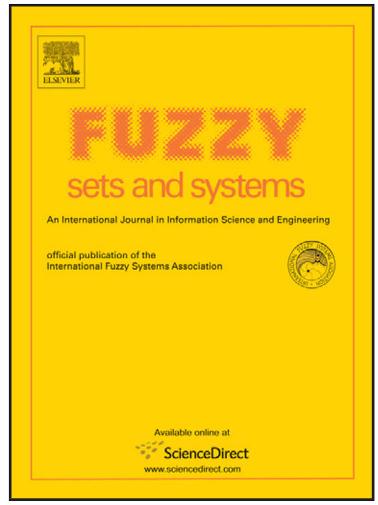

www.elsevier.com/locate/fss

Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers

Shang-Ming Zhou, Francisco Chiclana, Robert I. John, Jonathan M. Garibaldi

PII: S0165-0114(08)00332-1
DOI: doi:10.1016/j.fss.2008.06.018
Reference: FSS 5360

To appear in: *Fuzzy Sets and Systems*

Received date: 16 October 2007
Revised date: 26 June 2008
Accepted date: 27 June 2008

Cite this article as: Shang-Ming Zhou, Francisco Chiclana, Robert I. John and Jonathan M. Garibaldi, Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers, *Fuzzy Sets and Systems* (2008), doi:10.1016/j.fss.2008.06.018

Type-1 OWA Operators for Aggregating Uncertain Information with Uncertain Weights Induced By Type-2 Linguistic Quantifiers

Shang-Ming Zhou [a,*], Francisco Chiclana [a], Robert I. John [a], Jonathan M. Garibaldi [b]

[a]*Centre for Computational Intelligence, School of Computing,De Montfort University, Leicester, LE1 9BH, UK.*

[b]*Intelligent Modelling and Analysis (IMA) Research Group, School of Computer Science, University of Nottingham, Nottingham, NG8 1BB, UK.*

Abstract

The OWA operator proposed by Yager has been widely used to aggregate experts' opinions or preferences in human decision making. Yager's traditional OWA operator focuses exclusively on the aggregation of crisp numbers. However, experts usually tend to express their opinions or preferences in a very natural way via linguistic terms. These linguistic terms can be modelled or expressed by (type-1) fuzzy sets. In this paper, we define a new type of OWA operator, the type-1 OWA operator that works as an uncertain OWA operator to aggregate type-1 fuzzy sets with type-1 fuzzy weights, which can be used to aggregate the linguistic opinions or preferences in human decision making with linguistic weights. The procedure for performing type-1 OWA operations is analysed. In order to identify the linguistic weights associated to the type-1 OWA operator, type-2 linguistic quantifiers are proposed. The problem of how to derive linguistic weights used in type-1 OWA aggregation given a such type of quantifier is solved. Examples are provided to illustrate the proposed concepts.

Key words: Aggregation, OWA operator, type-1 OWA operator, type-2 fuzzy sets, type-2 linguistic quantifiers, soft decision making

* Corresponding author

Email addresses: smzhou@ieee.org (Shang-Ming Zhou), chiclana@dmu.ac.uk (Francisco Chiclana), rij@dmu.ac.uk (Robert I. John), jmg@cs.nott.ac.uk (Jonathan M. Garibaldi).

1 Introduction

Decision making is one of the most significant and omnipresent human activities in business, manufacturing, service etc.. Existing decision making paradigms include multi-expert decision making (i.e. group decision making), multi-criteria decision making and multi-expert multi-criteria decision making. All these paradigms require aggregation methods. The objective of aggregation is to combine individual experts' preferences or criteria into an overall one in a proper way so that the final decision takes into account all the individual contributions [6]. It has become a subject of intensive research due to its practical and academic significance. Currently, at least 90 different families of aggregation operators have been studied [2, 3, 6, 7, 16, 24, 37, 38]. Among them the Ordered Weighted Averaging (OWA) operator proposed by Yager [38] is the most widely used. But most of the existing aggregation operators, including the OWA, focus on aggregating crisp numbers, while in real world decision applications human experts exhibit remarkable capability to manipulate perceptions without any measurements and any computations [48]. For example, in practice, people perceive the distance, size, weight, likelihood, and other characteristics of physical and mental objects in a very natural way via linguistic terms, like "very long","big", "very heavy", "good" etc., when they cannot provide exact numbers for expressing vague and imprecise opinions [46]. Thus, how to effectively aggregate linguistic judgments for decision makers is a problem that needs to be addressed.

Linguistic terms can be characterised as linguistic variables via type-1 fuzzy sets or type-2 fuzzy sets, where type-1 fuzzy sets are the traditional fuzzy sets proposed by Zadeh in 1965 [45], and type-2 fuzzy sets were proposed by Zadeh later in 1975 [46]. Currently, there are two main schemes to aggregate linguistic information in decision making. The first scheme is to work directly on linguistic labels without considering the (mathematical) expression of the linguistic terms. The only requirement of this scheme is that these linguistic labels should satisfy an order relation. Bordogna et al. [1] proposed a linguistic modelling of consensus in group decision making, in which both experts' evaluations of alternatives and degree of consensus are expressed linguistically and where the overall linguistic performance evaluation is computed via a linguistic OWA operator based aggregation. Another method defined in [13, 14] integrates the OWA operator [38] and a convex combination method of linguistic labels. One advantage of such a scheme of directly aggregating linguistic labels without considering the expression of the linguistic terms lies in its high computing efficiency due to its symbolic aggregation in nature. However, the precision of the linguistic operations is an issue: in some cases, this scheme may yield a solution set with multiple alternatives for decision makers to choose, rather than a single one. Another matter is that most of the existing methods based on this scheme use the traditional OWA operator in nature which aims at aggregating crisp numbers. The second scheme of aggregating linguistic information is via operations performed on their associated membership functions.

Zimmermann and Zysno developed a family of compensatory operators for aggregating type-1 fuzzy sets by combining a t-norm and a t-conorm to produce certain compensation between criteria [51, 52]. This family of compensatory operators has been extended to aggregate weighted fuzzy sets in heterogeneous decision making problems [28], in which different experts were assigned different importance weights in the form of crisp numbers. In order to evaluate an overall linguistic value, a weighted average of the membership function values associated with the linguistic labels was first computed, and then this aggregation result was translated into linguistic terms via a linguistic approximation. There is a common problem with these two schemes: the importance weights for different experts are assumed to be precise numerical values. This assumption implies that uncertain linguistic labels are aggregated in terms of certain precise crisp weights. However, in real world decision making, the weights reflecting the experts' desired agenda for aggregating opinions and preferences are usually uncertain rather than crisp in nature. Hence, a reasonable way of aggregating linguistic judgments in decision making should consider linguistic weights as well rather than crisp weight values.

Interestingly, Meyer and Roubens [27] proposed a fuzzified Choquet integral to aggregate type-1 fuzzy numbers (normal convex type-1 fuzzy sets) based on a Mobious transform of a fuzzy measure; a different version of fuzzified Choquet integral was suggested in [36, 44] for fuzzy-valued integrands. However, in these two versions of the fuzzified Choquet integral the importance weights used are precise real values rather than uncertain quantities. The major advantage of using the Choquet integral lies in that it can provide a profound theoretical analysis and background, but it suffers from the serious drawback of needing to assign real values to the importance of all possible combinations [27]. Also, there is no clear way of inducing real values as importance weights for the fuzzified Choquet integrals [27, 44]. Currently, the fuzzy weighted averaging operator has been investigated to aggregate type-1 fuzzy sets by type-1 fuzzy importance weights as well [5, 11, 21]. However, how to generate the fuzzy importance weights for the fuzzy weighted averaging operator was not addressed. The fuzzy weighted averaging operator implies preferential independence of the experts' points of view. Preference is an important issue in soft decision making [2, 7], and has nowadays found significant interest in various fields such as economic decision making, social choice theory, operations research, databases, and human-computer interaction. One way of avoiding this independence condition is to generalise the traditional OWA operator as an aggregation operator for type-1 fuzzy sets, which is the aim of this paper. Mitchell and Estrakh suggested a way of generalising Yager's OWA operator by extending the integer ranks used in the reordering step to the case of real-number or fuzzy ranks [30], however, this generalised OWA operator still focuses on aggregating crisp number values based on crisp weights rather than fuzzy sets.

In this paper, a new type of OWA operator, the *type-1 OWA operator*, is proposed to aggregate the linguistic information in the form of type-1 fuzzy sets by linguistic weights in the form of type-1 fuzzy sets as well. Because Yager's OWA operator is nonlinear as opposed to the weighted averaging operator, a linear operator, so the proposed type-1 OWA operator is significantly different from the fuzzy weighted average operator. Moreover, in this paper we further address the problem of how to obtain the linguistic weights used in the type-1 OWA operator based on new type of linguistic quantifiers. It is known that in the traditional OWA operator, the linguistic quantifiers used for identifying its weights are modelled via type-1 fuzzy sets. The limitation is that a type-1 fuzzy set actually uses precise values to characterise uncertainty. As a result, the weights derived by these linguistic quantifiers are precise and crisp. As Klir and Folger [24, page 12] point out: "... it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers". In this paper, in order to model a higher level of uncertainty associated to linguistic quantifiers, the concept of a type-2 quantifier is defined in the form of a type-2 fuzzy set [19, 26]. A method of deriving uncertain linguistic weights based on type-2 quantifiers for the proposed type-1 OWA operator is also presented.

The rest of the paper is set out as follows. Section 2 defines the type-1 OWA operator for aggregating type-1 fuzzy sets, in which the procedure for performing type-1 OWA operations is analysed. Section 3 introduces the concept of a type-2 linguistic quantifier and addresses the problem of how to derive linguistic weights used in type-1 OWA aggregation given such a type of quantifier. Finally, in Section 4 we draw our conclusions and suggest further research on type-1 OWAs.

2 Type-1 OWA operator for aggregating type-1 fuzzy sets

In this section, we present an uncertain OWA operator for aggregating uncertain information with uncertain weights, the type-1 OWA operator. We also provide a procedure for performing type-1 OWA operations in practice. First, we briefly review Yager's traditional OWA operator.

2.1 Yager's OWA operator

In 1988, Yager introduced an aggregation technique based on the order weighted averaging (OWA) scheme [38]. Since then, OWA based aggregation strategies have been widely investigated and have achieved successful applications in many domains, such as decision making [4, 13, 14, 28, 38, 39], fuzzy control [41, 42], market analysis [43], image compression [29], etc..

Definition 1 *Yager's OWA operator of dimension n is a mapping $\phi\colon \mathbb{R}^n \to \mathbb{R}$, which has an associated set of weights $W = (w_1, \cdots, w_n)^T$ to it, so that $w_i \in [0,\ 1]$ and*

$\sum_{i=1}^{n} w_i = 1$,

$$\phi(a) = \phi(a_1, \cdots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)}$$

where $\sigma\colon \{1, \cdots, n\} \to \{1, \cdots, n\}$ *is a permutation function such that* $a_{\sigma(i)}$ *is the i-th highest value in the set* $\{a_1, \cdots, a_n : a_{\sigma(i)} \geq a_{\sigma(i-1)}\}$.

Generally speaking, the OWA operator based aggregation process consists of three steps:

(1) The first step is to re-order the input arguments in descending order. The element a_i is not associated with a particular weight w_i, but rather w_i is associated with a particular ordered position of an aggregated object.
(2) The second step is to determine the weights for the operator in a proper way.
(3) Finally, the OWA weights are used to aggregate the re-ordered arguments.

Among the three steps, the first step introduces a nonlinearity into the aggregation process by re-ordering the input arguments, which make Yager's OWA operator significantly different from the weighted averaging operator, a linear aggregation operator.

2.2 Type-1 OWA operator

The departure point for suggesting such an uncertain OWA operator is to aggregate the linguistic variables (modelled as type-1 fuzzy sets) used to express human opinions or preferences in soft decision making. Let $F(X)$ be the set of type-1 fuzzy sets defined on the domain of discourse X. Based on Zadeh's extension principle, we extend Yager's OWA operator when both weights and aggregated objects are type-1 fuzzy sets.

Definition 2 *Given n linguistic weights* $\{W_i\}_{i=1}^{n}$ *in the form of type-1 fuzzy sets defined on the domain of discourse* $[0,1]$*, an associated type-1 OWA operator of dimension n is a mapping* Φ*,*

$$\begin{aligned} \Phi\colon F(X) \times \cdots \times F(X) &\longrightarrow F(X) \\ (A_1, \cdots, A_n) &\mapsto G \end{aligned}$$

that aggregates type-1 fuzzy sets $\{A_i\}_{i=1}^{n}$ *in the following way,*

$$\mu_G(y) = \sup_{\substack{\sum_{k=1}^{n} \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U, a_i \in X}} \left(\mu_{W_1}(w_1) * \cdots * \mu_{W_n}(w_n) * \mu_{A_1}(a_1) * \cdots * \mu_{A_n}(a_n) \right) \quad (1)$$

where $*$ is a t-norm operator, $\bar{w}_i = \frac{w_i}{\sum_{i=1}^{n} w_i}$, and $\sigma : \{ 1, \cdots, n \} \rightarrow \{ 1, \cdots, n \}$ a permutation function such that $a_{\sigma(i)}$ is the i-th largest element in the set $\{a_1, \cdots, a_n\}$.

From the above definition, it can be seen that $\Phi(A_1, \cdots, A_n) = G \in F(X)$ is a type-1 fuzzy set defined on X. When the associated weights of the type-1 OWA operator are intervals (i.e., special cases of type-1 fuzzy sets), its expression simplifies as follows:

Definition 3 *Given n interval weights $\left\{\widehat{W}_i\right\}_{i=1}^{n}$, $\widehat{W}_i \subseteq [0,1] (i = 1, \cdots, n)$, the associated type-1 OWA operator is*

$$\mu_G(y) = \sup_{\substack{\sum_{k=1}^{n} \bar{w}_i a_{\sigma(i)} = y \\ w_i \in \widehat{W}_i, a_i \in X}} \left(\mu_{A_1}(a_1) * \cdots * \mu_{A_n}(a_n) \right) \tag{2}$$

where $*$ is a t-norm operator, $\bar{w}_i = \frac{w_i}{\sum_{i=1}^{n} w_i}$, and $\sigma : \{ 1, \cdots, n \} \rightarrow \{ 1, \cdots, n \}$ a permutation function such that $a_{\sigma(i)}$ is the i-th largest element in the set $\{a_1, \cdots, a_n\}$.

2.3 Procedure for performing type-1 OWA operations

Given a set of linguistic weights $\{W_i\}_{i=1}^{n} \subset F(U)$, in order for the associated type-1 OWA operator to aggregate type-1 fuzzy sets $\{A_i\}_{i=1}^{n} \subset F(X)$ on computer in practical applications, as usual, the domains X and U have to be discretised for calculation. Let the discretised domains be $\hat{X} = \left\{\hat{x_1}, \cdots, \hat{x_p}\right\}$ and $\hat{U} = \{\hat{u_1}, \cdots, \hat{u_k}\}$, which are partitions of the spaces X and U respectively. The final aggregation result should be achieved on the discretised domain $\hat{X}$. However, with all the combinations of $(w_1, \cdots, w_n, a_1, \cdots, a_n)$, where $w_i \in \hat{U}, a_i \in \hat{X}, i = 1, \cdots, n$, the term $\sum_{k=1}^{n} \bar{w}_i a_{\sigma(i)}$ produces another partition of X :

$$\overline{X} = \left\{\bar{x}_j\right\} = \left\{ \sum_{k=1}^{n} \bar{w}_i a_{\sigma(i)} \middle| w_i \in \hat{U}, a_i \in \hat{X}, i = 1, \cdots, n \right\}$$

The problem is that $\hat{X} \neq \overline{X}$, i.e., the two discretised versions of X may be different, and the cardinality of $\overline{X}$ is greater than or equal to the cardinality of $\hat{X}$: $\left|\overline{X}\right| \geq \left|\hat{X}\right|$. In other words, there are many points in $\overline{X}$ that lie between the neighbouring points in $\hat{X}$. The set $\overline{X}$ is referred to as an *over-partition* of the input space given the used $\hat{X}$. The consequence is that the fuzzy set, $\overline{G}$, generated on $\overline{X}$ according to the extension principle is likely to be unreadable, because for some data points that are in $\overline{X}$ but not in $\hat{X}$, their membership grades may not be consistent with the membership grades of the corresponding nearest points in $\hat{X}$. Let us consider for example the

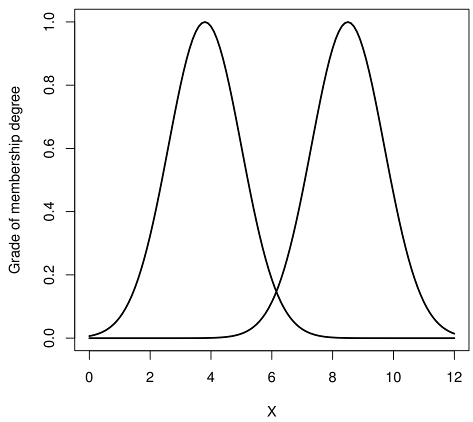

Fig. 1. Two type-1 fuzzy sets to be aggregated.

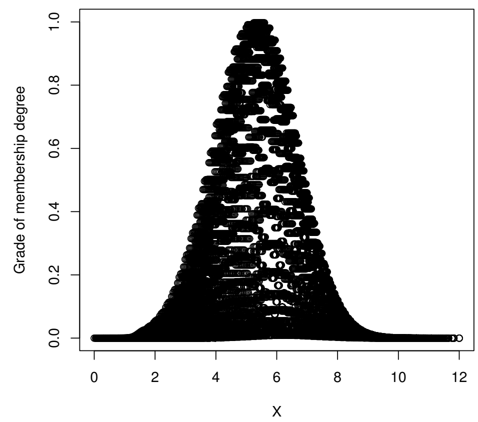

Fig. 2. Aggregation result $\overline{G}$ generated on the *over-partition* $\overline{X}$.

problem of aggregating the two type-1 fuzzy sets illustrated in Figure 1 by a type-1 OWA operator using the two interval weights $[0.25, 0.4]$ and $[1, 1]$. Let the discretised domains be $\hat{X} = \{0.24 \cdot k | k = 0, \cdots, 50\}$, $\widehat{W}_1 = \{0.25 + 0.01 \cdot k | k = 0, \cdots, 15\}$ and $\widehat{W}_2 = \{1\}$ respectively. The over-partition of the input space, $\overline{X}$, is obtained as $\overline{X} = \{0.0000, 0.0480, 0.0495, \cdots, 12.0000\} \neq \hat{X}$. Figure 2 shows the initial aggregation result $\overline{G}$ generated on the over-partition $\overline{X}$. Hence the final aggregation result, the type-1 fuzzy set G on the set $\hat{X}$, should be derived from $\overline{G}$. This can be conducted according to the extension principle as follows:

The sets $\hat{X}$ and $\overline{X}$ are two partitions of the domain X as shown in Figure 3, with $\overline{X}$ providing a finer resolution than $\hat{X}$. So the data points from the partition $\overline{X}$ lying between two neighbouring points in the coarse partition $\hat{X}$, for example the points $\hat{x}_i$ and $\hat{x}_{i+1}$ in $\hat{X}$, form a cluster: $\Theta_{\hat{x}_i} \triangleq \left\{\bar{x}_j \middle| \bar{x}_j \in \overline{X}, \hat{x}_i \leq \bar{x}_j < \hat{x}_{i+1}\right\}$, in which $\hat{x}_i$ is the cluster prototype.

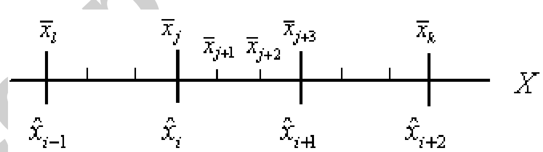

Fig. 3. The two partitions of domain X: $\overline{X}$ and $\hat{X}$.

This is analogous to a digital map with different resolutions: by zooming in, we can see a map with fine details, whilst by zooming out, all the details are displayed in a point, this point acts as one unit representing all the details behind it. Hence, the whole cluster $\Theta_{\hat{x}_i}$ with prototype $\hat{x}_i$ is treated as one unit, and all the membership grades of the data points in $\Theta_{\hat{x}_i}$ should be assigned to this unit. So according to the extension principle, the membership grade of this unit is obtained by maximising all its available membership grades. Thus, the membership grade of the resulting

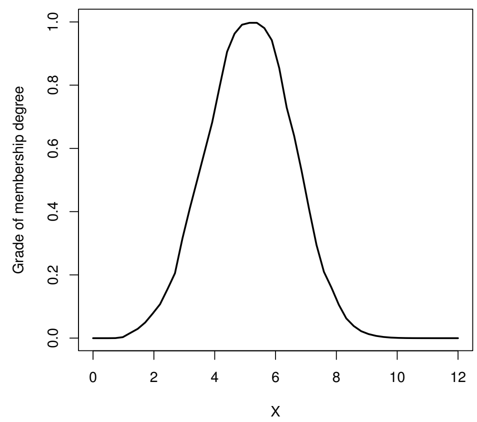

Fig. 4. The fuzzy set G on $\hat{X}$ derived from the fuzzy set $\overline{G}$ on $\overline{X}$ in Figure 2.

type-1 fuzzy set G at the prototype point $\hat{x}_i$ is obtained as

$$\mu_G(\hat{x}_i) = \sup_{\bar{x}_j \in \Theta_{\hat{x}_i}} \left(\mu_{\overline{G}}(\bar{x}_j)\right) \tag{3}$$

Figure 4 shows the resulting fuzzy set obtained by applying equation (3) to the initial aggregation result depicted in Figure 2. A Direct Approach to type-1 OWA operation is addressed as follows:

Step 1: Initialisation.
- Given the set of linguistic weights $\{W_i\}_{i=1}^n \subseteq F(U)$ for aggregating the set of type-1 fuzzy sets $\{A_i\}_{i=1}^n \subseteq F(X)$;
- Given the discretised domains of the linguistic weights, $\hat{U}$, and that of the aggregated objects, $\hat{X}$;
- Let the initial aggregation result $\overline{G} = (\overline{X}, \mu_{\overline{G}})$, where $\overline{X} = \{0\}$, and $\mu_{\overline{G}}(\bar{x}) = 0$.

Step 2: Obtain the initial result $\overline{G}$
(1) Select $w_1, \cdots, w_n \in \hat{U}$, $a_1, \cdots, a_n \in \hat{X}$;
(2) Normalise $(w_1, \cdots, w_n)$ as $\bar{w}_i = \frac{w_i}{\sum_{i=1}^{n} w_i}$;
(3) Perform the traditional OWA operation: $\bar{y} = \phi_{(\bar{w}_1, \cdots, \bar{w}_n)}(a_1, \cdots, a_n)$;
(4) Calculate $\mu_0 = \mu_{W_1}(w_1) * \cdots * \mu_{W_n}(w_n) * \mu_{A_1}(a_1) * \cdots * \mu_{A_n}(a_n)$;
(5) If there exists $y_k \in \overline{X}$ such that $\bar{y} = y_k$, update the potential membership grade $\mu_{\overline{G}}(y_k)$:

$$\mu_{\overline{G}}(y_k) \leftarrow max\left(\mu_{\overline{G}}(y_k), \mu_0\right)$$

Otherwise, $\bar{y}$ is added to $\overline{X}$, and $\mu_{\overline{G}}(\bar{y}) \triangleq \mu_0$;
(6) Go to ***Step 2***-(1), and continue until all weight vectors and aggregating points are selected.

Step 3: Derive the fuzzy set G on $\hat{X}$:

$$\mu_G(\hat{x}) = \sup_{\bar{x}_j \in \Theta_{\hat{x}}} \left(\mu_{\overline{G}}(\bar{x}_j)\right).$$

3 Linguistic quantifier guided type-1 OWA operator

The identification of an appropriate OWA operator, i.e., to determine the weighting vector associated to an OWA weights, is crucial in OWA based aggregation [25, 35, 40], because the OWA weights reflect the decision makers' desired agenda for aggregating the criteria. Yager [38, 40] proposed a popular method for identifying the traditional OWA weights via linguistic quantifiers like *"most"*, *"almost all"*, *"at least half"*, which is very useful in decision making, particularly in group decision making with an expected soft consensus solution [15, 17]. This "soft" majority (or minority) such as *"most"* (or *"least"*) is much closer to the real human perception in decision making. Interestingly, based on Zadeh's concept of linguistic quantifiers [47], some of the traditional OWA operators can be used to characterise these "soft" majority (or minority) operations [38, 40]. For example, given the linguistic quantifier *"most"* (Q), one can consider an aggregation of a set of criteria based on an imperative such as "*most of the criteria should be satisfied*". Yager called them as quantifier guided aggregations. As a matter of fact, the use of a quantifier guided aggregation can be considered as a manifestation of a semantically based aggregation consideration.

Given a non-decreasing quantifier, i.e., a function $Q : [0,1] \to [0,1]$ such that $Q(0) = 0$, $Q(1) = 1$ and if $x > y$ then $Q(x) \geq Q(y)$, an OWA aggregation guided by this function can be obtained as [38, 40]:

$$\phi_Q(a_1,\ldots,a_n) = \sum_{i=1}^{n} w_i \cdot a_{\sigma(i)},$$

where $a_{\sigma(i)}$ is the i-th largest value in the set $\{a_1,\ldots,a_n\}$; and

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right),\ i = 1,\ldots,n. \tag{4}$$

Figure 5 depicts the non-decreasing fuzzy quantifiers *"most"*, *"almost all"*, and *"at least half"* with membership function

$$Q(r) = \begin{cases} 0, & \text{if } r < a; \\ \frac{r-a}{b-a}, & \text{if } a \leq r \leq b; \\ 1, & \text{if } r > b. \end{cases}$$

characterised by parameters (a, b) =(0.3, 0.8), (0, 0.5) and (0.5, 1) respectively [20, 47].

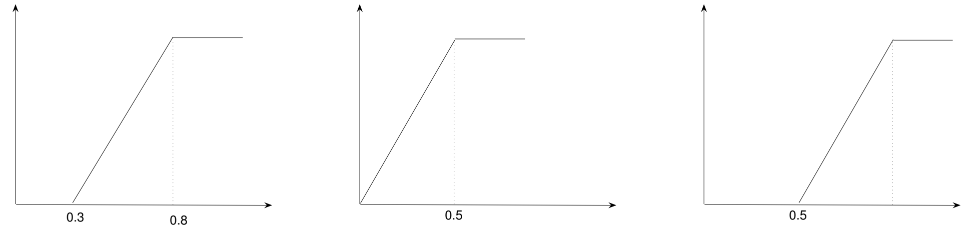

Fig. 5. Proportional fuzzy quantifiers (from left to right): (a)*" most"*, (b) *"at least half"*, (c) *"as many as possible"*.

3.1 Type-2 linguistic quantifiers

The existing definitions of linguistic quantifiers are all based on type-1 fuzzy sets. The limitation is that a type-1 fuzzy set actually uses precise values to characterise uncertainty, so if the linguistic quantifiers in the form of type-1 fuzzy sets are used to guide type-1 OWA aggregation, then certain precise crisp weights will be induced to characterise the uncertainty of linguistic variables. In order to characterise the higher level uncertainty associated to linguistic weights, it is more reasonable to define a linguistic quantifier by using type-2 fuzzy sets [19, 26]. Hence we propose type-2 quantifiers based on type-2 fuzzy sets and use them to determine the linguistic weights for the type-1 OWA operator.

Definition 4 *A type-2 quantifier $\widetilde{Q}$ is characterised by a type-2 fuzzy set defined on [0, 1], i.e.,*

$$\widetilde{Q} = \left\{ \left((r,u), \mu_{\widetilde{Q}}(r,u) \right) \middle| 0 \le \mu_{\widetilde{Q}}(r,u) \le 1 \; \forall r \in [0,1], \; \forall u \in J_r \subseteq [0,1] \right\}.$$

For example, the linguistic quantifier *"most"* can be defined by the following type-2 fuzzy set,

$$\widetilde{m} = \begin{pmatrix} 0 & 0.1 & 0.2 & 0.3 & 0.4 & \dots & 0.7 & 0.8 & 0.9 & 1 \\ trivial & trivial & trivial & trivial & very\ small & \dots & very\ big & utmost & utmost & utmost \end{pmatrix}$$

in which *"trivial"*, *"very small"*, ..., *"very big"*, and *"utmost"* can be represented as type-1 fuzzy numbers.

Before we define three different kinds of type-2 quantifiers: monotonically non-decreasing, monotonically non-increasing and unimodal type-2 quantifiers, we first introduce a partial ordering relation of type-1 fuzzy sets based on join and meet operations.

Given two type-1 fuzzy sets A and B, their meet ($A \sqcap B$) and join ($A \sqcup B$) are defined as follows [31, 46]:

$$\mu_{A \sqcap B}(z) = \sup_{\substack{x \wedge y = z \\ x \in D_A,\ y \in D_B}} \mu_A(x) * \mu_B(y)$$

$$\mu_{A \sqcup B}(z) = \sup_{\substack{x \vee y = z \\ x \in D_A,\ y \in D_B}} \mu_A(x) * \mu_B(y)$$

where $D_A, D_B \subseteq X$ represent the domains of A and B respectively; $*$ is a t-norm operator; $\wedge$ represents the minimum operation; and $\vee$ represents the maximum operation.

It is known that the set $\mathbb{R}$ of real numbers is linearly ordered and $(\mathbb{R}, \min, \max)$ is a lattice where min and max represent the minimum and maximum operators respectively, and for any $x, y \in \mathbb{R}$ the partial ordering relation $\geq$ ($\leq$) is defined as

$$x \geq y \iff \max(x, y) = x$$
$$y \leq x \iff \min(x, y) = y$$

Based on the extension principle, it can be seen that meet and join operators are the generalisation of the lattice min and max operators to type-1 fuzzy sets in nature. Hence $A \sqcap B$ and $A \sqcup B$ are referred to as the fuzzy minimum and fuzzy maximum of type-1 fuzzy sets A and B respectively, where $A, B \in F(X), X \subseteq \mathbb{R}$. $(F(X), \sqcap, \sqcup)$ forms a distributive lattice [22] [1] describing a partial ordering relation of type-1 fuzzy sets A and B as follows:

Definition 5 *Let A and B be type-1 fuzzy numbers on domain X. An ordering relation $\succcurlyeq$ is defined as*

$$\begin{aligned} A \succcurlyeq B &\iff A \sqcup B = A \\ &\iff A \sqcap B = B \end{aligned} \tag{5}$$

Ramik and Rimanek [32] [1] indicated that $A \succcurlyeq B$ if and only if there exist v_1, v_* and v_2 with $v_1 \geq v_* \geq v_2$, $\mu_A(v_1) = \mu_B(v_2) = 1$, $\mu_A(x) \leq \mu_B(x)$ for any $x < v_*$ and $\mu_A(x) \geq \mu_B(x)$ for any $x > v_*$.

We define the monotonically non-decreasing, monotonically non-increasing and unimodal type-2 quantifiers as follows:

[1] In [22] and [32], the defined $\widetilde{\min}$ and $\widetilde{\max}$, whose definitions are the same as the meet ($\sqcap$) and join ($\sqcup$), are used to represent the fuzzy minimum and maximum of type-1 fuzzy sets respectively.

Definition 6 *Given a partially ordinal relation $\succeq$ for type-1 fuzzy sets in $F([0,1])$, a type-2 quantifier $\widetilde{Q}$ is called*

(a) Regular monotonically non-decreasing if $\widetilde{Q}_0 = \widetilde{Q}(0,\ \cdot) = \dot{0}$, $\widetilde{Q}_1 = \widetilde{Q}(1,\ \cdot) = \dot{1}$ and $\widetilde{Q}_{r_1} = \widetilde{Q}(r_1,\ \cdot) \succeq \widetilde{Q}_{r_2} = \widetilde{Q}(r_2,\ \cdot)$ when $r_1 > r_2$.
(b) Regular monotonically non-increasing if $\widetilde{Q}_0 = \dot{1}$, $\widetilde{Q}_1 = \dot{0}$ and $\widetilde{Q}_{r_1} \succeq \widetilde{Q}_{r_2}$ when $r_1 < r_2$.
(c) Unimodal if $\tilde{Q}_0 = \widetilde{Q}_1 = \dot{0}$ and $\widetilde{Q}_r = \dot{1}$ for $a \le r \le b$, and $\widetilde{Q}_{r_2} \succeq \widetilde{Q}_{r_1}$ when $a \ge r_2 \ge r_1$, and $\widetilde{Q}_{r_1} \succeq \widetilde{Q}_{r_2}$ when $b \le r_1 \le r_2$.

where $\dot{1}$ and $\dot{0}$ are the singleton type-1 fuzzy sets

$$\dot{1}(w) = \begin{cases} 1, \text{ if } w = 1; \\ 0, \text{ otherwise.} \end{cases} \qquad \dot{0}(w) = \begin{cases} 1, \text{ if } w = 0; \\ 0, \text{ otherwise.} \end{cases}$$

For example, the fuzzy max partially ordinal relation $\succcurlyeq$ defined in (5) is used, then the type-2 quantifier *"most"* with parameters $a = 0.3, b = 0.5$ and $c = 0.8$ as depicted in Figure 6 is regular monotonically non-decreasing.

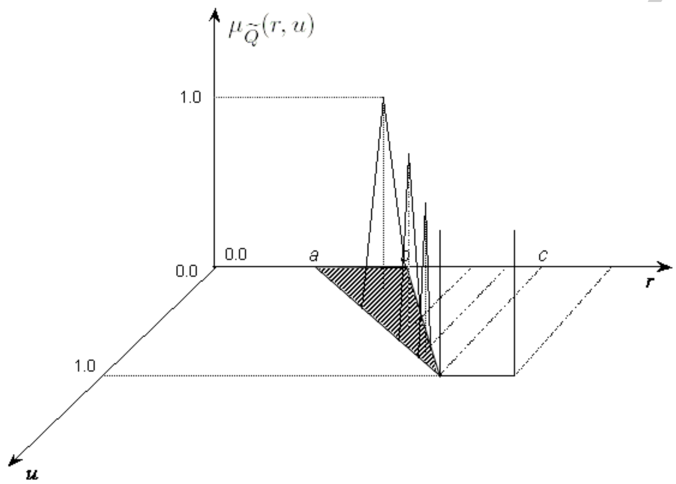

Fig. 6. Type-2 quantifier *"most"*

3.2 Type-2 linguistic quantifier guided type-1 OWA operators

Non-decreasing type-2 linguistic quantifiers can be used to determine the linguistic weights used in type-1 OWA aggregations. Indeed, given a type-2 linguistic quantifier $\widetilde{Q}$, for $r \in [0,1], Q_r = \mu_{\widetilde{Q}}(r,\cdot)$ is a type-1 fuzzy set and $\mu_{Q_r}(u) = \mu_{\widetilde{Q}}(r,u)$. Then the linguistic weights $\{W_i\}_{i=1}^n$ are induced by $\widetilde{Q}$ via fuzzy subtraction as follows:

$$W_i \triangleq Q_{i/n} - Q_{(i-1)/n} \tag{6}$$

However, it is noted that a well-known defective property of classical interval arithmetic and fuzzy interval arithmetic resides in the overestimation of the range [12,23,33]. The overestimation is often caused by the dependency problem and the wrapping effect. The dependency problem is the incapacity of fuzzy arithmetic to identify different occurrences of the same variable, while the wrapping effect appears when in each stage of an interval process, the intermediate results have to be wrapped into an interval. Interval arithmetic is conservative in that it guarantees to contain the set of all possible results including those where rounding errors combine in an unfavourable way, so the worst case assumption is implicitly made in interval arithmetic or fuzzy interval arithmetic, namely that: all intervals or fuzzy intervals are independent although most of them are not [12]. If two intervals are known to be dependent, this information can be used to compute narrower intervals that are still bounds on the set of all possible results [34]. For example, a very promising approach to reducing the effect of overestimation is to introduce the "requisite equality constraints" suggested by Klir [23]. In this paper, we apply Klir's scheme to the fuzzy interval subtraction $W_i = Q_{i/n} - Q_{(i-1)/n}$:

- If $Q_{i/n} = Q_{(i-1)/n}$, then $W_i = \dot{0}$.
- If $Q_{i/n} \neq Q_{(i-1)/n}$, then $\forall w \in [0,1]$,

$$\mu_{W_i}(w) = \sup_{\substack{u-s=w \\ u \in J_{i/n} \\ s \in J_{(i-1)/n}}} \mu_{Q_{i/n}}(u) * \mu_{Q_{(i-1)/n}}(s)$$

where $*$ is a t-norm operator, $J_{i/n}, J_{(i-1)/n}$ are primary membership grades of $\widetilde{Q}$ at $i/n, (i-1)/n$ respectively.

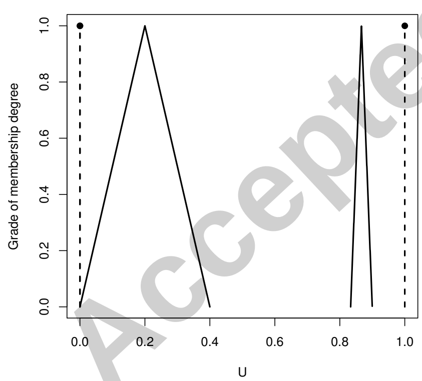

Fig. 7. Secondary membership functions of the type-2 quantifier *"most"* on the FOUs J_0 and $J_{0.25}$, $J_{0.5}$, $J_{0.75}$, and J_1

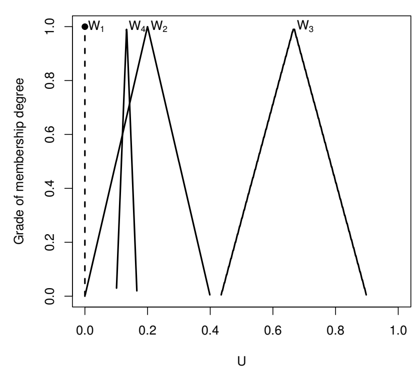

Fig. 8. Four linguistic weights induced by the type-2 quantifier-*"most"* (from left to right): W_1, W_4, W_2 and W_3

Example 1 *In the following, we address the procedure for inducing four linguistic weights in the form of type-1 fuzzy sets from the type-2 quantifier "most" illustrated* *in Figure 6:*

(1) *Determine the footprints of uncertainty (FOUs) [26] at* $r = 0, 0.25, 0.5, 0.75, 1$: $J_0 = [0,0], J_{0.25} = [0,0], J_{0.5} = [0,0.4], J_{0.75} = [0.833,0.9]$, *and* $J_1 = [1,1]$.
(2) *Determine the secondary membership functions defined on the FOUs* J_0, $J_{0.25}$, $J_{0.5}$, $J_{0.75}$, *and* J_1 *as illustrated in Figure 7 respectively. The secondary membership functions on the FOUs* J_0 *and* $J_{0.25}$ *are equal to the singleton fuzzy set* $\dot{0}$ *(first fuzzy set from left to right), while the secondary membership function on the FOU* J_1 *is the singleton fuzzy set* $\dot{1}$ *(last fuzzy set from left to right).*
(3) *Induce the four linguistic weights as shown in Figure 8 by performing the above fuzzy subtraction.*

A type-1 OWA operator associated with the above four induced linguistic weights can be defined and used to aggregate type-1 fuzzy sets, Figure 9 shows an example of aggregating four type-1 fuzzy sets using this induced type-1 OWA operator.

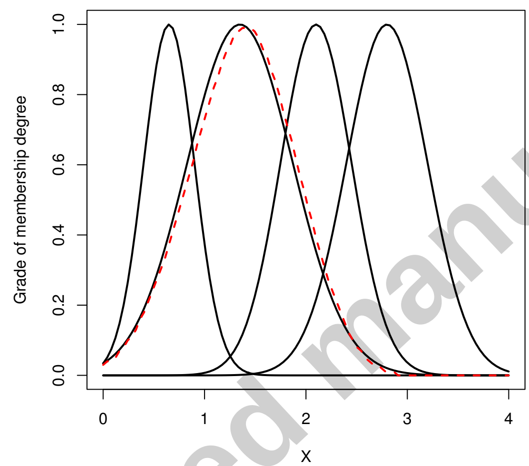

Fig. 9. Aggregation of type-1 OWA operator associated with the linguistic weights in Figure 8 - solid lines: fuzzy sets to be aggregated; dashed line: aggregation result

When the secondary grades of a type-2 quantifier are unity, the type-2 quantifier is called an *interval type-2 quantifier*, and therefore is fully characterised by its FOU. An example of the FOU of the interval type-2 quantifier *"most"* is depicted in Figure 10, in which $a = 0.3, b = 0.5$ and $c = 0.8$.

For an interval type-2 quantifier $\widetilde{Q}$, we have $\mu_{\widetilde{Q}_r}(u) = 1$, $\forall r$, and therefore $\mu_{W_i}(w) = 1$. This implies that the domain of W_i fully depends on the primary membership grades $J_{i/n}$ and $J_{(i-1)/n}$. Because the common case is that $J_{i/n}$ is an interval, so the domain of W_i can be fully characterised by the interval obtained by performing the interval subtraction:

$$\overline{W}_i = J_{i/n} - J_{(i-1)/n} \tag{7}$$

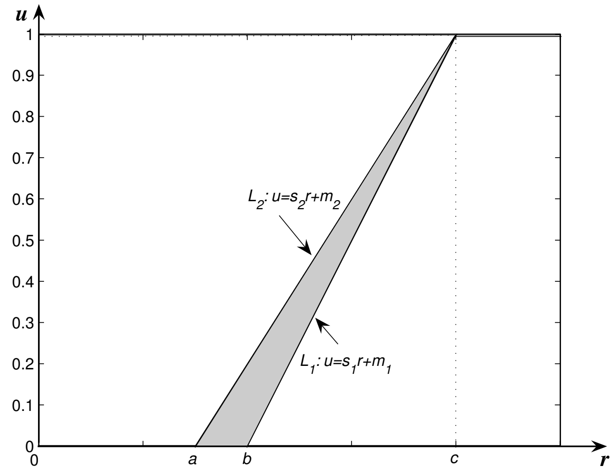

Fig. 10. FOU of interval type-2 quantifier *"most"*

However, as stated above, a drawback of classical interval arithmetic is the overestimation of the range. In interval arithmetic, the computation of bounds that are as narrow as possible is a member of the class of extremely difficult computing problems, called NP-hard [34]. A wide interval does not prove that a conventionally computed result is wrong, but it does indicate a risk [18]. Therefore, in this paper, Klir's scheme [23] is used to conduct the interval subtraction $\overline{W}_i$ to avoid overestimation: when two operand intervals are equal the resulting subtraction of the two intervals is zero.

In the following we derive the interval weights induced from the non-decreasing interval type-2 quantifier given in Figure 10. We note that if the two straight line equations used are

$$L_1 : u = s_1 r + m_1$$
$$L_2 : u = s_2 r + m_2$$

with $0 \leq a \leq b \leq c \leq 1$, where $a = -m_2/s_2$, $b = -m_1/s_1$, and $c = (1-m_1)/s_1 = (1-m_2)/s_2$, then it can be easily proved that $s_1 \geq s_2 > 0$, $m_1 \leq m_2 \leq 0$, $s_1 = 1/(c-b), m_1 = -b/(c-b)$, $s_2 = 1/(c-a)$ and $m_2 = -a/(c-a)$. We have the following cases:

- If $\dfrac{i}{n} \leq a$ or $\dfrac{i-1}{n} \geq c$, then $J_{i/n} = J_{(i-1)/n} = [0,0]$ or $[1,1]$ and therefore $\overline{W}_i = 0$.
- If $\dfrac{i-1}{n} \leq a \leq \dfrac{i}{n}$, then $J_{(i-1)/n} = [0,0]$ and therefore $\overline{W}_i = J_{i/n}$.
- If $a \leq \dfrac{i-1}{n} < \dfrac{i}{n} \leq b$, then $J_{(i-1)/n} = \left[0, L_2\left(\frac{i-1}{n}\right)\right]$, $J_{i/n} = \left[0, L_2\left(\frac{i}{n}\right)\right]$ and therefore

$$J_{i/n} = \left[0, L_2\left(\frac{i-1}{n}\right) + \frac{s_2}{n}\right] = \left[0, L_2\left(\frac{i-1}{n}\right)\right] + \left[0, \frac{s_2}{n}\right] = J_{(i-1)/n} + \frac{1}{n}[0, s_2]$$

By considering the "requisite equality constraints" [23], we obtain $\overline{W}_i$ as follows [2]

$$\overline{W}_i = J_{i/n} - J_{(i-1)/n} = \left(J_{(i-1)/n} - J_{(i-1)/n}\right) + \frac{1}{n}[0, s_2] = \frac{1}{n}[0, s_2]$$

- If $a \le \dfrac{i-1}{n} \le b \le \dfrac{i}{n} \le c$, then $J_{(i-1)/n} = \left[0, L_2\left(\frac{i-1}{n}\right)\right]$, $J_{i/n} = \left[L_1\left(\frac{i}{n}\right), L_2\left(\frac{i}{n}\right)\right]$ and therefore

$$\overline{W}_i = J_{i/n} - J_{(i-1)/n} = \left[L_1\left(\frac{i}{n}\right) - L_2\left(\frac{i-1}{n}\right), L_2\left(\frac{i}{n}\right)\right]$$

- If $b \le \dfrac{i-1}{n} < \dfrac{i}{n} \le c$, then $J_{(i-1)/n} = \left[L_1\left(\frac{i-1}{n}\right), L_2\left(\frac{i-1}{n}\right)\right]$, $J_{i/n} = \left[L_1\left(\frac{i}{n}\right), L_2\left(\frac{i}{n}\right)\right]$ and therefore

$$\begin{aligned} J_{i/n} - \frac{1}{n}[s_2, s_1] &= \left[L_1\left(\frac{i}{n}\right) - \frac{s_1}{n}, L_2\left(\frac{i}{n}\right) - \frac{s_2}{n}\right] \\ &= \left[L_1\left(\frac{i-1}{n}\right), L_2\left(\frac{i-1}{n}\right)\right] = J_{(i-1)/n} \end{aligned}$$

By considering the "requisite equality constraints" [23], $\overline{W}_i = \frac{1}{n}[s_2, s_1]$ is obtained [3]

- If $\dfrac{i-1}{n} \le c \le \dfrac{i}{n}$, then $J_{(i-1)/n} = \left[L_1\left(\frac{i-1}{n}\right), L_2\left(\frac{i-1}{n}\right)\right]$, $J_{i/n} = [1, 1]$ and therefore

$$\overline{W}_i = \left[1 - L_2\left(\frac{i-1}{n}\right), 1 - L_1\left(\frac{i-1}{n}\right)\right]$$

Then the domains of the linguistic weights W_i, $\widehat{W}_i$, used in type-1 OWA aggregation are identified as follows:

$$\widehat{W}_i = \overline{W}_i \cap [0, 1]$$

Example 2 *The interval weights induced from the interval type-2 quantifier "most" are obtained as follows:*

(1) Determine the FOUs at $r = 0, 0.25, 0.5, 0.75, 1$*:* $J_0 = [0, 0], J_{0.25} = [0, 0], J_{0.5} = [0, 0.4], J_{0.75} = [0.833, 0.9]$*, and* $J_1 = [1, 1]$.

---

[2] If the "requisite equality constraints" are ignored, certain overestimated intervals would be induced as $J_{i/n} - J_{(i-1)/n} = \left[-L_2\left(\frac{i-1}{n}\right), L_2\left(\frac{i}{n}\right)\right] = \left[-L_2\left(\frac{i-1}{n}\right), L_2\left(\frac{i-1}{n}\right)\right] + \frac{1}{n}[0, s_2] \supseteq \frac{1}{n}[0, s_2]$

[3] If the "requisite equality constraints" are ignored, certain overestimated intervals would be induced as $J_{i/n} - J_{(i-1)/n} = \left[L_1\left(\frac{i}{n}\right) - L_2\left(\frac{i-1}{n}\right), L_2\left(\frac{i}{n}\right) - L_1\left(\frac{i-1}{n}\right)\right] = \left[L_1\left(\frac{i}{n}\right) - L_2\left(\frac{i}{n}\right) + \frac{s_2}{n}, L_2\left(\frac{i}{n}\right) - L_1\left(\frac{i}{n}\right) + \frac{s_1}{n}\right] = \left[L_1\left(\frac{i}{n}\right) - L_2\left(\frac{i}{n}\right), L_2\left(\frac{i}{n}\right) - L_1\left(\frac{i}{n}\right)\right] + \frac{1}{n}[s_2, s_1] \supseteq \frac{1}{n}[s_2, s_1]$

(2) Determine the interval weights $\overline{W}_i = J_{i/n} - J_{(i-1)/n}$:

$$\overline{W}_1 = [0,0], \overline{W}_2 = [0,0.4], \overline{W}_3 = [0.433,0.9], \overline{W}_4 = [0.1,0.167].$$

Then the above induced interval weights can be used to define a type-1 OWA operator, Figure 11 shows an example of aggregating four type-1 fuzzy sets using this induced operator.

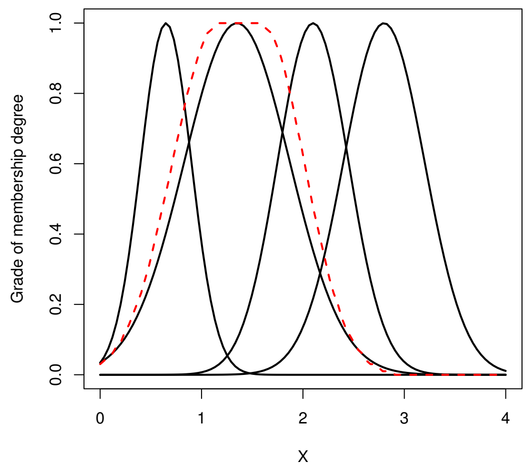

Fig. 11. Aggregation of type-1 OWA operator associated with the induced interval weights - solid lines: fuzzy sets to be aggregated; dashed line: aggregation result

4 Conclusion

In this paper, by extending Yager's OWA operator, a new type of OWA operator, called the type-1 OWA operator, is proposed in the interests of aggregating linguistic information via OWA mechanism for decision making. The type-1 OWA operator is capable of aggregating type-1 fuzzy sets with type-1 fuzzy set weights in the aggregation process. Moreover, type-2 linguistic quantifiers are suggested to identify the linguistic weights used in the OWA aggregation of linguistic information. The procedure for performing quantifier guided type-1 OWA operations is provided.

We believe that the proposed new type of OWA operator will lead to some new research areas. These include

- the relationship between the aggregation of the α-cuts of the fuzzy sets to be aggregated via a type-1 OWA operator and the resulting fuzzy set obtained via the procedure presented in the present paper. Because interval analysis techniques [8] has been adapted to fuzzy interval computation where ending-points of intervals are changed into left and right fuzzy bounds, the α-cut based interval analysis techniques would also possess the great potentials of being applied to improving the efficiency of type-1 OWA operation for real-time decision making.

- whether the properties of the traditional OWA operator remain in the proposed new type of OWA operator;
- how to aggregate type-2 fuzzy sets [19, 26] and non-stationary fuzzy sets [9, 10] via OWA mechanism;
- the possibility of applying the type-1 OWA operator to merging similar fuzzy sets for improving fuzzy model interpretability/transparency and parsimony [49, 50].

Additionally, the new type-1 OWA operator could have great potential in being applied to multi-expert decision making and multi-criteria decision making. These topics will receive much attention in our future research.

Acknowledgment

We are grateful to the anonymous reviewers who have helped us improve this paper. This work has been supported by the EPSRC Research Grant EP/C542215/1.

References

[1] G. M. Bordogna, Fedrizzi, and G. Pasi, A linguistic modelling of consensus in group decision making based on OWA operators, IEEE Trans. on Systems, Man and Cybernetics-part A 27 (1997) 126-133.

[2] F. Chiclana, E. Herrera-Viedma, F. Herrera, S. Alonso, Induced ordered weighted geometric operators and their use in the aggregation of multiplicative preference relations, International Journal of Intelligent Systems 19 (2004) 233-255.

[3] F. Chiclana, E. Herrera-Viedma, F. Herrera, S. Alonso, Some induced ordered weighted averaging operators and their use for solving group decision-making problems based on fuzzy preference relations, European Journal of Operational Research 182 (2007) 383-399,.

[4] V. Cutello and J. Montero, Hierarchies of aggregation operators, International Journal of Intelligent Systems 9 (1994) 1025-1045.

[5] W. M. Dong and F. S. Wong, Fuzzy weighted averages and implementation of the extension principle, Fuzzy Sets and Systems 21 (1987) 183-199.

[6] D. Dubois and H. Prade, A review of fuzzy set aggregation connectives, Information Sciences 36 (1985) 85-121.

[7] J. Fodor and M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer, Dordrecht, 1994.

[8] J. Fortin, D. Dubois and H. Fargier, Gradual numbers and their application to fuzzy interval analysis, IEEE Trans. on Fuzzy Systems 16(2)(2008) 388-402.

[9] J. M. Garibaldi, T. Ozen, Uncertain fuzzy reasoning: a case study in modelling expert decision making, IEEE Trans. on Fuzzy Systems 15 (2007) 16-30.

[10] J. M. Garibaldi, S. Musikasuwan and T. Ozen, The association between non-stationary and interval type-2 fuzzy sets: a case study, Proc. of IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2005), Reno, Nevada, USA, 22-25 May 2005, 224 - 229.

[11] Y. -Y. Guh, C. -C. Hon, and E. S. Lee, Fuzzy weighted average: the linear programming approach via Charnes and Cooper's rule, Fuzzy Sets and Systems 117 (2001) 157-160.

[12] M. Hanss, A nearly strict fuzzy arithmetic for solving problems with uncertainties, Proc. of the 19th NAFIPS Conference, Atlanta, GA, USA, 2000, 439-443.

[13] F. Herrera and E. Herrera-Viedma, Aggregation operators for linguistic weighted information, IEEE Trans. on Systems, Man and Cybernetics-part A 27 (1997) 646-656.

[14] F. Herrera and E. Herrera-Viedma, Linguistic decision analysis: steps for solving decision problems under linguistic information, Fuzzy Sets and Systems 115 (2000) 67-82.

[15] E. Herrera-Viedma, F. Herrera and F. Chiclana, A consensus model for multiperson decision making with different preference structures, IEEE Trans. on Systems, Man and Cybernetics Part-A 32 (2002) 394-402.

[16] F. Herrera, E. Herrera-Viedma and F. Chiclana, A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making, International Journal of Intelligent Systems 18 (2003) 689-707.

[17] E. Herrera-Viedma, L. Martinez, F. Mata and F. Chiclana, A consensus support system model for group decision-making problems with multigranular linguistic preference relations, IEEE Trans. on Fuzzy Systems 13 (2005) 644-658.

[18] T. Hickey, Q. Ju and M. H. Van Emden, Interval arithmetic: from principles to implementation, Journal of the ACM, 48 (5) (2001) 1038 - 1068.

[19] R. I. John and P. Innocent, Modelling uncertainty in clinical diagnosis using fuzzy logic, IEEE Trans. on Systems, Man and Cybernetics-part B 35(2005) 1340-1350.

[20] J. Kacprzyk and M. Fedrizzi, A 'soft' measure of consensus in the setting of partial (fuzzy) preferecnes, European Journal of Operational Research 34 (1988) 316-325.

[21] C. Kao, and S.-T. Liu, Fractional programming approach to fuzzy weighted average, Fuzzy Sets and Systems 120 (2001) 435-444.

[22] A. Kaufmann and M. M. Gupta, Introduction to Fuzzy Arithmetic, Theory and Applications, Van Nostrand Reinhold, New York, 1985.

[23] G. J. Klir, Fuzzy arithmetic with requisite constraints, Fuzzy Sets and Systems, 91 (1997) 165-175.

[24] G. J. Klir and T.A. Folger, Fuzzy Sets, Uncertainty and Information, Singapore: Prentice-Hall, 1988.

[25] B. Llamazares, Choosing OWA operator weights in the field of Social Choice, Information Sciences 177 (2007) 4745-4756.

[26] J. M. Mendel, R. I. John and F. Liu, Interval type-2 fuzzy logic systems made simple, IEEE Trans. on Fuzzy Systems, 14(2006), 808-821.

[27] P. Meyer and M. Roubens, On the use of the Choquet integral with fuzzy numbers in multiple criteria decision support, Fuzzy Sets and Systems 157 (7) (2006) 927-938

[28] L. Mich, M. Fedrizzi, and L. Gaio, Approximate reasoning in the modeling of consensus in group decisions, in E. P. Klement and W. Slany (eds.), Fuzzy Logic in Artificial Intelligence, Berlin, Germany: Springer-Verlag, 91-102, 1993.

[29] H. B. Mitchell and D. D. Estrakh, A modified OWA operator and its use in lossless DPCM image compression, International Journal of Uncertain Fuzziness, Knowledge Based Systems 5 (1997) 429-436.

[30] H. B. Mitchell and D. D. Estrakh, An OWA operator with fuzzy ranks, International Journal of Intelligent Systems 13 (1998) 69-81.

[31] M. Mizumoto and K. Tanaka, Some Properties of fuzzy sets of type 2, Information and Control 31(1976) 312-40.

[32] J. Ramik and J. Rimamek, Inequality relation between fuzzy numbers and its use in fuzzy operation, Fuzzy Sets and Systems 16 (1985) 123-138.

[33] V. Stahl, A sufficient condition for non-overestimation in interval arithmetic, Computing, 59 (4) (1997) 349-363.

[34] Sun Microsystems Inc., Interval Arithmetic in High Performance Technical Computing, White Paper, September 2002.

[35] Y. -M. Wang, and C. Parkan, A preemptive goal programming method for aggregating OWA operator weights in group decision making, Information Sciences 177 (2007) 1867-1877.

[36] Z. Wang, R. Yang, P.-A. Heng and K.-S. Leung, Real-valued Choquet integrals with fuzzy-valued integrands, Fuzzy Sets and Systems 157 (2) (2006) 256-269

[37] Z. S. Xu and Q. L. Da, An overview of operators for aggregating information, International Journal of Intelligent Systems 18 (2003) 953-969.

[38] R. R. Yager, On ordered weighted averaging aggregation operators in multi-criteria decision making, IEEE Trans. on Systems, Man and Cybernetics 18 (1988) 183-190.

[39] R. R. Yager, A general approach to criteria aggregation using fuzzy measures, International Journal of Man-Machine Study 38 (1993) 187-213.

[40] R. R. Yager, Families of OWA operators, Fuzzy Sets and Systems 59 (1993) 125-148.

[41] R. R. Yager and D. P. Filev, Generalizing the modeling of fuzzy logic controllers by parameterized aggregation operators, Fuzzy Sets and Systems 70 (1995) 303-313.

[42] R. R. Yager, D. P. Filev, and T. Sadeghi, Analysis of flexible structured fuzzy logic controllers, IEEE Trans. on Systems, Man and Cybernetics 24 (1994) 1035-1043.

[43] R. R. Yager, L. S. Goldstein, E. Mendels, FUZMAR: an approach to aggregating market research data based on fuzzy reasoning, Fuzzy Sets and Systems 68 (1994) 1-11.

[44] R. Yang, Z. Wang, P.-A. Heng and K.-S. Leung, Fuzzy numbers and fuzzification of the Choquet integral, Fuzzy Sets and Systems 153 (1) (2005) 95-113.

[45] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.

[46] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-1, Information Sciences 8 (1975) 199-249.

[47] L. A Zadeh, A computational approach to fuzzy quantifiers in natural languages, Comput Math Appl 9 (1983) 149-184.

[48] L. A. Zadeh, From computing with numbers to computing with words - from manipulation of measurements to manipulation of perceptions, IEEE Trans. on Circuits and Systems 45 (1999) 105-119.

[49] S. -M. Zhou, and J. Q. Gan, Constructing accurate and parsimonious fuzzy models with distinguishable fuzzy sets based on an entropy measure, Fuzzy Sets and Systems, 157 (2006) 1057-1074.

[50] S. -M. Zhou, and J. Q. Gan, Constructing parsimonious fuzzy classifiers based on L2-SVM in high-dimensional space with automatic model selection and fuzzy rule ranking, IEEE Trans. on Fuzzy Systems 15 (2007) 398-409.

[51] H. -J. Zimmermann, Fuzzy Sets, Decision Making and Expert Systems, Kluwer Academic Publishers, Boston, 1987.

[52] H. -J. Zimmermann and P. Zysno, Latent connectives in human decision making, Fuzzy Sets and Systems 4 (1980) 37-51.