mathematics

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On Grill S_β-Open Set in Grill Topological Spaces

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Abstract: In this article we originate a new class of Grill Set, namely GS_β-Open Set, which is parallel to the β Open Set in Grill Topological Space $(X,\ \theta,\ G)$. In addition, we entitle GS_β-continuous and GS_β-open functions by applying a GS_β-Open Set and we review some of its important properties. Many examples are given to explain the concept lucidly. The properties of GS_β open sets are investigated and studied. The theorems based on the arbitrary union and finite intersections are discussed with counter examples. Moreover, some operators like $GS_\beta - closure$ and $GS_\beta - interior$ are introduced and investigated. The concept of $GS_\beta - continuous$ functions are compared with the idea of $G - Semi\ Continuous$ function. The theorems based on $GS_\beta - continunity$ have been proved.

Keywords: GS_β-open sets; $GS_\beta O(X)$; GS_β-continuous function; GS_β-open function

MSC: 18F60

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Citation: Kalaivani, N.; Rahman, K.F.U.; Čepová, L.; Čep, R. On Grill S_β-Open Set in Grill Topological Spaces. *Mathematics* **2022**, *10*, 4626. https://doi.org/10.3390/math10234626

Academic Editor: Manuel Sanchis

Received: 4 November 2022
Accepted: 2 December 2022
Published: 6 December 2022

1. Introduction

In [1,2] the concept based on Grl had been a useful tool like nets and filters for getting rooted deviation in further studying some topological properties like compactifications, along with extension problems of different kinds. Many more analyses, such as Al Hawary et al. [3–7], had characterized and entrenched the properties based on Gene OS in the classical topo. The study of Grl on a TS was going on from 1930 and 1947 correspondingly until now. Mathematicians like Al Omari and Noiri along with Dasan and Thivagar had enriched this field and contribution in this field was worthy. Al Omari and Noiri [8] defined a new topology and they proposed generalized space in GTS. It was proved that Grls, nets along with filters, were useful and important for studying some tpl concepts such as proximity spaces, closure spaces, the theory of compactifications and other similar extension problems. The supremacy of mathematics was upheld by the interpolation of concepts like Grl N topology. Choquet [9] was the first one to develop Grl topology. Choquet [9] originated the philosophy of Grl on a TS and the thought of Grl was revealed to be an important manoeuvre for examining some topological properties. Dasan and Thivagar [10] proposed the concept of N-TS and also established the N-Tpl OS.

As noted from the literature [11], there had been a growing trend among topologists to propose and study different allied or weaker forms of OS, motivating the investigation of the corresponding types of cts-like functions between TS. This again had given rise to different decompositions of cts functions. Ganesan [12–14] utilized the operator φ to accomplish their decomposition of cty. Using the idea of Grl and many interesting constructions, properties and depictions had been deduced. Tpl developments were directly applied in topical fields such as artificial intelligence and information systems along with data analysis. Hatir and Jafari [15], Kanchana et al. [16] and Kuratowski [17] characterized new classes of sets in a GTS and obtained new composition of Cty in terms of Grl. A classical prototype for decomposition based on Cty along with Semi Cty was the article of

Levine [18,19]. During the past ten years, the study of Cty along with Compactness, nano CS and irresolute function has been generalized. Levine proposed the notion of generalized CS in TS and showed that compactness, countably compactness, para compactness and normality are all g-csd hereditary. Mandal and Mukherjee [20] fabricated the faintly Cty and weak Cty functions via tpl Grls.

Mashour [21] and Njastad [22] introduced and inspected semi pre-OS, generalized semi-OS, semi-generalized OS, generalized OS, SO sets and PO sets which are some of the weaker forms of the OS, and complements of these sets are labeled as CS correspondingly. Nagaveni proposed the weakly generalized CS and semi weakly generalized CS in GTS. Roy and Mukherjee [23,24] declared a new tpl opr via Grl and also discussed a type of compactness via Grl. Roy and Mukherjee [23] have used Grl on TS with a different attitude. Roy and Mukherjee [24] elongated this idea further and constructed a topology for corresponding Grl in a given TS. The notion of soft Grl, soft operators, precontinuity and soft topology τ_G were defined and discussed by Saif and Al-Muntaser [25]. The idea of disintegration of Cty on a GTS and some families of sets was characterized to Grl in [26–28]. Thorn [29] proved that Grls are always a union of ultra-filters. The idea of N TS was initiated by Veliko [30], and he also extended Grl topology to Grl N TS when further topological H-closed space was introduced.

Voskoglou [31] inspected the weaker and stronger forms of g-irresolute functions and Fuzzy topology in GTS. Song proposed the concept of absolutely countably compact and also inspected the relationship between these spaces along with other star compact spaces. Hatir and Jafari [15], with the same motivation, culminated in the interpolation and study of φ OS, where φ is a suitable operator. Zhong et al. [32] proposed a class of submeta compactness in L-TS. Devi et al. introduced a class of generalized semi opn-compact along with semi-generalized opn-compact in GTS, Pseudo metric topo, and investigated some of its theorems. Al Ghour [33] introduced the class of soft ω_p open sets and proved they closed under soft union and do not form a soft topology. In addition, decomposition of soft ω_p continuity has been defined and investigated. Al-shami et al. [34] introduced the concept of sum of soft topological spaces using pair wise disjoint soft topological spaces and studied some of its basic properties. Mahafzah et al. [35] designed some electronic architecture using a topological approach. Grill topology has diverse applications in science and engineering that comprise camouflage filters, categorization, digital image processing, forgery detection, Hausdorff raster spaces, image analysis, microscopy, paleontology, pattern recognition, population dynamics, stem cell biology, and topological psychology, along with visual merchandising.

In this article we propose a new class of set, namely GS_β-Ops, GS_β Csd set, and GS_β-Cty along with $GS_\beta - opn$ functions are investigated and some of their properties have been investigated. Many illustrations are given to explain the concept details. The concept of GS_β clos and GS_β int are investigated and studied. In addition to that, some properties are also investigated with some illustrations. The concept of G Semi continuous and GS_β continuity is independent if proved with a proper example. In addition to this theory, the concept of GS_β continuous mapping has been defined and investigated. Equivalence relationships between GS_β open function, GS_β closed function and GS_β continuous functions are investigated and studied. Many theorems based on $GS_\beta -$ cts functions have been proved.

2. Preliminaries

A collection G of nonempty Sbt based on a TS (X, θ) is said to be a Grl on X if: (i) $C \in G$ along with $C \subseteq D$ implies that $D \in G$; and in addition (ii) $C, D \subseteq X$ then $C \cup D \in G$ implies that $C \in G$ or $D \in G$. A triplet (X, θ, G) is labeled as a GTS.

Roy and Mukherjee [23] designated a similar topo by a Grl and they examined some tpl properties. For any point t of a TS (X, θ), $\theta(t)$ indicate the number of all opn nbd of t. We define the function $\varphi : P(X) \to P(X)$ as $\varphi(A) = \{t \in X : A \cap U \in G \text{ for all } U \in \theta(t)\}$

for every $A \in P(X)$. Similarly, $\mu(A) = A \cup \varphi(A)$ for all $A \in P(X)$ can be defined. The mapping μ satisfies Kuratowski closure axioms:

(i) $\mu(\phi) = \phi$;
(ii) if $C \subseteq D$, then $\mu(C) \subseteq \mu(D)$;
(iii) if $C \subseteq X$, then $\mu(\mu(C)) = \mu(C)$;
(iv) if $C, D \subseteq X$, then $\mu(C \cup D) = \mu(C) \cup \mu(D)$.

Analogous to a Grl G on a TS (X, θ), there exists a similar topo τ_G (say) on X denoted by $\tau_G = \{U \subseteq X : \mu\,(X - U) = X - U\}$, where for each and every $C \subseteq X$, $\mu(C) = C \cup \mu(C) = \tau_G\text{-}cl(C)$ and $\tau \subseteq \tau_G$.

The idea of disintegration of Cty on a GTS and some families of sets were characterized to Grl in [26–28].

A Sbt E in X is defined to be:

(i) φ-opn if $E \subseteq int(\varphi(E))$;
(ii) G-α.opn if $E \subseteq int(\mu(int(E)))$;
(iii) G-PO if $E \subseteq int(\mu(E))$;
(iv) G-SO, if $E \subseteq \mu(int(E))$;
(v) G-β. opn if $E \subseteq cl(int(\mu(E)))$;
(vi) β. opn if $E \subseteq cl(int(cl(E)))$.

The collection of all G-α.opn (resp. G-preopn, G-semiopn, G-β. opn) sets in a GTS (X, τ, G) is denoted as $G\alpha O(X)$ (res. $GPO(X)$, $GSO(X)$, $G\beta O(X)$), $\beta O(X)$. One says that a function $f : (X, \tau, G) \to (Y, \sigma)$ is supposed to be G-Semicontinous if $f^{-1}(M) \in GSO(X)$ for respective $M \in \sigma$.

Using the theory of semi interior and semi closure we have defined β-interior and β-closure sets. For each sbt D of X, (i) β int $(D) = \cup\, \{E : E \in \beta O(X)$ and $E \subseteq D\}$, and (ii) β cl $(D) = \cap\, \{M : X - M \in \beta O(X)$ and $D \subseteq M\}$.

In this article, we have characterized a GS_β-Ops in a GTS (X, θ, G) and we have investigated some basic properties. In addition to this, we have characterized $GS_\beta -$ cts, $GS_\beta - opn$, $GS_\beta - csd$ and $GS^*_\beta - cts$ function on a GTS (X, θ, G) and we have studied some of their important properties.

3. GS_β-Open Sets

Definition 1. *Accredit $(X, \theta\,,\ G)$ be a GTS along with B be a sbt of X. Then B is called $GS_\beta - opn$ in the case that there exists a $U \in \beta O(X)$ such that $U \subseteq B \subseteq \mu(U)$. The class of all GS_β-ops is expressed as $GS_\beta O(X)$. The complement of $X - B$ is called $GS_\beta C(X)$.*

Example 1. *Let $X = \{d, e, f\}$, $\theta = \{(\phi,\ X,\ \{d\},\ \{f\},\ \{d, f\}\}$ and $G = \{X,\ \{d\},\ \{d, f\}\}$. Then $GS_\beta O(X) = \{\phi, \{d\},\ \{f\},\ \{d, e\},\ \{e, f\},\ X\}$.*

Example 2. *Let $X = \{1,\ 2,\ 3\}$, $\theta = \{\phi,\ X, \{1\}, \{2\}, \{1{,}2\}\}$ and $G = \{X, \{2\}, \{2{,}3\}\ \}$. Then $GS_\beta O(X) = \{\phi,\ \{1\}, \{2\}, \{3\}, \{1{,}2\}, \{1{,}3\}, \{2{,}3\},\ X\}$.*

Theorem 1. *Assume that $(X, \theta,\ G)$ be a GTS along with let $C \subseteq X$. Then $C \in GS_\beta O(X)$ in the case that $B \subseteq \mu(\beta\, int(A))$.*

Proof. If $C \in\ GS_\beta O(X)$, all at once there occurs a $V \in\ \beta O(X)$ so that $V \subseteq C \subseteq \mu(V)$. However, $V \subseteq C$ entails $V \subseteq \beta int\ (C)$. Thus $\mu(V) \subseteq \mu(\beta\, int(C))$. Consequently $C \subseteq\ \mu(\beta\, int(C))$. Inversely, let $C \subseteq\ \mu(\beta\, int(C))$. To justify that $C \in\ GS_\beta O(X)$, take $V = \beta\, int\ (A)$, then $V \subseteq C \subseteq\ \mu(V)$ and $C \in\ GS_\beta O(X)$. □

Corollary 1. *If $B \subseteq X$, then $B \in\ GS_\beta O(X)$ iff $\mu(B) = \mu(\beta\, int(B))$.*

Proof. Given that $B \in GS_\beta O(X)$. Then μ is monotonic and idempotent, $\mu(B) \subseteq \mu(\mu(\beta\,\text{int}\,(B))) = \mu(\beta\,\text{int}\,(B)) \subseteq \mu\,(B)$ implies that $\mu(B) = \mu(\beta\,\text{int}(B))$. Since

$\mu(B) = \mu(\beta\, int\,(B))$ and hence μ is monotonic and idempotent, $\mu(\mu(\beta\, int\,(B))) \supseteq \mu(B)$, therefore $B \subseteq \mu(B)$ is proved. □

Theorem 2. *Let (X, θ, G) be a GTS. If $A \in GS_\beta O(X)$ and $B \subseteq X$ such that $\subseteq B \subseteq \mu(\beta\, int(A))$, then $B \in GS_\beta O(X)$.*

Proof. Given that $A \in GS_\beta\, O(X)$. Hence by the above Theorem 1, $A \subseteq \mu(\beta\, int(A))$, but $A \subseteq B$ implies that $\beta\, int(A) \subseteq \beta\, int(B)$ then consequently by Theorem 2.4 [17], $\mu(\beta\, int(A)) \subseteq \mu(\beta\, int(B))$. Accordingly, $B \subseteq \mu(\beta\, int(A)) \subseteq \mu(\beta\, int(B))$. Hence $B \in GS_\beta\, O(X)$. □

Corollary 2. *If $C \in GS_\beta\, O(X)$ and $D \subseteq X$ such that $C \subseteq D \subseteq \mu(C)$, then $D \in GS_\beta\, O(X)$.*

Proof. Proof follows directly from Theorem 2 and Corollary 1. □

Proposition 1. *If $U \in \beta O(X)$, then $U \in GS_\beta O(X)$.*

Proof. Let $U \in \beta O(X)$, it implies that $U = \beta\ int(U) \subseteq \mu(\beta int(U))$. Thus $U \in GS_\beta O(X)$. □

Note that the inverse of the above proposition need not be accurate.

Accredit $X = \{f, g, h\}$, $\theta = \{\varnothing, X, \{f\}, \{f, g\}\}$, $G = \{X, \{f\}, \{f, g\}\}$. Then $\beta\, O(X) = \{\varnothing, \{f\}, \{h\}, \{f, g\}, \{g, h\}, X\}$ then $GS_\beta\ O(X) = \{\varnothing, \{f\}, \{g\}, \{h\}, \{f, g\}, \{g, h\}, \{f, h\}, X\}$. Here $\{g\}$ and $\{f, h\}$ are GS_β ops but not β ops.

Theorem 3. *Given (X, θ, G) be a GTS. If $B \in GSO(X)$ then, $B \in GS_\beta O(X)$.*

Proof. Given that $B \in GSO(X)$, then $B \subseteq \mu(int(B))$. Therefore $int\,(B) \subseteq \beta\, int\,(B)$, we have $\mu(int\,(B)) \subseteq \mu(\beta\, int\,(B))$. By propo 3.1 $\mu(B) \subseteq cl(B)$, from the above two thms we get $\mu(int\,(B)) \subseteq cl(B)$. Since $B \subseteq cl(int\,(\mu(B))$. It follows that $B \in GS_\beta O(X)$. □

Note that the inverse of above Theorem need not be accurate. Through Example 2 it is obvious that $GSO(X) = \{\varnothing, \{1\}, \{2\}, \{3\}, X\}$. Therefore $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are GS_β open but not GSO.

Theorem 4. *Given that (X, θ, G)be a GTS:*

(i) If $A_i \in GS_\beta O(X)$ for all $i \in J$, then $\cup_{i \in J} A_i \in GS_\beta O(X)$;
(ii) If $D \in GS_\beta O(X)$ and $U \in \beta\, O(X)$ then $D \cap U \in GS_\beta O(X)$.

Proof. (i) Since $A_i \in GS_\beta O(X)$, there exist $A_i \subseteq \mu(\beta\, int\,(A_i))$ for all $i \in J$. Hence, we obtain $A_i \subseteq \mu(\beta int\,(A_i)) \subseteq \mu(\beta\, int\,(\cup_{i \in J} A_i)$ and hence $\cup_{i \in J} A_i \subseteq \mu(\beta\, int\,(\cup_{i \in J} A_i))$. This implies that $\cup_{i \in J} A_i \in GS_\beta O(X)$.
(ii) Accredit $D \in GS_\beta O(X)$ along with $U \in \beta\, O(X)$, then $D \subseteq \mu(\beta\, int\,(D))$ along with $\beta\ int\ (U) = U$. Now $D \cap U \subseteq \mu(\beta\, int\,(D)) \cap U = (\beta\ int\ (D) \cup \varphi(\beta\, int\,(D)) \cap U = (\beta int\,(D) \cap U) \cup (\varphi(\beta\, int\,(D) \cap U))$ (by Theorem 2.10 [17]) = $\beta int\,(D \cap U) \cup \varphi(\beta\, int\,(D \cap U) = \mu(\beta\, int\,(D \cap U))$. Hence, $D \cap U \in GS_\beta O(X)$. □

Remark 1. *The following example displays that if $E, F \in GS_\beta\, O(X)$, then $E \cap F \notin GS_\beta\, O(X)$.*

From Example 1, take $E = \{e, f\}$ and $F = \{d, e\}$, then $E, F \in GS_\beta\ O(X)$ but $E \cap F = \{e\} \notin GS_\beta\, O(X)$.

Theorem 5. *Let (X, θ, G) be a GTS and $B \subseteq X$. If $\in GS_\beta\, C(X)$, then $\beta\, int\,(\mu(B)) \subseteq B$.*

Proof. Suppose $B \in GS_\beta\ C(X)$. Accredit $X - B \in GS_\beta O(X)$ and so $X - B \subseteq \mu(\beta\ int\ (X - B)) \subseteq \beta\ cl(\beta\ int\ (X - B)) = X - \beta\ int\ (\ \beta cl(B)) \subseteq X - \beta\ int\ (\mu(B))$ implies that $\beta\ int(\mu(B)) \subseteq B$. □

Theorem 6. *Let $(X, \theta,\ G)$ be a GTS and $B \subseteq X$ such that $X - \beta\ int(\mu(B)) = \mu(\beta\ int(X - B))$. Then $B \in GS_\beta\ C(X)$ if and only if $\beta\ int(\mu(B)) \subseteq B$.*

Proof. The fundamental part is proved in Theorem 5. Conversely, suppose that $\beta int(\mu(B)) \subseteq B$, then $X - B \subseteq X - \beta int\ (\mu(B)) = \mu(\beta int(X - B))$ implies that $X - B \in GS_\beta\ O(X)$. Hence $B \in GS_\beta\ C(X)$. □

Definition 2. *Let $(X,\ \tau,\ G)$ be a GTS and $B \subseteq X$. Then:*

(i) *GS_β-int of B is defined as union of all GS_β-OS contained in B. Then $GS_\beta\ int(B) = \cup \{U : U \in GS_\beta O(X)$ and $U \subseteq B\}$;*
(ii) *GS_β-clos of B is defined as intersection of all GS_β-Cs containing B. Then $GS_\beta\ cl(B) = \cap \{F : X - F \in GS_\beta O(X)$ and $B \subseteq F\}$.*

Theorem 7. *Let $(X,\ \theta,\ G)$ be a GTS and $E \subseteq X$. Then:*

(i) *$GS_\beta\ int(E)$ is a GS_β-ops contained in E;*
(ii) *$GS_\beta\ cl(E)$ is a GS_β-csd containing E;*
(iii) *E is GS_β-csd if $GS_\beta\ cl(E) = E$;*
(iv) *E is $GGS_\beta - opn$ if $GS_\beta\ int(E) = E$;*
(v) *$GS_\beta\ int(E) = X - GS_\beta\ cl(X - E)$;*
(vi) *$GS_\beta\ cl(E) = X - GS_\beta\ int(X - E)$*

Proof . Proof follows from the Definition 2 and Theorem 4 (i). □

Theorem 8. *Accredit $(X,\ \theta,\ G)$ be a GTS and $A, B \subseteq X$. Then the following is correct.*

(i) *$A \subseteq B$, then $GS_\beta\ int(A) \subseteq GS_\beta int(B)$;*
(ii) *$GS_\beta\ int(A \cup B) \subseteq GS_\beta\ int(A) \cup GS_\beta int(B)$;*
(iii) *$GS_\beta\ int(A \cap B) = GS_\beta\ int(A) \cap GS_\beta\ int(B)$.*

Proof. Proof follows by the Definition 2. □

Definition 3. *A function $f : (X, \theta, G) \to (Y, \sigma)$ is said to be $GS_\beta - cts$ if $f^{-1}(V) \in GS_\beta O(X)$ for every $V \in \beta O(Y)$.*

Example 3. *Let $X = \{w,\ x,\ y,\ z\}$, $Y = \{1,\ 2\ 3,\ 4\}$,$\theta = \{\varnothing,\ X,\ \{w\},\ \{x\},\ \{w, x\}\}$,$\sigma = \{\varnothing,\ Y,\ \{1\},\ \{3\},\ \{1,3\}\}$ and $G = \{\{x,\ z\},\ X\}$. Then $GS_\beta\ O(X) = P(X)$ and $\beta\ O(Y) = \{\varnothing,\ Y,\ \{1\},\ \{2\},\ \{4\},\ \{1,\ 2\},\ \{2,4\},\ \{1,2,\ 3\}\}$. Define $f : (X,\ \theta,\ G) \to (Y,\ \sigma)$ by $f(w) = 2$, $f(x) = 4$, $f(y) = 1$, $f(z) = 3$. Then, inverse image of every β ops in Y is $GS_\beta - opn$ in X. Therefore f is $GS_\beta - cts$.*

Remark 2. *The idea of G-Semi cont [12] along with $GS_\beta - cts$ is independent.*

(i) *From Example 3, we have that $GSO(X) = \{\varnothing,\ X,\ \{w\},\ \{y\},\ \{w, x\},\ \{w, z\}\}$. Moreover, the function f is $GS_\beta - cts$. Further $f^{-1}(\{1,\ 2,\ 3\}) = \{w,\ x,\ z\}$ is not GSO in X for the ops $\{1,\ 2,\ 3\}$ of Y. Hence f is not G-Semi cont.*
(ii) *Accredit $X = \{l,\ m, n,\ o\}$, $Y = \{5,\ 6,\ 7,\ 8\}$, $\theta = \{\varnothing,\ X, \{l\}, \{n\}, \{o\}, \{l, n\}, \{l, o\}, \{n,\ o\}, \{l,\ n,\ o\}\}$, $\sigma = \{\varnothing,\ Y,\ \{6\},\ \{5,\ 6\}, \{6,\ 7\},\ \{5,\ 6,\ 7\}\}$ and $G = \{\{l\},\ \{n\},\ \{o\}, \{m, n\}, \{l,\ n,\ o\},\ \{m,\ n,\ o\},\ \{l,\ m,\ o\},\ X\}$. Then $GSO(X) = \theta$ and $GS_\beta\ O(X) = \{\varnothing,\ X,\ \{l\},\ \{n\}, \{o\},\ \{l,\ n\},\ \{l,\ o\},\ \{l,\ n,\ o\},\ \{m,\ n,\ o\},\ \{l,\ m,\ o\}\}$, $\beta\, O(Y) = P(Y)$. Define $f : (X, \theta,\ G) \to (Y, \sigma)$ by $f(l) = 6$, $f(m) = 8$, $f(n) = 7$, $f(o) = 5$. Then the*

function f is G-Semi cont. Correspondingly, the inverse image $f^{-1}(8) = \{m\}$ *is not* $GS_\beta - opn$ *in X for β-opn set,* $\{8\}$ *of Y. Later f is not* $GS_\beta - cts$.

From (i) and (ii) we clinch that the idea of G-Semi cont and $GS_\beta -$ cts are independent.

Theorem 9. *Considering a function* $m : (X, \theta, G) \to (Y, \sigma)$, *the subsequent conditions are equivalent:*

(i) m *is* $GS_\beta - cts$;
(ii) *For all* $H \in \beta\, C(Y)$, $m^{-1}(H) \in GS_\beta C(X)$;
(iii) *For all* $n \in X$*and each* $V \in \beta O(Y)$ *containing m(n), there occurs an* $U \in GS_\beta O(X)$ *containing n such that* $m(U) \subseteq V$ *.*

Proof . (i) $\Rightarrow$ (ii) Obvious from Definition 3.
(i) $\Rightarrow$ (iii) Let $V \in \beta O(Y)$ and $m(n) \in V$. Then by (i) $m^{-1}(V) \in GS_\beta O(X)$ containing n. Hence, taking $m^{-1}(V) = U$, we acquire $n \in U$ and $m(U) \subseteq V$.
(iii) $\Rightarrow$ (i) Let $V \in \beta O(Y)$ along with $n \in m^{-1}(V)$. Then $m(n) \in V \in \beta O(Y)$ and hence by (iii) there exist $U \in GS_\beta O(X)$ containing n such that $m(U) \subseteq V$. Then, we get $n \in U \subseteq \mu(\beta\, int(U)) \subseteq \mu(\beta\, int(m^{-1}(V))$. It shows that $m^{-1}(V) \subseteq \mu(\beta\, int(m^{-1}(V))$. Hence, m is $GS_\beta - cts$. □

Theorem 10. *A function* $m : (X, \theta, G) \to (Y, \sigma)$ *is* $GS_\beta - cts$ *in the case that the graph function* $n : X \to X \times Y$ *, categorized by* $n(z) = (z, f(z))$ *for apiece* $z \in X$*, is* $GS_\beta - cts$.

Proof. Assume that m is $GS_\beta - cts$. Accredit $z \in X$ also $w \in \beta O(X \times Y)$ containing $n(z)$. Then, there exist a $U \in \beta O(X)$ along with $V \in \beta O(Y)$, so that $n(z) = (z, f(z)) \in U \times V \subseteq W$. Since m is $GS_\beta - cts$, there exist a $G \in GS_\beta O(X)$ containing z such that $m(G) \subseteq V$. By Theorem 4(ii), $G \cap V \in GS_\beta O(X)$ along with $n(G \cap U) \subseteq U \times V \subseteq W$. This implies that n is $GS_\beta - cts$. Inversely, suppose that n is $GS_\beta - cts$.Accredit $z \in X$ and $V \in \alpha(Y)$ containing $f(z)$. Then $X \times V \in \beta O(X \times Y)$ and by GS_β-cty of n, there exist a $U \in GS_\beta O(X)$ containing z such that $n(U) \subseteq X \times V$. Then we got $m(U) \subseteq V$ and hence m is $GS_\beta - cts$. □

Definition 4. *Accredit* (X, θ) *be a TS along with let* (Y, σ, G) *be a GTS. A function* $m : (X, \theta) \to (Y, \sigma, G)$ *is said to be* $GS_\beta - opn$ *if for every* $U \in \beta O(X)$, $m(U)$ *is* $GS_\beta - opn$ *in* (Y, σ, G).

Theorem 11. *A function* $m : (X, \theta) \to (Y, \sigma, G)$ *is* $GS_\beta - opn$ *if for every* $r \in X$ *and each pre-nbd U of r, consists of a* $V \in GS_\beta O(Y)$ *such that* $m(r) \in V \subseteq m(U)$.

Proof. Suppose that m is $GS_\beta - opn$ function and let $\in X$. Accredit U be any pre-nbd of r. Then there occurs $G \in \beta O(X)$ so that $r \in G \subseteq U$. Therefore m is $GS_\beta - opn$, $m(G) = V$ (say) $\in GS_\beta O(Y)$ and $m(r) \in V \subseteq m(U)$. Inversely, suppose that $U \in \beta O(X)$. Then every $r \in U$, there occurs a $V_z \in GS_\beta O(X)$ such that $m(r) \in V_z \subseteq m(U)$. Thus $m(U) = \cup\{V_r : r \in U\}$ and hence by Theorem 4(i), $m(U) \in GS_\beta O(Y)$. This implies that m is $GS_\beta - opn$. □

Theorem 12. *Let* $m : (X, \theta) \to (Y,\ \sigma,\ G)$ *be a* $GS_\beta - opn$ *function. If* $D \subseteq Y$ *and* $F \in \beta\, C(X)$ *containing* $m^{-1}(D)$*, then there exists a* $H \in GS_\beta\, O(Y)$ *containing U such that* $m^{-1}(H) \subseteq F$.

Proof. Suppose that m is GS_β-open. Let $D \subseteq Y$ and $F \in \beta\ C(X)$ containing $m^{-1}(U)$. Then $X - F \in \beta\ O(X)$ and by GS_β-openness of m, $m(X - F) \in GS_\beta O(X)$. Thus $H = Y - m(X - F) \in GS_\beta C(Y)$. Consequently, $m^{-1}(D) \subseteq F$ implies that $D \subseteq H$. Further we obtain that $m^{-1}(H) \subseteq F$. □

Theorem 13. *For any bijection* $m : (X, \theta) \to (Y, \sigma, G)$ *the following conditions are equivalent:*

(i) $m^{-1} : (Y, \sigma, G) \to (X, \tau)$ *is* $GS_\beta - cts$

(ii) *m is* $GS_\beta - opn$*;*

(iii) *m is* GS_β*-csd.*

Proof. Proof follows from Definition 4. □

Definition 5. *Let* (X, θ, G) *be a GTS. A sbt E of X is defined as* GS_β^* *set if* $E = L \cap M$*, where* $L \in \beta O(X)$ *and* $\mu(\beta\, int(M)) = \beta\, int(M)$*.*

Theorem 14. *Let* (X, θ, G) *be a GTS and let* $B \subseteq X$*. Then* $B \in \beta O(X)$ *iff* $B \in GS_\beta O(X)$ *and B is* GS_β^**-set in* (X, θ, G)*.*

Proof. Let $B \in \beta O(X)$, Then $B \in GS_\beta O(X)$, implies that $B \subseteq \mu(\beta\, int(B))$. Also B can be expressed as $B = B \cap X$, where $B \in \beta O(X)$ and $\mu(\beta int\,(X)) = \beta\, int(X)$. Thus B is a GS_β^*-set. Inversely, let $B \in GS_\beta O(X)$ and B be GS_β^*-set. Then $B \subseteq \mu(\beta\, int(B)) = \mu(\beta\, int(U \cap V))$, where $U \in \beta O(X)$ and $\mu(\beta\, int(V)) = \beta\, int(V)$. Now $B \subseteq U \cap B \subseteq U \cap \mu(\beta\, int(U \cap V)) = U \cap (U \cap \mu(U) \cap \mu\beta(\, int(V)) = U \cap \beta\, int(V) = \beta\, int(B)$. Hence we get $B \in \beta O(X)$. □

Definition 6. *A function* $m : (X, \theta, G) \to (Y, \sigma)$ *is* $GS_\beta^* - cts$ *if for each* $D \in \beta O(Y)$, $m^{-1}(D)$ *is* GS_β^**-set in* (X, θ, G)*.*

Theorem 15. *Let* (X, θ, G) *be a GTS. Then for a function* $m : (X, \theta, G) \to (Y, \sigma)$ *the subsequent statement is equivalent:*

(i) *m is pre cts;*

(ii) *m is* $GS_\beta - cts$*and* $GS_\beta^* - cts$ *.*

Proof. Proof follows directly from the Definition 6. □

Example 4. *Let* $X = \{1, 2, 3, 4\}$, $\theta = \{\varphi, X, \{1\}, \{3\}, \{1,3\}\}$, $Y = \{a, b, c, d\}$, $\sigma = \{\varphi, Y, \{a\}, \{b\}, \{a,b\}\}$ *and* $G = \{\{1,2\}, X\}$. $GS_\beta O(X) = \{\{1\}, \{2\}, \{1,2\}, \{2,3\}\}$*. Define a function* $f : (X, \theta, G) \to (Y, \sigma)$ *by* $f(1) = a$, $f(2) = d$, $f(3) = c$, $f(4) = b$*. Hence function f is* GS_β^* *continuous because for each* $D \in \beta O(Y)$, $m^{-1}(D)$ *is* GS_β^* *Continuous (Figure 1).*

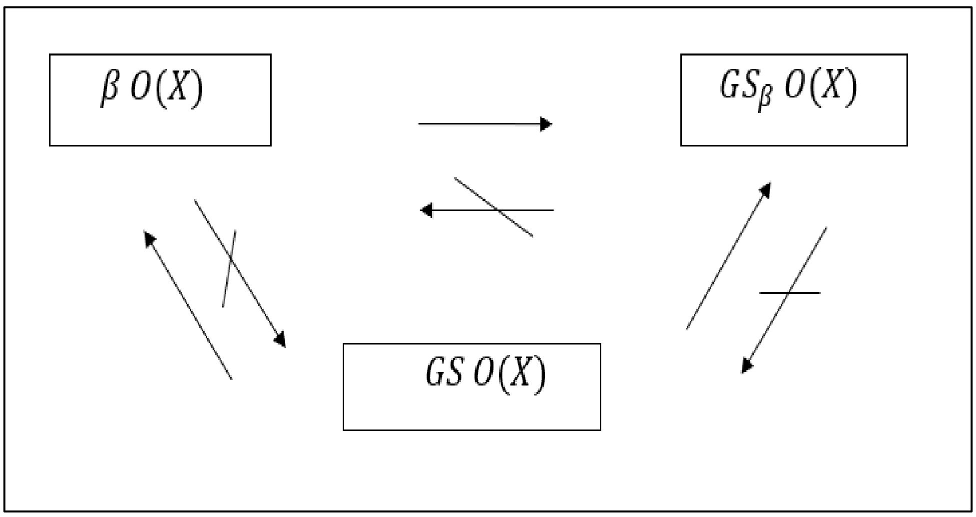

Figure 1. Relationship connecting β open set, GS_β open set and Gril Semi open set.

4. Conclusions

This research article investigated GS_β open set, GS_β closed set, GS_β continuous function, GS_β interior and closure and GS_β open function. The concept of GS_β open set along with the concept of β open set is compared and discussed. Many theorems are discussed besides the counter examples. Some significant characteristics and key properties which are associated with these GS_β open sets are proved with the help of GS_β interior and GS_β closure. In addition to this, the theory of GS_β Continuous mappings has been introduced and some theorems are provided. Finally, the concept of GS^*_β has been introduced and discussed in detail.

Author Contributions: Conceptualization, N.K. and K.F.U.R.; Data compilation, K.F.U.R.; Formal analysis, N.K. and R.Č.; Investigation, K.F.U.R. and L.Č.; Methodology, K.F.U.R., N.K., L.Č. and R.Č.; Supervision, N.K., R.Č. and L.Č.; Visualization, K.F.U.R. and N.K.; Writing—original draft, K.F.U.R. and N.K.; Writing—review and editing, N.K., L.Č. and R.Č. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data presented in this study are available through email upon request to the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

GTS	Grill Topological Space
Nbd	Neighborhood
Ops	Open Set
TS	Topological space
Cty	Continuity
Sbt	Subsets
opn	Open
PO	Preopen
SO	semi open
Semicont	semi continuous
cts	Continuous
int	Interior
clos	closure
Cs	closed set
GS	Grill Set
Thm	Theorem
Coro	Corollary
Gene	Generalized
Grl	Grill
Tpl opr	Topological operator

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