On Cauchy problems with Caputo Hadamard fractional derivatives

March 2, 2015

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Abstract

The current work is motivated by the so-called Caputo-type modification of the Hadamard or Caputo Hadamard fractional derivative discussed in [4]. The main aim of this paper is to study Cauchy problems for a differential equation with a left Caputo Hadamard fractional derivative in spaces of continuously differentiable functions. The equivalence of this problem to a nonlinear Volterra type integral equation of the second kind is shown. On the basis of the obtained results, the existence and uniqueness of the solution to the considered Cauchy problem is proved by using Banach's fixed point theorem. Finally, two examples are provided to explain the applications of the results.

MSC 2010**:** 26A33, 34A08, 34A12, 47B38.
Keywords: Caputo Hadamard fractional derivatives, Cauchy problem, Volterra integral equation, continuously differentiable function, fixed point theorem.

1 Introduction

Fractional calculus, that is, the theory of derivatives and integrals of fractional non-integer order, are used in many fields like: mathematics, physics, chemistry, engineering, and other sciences.

Few years ago, many scholars started making deeper researches on fractional differential equations. Intensive development of this latter and its applications led to that. (e.g.; [1, 2, 3, 10, 11, 12]). Many definitions were supplied for the Fractional order differential operators and many reports on the existence and uniqueness of solutions to differential equations in the frame of these operators appeared. (see for example [14] and the references therein).

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J. Hadamard [6] in 1892, introduced a new definition of fractional derivatives and integrals in which he claims:

$$\left(\mathcal{J}_{a_+}^{\alpha} g\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau}, \ (0 < a < t), \ Re(\alpha) > 0, \tag{1}$$

for suitable functions g, where Γ represents gamma function. This is the generalization of the n^{th} integral

$$\left(\mathcal{J}_{a_+}^{n} g\right)(t) = \int_a^t \frac{d\tau_1}{\tau_1} \int_a^{\tau_1} \frac{d\tau_2}{\tau_2} ... \int_a^{\tau_{n-1}} g(\tau_n) \frac{d\tau_n}{\tau_n} \equiv \frac{1}{\Gamma(n)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-1} g(\tau) \frac{d\tau}{\tau}, \tag{2}$$

where $n = [Re(\alpha)] + 1$ and $[Re(\alpha)]$ means the integer part of $Re(\alpha)$.

The corresponding left-sided Hadamard fractional derivative of order α is defined by

$$\left(\mathcal{D}_{a_+}^{\alpha} g\right)(t) = \delta^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-\alpha-1} g(\tau) \frac{d\tau}{\tau}, \ \alpha \in [n-1, n), \tag{3}$$

where $\delta = t\frac{d}{dt}$. The main difference between Hadamard's definition and the previous ones is that the kernel integral contains logarithmic function of arbitrary exponent. The present paper follows the Caputo-type definition based on the modification of Hadamard fractional derivatives. This approach is given by the equality,

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha} g\right)(t) = \left(\mathcal{D}_{a_+}^{\alpha}\right) \left[g(\tau) - \sum_{k=0}^{n-1} \frac{\delta^k g(a)}{k!} \left(\ln \frac{\tau}{a}\right)^k\right](t), \ (0 < a < t). \tag{4}$$

We can use the following equivalent representation, which follows from (3) and (4)

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha} g\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{n-\alpha-1} \delta^n g(\tau) \frac{d\tau}{\tau}. \tag{5}$$

The Caputo Hadamard derivative is obtained from the Hadamard derivative by changing the order of its differential and integral parts. Despite the different requirements on the function itself, the main difference between the Caputo Hadamard fractional derivative and the Hadamard fractional derivative is that the Caputo Hadamard derivative of a constant is zero [4]. The most important advantage of Caputo Hadamard is that it brought a new definition through which the integer order initial conditions can be defined for fractional order differential equations in the frame of the Hadamard fractional derivative.

In this article, we extend the approach of Kilbas et al. [10] to fractional Cauchy problems with a left Caputo Hadamard in spaces of continuously differentiable functions and prove the existence and uniqueness of solutions to these problems.

To get to our aim, the equivalence of the Cauchy type problems to a nonlinear Volterra type integral equation of the second kind is first proved. Once that is done, Banach's fixed point theorem is applied. By the end, some examples are given to illustrate the obtained results.

2 Preliminaries

Below, we recall some basic definitions, properties, theorems and lemmas needed in the rest of this paper.

Let $C^n([a,b], \mathbb{R})$ be the Banach space of all continuously differentiable functions from $[a,b]$ to $\mathbb{R}$. We will introduce the weighted space $C_{\gamma,\ln}[a,b]$, $C_{\delta,\gamma,\ln}^{n}[a,b]$ and $C_{\delta,\gamma,\ln}^{\alpha,r}[a,b]$ of the function g on the finite interval $[a,b]$.

Definition 2.1. If $\alpha \in (n-1, n]$ and $\gamma \in (0,1]$, then

(1) The space $C_{\gamma,\ln}[a,b]$ is defined by

$$C_{\gamma,\ln}[a,b] = \left\{ g : \left(\ln \frac{t}{a}\right)^{\gamma} g(t) \in C[a,b] \right\}, C_{0,\ln}[a,b] = C[a,b],$$

and on this space we define the norm $\|.\|_{C_{\gamma,\ln}}$ by

$$\|g\|_{C_{\gamma,\ln}} = \left\| \left(\ln \frac{t}{a}\right)^{\gamma} g(t) \right\|_{C} = \max_{t\in[a,b]} \left| \left(\ln \frac{t}{a}\right)^{\gamma} g(t) \right|.$$

(2) The space $C^{n}_{\delta,\gamma,\ln}[a,b]$ is defined by

$$C^{n}_{\delta,\gamma,\ln}[a,b] = \left\{ g : \delta^{k} g \in C[a,b],\ k = 0,..,n-1 \text{ and } \delta^{n} g \in C_{\gamma,\ln}[a,b] \right\},$$

and on this space we define the norm $\|.\|_{C^{n}_{\delta,\gamma,\ln}}$ by

$$\|g\|_{C^{n}_{\delta,\gamma,\ln}} = \sum_{k=0}^{n-1} \left\| \delta^{k} g \right\|_{C} + \|\delta^{n} g\|_{C_{\gamma,\ln}},\ \|g\|_{C^{n}_{\delta}} = \sum_{k=0}^{n} \max_{t\in[a,b]} \left| \delta^{k} g(t) \right|.$$

(3) We denote by $C^{\alpha,r}_{\delta,\gamma,\ln}[a,b]$ the space of functions g given on $[a,b]$ and such that

$$C^{\alpha,r}_{\delta,\gamma,\ln}[a,b] = \left\{ g \in C^{r}_{\delta}[a,b] : \left({}^{c}\mathcal{D}^{\alpha}_{a_+} g\right) \in C_{\gamma,\ln}[a,b],\ r \in \mathbb{N} \right\},$$
$$C^{r,r}_{\delta,\gamma,\ln}[a,b] = C^{r}_{\delta,\gamma,\ln}[a,b].$$

Property 2.2 ([10]). *The fractional integral operators* $\left(\mathcal{J}^{\alpha}_{a_+}\right)$ *satisfy the semigroup property*

$$\left(\mathcal{J}^{\alpha}_{a_+} \mathcal{J}^{\beta}_{a_+} g\right)(t) = \left(\mathcal{J}^{\alpha+\beta}_{a_+} g\right)(t),\ Re(\alpha) > 0,\ Re(\beta) > 0.$$

The fractional derivative operators $\left(\mathcal{D}^{\alpha}_{a_+}\right)$ *fullfil the semigroup property*

$$\left(\mathcal{D}^{\alpha}_{a_+} \mathcal{J}^{\beta}_{a_+} g\right)(t) = \left(\mathcal{J}^{\beta-\alpha}_{a_+} g\right)(t).$$

Property 2.3 ([4]). *Let* $Re(\alpha) \geq 0,\ n = [Re(\alpha)] + 1$ *and*$Re(\beta) > 0$, *then*

$$({}^{c}\mathcal{D}^{\alpha}_{a_+} \left(\ln \frac{t}{a}\right)^{\beta-1}) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{t}{a}\right)^{\beta-\alpha-1}, Re(\beta) > n.$$

On the other hand, for $k = 0, 1, .., n-1$,

$$({}^{c}\mathcal{D}^{\alpha}_{a_+} \left(\ln \frac{t}{a}\right)^{k}) = 0.$$

Lemma 2.4 ([4]). *Let* $\alpha \in \mathbb{C}$, $n = [Re(\alpha)] + 1$, *let* $g(t) \in AC^{n}_{\delta}[a,b]$ *or* $C^{n}_{\delta}[a,b]$, *then*

$$\left(\mathcal{J}^{\alpha}_{a_+} \left({}^{c}\mathcal{D}^{\alpha}_{a_+} g\right)\right)(t) = g(t) - \sum_{k=0}^{n-1} \frac{(\delta^{k} g)(a)}{k!} \left(\ln \frac{t}{a}\right)^{k}.$$

Lemma 2.5 ([10]). *Let $n \in \mathbb{N}$ and $0 \leq \gamma < 1$.The space $C^n_{\delta,\gamma,\ln}[a,\ b]$ consists of those and only those functions g which are represented in the form*

$$g(t) = \frac{1}{(n-1)!} \int_a^t \left(\ln \frac{t}{\tau} \right)^{n-1} \varphi(\tau) \frac{d\tau}{\tau} + \sum_{k=0}^{n-1} d_k \left(\ln \frac{t}{a} \right)^k,$$

where $\varphi \in C_{\gamma,\ln}[a,\ b]$ and d_k $(k = 0, 1, ..., n-1)$ are arbitrary constants, such that

$$\varphi(t) = \delta^n g(t),\ d_k = \frac{\delta^k g(a)}{k!} \quad (k = 0, 1, ..., n-1).$$

Lemma 2.6 ([10]). *Let $0 < a < b < +\infty$, $Re(\alpha) > 0$, and $0 \leq \gamma < 1$, then*

a. *If $\gamma > \alpha > 0$, then $\left(\mathcal{J}^\alpha_{a+}\right)$ is bounded from $C_{\gamma,\ln}[a,b]$ into $C_{\gamma-\alpha,\ln}[a,b]$:*

$$\left\| \mathcal{J}^\alpha_{a+} g \right\|_{C_{\gamma-\alpha,\ln}} \leq k \|g\|_{C_{\gamma,\ln}},\ k = \left(\ln \frac{b}{a} \right)^{Re(\alpha)} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular $\left(\mathcal{J}^\alpha_{a+}\right)$ is bounded in $C_{\gamma,\ln}[a,b]$.

b. *If $\gamma \leq \alpha$, then $\left(\mathcal{J}^\alpha_{a+}\right)$ is bounded from $C_{\gamma,\ln}[a,b]$ into $C[a,b]$:*

$$\left\| \mathcal{J}^\alpha_{a+} g \right\|_C \leq k \|g\|_{C_{\gamma,\ln}},\ k = \left(\ln \frac{b}{a} \right)^{Re(\alpha)-\gamma} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular $\left(\mathcal{J}^\alpha_{a+}\right)$ is bounded in $C_{\gamma,\ln}[a,b]$.

Lemma 2.7 ([10]). *The fractional operator $\left(\mathcal{J}^\alpha_{a+}\right)$ represents a mapping from $C[a,b]$ to $C[a,b]$ and*

$$\left\| \mathcal{J}^\alpha_{a+} g \right\|_C \leq \frac{1}{Re(\alpha)\Gamma(\alpha)} \left(\ln \frac{b}{a} \right)^{Re(\alpha)} \|g\|_C.$$

Theorem 2.8 (Banach fixed point Theorem, [10]). *Let $(X,\ d)$ be a nonempty complete metric space, let $0 \leq w < 1$, and let $T : X \longrightarrow X$ be a map such that for every $x,\ \tilde{x} \in X$, the relation*

$$d(Tx,\ T\tilde{x}) \leq wd(x,\ \tilde{x}),$$

holds. Then the operator T has a uniquely defined fixed point $x^* \in X$.

Furthermore, if T^k $(k \in \mathbb{N})$ is the sequence defined by

$$T^1 = T,\ T^k = TT^{k-1}(k \in \mathbb{N} - \{1\}),$$

then, for any $x_0 \in X$ $\left\{T^k x_0\right\}_{k=1}^{k=\infty}$ converges to the above fixed point x^*.

Definition 2.9 ([10]). Let $l \in \mathbb{N},\ G \subset \mathbb{R}^l,\ [a,b] \subset \mathbb{R},\ g : [a,b] \times G \longrightarrow \mathbb{R}$ be a function such that, for any $(x_1, ..., x_l),\ (\tilde{x}_1, ..., \tilde{x}_l) \in G,\ g$ satisfies generalized Lipschitizian condition:

$$|g[t, x_1, ..., x_l] - g[t, \tilde{x}, \tilde{x}_1, ..., \tilde{x}_l]| \leq A_1 |x_1 - \tilde{x}_1| + ... + A_l |x_l - \tilde{x}_l|,\ A_j \geq 0,\ j = 1, ..., l. \quad (6)$$

In particular,g satisfies the Lipschitzian condition with respect to the second variable if for all $t \in (a,b]$ and for any $x,\ \tilde{x} \in G$ one has

$$|g[t,x] - g[t,\tilde{x}]| \leq A|x - \tilde{x}|,\ A > 0. \quad (7)$$

3 Nonlinear Cauchy problem

In this section, we present the existence and uniqueness results in the space $C^{\alpha,r}_{\delta,\gamma,\ln}[a,b]$ of the Cauchy problem for the nonlinear fractional differential equation in the frame of Caputo Hadamarad fractional derivative. That is we consider the equation

$$\left({}^{C}\mathcal{D}^{\alpha}_{a_+}x\right)(t) = h\left[t, x(t)\right],\ Re(\alpha) > 0,\ t > a > 0, \tag{8}$$

subject to the initial conditions

$$\left(\delta^k x\right)(a_+) = d_k,\ d_k \in \mathbb{R},\ k = 0, ..., n-1,\ n = [Re(\alpha)] + 1. \tag{9}$$

The Volterra type integral equation corresponding to problem (8)-(9) is :

$$x(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} h\left[\tau, x(\tau)\right]\frac{d\tau}{\tau},\ a \le t \le b. \tag{10}$$

In partuclar, if $\alpha = n \in \mathbb{N}$ then the problem (8)-(9) is as follows:

$$(\delta^n x)(t) = h\left[t, x(t)\right],\ a \le t \le b,\ \left(\delta^k x\right)(a_+) = d_k \in \mathbb{R},\ k = 0, 1, ...n-1. \tag{11}$$

The corresponding integral equation to the problem (11) has the form:

$$x(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j + \left(\mathcal{J}^n_{a_+}h\right)(t),\ a \le t \le b. \tag{12}$$

Firstly, we we have to prove the equivalence of the Cauchy problem to the Volterra type integral equation in the sense that, if $x \in C^r_\delta[a,b]$ satisfies one of them, then it also satisfies the other one.

Theorem 3.1. *Let $Re(\alpha) > 0$, $n = [Re(\alpha)] + 1$, $(0 < a < b < +\infty)$, and $0 \le \gamma < 1$ be such that $\alpha \ge \gamma$. Let G be an open set in $\mathbb{R}$ and let $h : [a,b] \times G \longrightarrow \mathbb{R}$ be a function such that $h[t,x] \in C_{\gamma,\ln}[a,b]$ for any $x \in C_{\gamma,\ln}[a,b]$.*

- **(i)** *Let $r = n-1$ for $\alpha \notin \mathbb{N}$, if $x \in C^{n-1}_\delta[a,b]$ then x satisfies the relations (8) and (9) iff x satisfies equation (10).*
- **(ii)** *Let $r = n$ for $\alpha \in \mathbb{N}$, if $x \in C^n_\delta[a,b]$ then x satisfies the relation (11) if and only if, x satisfies equation (12).*

Proof. **(i)** Let $\alpha \notin \mathbb{N}$, $n-1 < \alpha < n$ and $x \in C^{n-1}_\delta[a,b]$.

(i.a) Here we prove the necessity. From definition of ${}^{C}\mathcal{D}^{\alpha}_{a_+}$ and (3) we obtain

$${}^{C}\mathcal{D}^{\alpha}_{a_+}x(t) = (\delta^n)\left(\mathcal{J}^{n-\alpha}_{a_+}\left[x(\tau) - \sum_{j=0}^{n-1}\frac{\delta^j x(a)}{j!}\left(\ln\frac{\tau}{a}\right)^j\right]\right)(t).$$

By hypothesis, $h[t,x] \in C_{\gamma,\ln}[a,b]$ and it follows from (8) that ${}^{C}\mathcal{D}^{\alpha}_{a_+}x(t) \in C_{\gamma,\ln}[a,b]$, and hence, by applying Lemma 2.5, we have

$$\left(\mathcal{J}^{n-\alpha}_{a_+}\left[x(\tau) - \sum_{j=0}^{n-1}\frac{\delta^j x(a)}{j!}\left(\ln\frac{t}{\tau}\right)^j\right]\right)(t) \in C^n_{\delta,\gamma,\ln}[a,b].$$

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By using Lemma 2.4, we obtain

$$\mathcal{J}_{a_+}^{\alpha}\left({}^{C}\mathcal{D}_{a_+}^{\alpha}x\right)(t)=x(t)-\sum_{j=0}^{n-1}\frac{\delta^{j}x(a)}{j!}\left(\ln\frac{t}{a}\right)^{j}. \tag{13}$$

In view of Lemma 2.6-(b), $\mathcal{J}_{a_+}^{\alpha}h[t,x]$ belongs to the $C[a,b]$ space, Applying $\left(\mathcal{J}_{a_+}^{\alpha}\right)$ to the both sides of (8) and utilizing (13), with respect to the initial conditions (9), we deduce that there exists a unique solution $x\in C_{\delta}^{n-1}[a,b]$ to equation (10).

(i.b) Let $x\in C_{\delta}^{n-1}[a,b]$ satisfies the equation (10).

– We want to show that x satisfies equation (8). Applying $\left(\mathcal{D}_{a_+}^{\alpha}\right)$ to both sides of (10), and taking into account (4), (9), Property 2.2 and Property 2.3, we get

$$\mathcal{D}_{a_+}^{\alpha}\left(x(t)-\sum_{j=0}^{n-1}\frac{\delta^{j}x(a)}{j!}\left(\ln\frac{t}{a}\right)^{j}\right)=\mathcal{D}_{a_+}^{\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}h[\tau,x(\tau)]\frac{d\tau}{\tau}\right),$$

then

$$\left({}^{C}\mathcal{D}_{a_+}^{\alpha}x\right)(t)=\left(\mathcal{D}_{a_+}^{\alpha}\right)\left(\mathcal{J}_{a_+}^{\alpha}h\right)(t)\equiv h[t,x(t)].$$

– Now, we show that x satisfies the initial relations (9). We obtain by differentiation both sides of (10) that,

$$\delta^{k}x(t)=\sum_{j=k}^{n-1}\frac{d_j}{(j-k)!}\left(\ln\frac{t}{a}\right)^{j-k}+\frac{1}{\Gamma(\alpha-k)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-k-1}h[\tau,x(\tau)]\,d\tau.$$

Changing the variable $\tau=a\left(\frac{t}{a}\right)^{s}$, yieldys

$$\begin{aligned}\delta^{k}x(t) &= \sum_{j=k}^{n-1}\frac{d_j}{(j-k)!}\left(\ln\frac{t}{a}\right)^{j-k}+\frac{1}{\Gamma(\alpha-k)}\int_0^1\left(\ln\frac{t}{a\left(\frac{t}{a}\right)^{s}}\right)^{\alpha-k-1}\\ &\quad\times h\left[a\left(\frac{t}{a}\right)^{s},x\left(a\left(\frac{t}{a}\right)^{s}\right)\right]a\ln\left(\frac{t}{a}\right)\left(\frac{t}{a}\right)^{s}ds\\ &= \sum_{j=k}^{n-1}\frac{d_j}{(j-k)!}\left(\ln\frac{t}{a}\right)^{j-k}+\frac{\ln\left(\frac{t}{a}\right)^{\alpha-k}}{\Gamma(\alpha-k)}\int_0^1(1-s)^{\alpha-k-1}h\left[a\left(\frac{t}{a}\right)^{s},x\left(a\left(\frac{t}{a}\right)^{s}\right)\right]ds.\end{aligned}$$

for $k=0,...,n-1$. Because $\alpha-k>n-1-k\geq 0$, using the continuity of h, Property 2.3 and Lemma 2.7 we get $\mathcal{J}_{a_+}^{\alpha}h[t,x]\in C[a,b]$, and taking a limit as $t\longrightarrow a_+$, we obtain $\delta^{k}x(a_+)=d_k$.

(ii) For $\alpha\in\mathbb{N}$ and $x(t)\in C_{\delta}^{n}[a,b]$ be the solution to the Cauchy problem (11).

(ii.a) Firstly, we prove the necessity. Applying $\left(\mathcal{J}_{a_+}^{n}\right)$ to both sides of equation (11), using (4) and Lemma 2.4, we have

$$\mathcal{J}_{a_+}^{n}\delta^{n}x(t)=x(t)-\sum_{k=0}^{n-1}\frac{\delta^{k}x(a)}{k!}\left(\ln\frac{t}{a}\right)^{k}=\mathcal{J}_{a_+}^{n}h(t),$$

since $\delta^{k}x(a_+)=d_k$, we arrive at equation (12) and hence the necessity is proved.

(ii.b) If $x \in C_{\delta}^{n}[a,b]$ satisfies the equation (12), in addition, by term-by-term differentiation of (12) in the usual sense k times, we get

$$\delta^{k}x(t)=\sum_{j=k}^{n-1}\frac{d_j}{(j-k)!}\left(\ln\frac{t}{a}\right)^{j-k}+\frac{1}{(n-k-1)!}\int_a^t\left(\ln\frac{t}{\tau}\right)^{n-k-1}h\left[\tau,x(\tau)\right]\frac{d\tau}{\tau},$$

for $k=0,...,n$. Using Property 2.3 , taking the limit as $t\longrightarrow a_+$, we obtain $\delta^{k}x(a_+)=d_k$, and $\delta^{n}x(t)=h\left[t,x(t)\right]$. Thus the Theorem 3.1 is proved for $\alpha\in\mathbb{N}$.

This completes the proof of the theorem. □

Corollary 3.2. *Under the hypotheses of Theorem* 3.1*, with* $0 < Re(\alpha) < 1$*, if* $x \in C_{\delta}[a,b]$ *then* $x(t)$ *satisfies the relation*

$$\left({}^{C}\mathcal{D}_{a_+}^{\alpha}x\right)(t)=h\left[t,x(t)\right],\ t>a>0,\ x(a)=d_0,$$

if and only if, x *satisfies the equation*

$$x(t)=d_0+\left(\mathcal{J}_{a_+}^{\alpha}h\right)(t),\ a\leq t\leq b.$$

The next step is to prove the existence of a unique solution to the Cauchy problem (8)-(9) in the space of functions $C_{\delta,\gamma,\ln}^{\alpha,r}[a,b]$ by using the Banach's fixed point theorem.

Theorem 3.3. *Let* $\alpha>0$*, ad* $n=[\Re(\alpha)]+1$*,* $0\leq\gamma<1$ *be such that* $\alpha\geq\gamma$*. Let* G *be an open set in* $\mathbb{R}$ *and* $h:]a,b]\times G\longrightarrow\mathbb{C}$ *be a function such that, for any* $x\in G$*,* $h[t,x]\in C_{\gamma,\ln}[a,b]$*,* $x\in C_{\gamma,\ln}[a,b]$*, and the Lipshitz condition* (7) *holds with respect to the second variable.*

(i) *If* $n-1<\alpha<n$*, then there exists a unique solution* x *to* (8)-(9) *in the space* $C_{\delta,\gamma,\ln}^{\alpha,n-1}[a,b]$.

(ii) *If* $\alpha=n$*, then there exists a unique solution* $x\in C_{\delta,\gamma,\ln}^{n}[a,b]$.

Since the problem (8)-(9) and the equation (10) are equivalent, it is enough to prove that there exists only one solution to (10).

Proof. Here we prove (i) only as (ii) can be proved similarly.

Step 1. First we show that there exists a unique solution $x\in C_{\delta}^{n-1}[a,b]$.

Divide the interval $[a,b]$ into M subdivisions $[a,t_1],\ [t_1,t_2],...,[t_{M-1},b]$ such that $a<t_1<t_2<...<t_{M-1}<b$.

(a) Choose $t_1\in]a,b]$ such that the inequality

$$w_1=A\sum_{k=0}^{n-1}\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)}\left(\ln\frac{t_1}{a}\right)^{Re(\alpha)-k}<1,\ A>0, \tag{14}$$

holds. Now we prove that there exists a unique solution $x(t)\in C_{\delta}^{n-1}[a,t_1]$ to equation (10) in the interval $[a,t_1]$.

It is easy to see that $C_\delta^{n-1}[a,t_1]$ is a complete metric space equipped with the distance

$$d(x_1,x_2)=\|x_1-x_2\|_{C_\delta^{n-1}[a,t_1]}=\sum_{k=0}^{n-1}\left\|\left(\delta^k x_1-\delta^k x_2\right)\right\|_{C[a,t_1]}.$$

Now, for any $x\in C_\delta^{n-1}[a,t_1]$, define operator T as follows

$$(Tx)(t)\equiv Tx(t)=x_0(t)+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{\tau})\right)^{\alpha-1}h[\tau,x(\tau)]\frac{d\tau}{\tau}, \tag{15}$$

with

$$x_0(t)=\sum_{j=0}^{n-1}\frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j. \tag{16}$$

Transforming the problem (10) into a fixed point problem, $x(t)=Tx(t)$, where T is defined by (15). One can see that the fixed points of T are nothing but solutions to problem (8)-(9). Applying the Banach contraction mapping, we shall prove that T has a unique fixed point.

Firstly, we have to show that:

(a.i) if $x(t)\in C_\delta^{n-1}[a,t_1]$, then $(Tx)(t)\in C_\delta^{n-1}[a,t_1]$.

(a.ii) $\forall x_1,\ x_2\in C_\delta^{n-1}[a,t_1]$ the following inequality holds:

$$\|Tx_1-Tx_2\|_{C_\delta^{n-1}[a,t_1]}\le w_1\|x_1-x_2\|_{C_\delta^{n-1}[a,t_1]},\ 0<w_1<1.$$

(a.i) Let us prove that $Tx:C_\delta^{n-1}[a,t_1]\longrightarrow C_\delta^{n-1}[a,t_1]$ is a continuous operator. Differentiating (15) k $(k=0,...,n-1)$ times, we arrive at the equality

$$\left(\delta^k Tx\right)(t)=\delta^k x_0(t)+\frac{1}{\Gamma(\alpha-k)}\int_a^t\left(\ln(\frac{t}{\tau}\right)^{\alpha-1-k}h[\tau,x(\tau)]\frac{d\tau}{\tau},$$

with

$$\delta^k x_0(t)=\sum_{j=k}^{n-1}\frac{d_j}{(j-k)!}\left(\ln\frac{t}{a}\right)^{j-k}.$$

It follows that $\delta^k x_0(t)\in C_\delta[a,t_1]$ because $x_0(t)$ might be further decomposed as a finite sum of functions in $C_\delta^{n-1}[a,t_1]$. When $x_0(t)\in C_\delta^{n-1}[a,t_1]$ then

$$\|x_0(t)\|_{C[a,t_1]}\le\|x_0(t)\|_{C_\delta^{n-1}[a,t_1]}=\sum_{k=1}^{n-1}\left\|\left(\delta^k x_0(t)\right)\right\|_{C[a,t_1]}+\|x_0(t)\|_{C[a,t_1]}.$$

On the other hand, we can apply Lemma 2.6-(b) with $\alpha\ge\gamma$, and α being replaced by $(\alpha-k)$, we have

$$\mathcal{J}_{a_+}^{\alpha-k}h[\tau,x(\tau)](t)\in C_\delta[a,t_1].$$

In view of Lemma 2.6 and (7), for all $k=0,...,n-1$, we have

$$\begin{aligned}\left\|\mathcal{J}_{a_+}^{\alpha-k}h[\tau,x(\tau)]\right\|_{C[a,t_1]} &\le \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-k-\gamma)}\left(\ln\frac{t_1}{a}\right)^{Re(\alpha)-k-\gamma}\|h[t,x(t)]\|_{C_{\gamma,\ln}[a,t_1]}\\ &\le A\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-k-\gamma)}\left(\ln\frac{t_1}{a}\right)^{Re(\alpha)-k-\gamma}\|x(t)\|_{C_{\gamma,\ln}[a,t_1]}\\ &\le A\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-k-\gamma)}\left(\ln\frac{t_1}{a}\right)^{Re(\alpha)-k}\|x(t)\|_{C[a,t_1]}.\end{aligned}$$

As fractional integrals are bounded in the space of functions continuous in interval $[a, t_1]$. The above implies that $Tx(t)$ belongs to the $C_\delta^{n-1}[a, t_1]$ space.

(a.ii) Next, we let $x_1, x_2 \in C_\delta^{n-1}[a, t_1]$ the following estimate holds:

$$\begin{aligned}\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a,t_1]} &= \left\| \mathcal{J}_{a_+}^{\alpha} \left(h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)] \right)(t) \right\|_{C_\delta^{n-1}[a,t_1]} \\ &= \sum_{k=0}^{n-1} \left\| \mathcal{J}_{a_+}^{\alpha-k} \left(h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)] \right)(t) \right\|_{C[a,t_1]} \\ &\le \sum_{k=0}^{n-1} \tfrac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \tfrac{t_1}{a}\right)^{Re(\alpha)-k-\gamma} \|h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)]\|_{C_{\gamma,\ln}[a,t_1]} \\ &\le A \sum_{k=0}^{n-1} \tfrac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \tfrac{t_1}{a}\right)^{Re(\alpha)-k-\gamma} \|x_1(t) - x_2(t)\|_{C_{\gamma,\ln}[a,t_1]} \\ &\le A \sum_{k=0}^{n-1} \tfrac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \tfrac{t_1}{a}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C[a,t_1]} \\ &\le A \sum_{k=0}^{n-1} \tfrac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \tfrac{t_1}{a}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[a,t_1]}.\end{aligned}$$

Thus

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a,t_1]} \le A \sum_{k=0}^{n-1} \tfrac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \tfrac{t_1}{a}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[a,t_1]}.$$

The last estimate shows that the operator T is a contraction mapping from $C_\delta^{n-1}[a, t_1]$Thus, the Banach fixed point theorem implies that there exists a unique function (solution) $x_0^* \in C_\delta^{n-1}[a, t_1]$ and this given as:

$$x_0^* = \lim_{m \to +\infty} T^m x_{00}^*, \ (m \in \mathbb{N}^*),$$

where

$$(T^m x_{00}^*)(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} h\left[\tau, \left(T^{m-1} x_{00}^*\right)(\tau)\right] \frac{d\tau}{\tau},$$

with $x_{00}^* \in C_\delta^{n-1}[a, t_1]$ is an arbitrary starting function.

Let us take $x_{00}^*(t) = x_0(t)$ when $d_k \neq 0$ with $x_0(t)$ defined by (16), if we denote by

$$x_m(t) = (T^m x_{00}^*)(t), (m \in \mathbb{N}^*),$$

then

$$\lim_{m \to +\infty} \|x_m(t) - x_0^*(t)\|_{C_\delta^{n-1}[a,t_1]} = 0.$$

Now we show that this solution $x_0^*(t)$ is unique. Suppose that there exist two solutions $x_0^*(t)$, $\tilde{x}_0^*(t)$ of equation (10) on $[a, t_1]$. Using Lemma 2.6 and substituting them into (10), we get

$$\|x_0^*(t) - \tilde{x}_0^*(t)\|_{C_\delta^{n-1}[a,t_1]} \le A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k} \|x_0^*(t) - \tilde{x}_0^*(t)\|_{C_\delta^{n-1}[a,t_1]}.$$

This relation yields

$$A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k} \ge 1,$$

which contradicts the assumption (14). Thus there is a unique solution $x_0^*(t) \in C_\delta^{n-1}[a, t_1]$.

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(b) We prove the existence of an unique solution $x(t) \in C_{\delta}^{n-1}[t_1, b]$. analogously

Further, if we consider the closed interval $[t_{1,}b]$, we can rewrite equation (10) in the form $x(t) = (Tx)(t)$ where

$$(Tx)(t) = x_{01}(t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{h[\tau, x(\tau)]}{\tau} d\tau, \tag{17}$$

where $x_{01}(t)$ defined by

$$x_{01}(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_{a}^{t_1} \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{h[\tau, x(\tau)]}{\tau} d\tau,$$

is a known function.

We note that $x_{01}(t) \in C_{\delta}^{n-1}[t_1, b]$. Differentiating (17) k $(k = 0, ..., n-1)$ times, we arrive at the equality

$$\left(\delta^k Tx\right)(t) = \delta^k x_{01}(t) + \frac{1}{\Gamma(\alpha - k)} \int_{a}^{t} \left(\ln \frac{t}{\tau}\right)^{\alpha-k-1} h[\tau, x(\tau)] \frac{d\tau}{\tau}.$$

It follows that $\delta^k x_{01}(t) \in C_{\delta}[t_1, b]$ and $\mathcal{J}_{a+}^{\alpha-k} h[\tau, x(\tau)] \in C_{\delta}[t_1, b]$ thus $(Tx)(t) \in C_{\delta}^{n-1}[t_1, b]$.

(b.i) Choose $t_2 \in]t_1, b]$ such that the inequality

$$w_2 = A \sum_{k=1}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha - k - \gamma + 1)} \left(\ln \frac{t_2}{t_1}\right)^{Re(\alpha)-k} < 1,$$

hold. Let $x_1, x_2 \in C_{\delta}^{n-1}[t_1, t_2]$ the following estimate holds:

$$\begin{aligned}
\|Tx_1 - Tx_2\|_{C_{\delta}^{n-1}[t_1,t_2]} &\leq \sum_{k=0}^{n-1} \left\| \mathcal{J}_{a+}^{\alpha-k} \left(h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)]\right)(t) \right\|_{C[t_1,t_2]} \\
&\leq A \sum_{k=0}^{n} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha - k + 1)} \left(\ln \frac{t_2}{t_1}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_{\delta}^{n-1}[t_1,t_2]}.
\end{aligned}$$

Hence Tx is a contraction in $C_{\delta}^{n-1}[t_1, t_2]$.

By Lemma 2.6-(b) and α being replaced by $\alpha - k$, we obtain that $\mathcal{J}_{t_1+}^{\alpha-k}(h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])$ is continuous in $[t_1, t_2]$. Then, the Banach fixed point theorem implies that there exists a unique solution $x_1^* \in C_{\delta}^{n-1}[t_1, t_2]$ to the equation (10) on the interval $[t_1, t_2]$.

Notice that $x_1^*(t_1) = x_0^*(t_1) = x_{01}(t_1)$. Further, Theorem 2.8 guarantees that this solution $x_1^*(t)$ is the limit of the convergent sequence $T^m x_{01}^*$. Thus, we have

$$\lim_{m \to +\infty} \|T^m x_{01}^* - x_1^*\|_{C_{\delta}^{n-1}[t_1,t_2]} = 0,$$

with

$$\left(T^m x_{01}^*\right)(t) = x_{01}(t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \left(\ln(\frac{t}{\tau})\right)^{\alpha-1} h\left[\tau, \left(T^{m-1} x_{01}^*\right)(\tau)\right] \frac{d\tau}{\tau}, (m \in \mathbb{N}^*).$$

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If $x_0(t) \neq 0$ then we can take $x_{01}^*(t) = x_0(t)$, therefore,

$$\lim_{m\to+\infty} \|x_m(t) - x_1^*(t)\|_{C_\delta^{n-1}[t_1,t_2]} = 0,\ x_m(t) = (T^m x_{01}^*)(t).$$

Now let

$$x^*(t) = \begin{cases} x_0^*(t) & t \in [t_1, t_2], \\ x_1^*(t) & t \in [a, t_1]. \end{cases}$$

Moreover, since $x^* \in C_\delta^{n-1}[a, t_1]$ and $x^* \in C_\delta^{n-1}[t_1, t_2]$, we have $x^* \in C_\delta^{n-1}[a, t_2]$, and hence there is a unique solution $x^* \in C_\delta^{n-1}[a, t_2]$ to the equation (10) on the interval $[a, t_2]$.

(b.ii) Finally, we prove that a unique solution $x(t) \in C_\delta^{n-1}[t_2, b]$ exists.

If $t_2 \neq b$, we choose $t_{i+1} \in]t_i, b]$ such that the relation

$$w_{i+1} = A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha - k - \gamma + 1)} \left(\ln \frac{t_{i+1}}{t_i}\right)^{Re(\alpha)-k} < 1,\ i = 2, 3, ..., M,\ b = t_M.$$

Repeating the above process i times, we also deduce that there exists a unique solution $x_i^* \in C_\delta^{n-1}[t_i, t_{i+1}]$ given as a limit of a convergent sequence $T^m x_{0i}^*$ i.e.,

$$\lim_{m\longrightarrow+\infty} \|T^m x_{0i}^* - x_i^*\|_{C_\delta^{n-1}[t_i,t_{i+1}]} = 0,\ i = 2, 3, ..., M.$$

Consequently, the previous relation can be rewritten as

$$\lim_{m\to+\infty} \|x_m(t) - x^*(t)\|_{C_\delta^{n-1}[a,b]} = 0, \tag{18}$$

with

$$x_m(t) = T^m x_{0i}^*,\ x_{0i}^*(t) = x_0(t),\ x^*(t) = x_i^*(t),\ i = 0, 1, ..., M,$$

and

$$x_i^*(t_{i+1}) = x_{i+1}^*(t_{i+1}),\ [a, b] = \cup [t_i, t_{i+1}],\ a = t_0 < ... < t_M = b.$$

Step 2. Now we show that $\left({}^{\mathcal{C}}\mathcal{D}_{a+}^{\alpha} x^*\right)(t) \in C_{\gamma,\ln}[a, b]$.

By (8), (18) and the Lipschitzian condition (7), we have that

$$\begin{aligned}\lim_{m\to+\infty} \left\| \left({}^{\mathcal{C}}\mathcal{D}_{a+}^{\alpha} x_m\right)(t) - \left({}^{\mathcal{C}}\mathcal{D}_{a+}^{\alpha} x^*\right)(t) \right\|_{C_{\gamma,\ln}[a,b]} &= \lim_{m\to+\infty} \|h[t, x_m(t)] - h[t, x^*(t)]\|_{C_{\gamma,\ln}[a,b]} \\ &\leq A \lim_{m\to+\infty} \|x_m(t) - x^*(t)\|_{C_{\gamma,\ln}[a,b]} \\ &\leq A \left(\ln \tfrac{b}{a}\right)^\gamma \lim_{m\to+\infty} \|x_m(t) - x^*(t)\|_{C[a,b]} \\ &\leq A \left(\ln \tfrac{b}{a}\right)^\gamma \lim_{m\to+\infty} \|x_m(t) - x^*(t)\|_{C_\delta^{n-1}[a,b]}.\end{aligned}$$

It is obvious that the right hand side of the above inequality approaches to zero independently, thus

$$\lim_{m\to+\infty} \left\| \left({}^{\mathcal{C}}\mathcal{D}_{a+}^{\alpha} x_m\right)(t) - \left({}^{\mathcal{C}}\mathcal{D}_{a+}^{\alpha} x^*\right)(t) \right\|_{C_{\gamma,\ln}[a,b]} = 0.$$

By hypothesis, $\left({}^{\mathcal{C}}\mathcal{D}_{a+}^{\alpha} x_m\right)(t) = h[t, x_m(t)]$ and $h[t, x(t)] \in C_{\gamma,\ln}[a, b]$ for $x \in C_\delta^{n-1}[a, b]$, we have $\left({}^{\mathcal{C}}\mathcal{D}_{a+}^{\alpha} x^*\right)(t) \in C_{\gamma,\ln}[a, b]$.

Consequently, $x^* \in C_{\delta,\gamma,\ln}^{\alpha,n-1}[a, b]$ is the unique solution to the problem (8)-(9). □

Corollary 3.4. *Under the hypotheses of Theorem* 3.3, *with* $\gamma = 0$, *there exists a unique solution* x *to the problem* (8)-(9) *in the space* $C_\delta^{\alpha,n-1}[a,b]$ *and to the problem* (11) *in the space* $C_\delta^n[a,b]$.

Proof. The above Corollary can be demonstrated in a similar way to that of Theorem 3.3, using the following inequality

$$w_{i+1} = A\sum_{k=0}^{n-1}\frac{1}{Re(\alpha-k)\Gamma(\alpha-k+1)}\left(\ln\frac{t_{i+1}}{t_i}\right)^{Re(\alpha)-k} < 1,\ i=0,...,M,\ a=t_0,\ b=t_M,$$

where $t_i \in [a,b]$ and we observe that T is a contractive mapping when the following inequality holds, indeed, for any $x_1,\ x_2 \in C_\delta^{n-1}[t_i,t_{i+1}]$

$$\begin{aligned}\|Tx_1-Tx_2\|_{C_\delta^{n-1}[t_i,t_{i+1}]} &= \textstyle\sum_{k=0}^{n-1}\left\|\mathcal{J}_{t_i+}^{\alpha-k}\left(h[\tau,x_1(\tau)]-h[\tau,x_2(\tau)]\right)(t)\right\|_{C[t_i,t_{i+1}]}\\ &\le \textstyle\sum_{k=0}^{n-1}\frac{\left(\ln\frac{t_{i+1}}{t_i}\right)^{Re(\alpha)-k}}{Re(\alpha-k)\Gamma(\alpha-k+1)}\|h[t,x_1(t)]-h[t,x_2(t)]\|_{C[t_i,t_{i+1}]}\\ &\le A\textstyle\sum_{k=0}^{n-1}\frac{\left(\ln\frac{t_{i+1}}{t_i}\right)^{Re(\alpha)-k}}{Re(\alpha-k)\Gamma(\alpha-k+1)}\|x_1(t)-x_2(t)\|_{C[t_i,t_{i+1}]}\\ &\le A\textstyle\sum_{k=0}^{n-1}\frac{\left(\ln\frac{t_{i+1}}{t_i}\right)^{Re(\alpha)-k}}{Re(\alpha-k)\Gamma(\alpha-k+1)}\|x_1(t)-x_2(t)\|_{C_\delta^{n-1}[t_i,t_{i+1}]}.\end{aligned}$$

□

4 The Generalized Cauchy type problem

The results in the previous section can be extended to the following equation, which is more general than (8) :

$$\left({}^c\mathcal{D}_{a+}^{\alpha}x\right)(t) = h\left[t,x(t),\left({}^c\mathcal{D}_{a+}^{\alpha_1}x\right)(t),...,\left({}^c\mathcal{D}_{a+}^{\alpha_l}x\right)(t)\right], \tag{19}$$

with $\alpha_j \in (j-1,\ j]$, $j=1,2,...,l$, $\alpha_0 = 0$, and $\left({}^c\mathcal{D}_{a+}^{\alpha_j}\right)$ denotes the Caputo Hadamard operator of order α_j.

The initial conditions for (19) are

$$\left(\delta^k x\right)(a_+) = d_k,\ d_k \in \mathbb{R}\ \ (k=0,...,n-1). \tag{20}$$

For simplicity, we denote by $h[t,\varphi(t,x)]$ instead of $h\left[t,x(t),\left({}^c\mathcal{D}_{a+}^{\alpha_1}x\right)(t),...,\left({}^c\mathcal{D}_{a+}^{\alpha_l}x\right)(t)\right]$.

Similar to the things discussed in the previous, our investigations are based on reducing the problem (19)-(20) to the Volterra equation

$$x(t) = \sum_{j=0}^{n-1}\frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}h[\tau,\varphi(\tau,x)]\frac{d\tau}{\tau},\ (t>a). \tag{21}$$

Theorem 4.1. *Let* $\alpha > 0$, $n = [Re(\alpha)]+1$ *and* $\alpha_j \in \mathbb{C}$ $(j=0,...,l)$ *be such that*

$$0 = Re(\alpha_0) < Re(\alpha_1) < ... < Re(\alpha_l) < n-1. \tag{22}$$

Let $G \in \mathbb{R}^{l+1}$ *be open subsets and let* $h:(a,\ b]\times G \longrightarrow \mathbb{R}$ *be a function such that* $h[t,x,x_1,...,x_l] \in C_{\gamma,\ln}[a,b]$ *for arbitrary* $x,\ x_1,\ ...,\ x_l \in C_{\gamma,\ln}[a,b]$ *and the Lipschitz condition* (6) *is fulfilled.*

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(i) *If* $x \in C^{\alpha,n-1}_{\delta,\gamma,\ln}[a,b]$, *then* x *holds the relations* (19)-(20) *if and only if* x *holds the equation* (21).

(ii) *If* $0 < \alpha < 1$, *then* $x \in C^{\alpha}_{\delta,\gamma,\ln}[a,b]$ *satisfies the relations*

$$\left({}^c\mathcal{D}^{\alpha}_{a_+}x\right)(t) = h\left[t,\varphi(t,x)\right],\ x(a_+) = d_0,\ d_0 \in \mathbb{R}, \tag{23}$$

iff x *satisfies the equation*

$$x(t) = d_0 + \left(\mathcal{J}^{\alpha}_{a_+}\right)h\left[\tau,\varphi(\tau,x)\right](t),\ (t > a). \tag{24}$$

Proof. Let $\alpha \in (n-1,n]$ and $x \in C^{n-1}_{\delta}[a,b]$ satisfies the relations (19)-(20).

(i.a) According to (4) and (19),

$$\left({}^c\mathcal{D}^{\alpha}_{a_+}x\right)(t) = \left(\mathcal{D}^{\alpha}_{a_+}\right)\left[x(\tau) - \sum_{k=0}^{n-1}\frac{\delta^k x(a)}{k!}\left(\ln\frac{\tau}{a}\right)^k\right](t).$$

We have $\left({}^c\mathcal{D}^{\alpha}_{a_+}x\right)(t) \in C_{\gamma,\ln}[a,b]$ and hence

$$\delta^n\mathcal{J}^{n-\alpha}_{a_+}\left(x(\tau) - \sum_{j=0}^{n-1}\frac{\delta^j x(a)}{j!}\left(\ln\frac{\tau}{a}\right)^j\right) \in C_{\gamma,\ln}[a,b].$$

Thus,

$$\mathcal{J}^{n-\alpha}_{a_+}\left(x(\tau) - \sum_{j=0}^{n-1}\frac{\delta^j x(a)}{j!}\left(\ln\frac{\tau}{a}\right)^j\right) \in C^{n}_{\delta,\gamma,\ln}[a,b],$$

and by Lemma 2.4

$$\left(\mathcal{J}^{\alpha}_{a_+}\right)\left({}^c\mathcal{D}^{\alpha}_{a_+}\right)x(t) = x(t) - \sum_{j=1}^{n-1}\frac{\delta^j x(a)}{(j-1)!}\left(\ln\frac{t}{a}\right)^{j-1},$$

Then, from (19), (20) and the last relation, we obtain

$$x(t) = \sum_{j=0}^{n-1}\frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j + \left(\mathcal{J}^{\alpha}_{a_+}\right)h\left[\tau,\varphi(\tau,x)\right](t),\ (t > a).$$

That is $x \in C^{n-1}_{\delta}[a,b]$ satisfy the equation (21).

(i.b) Now we prove the sufficiency. Let $x \in C^{n-1}_{\delta}[a,b]$ satisfies equation (21).

– From (21) we have

$$x(t) - \sum_{j=0}^{n-1}\frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j = \left(\mathcal{J}^{\alpha}_{a_+}\right)h\left[\tau,x(\tau),\left({}^c\mathcal{D}^{\alpha_1}_{a_+}x\right)(\tau),...,\left({}^c\mathcal{D}^{\alpha_l}_{a_+}x\right)(\tau)\right](t).$$

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Applying $\left(\mathcal{D}_{a_+}^{\alpha}\right)$ on both sides of this relation, taking into acount the conditions for h and using Property 2.2, we get

$$\begin{aligned}\left(\mathcal{D}_{a_+}^{\alpha}\right)\left(x(t)-\sum_{j=0}^{n-1}\frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j\right) &= \left(\mathcal{D}_{a_+}^{\alpha}\right)\left(\mathcal{J}_{a_+}^{\alpha}\right)h\left[\tau,\varphi\left(\tau,x\right)\right](t)\\ &= h\left[t,\varphi\left(t,x\right)\right].\end{aligned}$$

By (4), the left hand of the above expression is $\left({}^{c}\mathcal{D}_{a_+}^{\alpha}\right)$ and thus

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha}\right)x\left(t\right)=h\left[t,x\left(t\right),\left({}^{c}\mathcal{D}_{a_+}^{\alpha_1}x\right)\left(t\right),...,\left({}^{c}\mathcal{D}_{a_+}^{\alpha_l}x\right)\left(t\right)\right].$$

Hence $x\in C_{\delta}^{n-1}\left[a,b\right]$ satisfies (19).

– Applying δ^k $(k=0,...,n-1)$ to both sides of (21), we have

$$\delta^k x(t)=\sum_{j=k}^{n-1}\frac{d_j}{(j-k)!}\left(\ln\frac{t}{a}\right)^{j-k}+\left(\delta^k\right)\left(\mathcal{J}_{a_+}^{\alpha}\right)h\left[\tau,\varphi\left(\tau,x\right)\right](t),\ (t>a), \tag{25}$$

Since $x\in C_{\delta}^{n-1}\left[a,b\right]$ for any $\left(\left({}^{c}\mathcal{D}_{a_+}^{\alpha_1}x\right),...,\left({}^{c}\mathcal{D}_{a_+}^{\alpha_l}x\right)\right)\in\mathbb{R}^{n-1}$ and $\alpha-k>\gamma-(n-1)>0$, we have

$$\left(\mathcal{J}_{a_+}^{\alpha-k}\right)h\left[\tau,x\left(\tau\right),\left({}^{c}\mathcal{D}_{a_+}^{\alpha_1}x\right)\left(\tau\right),...,\left({}^{c}\mathcal{D}_{a_+}^{\alpha_l}x\right)\left(\tau\right)\right]\in C\left[a,b\right]. \tag{26}$$

On the other hand, by Lemma 2.3, we let $\tau\longrightarrow a_+$ on the both sides of (25), then we obtain

$$\delta^k x(\tau)\big|_{\tau=a+}=d_k,\ k=0,...,n-1.$$

Hence, x satisfying (21) satisfies the initial condition (20). That is $x\in C_{\delta}^{n-1}\left[a,b\right]$ satisfies the Cauchy problem (19)-(20).

Similarly, we prove the second part of the Theorem. □

Theorem 4.2. *Let $\alpha\in\mathbb{C}$, $n=[Re(\alpha)]+1$, $0\leq\gamma<1$ be such that $\gamma\leq\alpha$. Let $\alpha_j>0$ $(j=1,...,l)$ be such that conditions in (22) are satisfied. Let G be an open set in $\mathbb{R}^{l+1}$ and let $h:(a,b]\times G\longrightarrow\mathbb{R}$ be a function such that $h\left[t,x,x_1,\ ...,x_l\right]\in C_{\gamma,\ln}\left[a,b\right]$ for any $x,x_1,\ ...,x_l\in C_{\gamma,\ln}\left[a,b\right]$ and the Lipschitz condition (6) is fulfilled.*

(i) *If $n-1<\alpha<n$, then there is a unique solution x to the problem (19)-(20) in the space $C_{\delta,\gamma,\ln}^{\alpha,n-1}\left[a,b\right]$.*

(ii) *If $0<\alpha<1$, then there is a unique solution $x\in C_{\delta,\gamma,\ln}^{\alpha}\left[a,b\right]$ to (19) with the condition*

$$x\left(a_+\right)=d_0\in\mathbb{R}.$$

Proof. By Theorem 4.1 it is sufficient to establish the existence of a unique solution $x\in C_{\delta,\gamma,\ln}^{\alpha,n-1}\left[a,b\right]$ to the integral equation (21).

Step 1. First we show that there exists a unique solution $x\in C_{\delta}^{n-1}\left[a,b\right]$.

(a) We choose $t_1 \in]a,b]$, we prove the existence of a unique solution $x \in C_\delta^{n-1}[a,t_1]$, so that the conditions

$$w_1 = \sum_{k=0}^{n-1} \sum_{j=0}^{l} A_j \left(\ln \tfrac{t_1}{a}\right)^{Re(\alpha-\alpha_j)-k} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha-\alpha_j-k)} < 1 \quad \text{if } \gamma \leq \alpha,$$

holds, and apply the Banach fixed point theorem to prove the existence of a unique solution $x \in C_\delta^{n-1}[a,t_1]$ of the integral equation (21).

We rewrite the equation (21) in the form $x(t) = (Tx)(t)$, where

$$(Tx)(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} h[\tau, \varphi(\tau, x)] \frac{d\tau}{\tau},$$

with

$$x_0(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a}\right)^j.$$

It follows that $x_0(t) \in C_\delta^{n-1}[a,t_1]$ because $x_0(t)$ my be further decomposed as a finite sum of functions in $C_\delta^{n-1}[a,t_1]$,

$$h[\tau, \varphi(\tau, x)] \in C_{\gamma,\ln}[a,b] \Longrightarrow h[\tau, \varphi(\tau, x)] \in C_{\gamma,\ln}[a,t_1],$$

and, from Lemma 2.6-(b), we have, using the fact that $\alpha > 0$ and $0 \leq \gamma < 1$,

$$\mathcal{J}_{a_+}^{\alpha} h[\tau, \varphi(\tau, x)] \in C[a,t_1] \text{ if } \gamma \leq \alpha.$$

Let $x \in C_\delta^{n-1}[a,t_1]$, by Lemma 2.7, the integral in the right-hand side of (21) also belongs to $C_\delta^{n-1}[a,t_1]$ i.e., $\mathcal{J}_{a_+}^{\alpha} h[\tau, \varphi(\tau, x)] \in C_\delta^{n-1}[a,t_1]$, and hence $Tx \in C_\delta^{n-1}[a,t_1]$, this proves T is continuous on $C_\delta^{n-1}[a,t_1]$.

To show that T is a contraction we have to prove that, for any $x_1, x_2 \in C_\delta^{n-1}[a,t_1]$ there exists $w_1 > 0$ such that

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a,t_1]} \leq w_1 \|x_1 - x_2\|_{C_\delta^{n-1}[a,t_1]}.$$

By Lipschitzian condition (6), Property 2.2 and Lemma 2.4, thus

$$\left\| \left(\mathcal{J}_{a_+}^{\alpha} \left(h\left[\tau, x_1, {}^c\mathcal{D}_{a_+}^{\alpha_1} x_1, ..., {}^c\mathcal{D}_{a_+}^{\alpha_l} x_1\right] - h\left[\tau, x_2, {}^c\mathcal{D}_{a_+}^{\alpha_1} x_2, ..., {}^c\mathcal{D}_{a_+}^{\alpha_l} x_2\right] \right) \right)(t) \right\|$$

$$\leq \mathcal{J}_{a_+}^{\alpha} \left(\left\| h\left[\tau, x_1, {}^c\mathcal{D}_{a_+}^{\alpha_1} x_1, ..., {}^c\mathcal{D}_{a_+}^{\alpha_l} x_1\right] - h\left[\tau, x_2, {}^c\mathcal{D}_{a_+}^{\alpha_1} x_2, ..., {}^c\mathcal{D}_{a_+}^{\alpha_l} x_2\right] \right\| \right)(t)$$

$$\leq \textstyle\sum_{j=0}^{l} A_j \left\| \left(\mathcal{J}_{a_+}^{\alpha-\alpha_j} \right) \mathcal{J}_{a_+}^{\alpha_j} \left({}^c\mathcal{D}_{a_+}^{\alpha_j} \right)(x_1 - x_2) \right\| (t)$$

$$= \left(\textstyle\sum_{j=0}^{l} A_j \mathcal{J}_{a_+}^{\alpha-\alpha_j} \left\| \mathcal{J}_{a_+}^{\alpha_j} \left({}^c\mathcal{D}_{a_+}^{\alpha_j} \right)(x_1 - x_2) \right\| \right)(t)$$

$$= \left[\left(\textstyle\sum_{j=0}^{l} A_j \mathcal{J}_{a_+}^{\alpha-\alpha_j} \|x_1 - x_2\| \right)(\tau) - \sum_{k_j=0}^{n_j-1} \frac{\delta^{k_j}(x_1-x_2)(a_+)}{k_j!} \left(\ln \tfrac{t}{a}\right)^{k_j} \right].$$

By the hypothesis and Lemma 2.4, $\delta^{k_j} x_1(a_+) = \delta^{k_j}(x_2)(a_+)$, $k_j = 0, ..., n_j - 1$, $n_j = Re(\alpha_j) + 1$, thus

$$\begin{aligned} \left\| \mathcal{J}_{a_+}^{\alpha_j} \left({}^c\mathcal{D}_{a_+}^{\alpha_j} \right)(x_1 - x_2)(t) \right\| &= \left\| (x_1 - x_2)(t) - \sum_{k_j=0}^{n_j-1} \frac{\delta^{k_j}(x_1 - x_2)(a_+)}{k_j!} \left(\ln \frac{t}{a}\right)^{k_j} \right\| \\ &= \|(x_1 - x_2)(t)\| \end{aligned}$$

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for arbitrary $t \in [a, t_1]$. Thus we may continue our estimation above according to

$$\left\| \left(\mathcal{J}_{a+}^{\alpha} \left\{ h\left[\tau, \varphi\left(\tau, x_1\right)\right] - h\left[\tau, \varphi\left(\tau, x_2\right)\right] \right\} \right)(t) \right\| \leq \sum_{j=0}^{l} A_j \left(\mathcal{J}_{a+}^{\alpha-\alpha_j} \left(\left\| x_1 - x_2 \right\| \right) \right)(t). \qquad (27)$$

Moreover by Lemma 2.6-(b), (27) and by (a.ii) in Theorem 3.3 the following holds, indeed, for $x_1,\ x_2 \in C_{\delta}^{n-1}[a, t_1]$

$$\left\| \mathcal{J}_{a+}^{\alpha} \left(h\left[\tau, \varphi\left(\tau, x_1\right)\right] - h\left[\tau, \varphi\left(\tau, x_2\right)\right] \right)(t) \right\|_{C_{\delta}^{n-1}[a,t_1]} \leq \left\| \sum_{k=0}^{n-1} \mathcal{J}_{a+}^{\alpha-k} \left(h\left[\tau, \varphi\left(\tau, x_1\right)\right] - h\left[\tau, \varphi\left(\tau, x_2\right)\right] \right)(t) \right\|_{C_{\delta}[a,t_1]}$$

$$\leq \sum_{k=0}^{n-1} \sum_{j=0}^{l} A_j \left(\ln \frac{t_1}{a} \right)^{Re(\alpha-\alpha_j)-k} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha-\alpha_j-k)} \left\| x_1(t) - x_2(t) \right\|_{C_{\delta}^{n-1}[a,t_1]}.$$

We conclude that mapping T satisfies

$$\left\| Tx_1 - Tx_2 \right\|_{C_{\delta}^{n-1}[a,t_1]} \leq w_1 \left\| x_1 - x_2 \right\|'_{C_{\delta}^{n-1}[a,t_1]}$$

for any functions $x_1,\ x_2 \in C_{\delta}^{n-1}[a, t_1]$. Hence, a unique fixed point in space $C_{\delta}^{n-1}[a, t_1]$ exists and it is explicitly given as a limit of iterations of the mapping T i.e., $\exists x_0^* \in C_{\delta}^{n-1}[a, t_1]$ such that

$$\lim_{m \to +\infty} \left\| x_m(t) - x_0^*(t) \right\|_{C_{\delta}^{n-1}[a,t_1]} = 0,$$

Thus we deduce that a unique solution $x^*(t) \in C_{\delta}^{n-1}[a, b]$ xists such that

$$\lim_{m \to +\infty} \left\| x_m(t) - x^*(t) \right\|_{C_{\delta}^{n-1}[a,b]} = 0,$$

where

$$x_m(t) = T^m x_{0i}^*,\ x_{0i}^*(t) = x_0(t),\ x^*(t) = x_i^*(t),\ i = 0, 1, ..., M,$$

and

$$x_i^*(t_{i+1}) = x_{i+1}^*(t_{i+1}),\ [a, b] = \cup [t_i, t_{i+1}],\ a = t_0 < ... < t_M = b.$$

Step 2. To complete the proof of Theorem 4.2, we show that this unique solution $x(t) = x^*(t) \in C_{\delta}^{n-1}[a, b]$ belongs to the space $C_{\delta,\gamma,\ln}^{\alpha,n-1}[a, b]$. It is sufficient to prove that $\left({}^{c}\mathcal{D}_{a+}^{\alpha} x \right)(t) \in C_{\delta,\gamma,\ln}^{\alpha}[a, b]$. Using the estimate (27), we have

$$\left\| \left({}^{c}\mathcal{D}_{a+}^{\alpha} x_m \right)(t) - \left({}^{c}\mathcal{D}_{a+}^{\alpha} x^* \right)(t) \right\|_{C_{\gamma,\ln}[a,b]} = \left\| h\left[t, \varphi\left(t, x_m\right)\right] - h\left[t, \varphi\left(t, x^*\right)\right] \right\|_{C_{\gamma,\ln}[a,b]}$$

$$\leq \sum_{j=0}^{l} A_j \left\| {}^{c}\mathcal{D}_{a+}^{\alpha_j} \left(x_m(t) - x^*(t) \right) \right\|_{C_{\gamma,\ln}[a,b]}$$

$$\leq \sum_{j=0}^{l} A_j \left\| \mathcal{J}_{a+}^{n-1-\alpha_j} \delta^{n-1} \left(x_m(t) - x^*(t) \right) \right\|_{C_{\gamma,\ln}[a,b]}$$

$$\leq \sum_{j=0}^{l} A_j \left(\ln \frac{b}{a} \right)^{\gamma} \left\| \mathcal{J}_{a+}^{n-1-\alpha_j} \delta^{n-1} \left(x_m(t) - x^*(t) \right) \right\|_{C[a,b]}$$

$$\leq \sum_{j=0}^{l} A_j \frac{\left(\ln \frac{b}{a} \right)^{\gamma}}{Re(n-1-\alpha_j)\Gamma(n-1-\alpha_j)} \left\| \delta^{n-1} \left(x_m(t) - x^*(t) \right) \right\|_{C[a,b]}$$

$$\leq \sum_{j=0}^{l} A_j \frac{\left(\ln \frac{b}{a} \right)^{\gamma}}{Re(n-1-\alpha_j)\Gamma(n-1-\alpha_j)} \left\| x_m(t) - x^*(t) \right\|_{C^{n-1}[a,b]},$$

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It is clear that the right hand side of the above inequality approaches to zero independently. Hence,

$$\lim_{m\to+\infty}\left\|\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x_m\right)(t)-\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x^*\right)(t)\right\|_{C_{\gamma,\ln}[a,b]}=0.$$

Consequently, a unique solution $x^*\in C_{\delta,\gamma,\ln}^{\alpha,n-1}[a,b]$ of equation (21) exists. The second part of the theorem can be proved analogously. □

Corollary 4.3. *Under the hypotheses of Theorem* 4.2 *with* $\gamma=0$. *Then there exists a unique solution* $x^*(t)\in C_{\delta}^{n-1}[a,b]$ *to the Cauchy problem* (19)-(20).

Proof. The above Corollary can be demonstrated in a similar way to that of Theorem 4.2, using the following inequality

$$\left\|\mathcal{J}_{a_+}^{\alpha}\left(h\left[\tau,\varphi\left(\tau,x_1\right)\right]-h\left[\tau,\varphi\left(\tau,x_2\right)\right]\right)(t)\right\|_{C[t_i,t_{i+1}]}$$
$$\leq\sum_{k=0}^{n-1}\sum_{j=0}^{l}A_j\frac{\left(\ln\frac{t_i}{t_{i+1}}\right)^{Re(\alpha-\alpha_j)-k}}{\Re(\alpha-\alpha_j-k)\Gamma(\alpha-\alpha_j-k)}\left\|x_1(t)-x_2(t)\right\|_{C[t_i,t_{i+1}]},$$

for $i=0,1,...,M,\ a=t_0,\ b=t_M$, and

$$\left\|\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x_m\right)(t)-\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x^*\right)(t)\right\|_{C_{\gamma,\ln}[a,b]}\leq$$
$$\sum_{j=0}^{l}A_j\frac{\left(\ln\frac{b}{a}\right)^{\gamma}}{Re\left(n-1-\alpha_j\right)\Gamma\left(n-1-\alpha_j\right)}\left\|x_m(t)-x^*(t)\right\|_{C^{n-1}[a,b]}.$$

□

We can derive the corresponding results for the Cauchy problems for linear fractional equations.

Corollary 4.4. *Let* $\alpha>0,\ n=[Re(\alpha)]+1$ *and* $0\leq\gamma<1$ *be such that* $\alpha\geq\gamma$. *Let* $l\in\mathbb{N},\ \alpha_j>0\ (j=1,...,l)$ *be such that conditions in* (22) *are satisfied and let* $d_j(t)\in C[a,b]$ $(j=1,...,l)$ *and* $f(t)\in C_{\gamma,\ln}[a,b]$.

Then the Cauchy problem for the following linear differential equation of order α

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x\right)(t)+\sum_{j=1}^{l}d_j(t)\left({}^{c}\mathcal{D}_{a_+}^{\alpha_j}x\right)(t)+d_0(t)\,x(t)=f(t)\ \ (t>a),$$

with the initial conditions (9) *has a unique solution* $x(t)$ *in the space* $C_{\delta,\gamma,\ln}^{\alpha,n-1}[a,\ b]$.

In particular, there exists a unique solution $x(t)$ *in the space* $C_{\delta,\gamma,\ln}^{\alpha,n-1}[a,b]$ *to the Cauchy problem for the equation with* $\lambda_j\in\mathbb{R}$ *and* $\beta_j\geqq 0$ $(j=1,...,l)$:

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x\right)(t)+\sum_{j=1}^{l}\lambda_j\left(\ln\frac{t}{a}\right)^{\beta_j}\left({}^{c}\mathcal{D}_{a_+}^{\alpha_j}x\right)(t)+\lambda_0\left(\ln\frac{t}{a}\right)^{\beta_0}x(t)=f(t)\ \ (t>a).$$

Proof. The proof is a direct consequence of Theorem 4.2. □

5 Illustrative Examples

We give here some applications of the above results to Cauchy problems with the Caputo Hadamard derivative.

Example 5.1. *We consider the fractional differential equation of the form*

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x\right)(t)=\lambda\left(\ln\frac{t}{a}\right)^{\beta}[x(t)]^{m};\ t>a>0;\ Re(\alpha)>0,\ m>0;\ m\neq 1, \tag{28}$$

with $\lambda,\ \beta\in\mathbb{R}\ (\lambda\neq 0)$, with the initial conditions

$$\left(\delta^{k}x\right)(a_+)=0,\quad k=0,...,\ n-1. \tag{29}$$

(a) Suppose that the solution has the folowing form:

$$x(t)=c\left(\ln\frac{t}{a}\right)^{\nu},$$

then, this equation has the explicit solution

$$x(t)=\left[\frac{\Gamma(\gamma-\alpha+1)}{\lambda\Gamma(\gamma+1)}\right]^{\frac{1}{(m-1)}}\left(\ln\frac{t}{a}\right)^{\alpha-\gamma},\ \gamma=\frac{(\beta+m\alpha)}{(m-1)}. \tag{30}$$

Moreover, the condition (29) is satisfied.

Hence $x(t)$ is an eigenfunction if both of $\gamma+1$ and $\gamma-\alpha+1$ are not equal to 0 or negative integer. also using Property 2.3 it is easily verified that if the condition

$$\frac{(\beta+\alpha)}{(m-1)}\geq -1, \tag{31}$$

holds, this solution $x(t)$ belongs to $C_{\gamma}[a,b]$ and to $C[a,b]$ in the respective cases $0\leq\alpha$ and $\gamma-\alpha\leq 0$.

$$\begin{aligned} &x(t)\in C_{\gamma}[a,b] \quad \text{if} \quad 0\leq\gamma<1 \text{ and } 0\leq\alpha,\\ &x(t)\in C[a,b] \quad \text{if} \quad \gamma-\alpha\leq 0. \end{aligned} \tag{32}$$

The right-hand side of the equation (28) takes the form

$$h[t,x(t)]=\left[\frac{\Gamma(\gamma-\alpha+1)}{\lambda\Gamma(\gamma+1)}\right]^{\frac{m}{(m-1)}}\left(\ln\frac{t}{a}\right)^{-\gamma}. \tag{33}$$

The function $h[t,x(t)]\in C_{\gamma}[a,b]$ when $0\leq\gamma<1$ and $h[t,x(t)]\in C[a,b]$ when $\gamma\leq 0$

$$\begin{aligned} &h[t,x(t)]\in C_{\gamma}[a,b] \quad \text{if} \quad 0\leq\gamma<1,\\ &h[t,x(t)]\in C[a,b] \quad \text{if} \quad \gamma\leq 0. \end{aligned} \tag{34}$$

In accordance with (31), the following case is possible for the space of the right-hand side (33) and of the solution (30) :

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1. When $\alpha > 0$ and

$$\begin{array}{l} m > 1,\ -m\alpha \leq \beta < m-1-m\alpha,\ \beta \leq -\alpha, \\ \text{or} \\ 0 < m < 1,\ m-1-m\alpha < \beta \leq -m\alpha,\ \beta \geq -\alpha. \end{array}$$

2. If $0 < \alpha < 1$ these conditions take the following forms

$$m > 1,\ -m\alpha \leq \beta \leq -\alpha \text{ or } 0 < m < 1,\ -\alpha \leq \beta \leq -m\alpha. \tag{35}$$

3. If $\alpha \geq 1$ then

$$m > 1,\ -m\alpha \leq \beta < m-1-m\alpha \text{ or } 0 < m < 1,\ m-1-m\alpha < \beta \leq -m\alpha. \tag{36}$$

(b) Now we establish the conditions for the uniqueness of the solution (30) to the above problem (28)-(29). For this we have to choose the domain G and check when the Lipschitz condition (7) with the right-hand side of (28) is valid.

We choose the following domain:

$$G = \left\{(t,\ x) \in \mathbb{R}^2 : 0 < a < t \leq b,\ 0 < x < p\left(\ln\frac{t}{a}\right)^q,\ q \in \mathbb{R},\ p > 0\right\}. \tag{37}$$

To prove the Lipschitz condition (7) with

$$h\left[t, x\left(t\right)\right] = \lambda \left(\ln\frac{t}{a}\right)^{\beta} \left(x\left(t\right)\right)^{m}, \tag{38}$$

we have, for any (t, x_1) $(t, x_2) \in G$:

$$\left|h\left[t, x_1\right] - h\left[t, x_2\right]\right| \leq |\lambda| \left(\ln\frac{t}{a}\right)^{\beta} \left|x_1^m - x_2^m\right|. \tag{39}$$

By definition (37), we have

$$\left|x_1^m - x_2^m\right| < mK\left(\ln\frac{t}{a}\right)^q \left|x_1 - x_2\right|,\ m > 0.$$

Substituting this estimate into (39), we obtain

$$\left|h\left[t, x_1\right] - h\left[t, x_2\right]\right| \leq |\lambda|\, mK\left(\ln\frac{t}{a}\right)^{\beta+(m-1)q} \left|x_1 - x_2\right|.$$

Then the functions $h\left[t, x\left(t\right)\right]$ fulfil the Lipschitizian condition provided that $\beta+(m-1)\,q \geq 0$.

Proposition 5.2. *Let $\lambda, \beta \in \mathbb{R}\,(\lambda \neq 0)$ and $m > 0\ (m \neq 1)$, $\gamma = (\beta + m\alpha) \setminus (m-1)$. Let G be the domain (37), where $q \in \mathbb{R}$ is such that $\beta + (m-1)\,q \geq 0$.*

(i) *Let $0 < \alpha < 1$, if either of the conditions (35) holds, then the Cauchy problem*

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha} x\right)(t) = \lambda \left(\ln\frac{t}{a}\right)^{\beta} \left[x\left(t\right)\right]^m \text{ and } x\left(a_+\right) = 0, \tag{40}$$

has a unique solution $x\left(t\right) \in C_{\delta,\gamma,\ln}^{\alpha}\left[a, b\right]$ and this solution is given by (30).

(ii) *Let $n-1<\alpha<n$ $(n\in\mathbb{N}\setminus\{1\})$, if either of the conditions* (36) *holds, then the problem*

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x\right)(t)=\lambda\left(\ln\frac{t}{a}\right)^{\beta}[x(t)]^{m}\ \text{and}\ \left(\delta^{k}x\right)(a_+)=0,\quad k=0,...,\ n-1,\tag{41}$$

has a unique solution $x(t)\in C_{\delta,\gamma,\ln}^{\alpha,n-1}[a,b]$ and this solution is given by (30).

Remark 5.3. *If $\beta=0,\ 0<Re(\alpha)<1$ then the Lipschitz condition is violated in the domain* (37). *The Cauchy problem* (41) *admits of two continuous solutions $x=0$ and*

$$x(t)=\left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\right]^{\frac{1}{(m-1)}}\left(\ln\frac{t}{a}\right)^{\gamma},\ \gamma=\frac{\alpha}{(1-m)}.$$

Example 5.4. *Let us consider the following problem of order α $(Re(\alpha)>0)$*

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x\right)(t)=\lambda\left(\ln\frac{t}{a}\right)^{\beta}[x(t)]^{m}+c\left(\ln\frac{t}{a}\right)^{\nu},\ \lambda,\ c\in\mathbb{R}\ (\lambda\neq0)\ \text{and}\ \nu,\ \beta\in\mathbb{R}.\tag{42}$$

Then it is verified that the equation (42) has the solution of the form

$$x(t)=\mu\left(\ln\frac{t}{a}\right)^{\gamma},\ \gamma=(\beta+\alpha)\setminus(1-m).\tag{43}$$

In this case the right-hand side of 42 takes the form

$$h[t,x(t)]=(\lambda+c)\left(\ln\frac{t}{a}\right)^{(\beta+\alpha m)\setminus(1-m)}.\tag{44}$$

Using the same arguments as in the proof of Proposition 5.2 we derive the uniqueness result for the Cauchy problem 42.

Proposition 5.5. *Let $\lambda,\beta\in\mathbb{R}\,(\lambda\neq0)$ and $m>0\,(m\neq1)$, $\gamma=(\beta+m\alpha)\setminus(m-1)$. Let G be the domain* (37), *where $q\in\mathbb{R}$ is such that $\beta+(m-1)\,q\geq0$. Let $\nu=-\gamma$ and let the transcendental equation*

$$\Gamma\left(\frac{\alpha+\beta}{1-m}+1-\alpha\right)[\lambda y^{m}+c]-\Gamma\left(\frac{\alpha+\beta}{1-m}+1\right)y=0,$$

have a unique solution $y=\mu$.

(i) *Let $0<\alpha<1$, if either of the conditions* (35) *holds, then the Cauchy problem*

$$\left({}^{c}\mathcal{D}_{a_+}^{\alpha}x\right)(t)=\lambda\left(\ln\frac{t}{a}\right)^{\beta}[x(t)]^{m}+c\left(\ln\frac{t}{a}\right)^{\nu},\ x(a_+)=0,$$

has a unique solution $x(t)\in C_{\delta,\gamma,\ln}^{\alpha}[a,b]$ and this solution is given by (43).

(ii) *Let $n-1<\alpha<n$, if either of the conditions* (36) *holds, then the problem* (42)-(29) *has a unique solution $x(t)\in C_{\delta,\gamma,\ln}^{\alpha,n-1}[a,b]$ and this solution is given by* (43).

Acknowledgments

This work is partially supported by the Scientific and Technical Research Council of Turkey.

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