2 + c^{\frac{|S|}{2}} + c^{\frac{|S|}{2}+1} \in O(c^{\frac{|S|}{2}+1})$. However, this running time is asymptotically strictly dominated by $c^{|S|}$ which would be the running time of the immediate application of the brute force algorithm. An analogous conclusion will be obtained considering general isolated suborders which can be computed in time polynomial in $|S|$ (this would only replace the term $|S|^2$ by another polynomial).

8. Conclusion and Outlook

As main result, we introduced Algorithm 7.2 which can simplify counting of closures if the ordered set under consideration contains certain kinds of isolated suborders. Naturally, the question arises whether more general or other structures than isolated suborders can by used with the same purpose. Concepts which come to mind are bisimulation which are used also for simplification (in the sense of reducing the number of states) in model checking in [BK08] or model refinement in [GMS10]. The present work generalized the ideas introduced in [Glü21] from lattices to general orders; a further generalization step could be to consider counting of monads on categories since monads on categories are a generalization of closure operators. Also, it would be of interest to search for more special cases than the ones from Subsection 7.1 wherefrom the Algorithm would clearly benefit. Lastly, the algorithm is still awaiting its implementation. Part of such an implementation would also be an optimal (i.e., linear time) algorithm for the computation of isolated suborders along the lines sketched at the end of Subsection 6.2. A combination of classical programming and specific system like Mace4 (see [Mac]) or RelView (see [Rel]) seems to be an interesting and promising approach. Alas, for computing the number of closure systems on a powerset the approach presented here will be of no help: it is easy to see that a powerset lattice contains no useful isolated suborders.

References

[AJS21] N. Alpay, P. Jipsen, and M. Sugimoto. Unary-determined distributive ℓ-magmas and bunched implication algebras. In U. Fahrenberg, M. Gehrke, L. Santocanale, and M. Winter, editors, *Relational and Algebraic Methods in Computer Science - 19th International Conference, RAMiCS 2021, Marseille, France, November 2-5, 2021, Proceedings*, volume 13027 of *Lecture Notes in Computer Science*, pages 19–36. Springer, 2021. doi:10.1007/978-3-030-88701-8_2.

[BBW21] R. Berghammer, S. Börm, and M. Winter. Algorithmic counting of zero-dimensional finite topological spaces with respect to the covering dimension. *Appl. Math. Comput.*, 389:125523, 2021.

[BD18] G. Brinkmann and R. Deklerck. Generation of union-closed sets and moore families. *Journal of Integer Sequences*, 21(1):18.1.7, 2018. URL: https://cs.uwaterloo.ca/journals/JIS/VOL21/Brinkmann/brink6.html.

[Bir67] G. Birkhoff. *Lattice Theory*. Amer. Math. Soc., 3rd edition, 1967.

[BK08] C. Baier and J-P. Katoen. *Principles of Model Checking*. MIT Press, 2008.

[BPBV18] S. Bonzio, M. Pra Baldi, and D. Valota. Counting finite linearly ordered involutive bisemilattices. In J. Desharnais, W. Guttmann, and S. Joosten, editors, *Relational and Algebraic Methods in Computer Science - 17th International Conference, RAMiCS 2018, Groningen, The Netherlands, October 29 - November 1, 2018, Proceedings*, volume 11194 of *Lecture Notes in Computer Science*, pages 166–183. Springer, 2018.

[CCMP20] A. Conte, P. Crescenzi, A. Marino, and G. Punzi. Enumeration of s-d separators in dags with application to reliability analysis in temporal graphs. In J. Esparza and D. Král', editors, *45th International Symposium on Mathematical Foundations of Computer Science, MFCS 2020, August 24-28, 2020, Prague, Czech Republic*, volume 170 of *LIPIcs*, pages 25:1–25:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.MFCS.2020.25.

[CIR10] P. Colomb, A. Irlande, and O. Raynaud. Counting of moore families for n=7. In L. Kwuida and B. Sertkaya, editors, *Formal Concept Analysis, 8th International Conference, ICFCA 2010, Agadir, Morocco, March 15-18, 2010. Proceedings*, volume 5986 of *Lecture Notes in Computer Science*, pages 72–87. Springer, 2010.

[CM03] N. Caspard and B. Monjardet. The lattices of closure systems, closure operators, and implicational systems on a finite set: A survey. *Discrete Applied Mathematics*, 127(2):241–269, 2003.

[DHLM92] J. Demetrovics, G. Hencsey, L. Libkin, and I. B. Muchnik. On the interaction between closure operations and choice functions with applications to relational database. *Acta Cybern.*, 10(3):129–139, 1992. URL: https://cyber.bibl.u-szeged.hu/index.php/actcybern/article/view/3401.

[DP02] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order, Second Edition*. Cambridge University Press, 2002.

[EBJ+14] S. Elloumi, B. Boulifa, A. Jaoua, M. Saleh, J. Al Otaibi, and M. F. Frias. Inference engine based on closure and join operators over truth table binary relations. *J. Log. Algebraic Methods Program.*, 83(2):180–193, 2014.

[FJST20] U. Fahrenberg, C. Johansen, G. Struth, and R.B. Thapa. Generating posets beyond N. In U. Fahrenberg, P. Jipsen, and M. Winter, editors, *Relational and Algebraic Methods in Computer Science - 18th International Conference, RAMiCS 2020, Palaiseau, France, April 8-11, 2020, Proceedings [postponed]*, volume 12062 of *Lecture Notes in Computer Science*, pages 82–99. Springer, 2020. doi:10.1007/978-3-030-43520-2_6.

[Glü17] R. Glück. Algebraic investigation of connected components. In P. Höfner, D. Pous, and G. Struth, editors, *Relational and Algebraic Methods in Computer Science - 16th International Conference, RAMiCS 2017, Lyon, France, May 15-18, 2017, Proceedings*, volume 10226 of *Lecture Notes in Computer Science*, pages 109–126, 2017.

[Glü21] R. Glück. Isolated sublattices and their application to counting closure operators. In U. Fahrenberg, M. Gehrke, L. Santocanale, and M. Winter, editors, *Relational and Algebraic Methods in Computer Science - 19th International Conference, RAMiCS 2021, Marseille, France, November 2-5, 2021, Proceedings*, volume 13027 of *Lecture Notes in Computer Science*, pages 192–208. Springer, 2021. doi:10.1007/978-3-030-88701-8_12.

[GMS10] R. Glück, B. Möller, and M. Sintzoff. Model refinement using bisimulation quotients. In M. Johnson and D. Pavlovic, editors, *Algebraic Methodology and Software Technology - 13th International Conference, AMAST 2010, Lac-Beauport, QC, Canada, June 23-25, 2010. Revised Selected Papers*, volume 6486 of *Lecture Notes in Computer Science*, pages 76–91. Springer, 2010.

[Grä11] G. Grätzer. *Lattice theory: Foundation*. Springer Basel, 2011.

[ILS12] G. F. Italiano, L. Laura, and F. Santaroni. Finding strong bridges and strong articulation points in linear time. *Theor. Comput. Sci.*, 447:74–84, 2012. doi:10.1016/j.tcs.2011.11.011.

[JR92] P. Jipsen and H. Rose. *Varieties of Lattices*. Springer, 1st edition, 1992.

[Mac] Mace4. https://www.cs.unm.edu/~mccune/mace4/. [Online; accessed 23-February-2023].

[QRRV20] S. Quintero, S. Ramírez, C. Rueda, and F. Valencia. Counting and computing join-endomorphisms in lattices. In U. Fahrenberg, P. Jipsen, and W. Winter, editors, *Relational and Algebraic Methods in Computer Science - 18th International Conference, RAMiCS 2020, Palaiseau, France, April 8-11, 2020, Proceedings [postponed]*, volume 12062 of *Lecture Notes in Computer Science*, pages 253–269. Springer, 2020.

[Rel] RelView. https://www.informatik.uni-kiel.de/~progsys/relview/. [Online; accessed 23-February-2023].

[Rom08] S. Roman. *Lattices and Ordered Sets*. Springer, 1st edition, 2008.

[Tar72] Robert Endre Tarjan. Depth-first search and linear graph algorithms. *SIAM J. Comput.*, 1(2):146–160, 1972.