# Combinatorial Invariants of Toric Arrangements

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# Abstract

An *arrangement* is a collection of subspaces of a topological space. For example, a set of codimension one affine subspaces in a finite dimensional vector space is an arrangement of hyperplanes. A general question in arrangement theory is to determine to what extent the combinatorial data of an arrangement determines the topology of the complement of the arrangement. Established combinatorial structures in this context are matroids and – for hyperplane arrangements in $\mathbb{R}^d$ – oriented matroids.

Let $X$ be $\mathbb{C}^*$ or $S^1$, and $a_1, ..., a_n \in \mathbb{Z}^d$. By interpreting the $a_i$ as characters of the torus $T = \mathrm{Hom}(\mathbb{Z}^d, X) \cong X^d$ we obtain a *toric arrangement* in $T$ by considering the set of kernels of the characters. A toric arrangement $\mathcal{A}$ is covered naturally by a periodic affine hyperplane arrangement $\mathcal{A}^\uparrow$ in $V = \mathbb{C}^d$ or $\mathbb{R}^d$ (according to whether $X = \mathbb{C}^*$ or $S^1$) as seen for example in [30]. Moreover, for $V = \mathbb{R}^d$ the stratification of $V$ given by a finite hyperplane arrangement can be combinatorially characterized by an affine oriented matroid.

Our main objective is to find an abstract combinatorial description for the stratification of $T$ given by the toric arrangement $\mathcal{A}$ in the case $X = S^1$ – and to develop a concept of *toric oriented matroids* as an abstract characterization of arrangements of topological subtori in the compact torus $(S^1)^d$. Part of our motivation comes from the possible generalization of known topological results about the complement of "complexified" toric arrangements [31] to such *toric pseudoarrangements*.

Towards this goal, we study abstract combinatorial descriptions of locally finite hyperplane arrangements and group actions thereon. First, we generalize the theory of semimatroids [1, 63] and geometric semilattices [100] to the case of an infinite ground set, and study their quotients under group actions from an enumerative and structural point of view. As a second step, we consider corresponding generalizations of affine *oriented* matroids in order to characterize the stratification of $\mathbb{R}^d$ given by a locally finite non-central arrangement in $\mathbb{R}^d$ in terms of sign vectors.

## Zusammenfassung

Ein *Arrangement* ist eine Familie von Unterräumen eines topologischen Raums. Ein Arrangement von Hyperebenen ist zum Beispiel gegeben als eine Menge von affinen Unterräume der Kodimension 1 in einem endlich-dimensionalen Vektorraum. Von allgemeinen Interesse in der Theorie von Arrangements ist die Frage, inwieweit die Topologie des Komplements eines Arrangements von den kombinatorischen Daten des Arrangements festgelegt wird. Etablierte kombinatorische Strukuren in disem Kontext sind Matroide und – für Hyperebenenarrangement in $\mathbb{R}^d$ – orientierte Matroide.

Sei $X$ entweder $\mathbb{C}^*$ oder $S^1$ und $a_1, ..., a_n \in \mathbb{Z}^d$. Bei Betrachtung der $a_i$ als Charaktere des Torus $T = \mathrm{Hom}(\mathbb{Z}^d, X) \cong X^d$ erhalten wir ein *torisches Arrangement* in $T$ als die Menge der Niveaumengen der Charaktere. Jedes torische Arrangement $\mathcal{A}$ ist auf natürliche Weise überlagert von einem periodischen affinen Hyperebenenarrangement $\mathcal{A}^\uparrow$ in $V = \mathbb{C}^d$ oder $\mathbb{R}^d$ (abhängig davon, ob $X = \mathbb{C}^*$ oder $S^1$) gemäß unter Anderen [30]. Des Weiteren kann für $V = \mathbb{R}^d$ die von einem endlichen Hyperebenenarrangement gegebene Schichtung von $V$ kombinatorisch durch ein affines orientiertes Matroid beschrieben werden.

Unser vorrangiges Ziel ist es für $X = S^1$ eine kombinatorische Beschreibung der Schichtung von $T$, welche durch ein torisches Arrangement $\mathcal{A}$ gegeben wird, zu finden – und ein Konzept von *torischen orientierten Matroiden* als eine abstrakte Charakterisierung von Arrangements topologischer Untertori im kompakten Torus $(S^1)^d$ zu entwickeln. Teil unserer Motivation ist gegeben durch die daraus entstehende mögliche Verallgemeinerung von bekannten topologischen Resultaten über das Komplement "komplexifizierter" torischer Arrangements [31] auf solche *torischen Pseudoarrangements*.

In Hinblick auf unsere Zielsetzung betrachten wir abstrakte kombinatorische Beschreibungen von Hyperebenenarrangements und studieren Gruppenwirkungen darauf. Als Erstes verallgemeinern wir die Theorie von Semimatroiden [1, 63] und geometrischen Halbverbänden [100] auf den Fall einer unendlichen Grundmenge $S$ und untersuchen deren Quotienten unter Gruppenwirkungen aus enumerativen und strukturellen Gesichtspunkten. Als zweiten Schritt betrachten wir entsprechende Verallgemeinerungen von affinen *orientierten* Matroiden um die durch ein lokal endliches periodisches Hyperebenenarrangement in $\mathbb{R}^d$ gegebene Schichtung von $\mathbb{R}^d$ mithilfe von Zeichenvektoren $Z \in \{+,-,0\}^S$ zu charakterisieren.

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# Contents

# Introduction

Consider an arrangement of hyperplanes in a vector space $V$, that is to say a collection $\mathscr{A} = \{H_i\}_{i\in I}$ of affine subspaces of codimension one in $V$. The arrangement $\mathscr{A}$ determines a stratification of $V$. A general question in arrangement theory is to determine to what extent the combinatorial data of this stratification determines the topology of $\bigcup\mathscr{A}$ and $\mathcal{M}(\mathscr{A}) := V \setminus \bigcup\mathscr{A}$.

Let $X$ be $\mathbb{C}^*$ or $S^1$, and $a_1, ..., a_n \in \mathbb{Z}^d$. By interpreting the $a_i$ as characters of the torus $T = \mathrm{Hom}(\mathbb{Z}^d, X) \cong X^d$ we obtain a *toric arrangement* in $T$ by considering the set of kernels of the characters. A toric arrangement is covered naturally by a periodic affine hyperplane arrangement $\mathcal{A}^\uparrow$ in $V = \mathbb{C}^d$ or $\mathbb{R}^d$ (according to whether $X = \mathbb{C}^*$ or $S^1$) as seen for example in [30]. The toric arrangement can be regarded as the orbit space of $\mathcal{A}^\uparrow$ under a suitable action on the vector space $V$. Recent work of De Concini, Procesi and Vergne [33, 34] on partition functions generated new interest in combinatorial invariants of the topology of the complement of a toric arrangement.

Our main objective is to find suitable abstract descriptions for the combinatorics of a toric arrangement. Towards this goal, we study (abstract combinatorial descriptions of) locally finite hyperplane arrangements and group actions thereon, trying to mimic the case of the action of $\mathbb{Z}^d$ by translations on $\mathcal{A}^\uparrow$.

We approach the objective in two steps. First, we consider locally finite hyperplane arrangements and generalize the concept of *semimatroids* as introduced by Ardila [1] (independently by Kawahara [63]) and geometric semilattices (defined by Wachs and Walker [100]) to the case of an infinite ground set, and group actions thereon. As a second step, we study generalizations of so-called "*affine oriented matroids*" in order to characterize the stratification of $\mathbb{R}^d$ given by a locally finite hyperplane arrangement.

The following will give an outline of the recent progress in the related fields.

**Periodic arrangements:** Some of the first steps in the theory of toric arrangements were made by Lehrer [70] in 1995. De Concini and Procesi generated a wave of new interest in the topic with [32]. Subsequently, progress was made by –among others– the work of Ehrenbourg–Readdy–Slone [43] and Lawrence [68] on enumeration on the torus, the work of De Concini–Procesi–Vergne [33, 34] on partition functions and box splines, and by Moci's work [79, 81] about the topology of the complement. By considering the aforementioned covering relation of $\mathcal{A}$ and the periodic hyperplane arrangement $\mathcal{A}^{\uparrow}$, d'Antonio–Delucchi [30, 31] gave a presentation of $\pi_1(\mathcal{M}(\mathcal{A}))$ and show the minimality of the complement $\mathcal{M}(\mathcal{A})$.

Further motivation for a systematic combinatorial study of periodic structures comes from the considerations on periodic hyperplane arrangements carried out by Kamiya, Takemura and Terao [60, 61], and from the study of complements of arrangements on products of elliptic curves by Bibby [6] which, combinatorially and topologically, can be seen as quotients of "doubly periodic" subspace arrangements.

**Combinatorics:** The combinatorial framework for the theory of hyperplane arrangements given by matroid theory has proved very useful ever since Zaslavsky's work [104] on the partition of a space by hyperplanes. An analogous combinatorial description of finite non-central affine hyperplane arrangements is given by geometric semilattices (Wachs–Walker [100]) and semimatroids (Ardila [1], independently Kawahara [63]). The rising interest in toric arrangements initiated a search for a variation of the concept of matroid that captures the algebraic data of this setting. This gave rise to *arithmetic matroids* (d'Adderio–Moci [28], Brändén–Moci [20]) with an associated *arithmetic Tutte polynomial* [80], and *matroids over rings* (Fink-Moci [45]). Other contexts of application of arithmetic matroids include the theory of spanning trees of simplicial complexes [42] and interpretations in graph theory [29]. Recently, Bruhn, Diestel, Kriesell, Pendavingh and Wollan introduced an axiomatization of infinite matroids in [21] as a generalization of matroids.

The literature on enumerative aspects of group actions is manifold, starting with Pólya's classical work [90] and reaching recent results on polynomial invariants of actions on graphs [24]. The chapter on group actions in Stanley's book [97] offers a survey of some of the results in this vein, together with a sizable literature list. Group actions on (finite) partially ordered sets have been studied from the point of view of representation theory [94] and of the poset's topology [2, 98].

**Topological representation:** A cornerstone in the theory of oriented matroids is the topological representation theorem by Folkman and Lawrence [47] which states that every oriented matroid has a representation by a pseudosphere arrangement and allows us to consider oriented matroids as topological objects. A later proof based on piecewise linear topology is given by Edmonds and Mandel [75]. Bohne and Dress [17, 41] revealed the connection of zonotopal tilings with oriented matroids. Moreover, Bohne [17] introduces the concept "*multiple oriented matroid*" corresponding to (possibly infinite) periodic arrangements. Other combinatorial concepts to describe geometric objects in terms of sign vector systems are given by affine sign vector systems as an analogon of affine oriented matroids (Karlander [62], Baum–Zhu [5]) and conditional oriented matroids (Bandelt, Chepoi and Knauer [3]).

Starting from the classical work of Grünbaum [57, 56] and Ringel [92] about pseudoline arrangements in dimension two, there were several approaches to generalize the theory towards "*pseudoarrangements*" in $\mathbb{R}^d$ [18, 40, 77, 48, 89]. Motivating the development of a combinatorial characterization of locally finite arrangements from a topological point of view.

## Overview

The set up of this text is as follows. In Part I the necessary preliminaries are discussed. Part II deals with group actions on semimatroids and geometric semilattices. Part III is dedicated to generalizations of oriented matroids and their topological representations.

**Part I:** Chapter 0 is intended to set the groundwork for the following and to fix notations.

**Part II:** In Chapter 1 group actions on combinatorial structures are discussed, in particular we consider semimatroids and geometric semilattices on infinite ground sets as an abstract combinatorial description of locally finite affine hyperplane arrangements. $G$-semimatroids are introduced, as a semimatroid together with a group action, which can be thought of as *periodic arrangements*. We study under which conditions a $G$-semimatroid gives rise to an underlying matroid and when they determine an arithmetic matroid. The first example for a natural class of non-realizable arithmetic matroids is given. Furthermore, for every $G$-semimatroid a two-variable polynomial is defined which satisfies a Tutte-Grothendieck recursion and a generalization of Crapo's basis activity decomposition. The results of this chapter are joint work with Emanuele Delucchi [36].

**Part III:** Chapter 2 is devoted to the theory of arrangements. We start with the theory of hyperplane arrangements and toric arrangements. It follows an introduction to piecewise linear topology, pseudosphere arrangements and pseudoline arrangements. We will end with a short survey of the current literature about more general arrangements.

Chapter 3 gives an introduction to the theory of oriented matroids with focus on their geometric aspects. The topological representation of an oriented matroid by a pseudosphere arrangement is given. Moreover, further combinatorial concepts to describe geometric objects in terms of sign vectors are discussed.

Chapter 4 is dedicated to develop a description of locally finite arrangements in terms of sign vectors. Oriented semimatroids are defined as a generalization of affine oriented matroids. We show that every oriented semimatroid possesses an underlying semimatroid and a notion of deletion and contraction. Their relation to given concepts is discussed. Furthermore, we prove a generalization of affine sign vector systems which are analogue to affine oriented matroids (see [62, 5]) in Section 4.4. The Section 4.4 is based on joint work with Emanuele Delucchi and Kolja Knauer [35].

# Part I

# Preliminaries

# Chapter 0

# Basics

In this chapter we will introduce all necessary basics needed to understand Part II and Part III of this thesis. Starting with the combinatorial concepts of partially ordered sets and matroids, ending with the topological concept of cell complexes and subdivisions thereof. The advanced reader may skip this chapter without loss and start immediately with Chapter 1.

## 0.1 Partially ordered sets

First we recall some basics on partially ordered sets, or posets, referring to Stanley's book [95] for a thorough treatment.

**Definition 0.1.1.** *A **partially ordered set** or **poset** (for short) is a set $P$ together with binary relation $\leq$ satisfying*

1. *$x \leq x$ for all $x \in P$;* *(reflexivity)*
2. *if $x \leq y$ and $y \leq x$, then $x = y$;* *(antisymmetry)*
3. *if $x \leq y$ and $y \leq z$, then $x \leq z$.* *(transitivity)*

We use the obvious notation $x \geq y$ to mean $y \leq x$, $x < y$ to mean $x \leq y$ and $x \neq y$, and $x > y$ to mean $y < x$. Two elements $x, y \in P$ are called **comparable** if $x \leq y$ or $y \leq x$, **incomparable** otherwise. An element $y$ **covers** $x$ if $x \leq y$ and there exists no element $z \in P$ such that $x < z < y$. We denote the covering relation by $\lessdot$.

The subposet $[x, y] = \{z \in P : x \leq z \leq y\}$ of $P$ is called an **interval** (defined whenever $x \leq y$). A poset $P$ is **locally finite** if all intervals are finte. The **Hasse diagram** of a poset is a graph whose vertices are the elements of $P$ and the edges correspond to the cover relations in $P$, such that if $x \lessdot y$ then $y$ is drawn "above" $x$ (see Figure 1). A **chain** (or totally ordered set) is a poset in which any two elements are comparable. A chain in $P$ is maximal if it is not contained in any larger chain in $P$. All posets considered here are **chain-finite**, i.e., all chains have finite length.

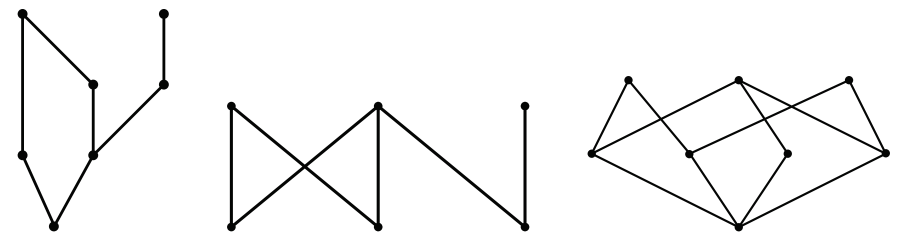


Figure 1: Some examples of posets.

**Definition 0.1.2.** *A* ***meet semilattice*** *is a poset $P$ such that all pairs $\{x,y\}$ in $P$ have a greatest lower bound or* **meet** *$x \wedge y \in P$, i.e., a unique element $x \wedge y \leq x,y$ such that $z \leq x \wedge y$ for any other $z \in P$ with $z \leq x,y$.*

We call a meet semilattice $P$ **complete** if for all subset $A \subseteq P$ there exists a meet $\wedge_{x\in A} x$ or for short $\wedge A$. All chain-finite meet semilattices are complete and have an unique minimal element $\hat{0}$ (see [100]). In this situation an **atom** is any element that covers $\hat{0}$. Dually, in a poset with an unique maximal element $\hat{1}$, the elements covered by $\hat{1}$ are accordingly called **coatom**.

Notice that if a set $A$ of elements of a meet semilattice has an upper bound, then it has a least upper bound $\vee A$ called the **join** of $A$, defined as the meet of the set of upper bounds of $A$.

**Definition 0.1.3.** *A* ***lattice*** *is a poset $P$ such for all pairs $x,y \in P$ there exist meet $x \wedge y$ and join $x \vee y$.*

Clearly, every meet semilattice with a unique maximal element $\hat{1}$ is a lattice. A poset $P$ is **ranked** with rank function $\mathrm{rk} : P \to \mathbb{N}$ if every unrefinable chain from a minimal element to a fixed element $x \in P$ has the same length $\mathrm{rk}(x)$. A poset is **pure** if all its maximal chains have the same length. Furthermore, a poset is called **bounded** if it has a bottom element $\hat{0}$ and a top element $\hat{1}$. If a finite poset is pure and bounded, we will call it **graded**. Note that the definition of a graded poset varies in literature. We use the same definition as Björner at al. in [12]. A set of atoms is called **independent** if its join exists and $\mathrm{rk}(\vee A) = |A|$.

**Definition 0.1.4.** *A* ***geometric lattice*** *is a ranked lattice $P$ which satisfies*

*(i) every element is a join of atoms;* *(atomic)*

*(ii)* $\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y)$ *for all $x,y \in P$.* *(semimodular)*

### 0.1.1 Polynomial invariants

For a poset $P$ let $\mathrm{Int}(P)$ denote the set of intervals in $P$.

**Definition 0.1.5.** *Let $P$ be a locally finite poset, then its **Möbius function** $\mu : \mathrm{Int}(P) \to \mathbb{Z}$ of $P$ is defined recursively by*

*(i)* $\mu(x,x) = 1$ *for all* $x \in P$,

*(ii)* $\sum_{x \leq y \leq z} \mu(x,y) = 0$ *for all* $x < z$ *in* $P$.

If $P$ has a minimal element $\hat{0}$ we write $\mu(x) = \mu(\hat{0}, x)$ for all $x \in P$. An **order ideal** or **down-set** is a subset $I \subseteq P$ such that $x \leq y$ and $y \in I$ implies $x \in I$. Dually an **up-set** or **filter** is a subset $F \subseteq P$ such that $x \leq y$ and $x \in F$ implies $x \in F$. An important application of the Möbius function is the Möbius inversion formula.

**Proposition 0.1.6** (Möbius inversion formula, see [95] Proposition 3.7.1)**.** *Let $P$ be a poset such that for all $x \in P$ the down-set $P_{\leq x}$ is finite and let $f, g : P \to K$, where $K$ is a field. Then the following two conditions are equivalent:*

$$g(x) = \sum_{y \leq x} f(y) \text{ for all } x \in P,$$

*if and only if*

$$f(x) = \sum_{y \leq x} g(y)\mu(y,x) \text{ for all } x \in P.$$

**Definition 0.1.7.** *Let $P$ be a finite ranked poset with $\hat{0}$, say of rank $d$. Define the **characteristic polynomial** $\chi_P(t)$ of $P$ by*

$$\chi_P(t) = \sum_{x \in P} \mu(\hat{0}, x) t^{d - \mathrm{rk}(x)}.$$

## 0.2 Matroids

For an introduction to matroid theory the books of Oxley [88] and Welsh [101] can be recommended. Matroids were developed around 1935 as abstract notion of dependencies trying to capture the common properties of graphs and matrices. Pioneering work was done among others by Whitney [102], Nakasawa [84, 85, 86], Birkhoff [7] and MacLane [74, 72]. A characteristic for matroids is that they can be defined in many different but equivalent ways. An interested reader can find the different axiom systems and the prove of their equivalence in [88, §1]. Furthermore, an important feature of matroid theory is that one can define a concept of duality. However, this notion depends highly on the fact that all considered structures are finite.

In the following, we will use the definition of a matroid via its rank function.

**Definition 0.2.1.** *A finite set $E$ together with a rank function $\mathrm{rk} : 2^E \to \mathbb{N}$ is a **matroid** $M = (E, \mathrm{rk})$ if it satisfies*

*(R1) If* $X \subseteq E$*, then* $0 \leq \mathrm{rk}(X) \leq |X|$.

*(R2) If* $X \subseteq Y \subseteq E$*, then* $\mathrm{rk}(X) \leq \mathrm{rk}(Y)$.

*(R3) If* $X$ *and* $Y$ *are subsets of* $E$*, then*

$$\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y).$$

A finite ground set $E$ together with a rank function $\mathrm{rk} : 2^E \to \mathbb{N}$ satisfying (R2), (R3) and $\mathrm{rk}(\varnothing) = 0$ is called a **polymatroid** (see [101, §18.2]).

For a matroid $M = (E, \mathrm{rk})$ a subset $X \subseteq E$ is called **independent** if $\mathrm{rk}(X) = |X|$ and **dependent** if $\mathrm{rk}(X) < |X|$. A **circuit** $C$ of $M$ is a minimal dependent set, that is to say for any $e \in C$ the set $C - \{e\}$ is independent and $C$ is dependent. A **basis** $B$ is a maximal independent set, that is to say $|B| = \mathrm{rk}(B) = \mathrm{rk}(E)$. The maximal value of the rank function rk, i.e. $\mathrm{rk}(E)$, is the **rank of** $M$, sometimes denoted as $\mathrm{rk}(M)$.

**Example 0.2.2** (Motivating example)**.**

*(1) Linear matroids: represented over a field* $\mathbb{K}$ *by vectors* $\{v_e\}_{e \in E}$

- *independent sets* $\mathcal{I}$ = *linearly independent subsets,*
- *bases* $\mathcal{B}$ = *bases for their span,*
- *circuits* $\mathcal{C}$ = *minimal dependent subsets.*

*(2) Graphic matroids: represented by a (connected) graph* $\mathcal{G} = (V, E)$

- *independent sets* $\mathcal{I}$ = *forests of edges,*
- *bases* $\mathcal{B}$ = *spanning trees,*
- *circuits* $\mathcal{C}$ = *cycles of the graph.*

Let $M = (E, \mathrm{rk})$ be a matroid, $x \in E$ and $X \subseteq E$. We will write $X \cup x$ instead $X \cup \{x\}$ when no confusion can arise. The **closure** of $X$ in $M$ is defined as $\mathrm{cl}_M(X) = \{x \in E : \mathrm{rk}(X \cup x) = \mathrm{rk}(X)\}$. A closed set $X \subseteq E$, that is to say $X = \mathrm{cl}_M(X)$, will be called a **flat** of $M$.

**Definition 0.2.3.** *A **loop** is an element of rank 0, the set of loops will be denoted by* $E_0$*. Two distinct elements* $e, f \in E - E_0$ *are called **parallel** if* $\mathrm{rk}(e, f) = 1$ $(= \mathrm{rk}(e) = \mathrm{rk}(f))$*. A matroid is called **simple** if it contains neither loops or nor parallel elements.*

**Theorem 0.2.4** (See [101], Section 3.3)**.** *A finite lattice is geometric if and only it is a lattice of flats of a matroid. Furthermore, each finite geometric lattice is the poset of flats of a unique simple matroid, up to isomorphism.*

**Definition 0.2.5.** *The **deletion** of $X$ from a matroid $M$ is given as the pair $M\backslash X = (E - X, \mathrm{rk}_{M\backslash X})$ with*

$$\mathrm{rk}_{M\backslash X}(A) = \mathrm{rk}(A)$$

*for $A \subseteq E - X$. The **contraction** of $M$ to $X$ is given as $M/X = (E - X, \mathrm{rk}_{M/X})$ with*

$$\mathrm{rk}_{M/X}(A) = \mathrm{rk}(A \cup X) - \mathrm{rk}(X)$$

*for $A \subseteq E - X$.*

The class of matroids is closed under contraction and deletion and a submatroid of $M$ obtained by a sequence of deletion and contraction will be called a **minor** of $M$. The **restriction** to $X$ is $M[X] := M\backslash(E - X)$.

**Proposition 0.2.6** (See [88], Chapter 2)**.** *For a matroid $M = (E, \mathrm{rk})$, the function $\mathrm{rk}^* : 2^E \to \mathbb{N}$ defined by*

$$\mathrm{rk}^*(X) = \mathrm{rk}(E - X) + |X| - \mathrm{rk}(E)$$

*satisfies (R1), (R2) and (R3). Thus $M^* = (E, \mathrm{rk}^*)$ is a matroid on $E$ and is called the **dual** of $M$. Furthermore, we have $(M^*)^* = M$.*

For the existence of duality in matroid theory the finiteness of the ground set is a crucial factor. Thus, by loosing the definition to an infinite ground set this property is lost if one doesn't adapt the notions of independent sets and bases adequately. Mathematicians struggled for a long time to find a well functioning generalization to an infinite ground set. For a reasonable definition see for example [21].

### 0.2.1 Tutte polynomial

As reference for the progress on Tutte polynomials we refer to Tutte [99], Crapo [26] and Brylawski and Oxley [22].

**Definition 0.2.7.** *Let $M = (E, \mathrm{rk})$ be a matroid than its **Tutte polynomial** is defined as*

$$T_M(x, y) = \sum_{X \subseteq E} (x - 1)^{\mathrm{rk}(E) - \mathrm{rk}(X)} (y - 1)^{|X| - \mathrm{rk}(X)}.$$

Given an order on $E$. Consider a basis $B \in \mathcal{B}$ of $M$. An element $e \in B$ is **internally active** if $e$ is the least element in the unique cocircuit contained in $(E \smallsetminus B) \cup e$. The number of internally active elements of $B$ is called the **internal activity** of $B$ and denoted by $\iota(B)$.

An element $f \in E \smallsetminus B$ is **externally active** if $f$ is the least element in the unique circuit contained in $B \cup f$. The number of externally active elements of $E \smallsetminus B$ is called the **external activity** of $B$ and denoted by $\varepsilon(B)$.

By Crapo's decomposition theorem (see [26]) the Tutte polynomial of a matroid can also be expressed in terms of its basis activities

$$T_M(x, y) = \sum_{B \in \mathcal{B}} x^{\iota(B)} y^{\varepsilon(B)}.$$

### 0.2.2 Matroids with more structure

In the articles [20] by Brändén and Moci and [28] by d'Adderio and Moci the notion of an arithmetic matroid was introduced. It arose from by the motivation to capture the linear algebraic and arithmetic information contained in a finite list of vectors in $\mathbb{Z}^n$ and corresponds to a matroid equipped with a multiplicity function $m$.

If $R \subseteq S \subseteq E$, let $[R,S] = \{A : R \subseteq A \subseteq S\}$ and say $[R,S]$ is a **molecule** if $S$ is the disjoint union $S = R \cup F \cup T$ and for each $A \in [R,S]$ we have

$$\operatorname{rk}(A) = \operatorname{rk}(R) + |A \cap F|.$$

**Definition 0.2.8** (See [20], Section 2)**.** *Let $(M,m)$ be a matroid equipped with a multiplicity function $m : 2^E \to \mathbb{R}$. If $(R,F,T)$ is a molecule, define*

$$\rho(R, R \cup F \cup T) := (-1)^{|T|} \sum_{A \in [R, R \cup F \cup T]} (-1)^{|R \cup F \cup T| - |A|} m(A).$$

*Then $(M,m)$ is **arithmetic** if the following axioms are satisfied:*

*(P) For every molecule $(R,F,T)$,*

$$\rho(R, R \cup F \cup T) \geq 0.$$

*(A1) For all $A \subseteq E$ and $e \in E$:*

*(A.1.1) If $\operatorname{rk}(A \cup e) = \operatorname{rk}(A)$ then $m(A \cup e)$ divides $m(A)$.*

*(A.1.2) If $\operatorname{rk}(A \cup e) > \operatorname{rk}(A)$ then $m(A)$ divides $m(A \cup e)$.*

*(A2) For every molecule $(R,F,T)$*

$$m(R)m(R \cup F \cup T) = m(R \cup F)m(R \cup T).$$

*We use **pseudo-arithmetic** to denote the case where $m$ only satisfies (P).*

We give the general definition and some properties of matroids over rings. Proofs and explanations can be found in [45].

**Definition 0.2.9** (Fink and Moci [45])**.** *Let $E$ be a finite set, $R$ a commutative ring and $M : 2^E \to R$-mod any function associating an $R$-module to each subset of $E$. This defines a **matroid over** $R$ if*

*(R) for any $A \subset E$, $e_1, e_2 \in E$, there is a pushout square*

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \cup \{e_1\}) \\ \downarrow & & \downarrow \\ M(A \cup \{e_2\}) & \longrightarrow & M(A \cup \{e_1, e_2\}) \end{array}$$

*such that all morphisms are surjections with cyclic kernel.*

The definition of a pushout can be found in [58].

**Remark 0.2.10** ([45, Section 6.1])**.** *Any matroid over the ring $R = \mathbb{Z}$ induces an arithmetic matroid on the ground set $E$ with rank function $\mathrm{rk}(A)$ equal the rank of $M(A)$ as a $\mathbb{Z}$-module, and $m(A)$ equal to the cardinality of the torsion part of $M(A)$.*

**Remark 0.2.11** (See Definition 2.2 in [45])**.** *A matroid $M$ over a ring $R$ is called **realizable** if there is a finitely generated $R$-module $N$ and a list $(x_e)_{e\in E}$ of elements of $N$ such that for all $A \subseteq E$ we have that $M(A)$ is isomorphic to the quotient $N/(\sum_{e\in A} Rx_e)$. Realizability is preserved under duality.*

## 0.3 Algebraic Topology

A complete introduction to algebraic topology and the theory of cell complexes may be found in the book [83] by Munkres. As well as from the combinatorial point of view the book of Kozlov [64], focusing on the combinatorial tools used in algebraic topology.

### 0.3.1 Cell complexes

**Definition 0.3.1.** *An (open) $k$-**cell** $\sigma^k$ is a topological space which is homeomorphic to the $k$-dimensional open ball $D^k$. A $0$-cell corresponds to a point.*

Roughly speaking, a cell complex is obtained by inductively glueing $k$-cells of increasing dimension. Begin with the discrete set of points, the 0-cells, then attach the $k$-cells of higher dimension along their boundaries. The construction of a CW complex was introduced by J. H. C. Whitehead.

**Definition 0.3.2.** *A **cell complex** or **CW complex** is a collection $\Delta$ of cells of a Hausdorff space $X$ constructed in the following way:*

- *Start with the discrete set $\Delta^0$ of $0$-cells in $\Delta$, the $0$-skeleton.*
- *Then attach the cells of greater dimension inductively. The $k$-**skeleton** $\Delta^k$ is obtained from $\Delta^{k-1}$ by attaching $k$-cells $\sigma_\alpha^k$ via maps $f_\alpha : S^{k-1} \to \Delta^k$. This means that $\Delta^k$ is the quotient space of the disjoint union $\Delta^{k-1} \sqcup_\alpha D_\alpha^k$ with a collection of $k$-disks $D_\alpha^k$ under the identifications $x \sim f_\alpha(x)$ for all $x \in S^{k-1} = \partial D_\alpha^k$. Hence, the set $X^k = X^{k-1} \sqcup_\alpha \sigma_\alpha^k$ where each $\sigma_\alpha^k$ is an open disk.*
- *Set $\Delta = \Delta^d$, if this process stops after some $d \in \mathbb{N}$. Then $\Delta$ is called $d$-dimensional. Otherwise, set $\Delta = \bigcup_{k\in\mathbb{N}} \Delta^k$ equipped with the weak topology, i.e., $A \subset \Delta$ is open if and only if $A \cap \Delta^k$ is open for all $k$, and call it infinite-dimensional.*

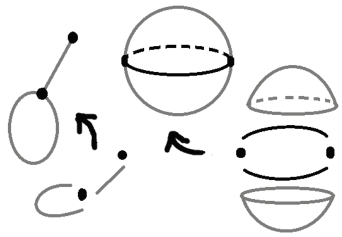


Figure 2: The construction of a cell complex.

A cell complex $\Delta$ is called **regular** if for each cell $\sigma_\alpha^k$ the restriction of the attaching map $f_\alpha : \partial D^k \to f_\alpha(\partial D^k)$ is a homeomorphism.

The space $||\Delta||$ is called the **underlying space**. If the underlying space correlates with the topological space, i.e. $T \cong ||\Delta||$, then the (regular) cell complex $\Delta$ is said to provide a **(regular) cell decomposition** of the space $T$. The **face poset** $\mathcal{F}(\Delta, \leq)$ is the set of closed cells ordered by containment. The **augemented face poset** $\hat{\mathcal{F}}(\Delta) = \mathcal{F}(\Delta) \cup \{\hat{0}, \hat{1}\}$ is enlarged by a minimal and maximal element.

Let $\Delta$ and $\Gamma$ be two regular cell complexes then $\Gamma$ is a **subdivision** of $\Delta$ if $||\Gamma|| = ||\Delta||$ and every closed cell of $\Gamma$ is a subset of some closed cell in $\Delta$.

### 0.3.2 Polyhedral Complexes

A **polytope** is the convex hull of a finite set of points in $\mathbb{R}^d$. A **polyhedron** is an intersection of finitely many closed halfspaces in $\mathbb{R}^d$. Thus, a polytope is a bounded polyhedron.

A **polyhedral complex** (see [107, Definition 5.1]) is a finite set $\mathcal{D}$ of polyhedra in $\mathbb{R}^d$ such that

(i) $\emptyset \in \mathcal{D}$,

(ii) if $P \in \mathcal{D}$, then all faces of $P$ are in $\mathcal{D}$ as well,

(iii) if $P, Q \in \mathcal{D}$, then $P \cap Q$ is a face both of $P$ and $Q$.

The dimension $\dim(\mathcal{D})$ is the largest dimension of a polyhedron in $\mathcal{D}$. The **underlying set** of $\mathcal{D}$ is the set $||\mathcal{D}|| = \bigcup_{P\in\mathcal{D}} P$. As above, a **polyhedral decomposition** of $||\mathcal{D}||$ is given by $\mathcal{D}$. A subset $\mathcal{D}' \subseteq \mathcal{D}$ is a **subcomplex** of $\mathcal{D}$ if it is itself a polyhedral complex. The set $\mathcal{D}$ is a **polytopal complex** if it contains only polytopes.

### 0.3.3 Simplicial complexes

Let $v_0, ..., v_k$ be affinely independent points in the Euclidean space $\mathbb{R}^d$, i.e., they do not lie in an affine subspace of dimension $k-1$. Then the convex hull of these $k+1$ points is a $k$-dimensional polytope, which is called a **$k$-simplex** and the points are the **vertices** of this simplex. Moreover, a non-empty subset of $\{v_0, ..., v_k\}$ spans a subsimplex, which is called **face** of the simplex.

**Definition 0.3.3** (See [83], Section 1.2)**.** *A set of simplices* $\Delta$ *in* $\mathbb{R}^d$ *is a* **(geometric) simplicial complex** *if every face of a simplex in* $\Delta$ *is a simplex in* $\Delta$ *as well and the intersection of any two simplices in* $\Delta$ *is a face of each of them.*

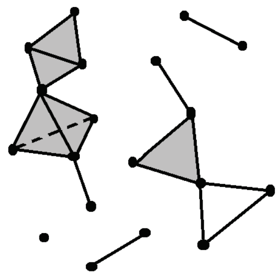


Figure 3: A 3-dimensional simplicial complex.

The subspace $||\Delta||$ of $\mathbb{R}^d$ is called the **underlying space** of $\Delta$. A simplicial complex gives a regular cell decomposition of its underlying space.

But since it is not particular convenient to deal with the specific underlying space all the time and its enough to consider the combinatorial data, we introduce the following notion.

**Definition 0.3.4** (See [83], Section 1.3)**.** *An* ***abstract simplicial complex*** *is a finite set* $S$ *and a collection* $\Delta$ *of subsets of* $S$*, such that if* $\sigma$ *is an element of* $\Delta$*, so is every subset of* $\sigma$*.*

An element $\sigma$ of $\Delta$ is called simplex and its dimension is one less its number of vertices. Calling it $k$**-simplex** if it has dimension $k$. Each non-empty subset of $\sigma$ is called a **face** of $\sigma$. A simplicial complex is **pure** if all its maximal faces have the same dimension. The dimension of a simplicial complex is the largest dimension of its simplicies, if there exists no maximum its dimension is infinite.

Furthermore, we can associate a topological space $||\Delta||$ to an abstract simplicial complex $\Delta$, called its **geometric realization**, which is a (geometric) simplicial complex. In the following when we speak about a simplicial complex we will consider an abstract simplicial complex.

**Definition 0.3.5.** *Let* $P$ *be a poset. The* ***order complex*** $\Delta(P)$ *of* $P$ *is the simplicial complex with the elements of* $P$ *as vertices and the* $k$*-faces corresponding to the* $k$*-dimensional chains in* $P$*.*

## 0.4 Categories

In order to work with categories later on we will introduce them briefly. The interested reader can find more information about categories in [73]. A short but convenient introduction can also be found in [64]. For us the following is sufficient.

A **class** is a collection of sets, not necessarily a set itself. A class that is not a set is called a **proper class**, and a class that is a set is called a **small class**. For instance, the class of all ordinal numbers, and the class of all sets, are proper classes. A **morphism** is a structure-preserving map from one mathematical structure to another.

**Definition 0.4.1** (See [64], Definition 4.1)**.** *A **category** $C$ consist of a class of objects* $\mathrm{ob}(C)$ *and a class of morphisms* $\hom(C)$ *between these objects. The class of morphisms is a disjoint union of sets* $\hom(a,b)$*, for every pair* $a, b \in \mathrm{ob}(C)$*, with a given composition rule*

$$\hom(a,b) \times \hom(b,c) \to \hom(a,c), (m_1, m_2) \mapsto m_2 \circ m_1$$

*satisfying the following axioms:*

- *the composition is associative, when defined;*
- *for each* $a \in \mathrm{ob}(C)$ *there exists a unique identity morphism* $\mathrm{id}_a \in \hom(a,a)$ *such that* $\mathrm{id}_a \circ f = f$ *and* $g \circ \mathrm{id}_a = g$*, whenever the compositions are defined.*

*If the classes are proper sets it is called **small category**.*

A morphism $m$ is called an **inverse** of $\tilde{m}$ if both compositions $m \circ \tilde{m}$ and $\tilde{m} \circ m$ exist, which are then both equal to identity morphisms. A small category is called **acyclic** if only identity morphisms have inverses, and any morphism from an object to itself is an identity.

**Example 0.4.2.** *Some examples for categories are*

*(i)* ***Set****: objects = sets, morphisms = functions;*

*(ii)* ***Grp****: objects = groups, morphisms = group homomorphisms;*

*(iii)* ***Top****: objects = topological spaces, morphisms = continuous maps;*

*(iv)* ***Vect***$_{\mathbb{K}}$*: objects = vector spaces over* $\mathbb{K}$*, morphisms =* $\mathbb{K}$*-linear maps.*

# Part II

# Dependency structures and group actions

# Chapter 1

# Group actions on semimatroids

This chapter is about group actions on combinatorial structures. There is an extensive literature on enumerative aspects of group actions, from Pólya's classical work [90] to, e.g., recent results on polynomial invariants of actions on graphs [24]. The chapter on group actions in Stanley's book [97] offers a survey of some of the results in this vein, together with a sizable literature list. Moreover, group actions on (finite) partially ordered sets have been studied from the point of view of representation theory [94] and of the poset's topology [2, 98].

Here we consider group actions on (possibly infinite) semimatroids and geometric semilattices from a structural perspective. We develop an abstract setting that fits different contexts arising in the literature, allowing us to unify and generalize many recent results.

**Motivation.** Our original motivation came from the desire to better understand the different new combinatorial structures that have been introduced in the wake of recent work of De Concini–Procesi–Vergne [33, 34] on partition functions, and have soon gained independent research interest. Our motivating goals are

– to organize these different structures into a unifying theoretical framework, in particular developing new combinatorial interpretations also in the non-realizable case;

– to understand the geometric side of this theory, in particular in terms of an abstract class of posets (an 'arithmetic' analogue of geometric lattices).

To be more precise, let us consider a list $a_1, \dots, a_n \in \mathbb{Z}^d$ of integer vectors. Such a list gives rise to an *arithmetic matroid* (d'Adderio-Moci [28] and Brändén-Moci [20]) with an associated *arithmetic Tutte polynomial* [80], and a *matroid over the ring* $\mathbb{Z}$ (Fink-Moci [45]). Moreover, by interpreting the $a_i$ as characters of the torus $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{C}^*) \simeq (\mathbb{C}^*)^d$ we obtain a *toric arrangement* in $(S^1)^d \subseteq (\mathbb{C}^*)^d$ defined by the kernels of the characters, with an associated *poset of connected* *components* of intersections of these hypersurfaces. In this case, the arithmetic Tutte polynomial computes the characteristic polynomial of the arrangement's poset and the Poincaré polynomial of the arrangement's complement, as well as the Ehrhart polynomial of the zonotope spanned by the $a_i$ and the dimension of the associated Dahmen-Micchelli space [80].

Other contexts of application of arithmetic matroids include the theory of spanning trees of simplicial complexes [42] and interpretations in graph theory [29].

On an abstract level, arithmetic matroids offer an abstract theory supporting some notable properties of the arithmetic Tutte polynomial, while matroids over rings are a very general and strongly algebraic theory with different applications for suitable choices of the "base ring" (e.g., to tropical geometry for matroids over discrete valuation rings). However, outside the case of lists of integer vectors in abelian groups, the arithmetic Tutte polynomial and arithmetic matroids have few combinatorial interpretations. For instance, the poset of connected components of intersections of a toric arrangement – which provides combinatorial interpretations for many an evaluation of arithmetic Tutte polynomials – has no counterpart in the case of non-realizable arithmetic matroids. Moreover, from a structural point of view it is striking (and unusual for matroidal objects) that there is no known cryptomorphism for arithmetic matroids, while for matroids over a ring a single one was recently presented [46]. In addition, some conceptual relationships between arithmetic matroids (which come in different variants, see [20, 28]) and matroids over rings are not yet cleared.

In research unrelated to arithmetic matroids – e.g. by Ehrenborg, Readdy and Slone [43] and Lawrence [68] on enumeration on the torus, and by Kamiya, Takemura and Terao [60, 61] on characteristic quasipolynomials of affine arrangements – posets and 'multiplicities' related to (but not satisfying the strict requirements of those arising with) arithmetic matroids were brought to light, calling for a systematic study of the abstract properties of "periodic" combinatorial structures.

Further motivation comes from recent progress in the study of complements of arrangements on products of elliptic curves [6] which, combinatorially and topologically, can be seen as quotients of "doubly periodic" subspace arrangements.

**Results.** We initiate the study of actions of groups by automorphisms on semimatroids (for short "$G$-semimatroids"). Helpful intuition comes, once again, from the case of integer vectors, where the associated toric arrangement is covered naturally by a periodic affine hyperplane arrangement: here semimatroids, introduced by Ardila [1] (independently Kawahara [63]), enter the picture as abstract combinatorial descriptions of affine hyperplane arrangements. In particular, we obtain the following results (see also Table 1.1 for a quick overview).

– An equivalence (a.k.a. *cryptomorphism*) between $G$-semimatroids, which are defined in terms of certain set systems, and group actions on geometric semilattices (in the sense of Walker and Wachs [100]), based on a theorem extending Ardila's equivalence between semimatroids and geometric semilattices to the infinite case (Theorem E).

– Under appropriate conditions every $G$-semimatroid gives rise to an "underlying" finite (poly)matroid (Theorem A). Additional conditions can be imposed so that orbit enumeration determines an arithmetic matroid, often non-realizable. In fact, we see that the defining properties of arithmetic matroids arise in a natural 'hierarchy' with stronger conditions on the action (Theorem B and Theorem C).

– In particular, we obtain the first natural class of examples of non-realizable arithmetic matroids.

– To every $G$-semimatroid is naturally associated a poset $\mathcal{P}$ obtained as a quotient of the geometric semilattice of the semimatroid acted upon. In particular, this gives a natural abstract generalization of the poset of connected components of intersections of a toric arrangement.

– To every $G$-semimatroid is associated a two-variable polynomial which evaluates as the characteristic polynomial of $\mathcal{P}$ (Theorem F) and, under mild conditions on the action, satisfies a natural Tutte-Grothendieck recursion (Theorem G) and a generalization of Crapo's basis-activity decomposition (Theorem H). In particular, for every arithmetic matroid arising from group actions we have a new combinatorial interpretation of the coefficients of the arithmetic Tutte polynomial in terms of enumeration on $\mathcal{P}$ subsuming Brändén and Moci's interpretation [20, Theorem 6.3] in the realizable case.

– To every action of a finitely generated abelian group is associated a family of $\mathbb{Z}$-modules, and we can characterize (Theorem D) when this gives rise to a representable matroid over $\mathbb{Z}$.

**Structure of this Chapter.** First, in Section 1.1 we recall the definitions of semimatroids, arithmetic matroids and matroids over a ring. Then we devote Section 1.2 to explaining our guiding example, namely the "realizable" case of a $\mathbb{Z}^d$ action by translations on an affine hyperplane arrangement. Then, Section 1.3 gives a panoramic run-through of the main definitions and results, in order to establish the 'Leitfaden' of our work. Before delving into the technicalities of the proofs, in Section 1.4 we will discuss some specific examples (mostly arising from actions on arrangements of pseudolines) in order to illustrate and distinguish the different concepts we introduce. Then we will move towards proving the announced results. First, in Section 1.5 we prove the cryptomorphism between finitary semimatroids and finitary geometric semilattices. Section 1.6 is devoted to the construction of the underlying (poly)matroid and semimatroid of an action. Then, in Section 1.7 we will focus on *translative* actions (Definition

1.3.1), for which the orbit-counting function gives rise to a *pseudo-arithmetic semimatroid* over the action's underlying semimatroid. Subsequently, in Section 1.8, we will further (but mildly) restrict to *almost-arithmetic* actions, and recover "most of" the properties required in the definition of arithmetic matroids. In Section 1.9 we will then discuss the much more restrictive condition on the action which ensures that our orbit-count function fully satisfies the definition of an arithmetic matroid and, for actions of abelian groups, we will derive a characterization of realizable matroids over $\mathbb{Z}$. The closing Section 1.10 is devoted to the study of certain "Tutte" polynomials associated to $G$-semimatroids.

The results in this chapter are joint work with Emanuele Delucchi. The preprint [36] is available on *ArXiv*.

## 1.1 The main characters

We start by recalling some definitions and results from the literature, modified in order to better fit our setting.

### 1.1.1 Finitary semimatroids

We start by recalling the definition of a semimatroid, which we state without finiteness assumptions on the ground set. This relaxation substantially impacts the theory developed by Ardila [1], much of which rests on the fact that any finite semimatroid can be viewed as a certain substructure of an 'ambient' matroid. Here we list the definition and some immediate observations, while Section 1.5 will be devoted to prove the cryptomorphism with geometric semilattices. We note that equivalent structures were also introduced by Kawahara [63] under the name quasi-matroids with a view on the study of the associated Orlik-Solomon algebra.

**Definition 1.1.1** (Compare [1, Definition 2.1]). *A **finitary semimatroid** is a triple $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk}_{\mathcal{C}})$ consisting of a (possibly infinite) set $S$, a non-empty finite dimensional simplicial complex $\mathcal{C}$ on $S$ and a bounded function $\mathrm{rk}_{\mathcal{C}} : \mathcal{C} \to \mathbb{N}$ satisfying the following conditions.*

*(R1) If $X \in \mathcal{C}$, then $0 \leq \mathrm{rk}_{\mathcal{C}}(X) \leq |X|$.*

*(R2) If $X, Y \in \mathcal{C}$ and $X \subseteq Y$, then $\mathrm{rk}_{\mathcal{C}}(X) \leq \mathrm{rk}_{\mathcal{C}}(Y)$.*

*(R3) If $X, Y \in \mathcal{C}$ and $X \cup Y \in \mathcal{C}$, then $\mathrm{rk}_{\mathcal{C}}(X) + \mathrm{rk}_{\mathcal{C}}(Y) \geq \mathrm{rk}_{\mathcal{C}}(X \cup Y) + \mathrm{rk}_{\mathcal{C}}(X \cap Y)$.*

*(CR1) If $X, Y \in \mathcal{C}$ and $\mathrm{rk}_{\mathcal{C}}(X) = \mathrm{rk}_{\mathcal{C}}(X \cap Y)$, then $X \cup Y \in \mathcal{C}$.*

*(CR2) If $X, Y \in \mathcal{C}$ and $\mathrm{rk}_{\mathcal{C}}(X) < \mathrm{rk}_{\mathcal{C}}(Y)$, then $X \cup y \in \mathcal{C}$ for some $y \in Y - X$.*

*Here and in the following, we will often write* $\mathrm{rk}$ *instead of* $\mathrm{rk}_{\mathcal{C}}$ *and omit braces when representing singleton sets, thus writing* $\mathrm{rk}(x)$ *for* $\mathrm{rk}(\{x\})$ *and* $X \cup x$ *for* $X \cup \{x\}$*, when no confusion can occur.*

We call $S$ the **ground set**, $\mathcal{C}$ the **collection of central sets** and rk the **rank function** of the finitary semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$, respectively. The **rank** of the semimatroid is the maximum value of rk on $\mathcal{C}$ and we will denote it by $\mathrm{rk}(\mathcal{S})$. A set $X \in \mathcal{C}$ is called **independent** if $|X| = \mathrm{rk}(X)$. A **basis** of $\mathcal{S}$ is an inclusion-maximal independent set.

**Remark 1.1.2.** *We adopt the convention that every $x \in S$ is a vertex of $\mathcal{C}$, i.e., $\{x\} \in \mathcal{C}$ for all $x \in S$. Although this is not required in [1], it will not affect our considerations while simplifying the formalism. See also Remark 1.1.13.*

**Definition 1.1.3.** *A finitary semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ is **simple** if $\mathrm{rk}(x) = 1$ for all $x \in S$ and $\mathrm{rk}(x, y) = 2$ for all $\{x, y\} \in \mathcal{C}$ with $x \neq y$. By a finite semimatroid we will mean a finitary semimatroid with a finite ground set.*

**Remark 1.1.4.** *Recall a polymatroid is given by a finite ground set and a rank function $\mathrm{rk} : 2^S \to \mathbb{N}$ satisfying (R2), (R3) and $\mathrm{rk}(\emptyset) = 0$. Polymatroids will appear furtively but naturally in our considerations, and we refer e.g. to [101, §18.2] for a broader account of these structures.*

**Definition 1.1.5.** *We call any $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ satisfying (R1), (R2), (R3) a **locally ranked triple**.*

We now recall some facts about semimatroids for later reference. Except where otherwise specified, the proofs are completely parallel to those given in [1, Section 2].

**Remark 1.1.6.** *A finitary semimatroid satisfies a 'local' version of (R1) and (R2) and a stronger version of (CR1) and (CR2), as well.*

*(R2')* *If $X \cup x \in \mathcal{C}$ then $\mathrm{rk}(X \cup x) - \mathrm{rk}(X)$ equals 0 or 1.*

*(CR1')* *If $X, Y \in \mathcal{C}$ and $\mathrm{rk}(X) = \mathrm{rk}(X \cap Y)$, then $X \cup Y \in \mathcal{C}$ and $\mathrm{rk}(X \cup Y) = \mathrm{rk}(Y)$.*

*(CR2')* *If $X, Y \in \mathcal{C}$ and $\mathrm{rk}(X) < \mathrm{rk}(Y)$, then $X \cup y \in \mathcal{C}$ and $\mathrm{rk}(X \cup y) = \mathrm{rk}(X) + 1$ for some $y \in Y - X$.*

In both [1] and [63] the main motivation for introducing semimatroids is the aim for a combinatorial study of affine hyperplane arrangements. We illustrate this connection in the following example.

**Example 1.1.7** (See Proposition 2.2 in [1])**.** *Given a positive integer $d$ and a field $\mathbb{K}$, an **affine hyperplane** is an affine subspace of dimension $d - 1$ in the vector space $\mathbb{K}^d$ (for more details see Section 2.1.1). A **hyperplane arrangement** in $\mathbb{K}^d$ is a collection $\mathscr{A}$ of affine hyperplanes in $\mathbb{K}^d$. The arrangement is called **locally finite** if every point in $\mathbb{K}^d$ has a neighbourhood that intersects only finitely many hyperplanes of $\mathscr{A}$. A subset $X \subseteq \mathscr{A}$ is **central** if $\cap X \neq \emptyset$. Let $\mathcal{C}_{\mathscr{A}}$ denote the set of central subsets of $\mathscr{A}$ and define the rank function $\mathrm{rk}_{\mathscr{A}} : \mathcal{C}_{\mathscr{A}} \to \mathbb{N}$ as $\mathrm{rk}_{\mathscr{A}}(X) = d - \dim \cap X$.*

*Then, the triple $(\mathscr{A}, \mathcal{C}_{\mathscr{A}}, \mathrm{rk}_{\mathscr{A}})$ is a finitary semimatroid.*

**Example 1.1.8** (Pseudoline arrangements)**.** *There are cases of non-representable semimatroids in which we can still take advantage of a pictorial representation – one such instance is given by **arrangements of pseudolines** which, in the setting e.g. of [57], are sets of homeomorphic images of $\mathbb{R}$ in $\mathbb{R}^2$ ("pseudolines") such that every point of $\mathbb{R}^2$ has a neighborhood intersecting only finitely many pseudolines, and any two pseudolines in the set intersect at most in one point (and if they intersect, they do so transversally). See also Section 2.3.*

*Figure 1.1 shows such an arrangement of pseudolines. The associated triple is $(S, \mathcal{C}, \mathrm{rk})$ with*

$$S = \{a_i \mid i \in \mathbb{Z}\} \cup \{b_i \mid i \in \mathbb{Z}\} \cup \{c_i \mid i \in \mathbb{Z}\} \cup \{d_i \mid i \in \mathbb{Z}\} \cup \{e_i \mid i \in \mathbb{Z}\},$$

$$\begin{aligned}\mathcal{C} = &\{\varnothing\} \cup \{a_i\}_i \cup \{b_i\}_i \cup \{c_i\}_i \cup \{d_i\}_i \cup \{e_i\}_i \cup \{a_i, b_j\}_{i,j} \cup \{a_i, c_j\}_{i,j} \\ &\cup \{a_i, d_j\}_{i,j} \cup \{a_i, e_j\}_{i,j} \cup \{b_i, c_j\}_{i,j} \cup \{b_i, d_j\}_{i,j} \cup \{b_i, e_j\}_{i,j} \cup \{c_i, d_j\}_{i,j} \\ &\cup \{d_i, e_j\}_{i,j} \cup \{a_{2i+k}, b_{2i-k}, c_k\}_{i,k} \cup \{a_{2i+k}, b_{2i-k}, d_k\}_{i,k} \cup \{a_k, b_{k-2i-1}, e_i\}_{i,k} \\ &\cup \{a_{2i+k}, c_k, d_i\}_{i,k} \cup \{b_{2i-k}, c_k, d_i\}_{i,k} \cup \{a_{2i+k}, b_{2i-k}, c_k, d_i\}_{i,k},\end{aligned}$$

$$\mathrm{rk}(X) = \mathrm{codim}(\cap X) \text{ for all } X \in \mathcal{C}$$

*Here and in all following examples we will, for readability's sake, omit to specify that all indices run over $\mathbb{Z}$ and that the union is taken over sets of sets, thus using the shorthand notation $\{a_i, b_j\}_{i,j}$ for $\{\{a_i, b_j\} \mid i, j \in \mathbb{Z}\}$.*

*Notice that this triple cannot be obtained from an arrangement of straight lines.*

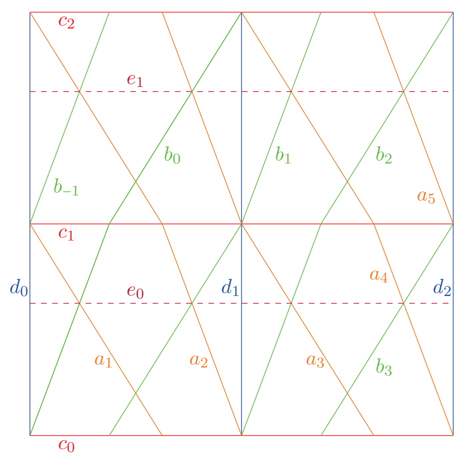



Figure 1.1: A non-stretchable pseudoline arrangement (it should be thought of as repeating and tiling the plane).

**Definition 1.1.9.** *Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be a finitary semimatroid and $X \in \mathcal{C}$. The **closure of** $X$ **in** $\mathcal{C}$ is*

$$\mathrm{cl}(X) := \{x \in S \mid X \cup x \in \mathcal{C}, \mathrm{rk}(X \cup x) = \mathrm{rk}(X)\}.$$

*A **flat** of a finitary semimatroid $\mathcal{S}$ is a set $X \in \mathcal{C}$ such that $\mathrm{cl}(X) = X$. The set of flats of $\mathcal{S}$ ordered by containment forms the **poset of flats of** $\mathcal{S}$, which we denote by $\mathcal{L}(\mathcal{S})$.*

**Remark 1.1.10.** *It is not difficult to show (for example following [1, Section 2]) that for all $X \in \mathcal{C}$ we have $\mathrm{cl}(X) = \max\{Y \supseteq X \mid X \in \mathcal{C}, \mathrm{rk}(X) = \mathrm{rk}(Y)\}$, i.e., the closure of $X$ is the maximal central set containing $X$ and having same rank as $X$. In particular, we have a monotone function $\mathrm{cl} : \mathcal{C} \to \mathcal{C}$.*

**Remark 1.1.11.** *A fundamental result in matroid theory states that a poset is the poset of flats of a matroid if and only if it is a geometric lattice (see Theorem 0.2.4 or [101, Section 3.3]). In Section 1.5 we will prove a similar correspondence between simple finitary semimatroids and geometric semilattices.*

**Definition 1.1.12.** *Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be a locally ranked triple. For every $T \subseteq S$ let $\mathcal{C}_{\setminus T} := \mathcal{C} \cap 2^{S-T}$ and define the **deletion** of $T$ from $\mathcal{S}$ as*

$$\mathcal{S}\backslash T := (S - T, \mathcal{C}_{\setminus T}, \mathrm{rk}),$$

*where we slightly abuse notation and write $\mathrm{rk}$ for $\mathrm{rk}|_{\mathcal{C}_{\setminus T}}$. Moreover, we will denote by $\mathcal{S}[T] := \mathcal{S}\backslash(S - T)$ the **restriction** to $T$.*

*Furthermore, for every central set $X \in \mathcal{C}$ let $\mathcal{C}_{/X} := \{Y \in \mathcal{C}_{\setminus X} \mid Y \cup X \in \mathcal{C}\}$, $S_{/X} := \{s \in S \mid \{s\} \in \mathcal{C}_{/X}\}$ and define the **contraction** of $X$ in $\mathcal{S}$ as*

$$\mathcal{S}/X := (S_{/X}, \mathcal{C}_{/X}, \mathrm{rk}_{/X}),$$

*where, for every $Y \in \mathcal{C}_{/X}$, $\mathrm{rk}_{/X}(Y) = \mathrm{rk}_{\mathcal{C}}(Y \cup X) - \mathrm{rk}_{\mathcal{C}}(X)$.*

**Remark 1.1.13.** *This definition applies in particular to the case where $\mathcal{S}$ is a semimatroid and, in this case, differs slightly from that given in [1]: since we assume every element of the ground set of a semimatroid to be contained in a central set, we need to further constrain the ground set of the contraction.*

**Example 1.1.14.** *Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be the semimatroid of Example 1.1.8, see Figure 1.1. If $T := \{e_i\}_{i \in \mathbb{Z}}$, then*

$$\mathcal{C}_{\setminus T} = \mathcal{C} - (\{e_i\}_i \cup \{a_i, e_j\}_{i,j} \cup \{b_i, e_j\}_{i,j} \cup \{d_i, e_j\}_{i,j} \cup \{a_k, b_{k-2i-1}, e_i\}_{i,k}),$$

*and the semimatroid $\mathcal{S}\backslash T$ is the one associated to the arrangement in Figure 1.2 (left-hand side).*

*The contraction of* $\mathcal{S}$ *to* $e_0 \in S$ *has ground set* $S_{/\{e_0\}} = S - (\{c_i\}_{i\in\mathbb{Z}} \cup \{e_i\}_{i\in\mathbb{Z}})$ *and family of central sets* $\mathcal{C}_{/\{e\}} = \{\varnothing\} \cup \{a_i\}_i \cup \{b_i\}_i \cup \{d_i\}_i \cup \{a_i, b_{i-1}\}_i$ *with rank function* $\mathrm{rk}_{/\{e_0\}}$ *given by*

$$\begin{aligned}
&\mathrm{rk}_{/\{e_0\}}(\varnothing) = \mathrm{rk}(\{e_0\}) - \mathrm{rk}(\{e_0\}) = 0,\\
&\mathrm{rk}_{/\{e_0\}}(\{a_i\}) = \mathrm{rk}(\{a_i, e_0\}) - \mathrm{rk}(\{e_0\}) = 2 - 1 = 1,\\
&\textit{similarly } \mathrm{rk}_{/\{e_0\}}(\{b_i\}) = \mathrm{rk}_{/\{e_0\}}(\{d_i\}) = 1,\\
&\mathrm{rk}_{/\{e_0\}}(\{a_i, b_{i-1}\}) = \mathrm{rk}(\{a_i, b_{i-1}, e_0\}) - \mathrm{rk}(\{e_0\}) = 1.
\end{aligned}$$

*This is represented by the arrangement of points on a line depicted in the right-hand side of Figure 1.2.*

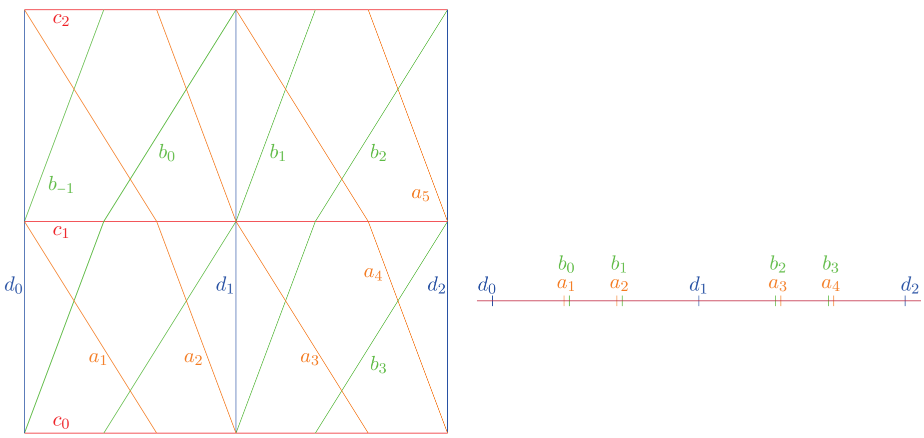



Figure 1.2: Arrangements of pseudolines corresponding to the deletion $\mathcal{S}\backslash\{e_i\}_i$ (l.h.s.), and the contraction $\mathcal{S}/\{e_0\}$ (r.h.s.), where $\mathcal{S}$ is the semimatroid of Example 1.1.8. Again, we show only local pieces of these infinite arrangements, and the pictures must be thought of as being repeated in order to fill the plane (resp. the line).

**Proposition 1.1.15.** *Let* $\mathcal{S} = (S, \mathrm{rk}, \mathcal{C})$ *be a finitary semimatroid. For every* $T \subset S$*,* $\mathcal{S}\backslash T$ *is a finitary semimatroid and, for every* $X \in \mathcal{C}$*,* $\mathcal{S}/X$ *is a finitary semimatroid.*

*Proof.* The proof of [1, Proposition 7.5 and 7.7] adapts straightforwardly. □

**Definition 1.1.16.** *A* ***loop*** *of a locally ranked triple* $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ *is any* $s \in S$ *with* $\mathrm{rk}(s) = 0$*. An* ***isthmus*** *of* $\mathcal{S}$ *is any* $s \in S$ *such that, for every* $X \in \mathcal{C}$*,* $X \cup s \in \mathcal{C}$ *and* $\mathrm{rk}(X \cup s) = \mathrm{rk}(X) + 1$*.*

To every locally ranked triple $(S, \mathcal{C}, \mathrm{rk})$ with a finite ground set $S$ we can associate the following polynomial.

$$T_{\mathcal{S}}(x, y) := \sum_{X \in \mathcal{C}} (x-1)^{\mathrm{rk}(\mathcal{S}) - \mathrm{rk}(X)} (y-1)^{|X| - \mathrm{rk}(X)}$$

**Remark 1.1.17.** *If $\mathcal{S}$ is a finite semimatroid, this is exactly the Tutte polynomial of $\mathcal{S}$ introduced and studied by Ardila [1]. In particular, if $\mathcal{S}$ is a matroid, this is the associated Tutte polynomial.*

One of the first and most famous results about Tutte polynomials of matroids is the following "activities decomposition theorem" first proved by Crapo (for terminology we refer to [88]).

**Proposition 1.1.18** ([26, Theorem 1])**.** *Let $\mathcal{S}$ be a matroid with set of bases $\mathcal{B}$ and fix a total ordering $<$ on $S$. Then,*

$$T_{\mathcal{S}}(x,y) = \sum_{B\in\mathcal{B}} x^{|I(B)|} y^{|E(B)|},$$

*where*

$I(B)$ *is the set of **internally active** elements of $B$, i.e., the set of all $b \in B$ which are $<$-minimal in some codependent subset of $S - (B - b)$.*

$E(B)$ *is the set of **externally active** elements of $B$, i.e., the set of all $e \in S - B$ that are $<$-minimal in some dependent subset of $B \cup e$.*

**Remark 1.1.19.** *One of the major results about arithmetic Tutte polynomials is an analogon to Crapo's theorem for realizable arithmetic matroids (see Remark 1.1.24). One of our results is the generalization of this theorem to all centred translative G-semimatroids (Theorem H).*

### 1.1.2 Arithmetic (semi)matroids and their Tutte polynomials

We extend the Definition 0.2.2 of arithmetic matroids given in [28] and [20] to include the case where the underlying structure is a (finite) semimatroid.

**Definition 1.1.20** (Compare Section 2 of [20])**.** *Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be a locally ranked triple. A **molecule** of $\mathcal{S}$ is any triple $(R, F, T)$ of disjoint sets with $R \cup F \cup T \in \mathcal{C}$ and such that, for every $A$ with $R \subseteq A \subseteq R \cup F \cup T$,*

$$\mathrm{rk}(A) = \mathrm{rk}(R) + |A \cap F|.$$

Here and in the following, given any two sets $X \subseteq Y$ we will denote by $[X, Y] = \{A \subseteq Y \mid X \subseteq A\}$ the interval between $X$ and $Y$ in the boolean poset of subsets of $Y$.

**Remark 1.1.21.** *The notion of basis activities for matroids briefly recapped in Proposition 1.1.18 above allows, once a total ordering of the ground set $S$ is fixed, to associate to every basis $B$ a molecule $(B - I(B), I(B), E(B))$.*

**Definition 1.1.22** (Extending Moci and Brändén [20])**.** *Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be a finite locally ranked triple and $m : \mathcal{C} \to \mathbb{R}$ any function. If $(R, F, T)$ is a molecule, define*

$$\rho(R, R \cup F \cup T) := (-1)^{|T|} \sum_{A \in [R, R \cup F \cup T]} (-1)^{|R \cup F \cup T| - |A|} m(A).$$

*The pair $(\mathcal{S}, m)$ is **arithmetic** if the following axioms are satisfied:*

*(P) For every molecule $(R, F, T)$,*

$$\rho(R, R \cup F \cup T) \geq 0.$$

*(A1) For all $A \subseteq S$ and $e \in S$ with $A \cup e \in \mathcal{C}$:*

*(A.1.1) If $\mathrm{rk}(A \cup \{e\}) = \mathrm{rk}(A)$ then $m(A \cup \{e\})$ divides $m(A)$.*

*(A.1.2) If $\mathrm{rk}(A \cup \{e\}) > \mathrm{rk}(A)$ then $m(A)$ divides $m(A \cup \{e\})$.*

*(A2) For every molecule $(R, F, T)$*

$$m(R) m(R \cup F \cup T) = m(R \cup F) m(R \cup T).$$

*Following [20] we use the expression **pseudo-arithmetic** to denote the case where $m$ only satisfies (P). An **arithmetic matroid** is an arithmetic pair $(\mathcal{S}, m)$ where $\mathcal{S}$ is a matroid.*

**Example 1.1.23.** *To every set of integer vectors, say $a_1, \ldots, a_n \in \mathbb{Z}^d$ is associated a matroid on the ground set $[n] := \{1, \ldots, n\}$ with rank function*

$$\mathrm{rk}(I) := \dim_{\mathbb{Q}}(\mathrm{span}(a_i)_{i \in I}),$$

*and a multiplicity function $m(I)$ defined for every $I \subseteq [n]$ as the greatest common divisor of the minors of the matrix with columns $(a_i)_{i \in I}$. These determine an arithmetic matroid [28]. We say that the vectors $a_i$ **realize** this arithmetic matroid which we call then **realizable**.*

To every arithmetic pair $(\mathcal{S}, m)$ we associate an arithmetic Tutte polynomial as a straightforward generalization of Moci's definition from [80].

$$T_{(\mathcal{S},m)}(x, y) := \sum_{X \in \mathcal{C}} m(X)(x-1)^{\mathrm{rk}(\mathcal{S}) - \mathrm{rk}(X)} (y-1)^{|X| - \mathrm{rk}(X)} \tag{1.1}$$

**Remark 1.1.24.** *When $(\mathcal{S}, m)$ is an arithmetic matroid, the Tutte polynomial $T_{(\mathcal{S},m)}(x, y)$ enjoys a rich structure theory, investigated for instance in [28, 20]. When this arithmetic matroid is realizable, say by a set of vectors $a_1, \ldots, a_n \in \mathbb{Z}^d$, the arithmetic Tutte polynomial specializes e.g. to the characteristic polynomial of the associated toric arrangement (see Section 1.2) and to the Ehrhart polynomial of the zonotope obtained as the Minkowski sum of the $a_i$. Moreover, always in the realizable case, Crapo's decomposition theorem (Proposition 1.1.18) has an analogue [20, Theorem 6.3] which gives a combinatorial interpretation of the coefficients of the polynomial in terms of counting integer points of zonotopes and intersections in the associated toric arrangement.*

### 1.1.3 Matroids over rings

We give the general definition and some properties of matroids over rings. Proofs and explanations can be found in [45].

**Definition 1.1.25** (Fink and Moci [45]). *Let $E$ be a finite set, $R$ a commutative ring and $M : 2^E \to R$-mod any function associating an $R$-module to each subset of $E$. This defines a **matroid over** $R$ if*

*(R) for any $A \subset E$, $e_1, e_2 \in E$ , there is a pushout square*

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \cup \{e_1\}) \\ \downarrow & & \downarrow \\ M(A \cup \{e_2\}) & \longrightarrow & M(A \cup \{e_1, e_2\}) \end{array}$$

*such that all morphisms are surjections with cyclic kernel.*

**Remark 1.1.26** ([45, Section 6.1]). *Any matroid over the ring $R = \mathbb{Z}$ induces an arithmetic matroid on the ground set $E$ with rank function $\mathrm{rk}(A)$ equal the rank of $M(A)$ as a $\mathbb{Z}$-module, and $m(A)$ equal to the cardinality of the torsion part of $M(A)$.*

**Remark 1.1.27** (See Definition 2.2 in [45]). *A matroid $M$ over a ring $R$ is called **realizable** if there is a finitely generated $R$-module $N$ and a list $(x_e)_{e \in E}$ of elements of $N$ such that for all $A \subseteq E$ we have that $M(A)$ is isomorphic to the quotient $N / (\sum_{e \in A} R x_e)$. Realizability is preserved under duality.*

## 1.2 Geometric intuition: Periodic arrangements

As an introductory example we describe the arithmetic matroid and the matroid over $\mathbb{Z}$ associated to periodic real arrangements. Our treatment is designed to highlight the structures we will encounter in the general theory later. Let $\mathbb{K}$ stand for either $\mathbb{R}$ or $\mathbb{C}$.

Recall that an **affine hyperplane arrangement** is a locally finite set $\mathscr{A}$ of hyperplanes in $\mathbb{K}^d$. It is called **periodic** if it is (globally) invariant under the action of a group acting on $\mathbb{K}^d$ by translations.

For simplicity, we will consider the standard action of $\mathbb{Z}^d$, with $k \in \mathbb{Z}^d$ acting as $t_k(x) = x + \sum_i k_i e_i$, where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{K}^d$, and we will suppose the arrangement $\mathscr{A}$ being given by a finite list of integer vectors $a_1, \ldots, a_n \in \mathbb{Z}^d$ (which we arrange as the columns of a matrix $A$) together with a corresponding list $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ of real numbers as follows.

For $X \subseteq [n]$ let $A[X]$ be the $d \times |X|$ matrix obtained restricting $A$ to the relevant columns. For $X \subseteq [n]$ and $k \in \mathbb{Z}^X$ we define

$$H(X, k) := \{x \in \mathbb{K}^d \mid \forall i \in X : a_i^T x = \alpha_i + k_i\}.$$

Then,

$$\mathscr{A} = \{H(\{i\}, j) \mid i \in [n], j \in \mathbb{Z}\}.$$

The poset of intersections of $\mathscr{A}$ is given by the set

$$\mathcal{L}(\mathscr{A}) := \{\cap\mathscr{K} \mid \mathscr{K} \subseteq \mathscr{A}\} - \{\varnothing\}$$

ordered by reverse inclusion. This is a geometric semilattice in the sense of Walker and Wachs [100], see also Definition 1.5.1.

**Remark 1.2.1.** *The* ***toric arrangement*** *associated to* $\mathscr{A}$ *is the set*

$$\underline{\mathscr{A}} := \{H/\mathbb{Z}^d \mid H \in \mathscr{A}/\mathbb{Z}^d\}$$

*of quotients of orbits of the action on* $\mathscr{A}$. *The* ***poset of layers*** *of* $\underline{\mathscr{A}}$ *is the set* $\mathcal{C}(\mathscr{A})$ *of connected components of the intersections of elements of* $\underline{\mathscr{A}}$, *ordered by reverse inclusion. This poset is an important feature of toric arrangements: in the case* $\mathbb{K} = \mathbb{C}$ *we have an arrangement in the complex torus* $\mathbb{C}^d/\mathbb{Z}^d$ *(customarily given as a family of level sets of characters, see Remark 1.3.16 and Section 2.1.2) and* $\mathcal{C}(\underline{\mathscr{A}})$ *encodes much of the homological data about the arrangement's complement (see e.g. [32, 23]). When* $\mathbb{K} = \mathbb{R}$, *this poset is related to enumeration of the induced cell structure on the compact torus* $\mathbb{R}^d/\mathbb{Z}^d \simeq (S^1)^d$ *[68, 43].*

**Remark 1.2.2.** *We see that,* $\mathcal{C}(\underline{\mathscr{A}})$ *is the quotient poset of* $\mathcal{L}(\mathscr{A})$ *(see Definition 1.3.17) under the induced action of* $\mathbb{Z}^d$ *(where the element* $e_l \in \mathbb{Z}^d$ *maps* $H(\{i\}, j)$ *to* $H(\{i\}, j + \langle e_l \mid a_i \rangle)$*).*

For $X \subseteq [n]$ and $k \in \mathbb{Z}^X$ define

$$W(X) := \{k \in \mathbb{Z}^X \mid H(X, k) \neq \varnothing\}.$$

**Remark 1.2.3.** *Notice that* $H(X, k)$ *is the preimage of* $\alpha + k$ *with respect to the linear function* $\mathbb{R}^d \to \mathbb{R}^X$, $x \mapsto A[X]^T x$, *thus* $H(X, k)$ *is connected whenever non-empty .*

We call $\mathscr{A}$ **centred** if $\alpha_i = 0$ for all $i = 1, \ldots, n$ (note that our notion differs from the one used in [87]) and assume this for simplicity throughout this section. Notice that the toric arrangements considered e.g. in [80] can be obtained from actions on centred arrangements.

**Remark 1.2.4.** *If* $\mathscr{A}$ *is centred, then* $W(X) = (A[X]^T \mathbb{R}^d) \cap \mathbb{Z}^X$ *for all* $X \subseteq [n]$, *thus* $W(X)$ *is a pure subgroup (hence a direct summand) of* $\mathbb{Z}^X$.

**Lemma 1.2.5.** *If* $\mathscr{A}$ *is centred, the map* $\varphi_X : k \mapsto H(X, k)$ *is a bijection between* $W(X)$ *and the connected components of* $H(X) := \bigcup_{k \in \mathbb{Z}^X} H(X, k)$.

*Proof.* The map is well-defined and surjective by definition of $W(X)$. It is injective by Remark 1.2.3, as preimages of different elements are disjoint. □

**Remark 1.2.6.** *We say that* $\mathbb{Z}^d$ *acts on* $\mathbb{Z}^{\{i\}}$ *by* $e_l(j) = j + \langle e_l \mid a_i \rangle$ *and, by coordinatewise extension, we obtain an action of* $\mathbb{Z}^d$ *on* $\mathbb{Z}^X$ *for all* $X \subseteq [n]$*. This induces an action of* $\mathbb{Z}^d$ *on* $W(X)$ *which is the action on* $W(X)$ *of its subgroup* $A[X]^T\mathbb{Z}^d$ *by addition and coincides with the 'natural' action described in Remark 1.2.2.*

**Definition 1.2.7.** *For* $X \subseteq [n]$ *let* $I(X) := A[X]^T\mathbb{Z}^d$ *and consider*

$$Z(X) := \mathbb{Z}^X / I(X).$$

**Lemma 1.2.8.** *We have a direct sum decomposition of abelian groups*

$$Z(X) \simeq \mathbb{Z}^\eta \oplus W(X)/I(X),$$

*where* $\eta = |X| - \operatorname{rk} A[X]^T$*, the* ***nullity*** *of* $X$*, is the rank of* $Z(X)$ *as a* $\mathbb{Z}$*-module.*

*Proof.* The decomposition is elementary algebra. For the claim on the rank, notice that both $W(X)$ and $I(X)$ are, by construction, free abelian groups of rank $\operatorname{rk} A[X]^T$, thus the quotient on the right hand side is pure torsion. □

**Remark 1.2.9.** *Arithmetic matroids were introduced by d'Adderio and Moci in [28] in order to study, in the centred case, the combinatorial properties of the rank and multiplicity functions on the subsets of* $[n]$*, where every* $X$ *has* $\operatorname{rk}(X) := \operatorname{rk} A[X]$ *and* $m(X) := [\mathbb{Z}^d \cap A[X]\mathbb{R}^X : A[X]\mathbb{Z}^X]$*. Since, by Remark 1.2.4 and Remark 1.2.6,*

$$|W(X)/I(X)| = [W(X) : I(X)] = [\mathbb{Z}^X \cap A[X]^T\mathbb{R}^d : A[X]^T\mathbb{Z}^d],$$

*classical work of McMullen [76] shows that* $m(X) = |W(X)/I(X)|$*, and we recover in a geometric way the multiplicity function from [28].*

**Remark 1.2.10.** *The function* $\varphi_X$ *induces a (natural) bijection between the elements of* $W(X)/I(X)$ *and the layers of* $H(X)/\mathbb{Z}^d$ *in the toric arrangement* $\underline{\mathscr{A}}$*. This bijection exhibits the enumerative results proved in [28].*

**Remark 1.2.11.** *As proved in [80], the arithmetic Tutte polynomial associated to this arithmetic matroid evaluates to many interesting invariants, for instance to the characteristic polynomial of the poset* $\mathcal{C}(\underline{\mathscr{A}})$ *and, thus, counts the number of chambers of the associated toric arrangement in* $(S^1)^d$*. Moreover, the quotient of the induced action on the complexification of* $\mathscr{A}$ *is an arrangement of subtori in* $(\mathbb{C}^*)^d$*, and the arithmetic Tutte polynomial specializes to the Poincaré polynomial of its complement.*

For $Y \subseteq X \subseteq [n]$ we consider $\mathbb{Z}^Y \subseteq \mathbb{Z}^X$ as an intersection of coordinate subspaces and let $\pi_{X,Y}$ denote the projection of $\mathbb{Z}^X$ onto $\mathbb{Z}^Y$. Since $I(Y) = I(X) \cap \mathbb{Z}^Y$, the map $\pi_{X,Y}$ restricts to a surjection $I(X) \to I(Y)$ and induces a map $\pi_{X,Y} : Z(X) \to Z(Y)$ which, if $|X \smallsetminus Y| = 1$, has cyclic kernel.

**Lemma 1.2.12.** *For $X \subseteq [n]$, $i, j \in X$, the diagram*

$$\begin{array}{ccc} Z(X) & \xrightarrow{\pi_{X,X\smallsetminus i}} & Z(X\smallsetminus i) \\ \downarrow{\scriptstyle \pi_{X,X\smallsetminus j}} & & \downarrow{\scriptstyle \pi_{X\smallsetminus i,X\smallsetminus i,j}} \\ Z(X\smallsetminus j) & \xrightarrow{\pi_{X\smallsetminus j,X\smallsetminus i,j}} & Z(X\smallsetminus\{i,j\}) \end{array}$$

*is a pushout square of epimorphisms with cyclic kernels.*

*Proof.* The square in the claim is obtained as the cokernel of a (mono)morphism of pushout squares of surjections with cyclic kernels and, as such, it is a pushout square of surjections with cyclic kernels. □

**Theorem 1.2.13.** *The assignment $A \mapsto Z([n] \smallsetminus A)$ defines a matroid over $\mathbb{Z}$. It is the dual of the matroid $M_X$ over $\mathbb{Z}$ associated to the list $X := \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d$ in [45].*

*Proof.* The previous lemma shows that this in fact defines a matroid over $\mathbb{Z}$. To see that it is the claimed dual, it is enough to follow the construction of the dual in [45]. □

## 1.3 Overview: setup and main results

Throughout we consider as given a finitary semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ on the ground set $S$ with set of central sets $\mathcal{C}$, rank function $\mathrm{rk} : \mathcal{C} \to \mathbb{N}$ and semilattice of flats $\mathcal{L}$.

Let $G$ be a group acting on $S$. Then, $G$ acts on the power set of $S$ by

$$gX := \{g(x) \mid x \in X\}.$$

Given such an action, for every $X \subseteq S$ we write

$$\underline{X} := \{Gx \mid x \in X\} \subseteq S/G$$

for the set of orbits of elements met by $X$.

### 1.3.1 Group actions on semimatroids

**Definition 1.3.1.** *An **action** of $G$ on a semimatroid $\mathcal{S} := (S, \mathcal{C}, \mathrm{rk})$ is an action of $G$ on $S$, whose induced action on the subsets of $S$ preserves rank and centrality. A $G$-semimatroid $\mathfrak{S} = {}^G(S, \mathcal{C}, \mathrm{rk})$ is a semimatroid together with a $G$-action. We define then*

$$E_{\mathfrak{S}} := S/G; \qquad \mathcal{C}_{\mathfrak{S}} = \mathcal{C}/G; \qquad \underline{\mathcal{C}} := \{\underline{X} \mid X \in \mathcal{C}\}.$$

*We call such an action*

- ***centred*** *if there is a $X \in \mathcal{C}$ with $\underline{X} = E_{\mathfrak{S}}$,*

– ***weakly translative*** *if, for all* $g \in G$ *and all* $x \in S$, $\{x, g(x)\} \in \mathcal{C}$ *implies* $\mathrm{rk}(\{x, g(x)\}) = \mathrm{rk}(x)$.

– ***translative*** *if, for all* $g \in G$ *and all* $x \in S$, $\{x, g(x)\} \in \mathcal{C}$ *implies* $g(x) = x$.

*Moreover, for* $A \subseteq E_{\mathfrak{S}}$ *define*

$$\underline{\mathrm{rk}}(A) := \max\{\mathrm{rk}_{\mathcal{C}}(X) \mid \underline{X} \subseteq A\}$$

*and write* $\mathrm{rk}(\mathfrak{S}) := \underline{\mathrm{rk}}(E_{\mathfrak{S}}) = \mathrm{rk}(\mathcal{S})$ *for the rank of the* $G$*-semimatroid* $\mathfrak{S}$.

**Remark 1.3.2.** *Notice that a translative action is, trivially, weakly translative. Moreover, any weakly translative action on a simple semimatroid is translative.*

**Remark 1.3.3.** *We will sometimes find it useful to consider the set system* $\mathcal{C}_{\mathfrak{S}}$ *as a poset, with the natural order defined by* $GX \leq GY$ *if* $X \subseteq gY$ *for some* $g \in G$ *(notice that this is the poset quotient of* $\mathcal{C}$ *ordered by inclusion).*

**Definition-assumption 1.3.4.** *The action is called* ***cofinite*** *if the set* $\mathcal{C}_{\mathfrak{S}}$ *is finite (in particular, then* $E_{\mathfrak{S}}$ *is finite). We will assume this throughout without further mention.*

**Theorem A.** *Every* $G$*-action on* $\mathcal{S}$ *gives rise to a polymatroid on the ground set* $E_{\mathfrak{S}}$ *with rank function* $\underline{\mathrm{rk}}$ *(see Remark 1.1.4). This polymatroid is a matroid if and only if the action is weakly translative, in which case the triple*

$$\mathcal{S}_{\mathfrak{S}} := (E_{\mathfrak{S}}, \underline{\mathcal{C}}, \underline{\mathrm{rk}})$$

*is locally ranked and satisfies (CR2). The triple* $\mathcal{S}_{\mathfrak{S}}$ *is a matroid if and only if* $\mathfrak{S}$ *is centred.*

*Proof.* The first part of the claim is Proposition 1.6.4. The second part follows from Proposition 1.6.4 and Proposition 1.6.7. □

**Example 1.3.5.** *As an illustration consider the semimatroid* $\mathcal{S}$ *described in Example 1.1.8 (and Figure 1.1) with an action of the group* $\mathbb{Z}^2$ *given by*

$$e_1(a_i) = a_{i+2}, e_1(b_i) = b_{i+2}, e_1(c_i) = c_i, e_1(d_i) = d_{i+1}, e_1(e_i) = e_i$$
$$e_2(a_i) = a_{i+1}, e_2(b_i) = b_{i-1}, e_2(c_i) = c_{i+1}, e_2(d_i) = d_i, e_2(e_i) = e_{i+1}$$

*where* $e_1, e_2$ *is the standard basis of* $\mathbb{Z}^2$.

*This action gives rise to a well-defined* $\mathbb{Z}^2$*-semimatroid* $\mathfrak{S}$*, with*

$$E_{\mathfrak{S}} = \{a, b, c, d, e\}, \ \underline{\mathcal{C}} = 2^{\{a,b,c,d\}} \cup 2^{\{a,b,e\}} \cup 2^{\{e,d\}}$$

*and rank function defined for* $A \subseteq E_{\mathfrak{S}}$ *via* $\underline{\mathrm{rk}}(\emptyset) = 0$, $\underline{\mathrm{rk}}(A) = 1$ *if* $|A| = 1$*, and* $\underline{\mathrm{rk}}(A) = 2$ *else. A pictorial representation of the fundamental region of this action is given in Figure 1.3, and the associated* $\mathcal{C}_{\mathfrak{S}}$ *is shown in Figure 1.4.*

*In this case, $\mathcal{S}_{\mathfrak{S}}$ does not satisfy (CR1). For instance, with $X := \{a, b, c\}$ and $Y := \{a, b, e\}$, we have $X, Y \in \underline{\mathcal{C}}$ with $\underline{\mathrm{rk}}(X \cap Y) = \underline{\mathrm{rk}}(\{a, b\}) = 2 = \underline{\mathrm{rk}}(X)$, but $X \cup Y = \{a, b, c, e\} \notin \underline{\mathcal{C}}$.*

*Notice that $\mathcal{S}_{\mathfrak{S}}$ not being a semimatroid is not a consequence of $\mathfrak{S}$ not being representable. In fact, Figure 1.5 shows that the properties of being representable, centred and $\mathcal{S}_{\mathfrak{S}}$ being a semimatroid can appear in any combination not explicitly covered in Theorem A.*

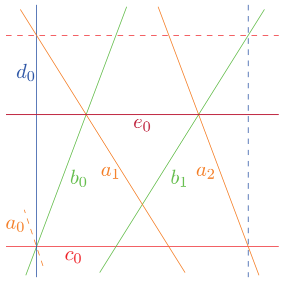



Figure 1.3: A picture of the fundamental region of the $\mathbb{Z}^2$-semimatroid of Example 1.3.5, obtained from the natural action by translations on the pseudoline arrangement of Figure 1.1.

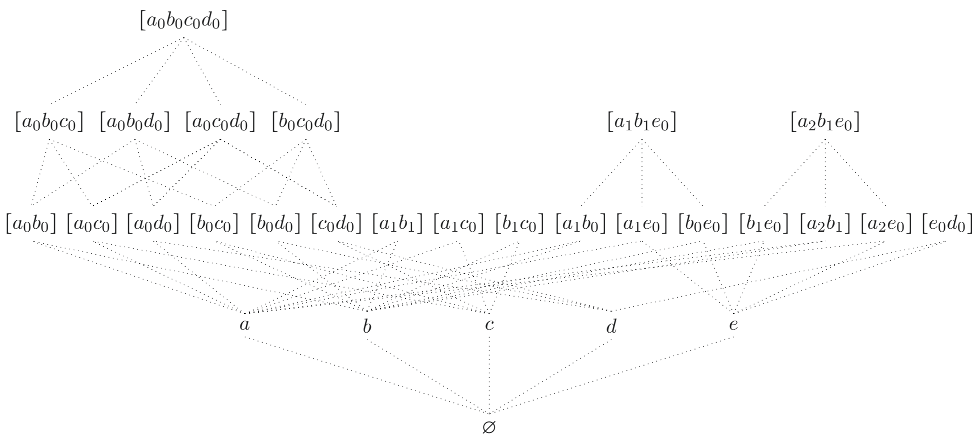



Figure 1.4: The set system $\mathcal{C}_{\mathfrak{S}}$, with dashed lines representing the Hasse diagram of the associated poset. We use shorthand notation, where we write, e.g. $[a_0b_0c_0]$ for the orbit $\mathbb{Z}^2\{a_0, b_0, c_0\}$.

**Definition 1.3.6.** *Let $\mathfrak{S}$ be a G-semimatroid. Given $A \subseteq E_{\mathfrak{S}}$ we define*

$$\lceil A \rceil^{\mathcal{C}} := \{X \in \mathcal{C} \mid \underline{X} = A\} \subseteq \mathcal{C}.$$

*For any given $A \subseteq E_{\mathfrak{S}}$, the set $\lceil A \rceil^{\mathcal{C}}$ carries a natural G-action, and we will be concerned with the study of its orbit set, i.e., the set*

$$\lceil A \rceil^{\mathcal{C}}/G = \{\mathcal{T} \in \mathcal{C}_{\mathfrak{S}} \mid \lfloor \mathcal{T} \rfloor = A\}$$

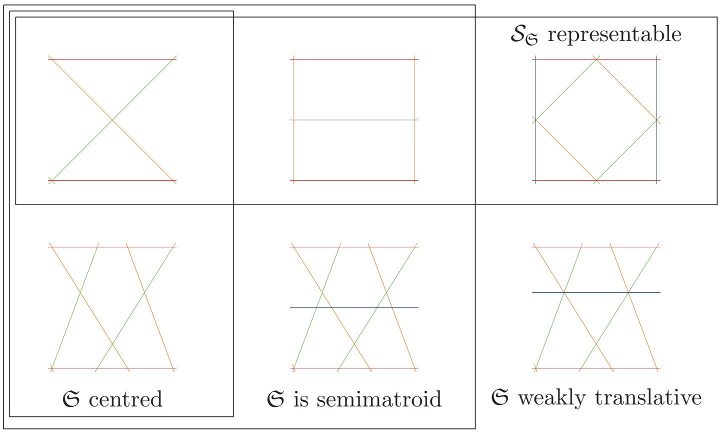



Figure 1.5: This diagram shows examples (arising from pseudoline arrangements) that realize every combination of centred, representable and "$\mathcal{S}_{\mathfrak{S}}$ is semimatroid", within weakly translative actions (with the only constraint that centred actions always afford $\mathcal{S}_{\mathfrak{S}}$ to be semimatroid - indeed in this case $\mathcal{S}_{\mathfrak{S}}$ is a matroid).

*where, for any orbit* $\mathcal{T} = G\{t_1, \dots, t_k\} \in \mathcal{C}_{\mathfrak{S}}$ *we write*

$$\lfloor \mathcal{T} \rfloor := \{Gt_1, \dots, Gt_k\},$$

*so that* $\lfloor \cdot \rfloor$ *defines a map* $\mathcal{C}_{\mathfrak{S}} \to \underline{\mathcal{C}}$. *For every* $A \subseteq E_{\mathfrak{S}}$, *let then*

$$m_{\mathfrak{S}}(A) := |\lceil A \rceil^{\mathcal{C}}/G|.$$

**Remark 1.3.7.** *We illustrate the relationships between the previous definitions by fitting them into a diagram.*

$$\begin{array}{ccccc}
\mathcal{C} & \xrightarrow{\ /G\ } & \mathcal{C}_{\mathfrak{S}} & \xrightarrow{\ \lfloor \cdot \rfloor\ } & \underline{\mathcal{C}} \subseteq 2^{E_{\mathfrak{S}}} \\
\cup\!\!| & & \cup\!\!| & & \cup\!\!| \\
\lceil A \rceil^{\mathcal{C}} & \overset{\textit{preimage of}}{\dashleftarrow} & \lceil A \rceil^{\mathcal{C}}/G & \overset{\textit{preimage of}}{\dashleftarrow} & A \\
\Psi & & \Psi & & \Psi \\
X & \longmapsto & GX & \longmapsto & \underline{X}
\end{array}$$

Obviously, the number $m_{\mathfrak{S}}(A)$ is non-zero if and only if $A \in \underline{\mathcal{C}}$. We will often tacitly consider the restriction of $m_{\mathfrak{S}}$ to its support, which in the cofinite case defines a multiplicity function $m_{\mathfrak{S}} : \underline{\mathcal{C}} \to \mathbb{N}_{>0}$.

**Example 1.3.8.** *We continue our running example and, in the* $\mathbb{Z}^2$*-semimatroid* $\mathfrak{S}$ *of Example 1.3.5, we consider for example the set* $\{a, b\} \in \underline{\mathcal{C}}$. *Then,*

$$\lceil \{a, b\} \rceil^{\mathcal{C}} = \{\{a_i, b_j\} \mid i, j \in \mathbb{Z}\}$$

*and so*

$$\lceil \{a, b\} \rceil^{\mathcal{C}}/\mathbb{Z}^2 = \left\{\mathbb{Z}^2\{a_0, b_0\}, \mathbb{Z}^2\{a_1, b_0\}, \mathbb{Z}^2\{a_1, b_1\}, \mathbb{Z}^2\{a_2, b_1\}\right\},$$

*thus* $m_{\mathfrak{S}}(\{a,b\}) = 4$. *Repeating this procedure for all elements of* $\underline{\mathcal{C}}$ *we obtain the multiplicities written as 'exponents' next to the corresponding sets in Figure 1.6.*

$$\underline{\mathcal{C}} = \left\{ \begin{array}{ccccccccc} & & & & \{a,b,c,d\}^{(1)} & & & & \\ & \{a,b,c\}^{(1)} & & \{a,b,d\}^{(1)} & \{a,c,d\}^{(1)} & \{b,c,d\}^{(1)} & & \{a,b,e\}^{(2)} & \\ \{a,b\}^{(4)} & \{a,c\}^{(2)} & \{b,c\}^{(2)} & \{a,d\}^{(1)} & \{b,c\}^{(1)} & \{b,d\}^{(1)} & \{a,e\}^{(2)} & \{b,e\}^{(2)} & \{e,d\}^{(1)} \\ a^{(1)} & & b^{(1)} & & c^{(1)} & & d^{(1)} & & e^{(1)} \\ & & & & \varnothing^{(1)} & & & & \end{array} \right\}$$

Figure 1.6:

**Definition 1.3.9.** *We call the action of* $G$

- ***normal*** *if, for all* $x \in S$, $\operatorname{stab}(x)$ *is a normal subgroup of* $G$,
- ***almost arithmetic*** *if it is translative and normal.*

**Theorem B.** *If* $\mathfrak{S}$ *is a* $G$*-semimatroid associated to an almost arithmetic action, then the pair* $(\mathcal{S}_{\mathfrak{S}}, m_{\mathfrak{S}})$ *is pseudo-arithmetic. If* $\mathcal{S}_{\mathfrak{S}}$ *is a semimatroid,* $m_{\mathfrak{S}}$ *defines a pseudo-arithmetic semimatroid whose arithmetic Tutte polynomial is* $T_{\mathfrak{S}}(x,y)$ *and satisfies an analogue of Crapo's decomposition formula (Theorem H) generalizing the combinatorial interpretation of [20, Theorem 6.3].*

**Remark 1.3.10.** *If* $\mathfrak{S}$ *is also centred, then* $\mathcal{S}_{\mathfrak{S}}$ *is a matroid, and* $m_{\mathfrak{S}}$ *defines a pseudo-arithmetic matroid on* $E_{\mathfrak{S}}$ *in the sense of [20]. Notice that this way we can produce the first natural class of non-realizable arithmetic matroids (e.g., by the action associated to the pseudoarrangement in Figure 1.3).*

**Notation 1.3.11.** *If the action of* $G$ *is translative, for every* $X \subseteq S$ *we have that* $\operatorname{stab}(X) = \cap_{x \in X} \operatorname{stab}(x)$. *If, moreover, the action is normal, it follows that, for every* $X \in \mathcal{C}$, $\operatorname{stab}(X)$ *is a normal subgroup of* $G$. *We can then define the group*

$$\Gamma(X) := G/\operatorname{stab}(X)$$

*and, for* $g \in G$, *write* $[g]_X := g + \operatorname{stab}(X) \in \Gamma(X)$. *For any* $X \subseteq S$ *consider then the group*

$$\Gamma^X := \prod_{x \in X} \Gamma(x)$$

*and given* $\gamma \in \Gamma^X$ *define*

$$\gamma.X := \{\gamma_{Gx}(x) \mid x \in X\}$$

*which allows us to define a subset*

$$W(X) := \{\gamma \in \Gamma^X \mid \gamma.X \in \mathcal{C}\}$$

*with the natural map*

$$h_X : G \to W(X), \quad h_X(g) = ([g]_x)_{x \in X}.$$

**Example 1.3.12.** *In our running example (from Example 1.1.8 and 1.3.5) we can illustrate the construction of $W(X)$ by taking e.g. $X = \{a_0, b_0, c_0\} \in \mathcal{C}$. We have*

$$\operatorname{stab}(a_0) = \mathbb{Z}\left(\begin{smallmatrix}-1\\2\end{smallmatrix}\right), \quad \operatorname{stab}(b_0) = \mathbb{Z}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right), \quad \operatorname{stab}(c_0) = \mathbb{Z}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$$

*hence*

$$\begin{aligned}
\Gamma(a_0) &= \mathbb{Z}^2/\operatorname{stab}(a_0) = \{\left(\begin{smallmatrix}0\\k\end{smallmatrix}\right) + \operatorname{stab}(a_0) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}\\
\Gamma(b_0) &= \mathbb{Z}^2/\operatorname{stab}(b_0) = \{\left(\begin{smallmatrix}0\\-k\end{smallmatrix}\right) + \operatorname{stab}(b_0) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}\\
\Gamma(c_0) &= \mathbb{Z}^2/\operatorname{stab}(c_0) = \{\left(\begin{smallmatrix}0\\k\end{smallmatrix}\right) + \operatorname{stab}(c_0) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}
\end{aligned}$$

*where we take the isomorphism with $\mathbb{Z}$ to send $k \in \mathbb{Z}$ to the element listed in the braces.*

*Then, $\Gamma^X = \Gamma(a_0) \times \Gamma(b_0) \times \Gamma(c_0) \simeq \mathbb{Z}^3$ and for $\gamma \in \Gamma^X$, say $\gamma = (i, j, l) \in \mathbb{Z}^3$, our choice of the isomorphisms with $\mathbb{Z}$ above implies that*

$$\gamma.\{a_0, b_0, c_0\} = \{a_i, b_j, c_l\}$$

*and thus we see that $\gamma.\{a_0, b_0, c_0\} \in \mathcal{C}$ if and only if $i - l = j + l$ is an even number (compare Example 1.1.8). Therefore*

$$W(X) = \{(2h + l, 2h - l, l) \mid h, l \in \mathbb{Z}\}$$

*is clearly seen to be a subgroup of $\Gamma^X$. We leave it to the reader to check that this applies to every $X$, thus the $\mathbb{Z}^2$-semimatroid $\mathfrak{S}$ is arithmetic (though not centred, neither representable, and $\mathcal{S}_{\mathfrak{S}}$ is not a semimatroid).*

**Definition 1.3.13.** *An almost-arithmetic action is called* ***arithmetic*** *if $W(X)$ is a subgroup of $\Gamma^X$ for all $X \in \mathcal{C}$.*

**Theorem C.** *If $\mathfrak{S}$ is an arithmetic $G$-semimatroid, then the pair $(\mathcal{S}_{\mathfrak{S}}, m_{\mathfrak{S}})$ is arithmetic. If $\mathfrak{S}$ is also centred, then $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ is an arithmetic matroid.*

### 1.3.2 Matroids over $\mathbb{Z}$

**Definition 1.3.14.** *Let $G$ be a finitely generated abelian group. If $\mathfrak{S}$ denotes the $G$-semimatroid associated to a centred (and cofinite) arithmetic action we define, for all $A \subseteq E_{\mathfrak{S}}$, $A^c := E_{\mathfrak{S}} - A$ and*

$$M_{\mathfrak{S}}(A) := \mathbb{Z}^{|A^c| - \underline{\mathrm{rk}}(A^c)} \oplus W(A^c)/h_{A^c}(G).$$

**Remark 1.3.15.** *The modules $M_{\mathfrak{S}}(A)$ are well-defined because, by Lemma 1.9.1, the group $W(X)$ does not depend on the choice of $X \in \lceil \underline{X} \rceil^{\mathcal{C}}$.*

**Theorem D.** *$M_{\mathfrak{S}}$ is a representable matroid over $\mathbb{Z}$ if and only if $W(A)$ is a direct summand of $\Gamma^A$.*

**Remark 1.3.16.** *In general, a toric arrangement in $(\mathbb{C}^*)^d$ is given as a family of level sets of characters of $(\mathbb{C}^*)^d$ (see Section 2.1.2). Of course, by lifting the toric arrangement to the universal covering space of the torus one recovers a periodic affine hyperplane arrangement. The matroid $M(A)$ associated to the action of $\mathbb{Z}^d$ on the intersection semilattice of $\mathcal{L}(\mathscr{A})$ is the dual of the one associated to the characters defining the toric arrangement (see Theorem 1.2.13).*

### 1.3.3 Group actions on finitary geometric semilattices

The main fact that allows us to establish poset-theoretic formulation of the theory of $G$-semimatroids is the following cryptomorphism result between finitary semimatroids and finitary geometric semilattices. Its proof is the object of Section 1.5.

**Theorem E.** *A poset $\mathcal{L}$ is a finitary geometric semilattice if and only if it is isomorphic to the poset of flats of a finitary semimatroid. Furthermore, each finitary geometric semilattice is the poset of flats of an unique simple finitary semimatroid (up to isomorphism).*

We now introduce and discuss some basics about group actions on finitary geometric semilattices.

**Definition 1.3.17.** *An action of $G$ on a geometric semilattice $\mathcal{L}$ is given by a group homomorphism of $G$ in the group of poset automorphisms of $\mathcal{L}$. For simplicity we will identify a group element in $G$ with the automorphism to which it corresponds. We define*

$$\mathcal{P}_{\mathfrak{S}} := \mathcal{L}/G,$$

*the set of orbits of elements of $\mathcal{L}$ ordered such that $GX \leq GY$ if there is $g$ with $X \leq gY$ (this is a common definition of a "quotient poset", e.g. see [98]).*

**Remark 1.3.18.** *If $\mathfrak{S}$ arises from a non-trivial $\mathbb{Z}^2$-action by translations on an arrangement of pseudolines then $\mathcal{P}_{\mathfrak{S}}$ is the poset of layers of the associated pseudoarrangement on the torus.*

**Remark 1.3.19.** *It is clear that every action on a semimatroid induces an action on its semilattice of flats, and every action on a geometric semilattice induces an action on the associated simple semimatroid. It is an exercise to reformulate the requirements of the different kinds of actions in terms of the poset - where, however, the distinction between weakly translative and translative does not show. In our proofs we will mostly use the semimatroid language, in* *order to treat the most general (i.e., non-simple) case, and will call an action on a geometric semilattice* ***cofinite, weakly translative, translative, normal, arithmetic****, etc., if the corresponding* $G$*-semimatroid is.*

**Example 1.3.20.** *The poset* $\mathcal{P}_{\mathfrak{S}}$ *for the* $\mathbb{Z}^2$*-semimatroid of Example 1.3.5 can be read off the picture of the fundamental region in Figure 1.3, and gives the poset depicted in Figure 1.7.*

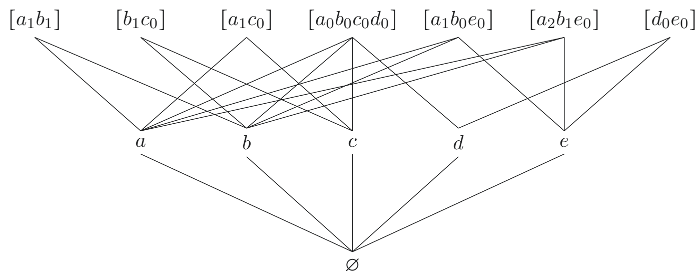



Figure 1.7: The poset $\mathcal{P}_{\mathfrak{S}}$ for the (non-representable) $\mathbb{Z}^2$-semimatroid $\mathfrak{S}$ of our running Example 1.3.5, where we use the same conventions as in Figure 1.4.

In order to highlight the parallelism with the formulation in terms of rank-preserving actions, we state one additional definition.

**Definition 1.3.21.** *Given a* $G$*-semimatroid* $\mathfrak{S}$*, define the following function*

$$\kappa_{\mathfrak{S}} : \mathcal{C}_{\mathfrak{S}} \to \mathcal{P}_{\mathfrak{S}}, \quad GX \mapsto G\,\mathrm{cl}_{\mathcal{C}}(X).$$

The function $\kappa_{\mathfrak{S}}$ is independent from the choice of representatives (since the action is rank-preserving) and thus defines a "closure operator" $\kappa_{\mathfrak{S}} : \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}_{\mathfrak{S}}$ whose closed sets are exactly the elements of $\mathcal{P}_{\mathfrak{S}}$. If we think of $\mathcal{C}_{\mathfrak{S}}$ as a poset with the natural order given by $GX \leq GY$ if there is $g \in G$ with $gX \subseteq Y$ and if we let $\mathcal{C}$ and $\underline{\mathcal{C}}$ be ordered by inclusion, then for every weakly translative $\mathfrak{S}$-semimatroid we have the following commutative diagram of order-preserving functions.

$$\begin{array}{ccccccc} \mathcal{C} & \xrightarrow{/G} & \mathcal{C}_{\mathfrak{S}} & \xrightarrow{\lfloor\cdot\rfloor} & \underline{\mathcal{C}} & \xrightarrow{\subseteq} & 2^{E_{\mathfrak{S}}} \\ \downarrow{\scriptstyle \mathrm{cl}_{\mathcal{C}}} & & \downarrow{\scriptstyle \kappa_{\mathfrak{S}}} & & & & \downarrow{\scriptstyle \underline{\mathrm{cl}}} \\ \mathcal{L} & \xrightarrow{/G} & \mathcal{P}_{\mathfrak{S}} & & \xrightarrow{\underline{\mathrm{cl}}\lfloor\cdot\rfloor} & & \mathcal{L}_0 \end{array}$$

### 1.3.4 Tutte polynomials of group actions

**Definition 1.3.22.** *Given a* $G$*-semimatroid* $\mathfrak{S}$ *we define the following polynomial*

$$T_{\mathfrak{S}}(x,y) := \sum_{A \in \underline{\mathcal{C}}} m_{\mathfrak{S}}(A)(x-1)^{\underline{\mathrm{rk}}(E_{\mathfrak{S}})-\underline{\mathrm{rk}}(A)}(y-1)^{|A|-\underline{\mathrm{rk}}(A)}. \tag{1.2}$$

This definition is natural in its own right, as can be seen in Section 1.10.1 and Section 1.10.2. If the action is centred, so that $\mathcal{S}_{\mathfrak{S}}$ is a matroid, then we recover Equation (1.1) and, in particular, the polynomial $T_{\mathfrak{S}}(x, y)$ generalizes Moci's arithmetic Tutte polynomial [80] (which we recover in the realizable, resp. arithmetic case). The first result we prove is valid in the full generality of weakly translative actions, and concerns the characteristic polynomial of the poset $\mathcal{P}_{\mathfrak{S}}$ (see e.g., [95] for background on characteristic polynomials of posets, and our Section 1.10.1 for the precise definition).

**Theorem F.** *Let $\mathfrak{S}$ be any weakly translative and loopless $G$-semimatroid, and let $\chi_{\mathfrak{S}}(t)$ denote the characteristic polynomial of the poset $\mathcal{P}_{\mathfrak{S}}$. Then,*

$$\chi_{\mathfrak{S}}(t) = (-1)^r T_{\mathfrak{S}}(1-t, 0).$$

**Example 1.3.23.** *For our running example we have (e.g., from Figure 1.6)*

$$\begin{aligned} T_{\mathfrak{S}}(x, y) &= (x-1)^2 + 5(x-1) + 16 + 6(y-1) + (y-1)^2 \\ &= x^2 + y^2 + 3x + 4y + 7 \end{aligned}$$

*and, from Figure 1.7,*

$$\chi_{\mathfrak{S}}(t) = t^2 - 5t + 11.$$

*An elementary computation now verifies Theorem F.*

The polynomials $T_{\mathfrak{S}}(x, y)$ associated to translative actions satisfy a deletion-contraction recursion. To this end, we need to introduce the operations of contraction and deletion for $G$-semimatroids.

**Definition 1.3.24.** *For every $T \subseteq E_{\mathfrak{S}}$, $G$ acts on $\mathcal{S} \backslash \cup T$. We denote the associated $G$-semimatroid by $\mathfrak{S} \backslash T$ and call this the **deletion** of $T$. We follow established matroid terminology and denote by $\mathfrak{S}[T] := \mathfrak{S} \backslash (S - \cup T)$ the **restriction** to $T$.*

**Remark 1.3.25.** *Comparing definitions one readily sees that $\mathcal{S}_{\mathfrak{S}[T]} = \mathcal{S}_{\mathfrak{S}}[T]$ and that, for every $A \subseteq E_{\mathfrak{S}}$, $m_{\mathfrak{S}[T]}(A) = m_{\mathfrak{S}}(A)$.*

**Definition 1.3.26.** *Recall $\mathcal{C}_{\mathfrak{S}} := \mathcal{C}/G$. For all $\mathcal{T} \in \mathcal{C}_{\mathfrak{S}}$ define the **contraction** of $\mathfrak{S}$ to $\mathcal{T}$ by choosing a representative $T \in \mathcal{T}$ and considering the action of* $\mathrm{stab}(T)$ *on the contraction $\mathcal{S}/T$. This defines the* $\mathrm{stab}(T)$*-semimatroid $\mathfrak{S}/\mathcal{T}$.*

Clearly $\mathfrak{S}/\mathcal{T}$ does not depend on the choice of the representative $T \in \mathcal{T}$.

**Remark 1.3.27.** *By Proposition 1.10.6, weak translativity, translativity, normality and atithmeticity of actions are preserved under taking minors.*

**Theorem G.** *Let $\mathfrak{S}$ be a translative $G$-semimatroid and let $e \in E_{\mathfrak{S}}$. Then*

*(1) if $e$ is neither a loop nor an isthmus of $\mathcal{S}_{\mathfrak{S}}$,*

$$T_{\mathfrak{S}}(x, y) = T_{\mathfrak{S}/e}(x, y) + T_{\mathfrak{S} \backslash e}(x, y);$$

*(2) if $e$ is an isthmus, $T_{\mathfrak{S}}(x,y) = (x-1)T_{\mathfrak{S}\backslash e}(x,y) + T_{\mathfrak{S}/e}(x,y)$;*

*(3) if $e$ is a loop, $T_{\mathfrak{S}}(x,y) = T_{\mathfrak{S}\backslash e}(x,y) + (y-1)T_{\mathfrak{S}/e}(x,y)$.*

**Example 1.3.28.** *If $\mathfrak{S}$ is the $\mathbb{Z}^2$-semimatroid of our running example, then $\mathfrak{S}\backslash e$ is given by the induced $\mathbb{Z}^2$-action on the semimatroid $\mathcal{S}\backslash\{e_i\}_{i\in\mathbb{Z}}$ associated to the periodic arrangement of Figure 1.2.(a). Moreover, $\mathfrak{S}/e$ is the $\mathbb{Z}$-semimatroid given by the action of $\mathrm{stab}(e_0) = \mathbb{Z}\binom{1}{0} \simeq \mathbb{Z}$ on the finitary semimatroid associated to the periodic arrangement of Figure 1.2.(b). A picture of the fundamental regions of these two actions is given in Figure 1.8, from which we can compute*

$$\begin{aligned} T_{\mathfrak{S}\backslash e}(x,y) &= (x-1)^2 + 4(x-1) + 11 + 4(y-1) + (y-1)^2 \\ &= x^2 + y^2 + 2x + 2y + 5 \\ T_{\mathfrak{S}/e}(x,y) &= (x-1) + 5 + 2(y-1) = x + 2y + 2 \end{aligned}$$

*and easily verify that the sum of these polynomials equals $T_{\mathfrak{S}}(x,y) = x^2 + y^2 + 3x + 4y + 7$ (Example 1.3.23).*

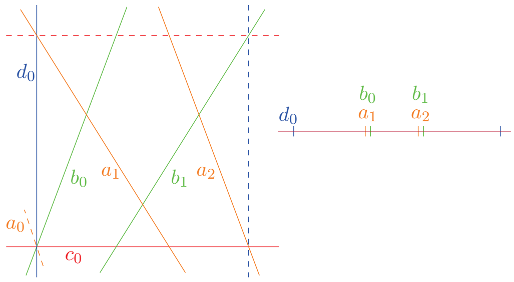



Figure 1.8: Fundamental regions of $\mathfrak{S}\backslash e$ and $\mathfrak{S}/e$ given by group actions (as described above).

## 1.4 Some examples

**Example 1.4.1** (Reflection groups)**.** *Let $G$ be a finite or affine complex reflection group acting on the intersection poset of its reflection arrangement. This setting has been considered extensively, especially in the finite case (see e.g. the treatment of Orlik and Terao [87]). These actions are not translative, and thus fall at the margins of our present treatment. Still, we would like to mention them as a token of the fact that further investigation of non-translative actions - e.g., the case where $(E, \underline{\mathrm{rk}})$ is a polymatroid - is clearly warranted, for instance in order to characterize a class of posets associated to representations of finite groups that were recently used in computations of Motivic classes [37].*

**Example 1.4.2** (Toric arrangements)**.** *The natural setting in order to develop a combinatorial framework for toric arrangements is that of $\mathbb{Z}^d$ acting on an affine hyperplane arrangement on $\mathbb{C}^d$ (see Section 1.2). Such actions will often*

| $G$-semimatroid $\mathfrak{S}$ | Loc. ranked triple $\mathcal{S}_{\mathfrak{S}}$ | Multiplicity $m_{\mathfrak{S}}$ | Poset $\mathcal{P}_{\mathfrak{S}}$ | Polynomial $T_{\mathfrak{S}}(x,y)$ | Modules $M_{\mathfrak{S}}$ |
|---|---|---|---|---|---|
| Weakly translative | well-defined (Theorem A) | | $\chi_{\mathcal{P}_{\mathfrak{S}}}(t) = (-1)^r T_{\mathfrak{S}}(1-t,0)$ (Theorem F) | | |
| Translative | Pseudo-arithmetic (Proposition 1.7.19) | | | Tutte-Grothendieck recursion (Theorem G) | |
| Translative and normal | Almost-arithmetic (P, A.1.2, A2) (Theorem B) | | | | |
| ...and $\mathcal{S}_{\mathfrak{S}}$ a semimatroid | | | | Activity decomposition (Theorem H) | |
| Arithmetic | Arithmetic (Theorem C) | | | | |
| centred | Matroid | | | | |
| Realizable and centred | Arithmetic matroid dual to that of [20] | | Poset of layers of toric arrangement | Arithmetic Tutte polynomial | Representable matroid over $\mathbb{Z}$ |

Table 1.1: A tabular overview of our setup and our results

*fail to be centred unless the toric arrangement at hand is defined by kernels of characters. Therefore we will try to state our results as much as possible without centrality assumptions, adding them only when needed in order to establish a link to the arithmetic and algebraic matroidal structures appeared in the literature.*

The next examples will refer to the pictures of Figure 1.9. These are to be interpreted as the depiction of a fundamental region for an action of $\mathbb{Z}^2$ by unit translations in orthogonal directions (vertical and horizontal) on an arrangement of pseudolines in $\mathbb{R}^2$ (e.g. in the sense of Grünbaum [57], see also Section 2.3) which, then, can be recovered by 'tiling' the plane by translates of the depicted squares. Notice that the intersection poset of any arrangement of pseudolines is trivially a geometric semilattice, and thus defines a simple semimatroid. We will call $a$, $b$, $c$, $d$ the orbits of the respective colors.

**Example 1.4.3.** *The $\mathbb{Z}^2$-semimatroid described in Figure 1.9 is clearly almost-arithmetic, but cannot be arithmetic, because the multiplicity $m_{\mathfrak{S}}(\{c,b,a\}) = 3$ does not divide $m_{\mathfrak{S}}(\{c,a\}) = 4$, violating (A.1.1).*

**Example 1.4.4.** *One readily verifies that the $\mathbb{Z}^2$-semimatroid described at the left-hand side of Figure 1.8 is arithmetic. However, $M_{\mathfrak{S}}$ is not a matroid over*

*$\mathbb{Z}$. Indeed, the requirement of Definition 1.1.25 fails for the square*

$$\begin{array}{ccc} M_{\mathfrak{S}}(\{b\}) \cong \mathbb{Z} & \xrightarrow{?} & M_{\mathfrak{S}}(\{b,c\}) \cong \mathbb{Z}_2 \\ {\scriptstyle ?}\downarrow & & {\scriptstyle ?}\downarrow \\ M_{\mathfrak{S}}(\{a,b\}) \cong \mathbb{Z}_4 & \xrightarrow{?} & M_{\mathfrak{S}}(\{a,b,c\}) \cong \{0\} \end{array}$$

*which clearly cannot be made to be a pushout square of surjections with cyclic kernel.*

**Remark 1.4.5.** *Examples where $M_{\mathfrak{S}}$ is a non-realizable matroid over $\mathbb{Z}$ can easily be generated in a trivial way, e.g. by considering trivial group actions on non-realizable matroids. We do not know whether there is a periodic pseudoarrangement for which $M_{\mathfrak{S}}$ is a non-realizable matroid over $\mathbb{Z}$.*

**Example 1.4.6.** *We close with the realizable case: the arrangement on the top left of Figure 1.5 is a periodic affine arrangement in the sense of Section 1.2, thus the associated $M_{\mathfrak{S}}$ is a realizable matroid over $\mathbb{Z}$.*

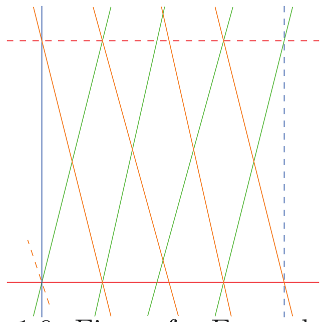


Figure 1.9: Figure for Example 1.4.3

## 1.5 Finitary geometric semilattices

In this section we study posets associated to finitary semimatroids. This leads us to consider geometric semilattices in the sense of Walker and Wachs [100]. Our goal here is to prove a finitary version of the equivalence between simple semimatroids and geometric semilattices given in [1]. The basics about partially ordered sets can be found in Section 0.1.

**Definition 1.5.1** (See Theorem 2.1 in [100])**.** *Let $\mathcal{L}$ be a (possibly infinite) ranked meet semilattice $\mathcal{L}$ with bounded rank function $\mathrm{rk}_{\mathcal{L}}$. If $\mathcal{L}$ satisfies the following conditions, it is called a* ***finitary geometric semilattice****.*

*(G3) There is $N \in \mathbb{N}$ such that every (maximal) interval is a geometric lattice with at most $N$ atoms.*

(G4) *If $A$ is an independent set of atoms and $x \in \mathcal{L}$ with $\mathrm{rk}_{\mathcal{L}}(x) < \mathrm{rk}_{\mathcal{L}}(\vee A)$, then there is an atom $a \in A$ with $a \nleq x$ and $x \vee a$ exists.*

**Remark 1.5.2.** *The definition given in [100] of a finite geometric semilattice is that of a finite ranked meet-semilattice which satisfies:*

(G1) *Every element is a join of atoms.*

(G2) *The collection of independent sets of atoms is the set of independent sets of a matroid.*

*In the finite case, Walker and Wachs prove that this is equivalent to Definition 1.5.1, which we choose to take as a definition because of its more immediate generalization to the infinite case. We nevertheless keep, for consistency, the labeling of the conditions as in [100].*

**Proposition 1.5.3.** *In a finitary geometric semilattice $\mathcal{L}$, the following properties are satisfied:*

(G1') *If $x, y \in \mathcal{L}$ and $y$ covers $x$ then there is an atom $a \in \mathcal{L}$ such that $x \vee a = y$.*

(G1") *Every element $x \in \mathcal{L}$ is a join of an independent set of atoms, which we call **basis for** $x$.*

*Proof.* In finitary geometric semilattices, the property (G1) is satisfied by (G3) and from there the proof follows [100]. □

*Proof of Theorem E.* Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk}_{\mathcal{C}})$ be a finitary semimatroid, $\mathcal{L}(\mathcal{S})$ its poset of flats (see Definition 1.1.9). We begin by showing that $\mathcal{L}(\mathcal{S})$ is a geometric semilattice.

- *$\mathcal{L}(\mathcal{S})$ is a ranked meet semilattice with bounded rank function.* For some arbitrary flats $X, Y$ of $\mathcal{S}$, their subset $X \cap Y$ is also central and its closure $\mathrm{cl}(X \cap Y) \in \mathcal{L}(\mathcal{S})$ is a lower bound of $X$ and $Y$ by Remark 1.1.10. Now suppose $A \in \mathcal{L}(\mathcal{S})$ is a lower bound of $X, Y$ in $\mathcal{L}(\mathcal{S})$, thus $A \subseteq X, Y$. In particular, this means $A \subseteq X \cap Y \subseteq \mathrm{cl}(X \cap Y)$. Therefore, the set $\mathrm{cl}(X \cap Y)$ is the meet of $X$ and $Y$ in $\mathcal{L}(\mathcal{S})$.

  Since we consider flats of $\mathcal{S}$, (CR2') implies that $\mathcal{L}(\mathcal{S})$ is ranked with rank function $\mathrm{rk}_{\mathcal{L}} := \mathrm{rk}_{\mathcal{C}}$.

- *Condition (G3).*

  If $X$ is a maximal flat of $\mathcal{S}$, then in particular $\mathrm{rk}_{\mathcal{C}}$ is defined for every subset of $X$ and satisfies axioms (R1-R3). Thus $\mathrm{rk}_{\mathcal{C}}$ defines a matroid $M$ on $X$ whose closure operator coincides with $\mathrm{cl}_{\mathcal{C}}$ (since $X$ is closed, $\mathrm{cl}_{\mathcal{C}}$ restricts to a function $2^X \to 2^X$), and thus the lattice of flats of $M$ is isomorphic to the interval $[\hat{0}, X]$ in $\mathcal{L}(\mathcal{S})$, proving that this interval is indeed a geometric lattice.

For the bound on the set of atoms, notice that a top simplex $X$ of $\mathcal{C}$ is a maximal flat of $\mathcal{S}$, hence its cardinality is at least the number of atoms in $\mathcal{L}(\mathcal{S})_{\leq X}$. Thus, if $d$ is the (finite) dimension of the simplicial complex $\mathcal{C}$, the poset $\mathcal{L}(\mathcal{S})$ satisfies (G3) with $N = d+1$.

- *Condition (G4).* Now let $A$ be an independent set of atoms in $\mathcal{L}(\mathcal{S})$ and $X$ a flat of $\mathcal{S}$ such that $\mathrm{rk}_{\mathcal{C}}(X) < \mathrm{rk}_{\mathcal{C}}(\vee A) = \mathrm{rk}_{\mathcal{C}}(\mathrm{cl}(\cup A)) = \mathrm{rk}_{\mathcal{C}}(\cup A)$. By (CR2), the set $X \cup a$ is central for some element $a$ in $\cup A - X$. In particular, its closure $\mathrm{cl}(a) \nleq X$ in $\mathcal{L}(\mathcal{S})$ and is an atom from $A$. Furthermore, the set $X \cup \mathrm{cl}(a)$ is a subset of $\mathrm{cl}(X \cup a)$ by Remark 1.1.10 and hence central as well. So the join $X \vee \mathrm{cl}(a) = \mathrm{cl}(X \cup \mathrm{cl}(a))$ exists and (G4) is satisfied. Thus $\mathcal{L}(\mathcal{S})$ is indeed a finitary geometric semilattice.

Conversely, let $\mathcal{L}$ be a finitary geometric semilattice. Let $S_{\mathcal{L}}$ denote the set of atoms of $\mathcal{L}$ and set

$$\mathcal{C}_{\mathcal{L}} = \{X \subseteq S_{\mathcal{L}} : \vee X \in \mathcal{L}\}.$$

Moreover, we define the function $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}} : \mathcal{C}_{\mathcal{L}} \to \mathbb{N}, X \mapsto \mathrm{rk}_{\mathcal{L}}(\vee X)$. Now suppose $Y \subseteq X \in \mathcal{C}_{\mathcal{L}}$, then $\vee X$ is also an upper bound for $Y$ and thus its join $\vee Y$ exists since $\mathcal{L}$ is a meet semilattice. Hence, the collection $\mathcal{C}_{\mathcal{L}}$ is an abstract simplicial complex. The cardinality of any simplex is bounded by $N$, thus $\mathcal{C}$ is finite-dimensional. We will show that $\mathcal{S}_{\mathcal{L}} := (S_{\mathcal{L}}, \mathcal{C}_{\mathcal{L}}, \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}})$ is a finitary semimatroid with semilattice of flats $\mathcal{L}(\mathcal{S}_{\mathcal{L}})$ isomorphic to $\mathcal{L}$.

- *Axioms (R1) - (R3).* For $X \in \mathcal{C}_{\mathcal{L}}$, then the join $\vee X$ exists and $[\hat{0}, \vee X]$ is a geometric lattice by (G3). Thus (e.g., by Remark 1.1.11) it defines a matroid $M_X$ with ground set the atoms in $[\hat{0}, \vee X]$ whose rank function is a restriction of $\mathrm{rk}_{\mathcal{L}}$ (resp. $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}$). Thus, (R1) is satisfied because it is satisfied in $M_X$. Moreover, for every $X \subseteq Y \in \mathcal{C}_{\mathcal{L}}$ the condition (R2) is satisfied since it is satisfied in $M_Y$, and for every $X, Y$ with $X \cup Y \in \mathcal{C}_{\mathcal{L}}$ the condition (R3) is satisfied because it is satisfied in $M_{X \cup Y}$.

- *Axiom (CR1).* Take $X, Y \in \mathcal{C}_{\mathcal{L}}$ with $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(X) = \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(X \cap Y)$. Evidently, the join of $X$ is also an upper bound of $X \cap Y$, thus $\vee(X \cap Y) \leq \vee X$. Since by assumption $\mathrm{rk}_{\mathcal{L}}(\vee X) = \mathrm{rk}_{\mathcal{L}}(\vee(X \cap Y))$ and $\mathcal{L}$ is ranked, $\vee(X \cap Y) = \vee X$. So

$$\vee X = \vee(X \cap Y) \leq \vee Y,$$

that is to say every upper bound of $Y$ is also an upper bound of $X$. Hence $\vee(X \cup Y) = \vee Y$ and $X \cup Y \in \mathcal{C}$, and (CR1) is satisfied.

- *Axiom (CR2).* Let $X, Y$ be in $\mathcal{C}_{\mathcal{L}}$ and such that $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(X) < \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(Y)$. Let then $A \subseteq Y$ be a basis for $\vee Y$ (see (G1") - since $[\hat{0}, \vee Y]$ is a geometric lattice we can find this basis in $Y$). Property (CR2) now follows applying (G4) to $X$ and $A$.

- *There is a poset isomorphism* $\mathcal{L} \simeq \mathcal{L}(\mathcal{S}_{\mathcal{L}})$. Let $\varphi : \mathcal{L} \to \mathcal{L}(\mathcal{S}_{\mathcal{L}})$ be defined by $\varphi(x) = \{a \in S_{\mathcal{L}} : a \leq x\}$. For $\varphi$ to be well-defined, we must check that $\varphi(x)$ is a flat of $\mathcal{S}_{\mathcal{L}}$. First, by (G3) we have $\vee\varphi(x) = x$ and thus, $\varphi(x) \in \mathcal{C}_{\mathcal{L}}$. Now suppose $b$ is an element of $S$ such that $\varphi(x) \cup b$ is a central set and $\mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(\varphi(x) \cup b) = \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}}(\varphi(x))$. Since by assumption $\mathrm{rk}_{\mathcal{L}}(\vee(\varphi(x) \cup b)) = \mathrm{rk}_{\mathcal{L}}(\vee\varphi(x))$ and $\vee(\varphi(x) \cup b) \leq \vee\varphi(x) = x$, we get equality. Then $b \leq x$, that is to say $b \in \varphi(x)$. Hence, the set $\varphi(x)$ is a flat of $\mathcal{S}_{\mathcal{L}}$.

  The function $\varphi$ clearly preserves order and is one-to-one. Let $Y$ be a flat of $\mathcal{S}_{\mathcal{L}}$, to show that $\varphi$ is onto we have to find some $x \in \mathcal{L}$ with $\varphi(x) = Y$. But this is the same as to say $x = \vee Y$, which exists by definition of $\mathcal{S}_{\mathcal{L}}$. Hence, $\varphi$ is an isomorphism and the theorem is proven.

The semimatroid $\mathcal{S}_{\mathcal{L}} = (S_{\mathcal{L}}, \mathcal{C}_{\mathcal{L}}, \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}})$ is simple by construction. We are left with showing that every other simple semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ with a poset-isomorphism $\varphi : \mathcal{L}(\mathcal{S}) \xrightarrow{\cong} \mathcal{L}$ is isomorphic to $\mathcal{S}_{\mathcal{L}}$. Since every element of a simple semimatroid is closed, $\varphi$ induces a natural bijection

$$\varphi_S : S \to S_{\mathcal{L}}, \qquad \{\varphi_S(x)\} = \varphi(\{x\})$$

For every $X = \{x_1, \ldots, x_k\} \in \mathcal{C}$, we have

$$\bigvee_{i=1}^{k} \{\varphi_S(x_i)\} = \bigvee_{i=1}^{k} \varphi(\{x_i\}) = \varphi(\bigvee_{i=1}^{k} \{x_i\}),$$

hence the right-hand side exists in $\mathcal{L}$, thus $\varphi_S(X) \in \mathcal{C}_{\mathcal{L}}$. An analogous argument using $\varphi_S^{-1}$ shows that in fact $\varphi_S$ induces an isomorphism $\mathcal{C} \cong \mathcal{C}_{\mathcal{L}}$.

It remains to show that $\varphi_S$ preserves rank of central sets. But this is an easy computation. For $X = \{x_1, \ldots, x_k\} \in \mathcal{C}$:

$$\mathrm{rk}_{\mathcal{C}}(X) = \mathrm{rk}_{\mathcal{L}(\mathcal{S})}(\bigvee_i \{x_i\}) = \mathrm{rk}_{\mathcal{L}}(\bigvee_i \varphi(\{x_i\})) = \mathrm{rk}_{\mathcal{C}_{\mathcal{L}}} \varphi_S(X).$$

□

## 1.6 The underlying matroid of a group action

This section is devoted to the proof of Theorem A. Let $\mathfrak{S}$ be a $G$-semimatroid associated to an action of $G$ on a semimatroid $(S, \mathcal{C}, \mathrm{rk})$. Recall from Section 1.3 the set $E_{\mathfrak{S}} := S/G$ of orbits of elements, the family $\underline{\mathcal{C}} = \{\underline{X} \subseteq E_{\mathfrak{S}} \mid X \in \mathcal{C}\}$, and that we only consider actions for which $E_{\mathfrak{S}}$ is finite.

For every $A \subseteq E_{\mathfrak{S}}$ define

$$J(A) := \{X \in \mathcal{C} | \underline{X} \subseteq A\}$$

and write $J_{\max}(A)$ for the set of inclusion-maximal elements of $J(A)$.

**Lemma 1.6.1.** *For every* $X, Y \in J_{\max}(A)$, $\mathrm{rk}(X) = \mathrm{rk}(Y)$.

*Proof.* By way of contradiction assume $\mathrm{rk}(X) > \mathrm{rk}(Y)$. Then with (CR2) we can find $x \in X - Y$ with $Y \cup x \in \mathcal{C}$ and clearly $\underline{Y \cup x} \subseteq A$ contradicting maximality of $Y$. □

**Definition 1.6.2.** *For any* $A \subseteq E_{\mathfrak{G}}$ *choose* $X \in J_{\max}(A)$ *and let* $\underline{\mathrm{rk}}(A) := \mathrm{rk}(X)$ *(Lemma 1.6.1 shows that this is well-defined and independent on the choice of* $X$*).*

**Remark 1.6.3.** *For all* $A \subseteq E_{\mathfrak{G}}$ *we have*

$$\underline{\mathrm{rk}}(A) = \max\{\underline{\mathrm{rk}}(A') \mid A' \subseteq A, A' \in \underline{\mathcal{C}}\},$$

*because for* $A' \subseteq A$ *clearly* $J(A') \subseteq J(A)$.

**Proposition 1.6.4.** *The pair* $(E_{\mathfrak{G}}, \underline{\mathrm{rk}})$ *always satisfies (R2) and (R3), and thus defines a polymatroid on* $E_{\mathfrak{G}}$ *when* $E_{\mathfrak{G}}$ *is finite.*
*Moreover,* $(E_{\mathfrak{G}}, \underline{\mathrm{rk}})$ *satisfies (R1) if and only if the action is weakly translative.*

*Proof.* Property (R2) is trivial, we check (R3). Consider $A, A' \subseteq E$, and choose $B_0 \in J_{\max}(A \cap A')$, hence by Lemma 1.6.1 $\mathrm{rk}(B_0) = \underline{\mathrm{rk}}(A \cap A')$. Then $B_0 \in J(A)$ and thus we can find $B_1 \in J(A)$ such that $B_0 \cup B_1 \in J_{\max}(A)$ and a maximal $B_2 \in J(A')$ such that $B_0 \cup B_1 \cup B_2$ is in $J(A' \cup A)$. Then, $B_0 \cup B_1 \cup B_2 \in J_{\max}(A' \cup A)$ because otherwise we would need to complete it with some $B_2' \in J(A)$ in order to get an element of $J_{\max}(A \cup A')$ – but then, $B_0 \cup B_1 \cup B_2' \supseteq B_0 \cup B_1 \in J_{\max}(A)$, thus $B_2' = \varnothing$ by the choice of $B_1$. Using (R3) in $(S, \mathcal{C}, \mathrm{rk})$ we obtain

$$\begin{aligned}\underline{\mathrm{rk}}(A \cap A') + \underline{\mathrm{rk}}(A \cup A') - \underline{\mathrm{rk}}(A) &= \mathrm{rk}(B_0) + \mathrm{rk}(B_0 \cup B_1 \cup B_2) - \mathrm{rk}(B_0 \cup B_1)\\ &\le \mathrm{rk}(B_0 \cup B_2) \le \underline{\mathrm{rk}}(A'),\end{aligned}$$

where the last inequality follows from $\underline{B_0 \cup B_2} \subseteq A'$, and so (R3) is satisfied for $\underline{\mathrm{rk}}$.

Now suppose that the action is weakly translative. For (R1) we need to show that $0 \le \underline{\mathrm{rk}}(A) \le |A|$ for every $A \subseteq E_{\mathfrak{G}}$. The left hand side inequality is trivial. Consider $A \subseteq E_{\mathfrak{G}}$ and choose $X \in J_{\max}(A)$. We claim that for every $x \in X$ with $g(x) \in X$ we have $\mathrm{rk}(X) = \mathrm{rk}(X - g(x))$; indeed, using (R3) in $(S, \mathcal{C}, \mathrm{rk})$ on the sets $X - g(x)$ and $\{x, g(x)\}$, we obtain

$$\mathrm{rk}(X) + \mathrm{rk}(x) \le \mathrm{rk}(X - g(x)) + \mathrm{rk}(\{x, g(x)\}) = \mathrm{rk}(X - g(x)) + \mathrm{rk}(x)$$

where in the last equality we used weak translativity of the action. Thus we get $\mathrm{rk}(X) \le \mathrm{rk}(X - g(x))$ and, the other inequality being trivial from (R2), we have the claimed equality. With it, we can then choose a system $X'$ of representatives of the orbits in $\underline{X}$ and write

$$\underline{\mathrm{rk}}(A) = \mathrm{rk}(X) = \mathrm{rk}(X') \le |X'| = |\underline{X}| \le |A|. \tag{1.3}$$

Conversely, if the action is not weakly translative, choose $x \in X$ and $g \in G$ violating the weak translativity condition and consider $A := Gx$. First notice that $x$ cannot be a loop, since if $\mathrm{rk}(x) = 0$ then $\mathrm{rk}(g(x)) = 0$ and $\mathrm{rk}(\{x, g(x)\})$ must equal 0 (otherwise it would contain an independent set of rank 1). Therefore, we have $\mathrm{rk}(\{x, g(x)\}) = \mathrm{rk}(x)$ in agreement with the weak translativity condition. Hence, it must be $\mathrm{rk}(x) = 1$, and we have $\underline{\mathrm{rk}}(A) \geq \mathrm{rk}(\{x, g(x)\}) > \mathrm{rk}(x) = 1 = |\{A\}|$, in clear violation of (R1). □

**Corollary 1.6.5.** *If the action is weakly translative, for all $X \in \mathcal{C}$ we have* $\mathrm{rk}(X) = \underline{\mathrm{rk}}(\underline{X})$.

*Proof.* This is a consequence of Equation (1.3) in the previous proof, and of the discussion preceding it. □

**Remark 1.6.6.** *The matroid $(E_{\mathfrak{G}}, \underline{\mathrm{rk}})$ is, in some sense an 'artificial' construct, although in some cases useful. For instance, when $(S, \mathcal{C}, \mathrm{rk})$ is the semimatroid of a periodic arrangement $\mathscr{A}$ of hyperplanes in real space, then $(E_{\mathfrak{G}}, \underline{\mathrm{rk}})$ is the matroid of the arrangement $\mathscr{A}_0$ which plays a key role in the techniques used in [30, 31, 23].*

**Proposition 1.6.7.** *Let $\mathfrak{G}$ be weakly translative. Then $\mathcal{S}_{\mathfrak{G}} := (E_{\mathfrak{G}}, \underline{\mathcal{C}}, \underline{\mathrm{rk}})$ is a locally ranked triple satisfying (CR2).*

*Proof.* Proposition 1.6.4 implies that (R1), (R2), (R3) hold.

For (CR2), let $A, B \in \underline{\mathcal{C}}$ with $\underline{\mathrm{rk}}(A) < \underline{\mathrm{rk}}(B)$ and choose $X \in \lceil A \rceil^{\mathcal{C}}$ and $Y \in \lceil B \rceil^{\mathcal{C}}$. Then, by Corollary 1.6.5, $\mathrm{rk}(X) < \mathrm{rk}(Y)$. Using (CR2') in $\mathcal{S}$ (cf. Remark 1.1.6) we find $y \in Y - X$ with $X \cup y \in \mathcal{C}$ and $\mathrm{rk}(X \cup y) > \mathrm{rk}(X)$. Set $b := \underline{y}$. Then, $A \cup b = \underline{X \cup y} \in \underline{\mathcal{C}}$ and $b \in B - A$ (otherwise $b \in A$, thus – using Corollary 1.6.5 – $\mathrm{rk}(X \cup y) = \underline{\mathrm{rk}}(A \cup b) = \underline{\mathrm{rk}}(A) = \mathrm{rk}(X)$, a contradiction). □

## 1.7 Translative actions

We now proceed towards establishing Theorem B. The main idea in this section is to associate a diagram of sets and injective maps to every molecule of the quotient triple $\mathcal{S}_{\mathfrak{G}}$. In the realizable case, this diagram specializes to the inclusion pattern of integer points in semiopen parallelepipeds as well as to that of layers of the associated toric arrangement. By considering some enumerative statistics of these diagrams, in later sections we obtain, in the general (non-representable) case, analogues and extensions of some combinatorial decompositions given in [28] for the realizable case, most notably Theorem H.

After a preliminary general lemma on translative actions, the proof that the above-mentioned diagrams are well defined will require the (rather technical) Section 1.7.1. After that, in Section 1.7.2 we will use the properties of these diagrams in order to prove the required statements about the function $m_{\mathfrak{G}}$, as well as some other facts used later on.

Recall the definitions in Section 1.3, and in particular that $\mathfrak{S}$ denotes a $G$-semimatroid corresponding to the action of a group $G$ on a semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$. In this section we suppose this action always to be cofinite and translative. In particular, we can consider the associated locally ranked triple $\mathcal{S}_{\mathfrak{S}} = (E_{\mathfrak{S}}, \underline{\mathcal{C}}, \underline{\mathrm{rk}})$ with multiplicity function $m_{\mathfrak{S}}$ (see Section 1.3).

**Definition 1.7.1.** *Given $A \in \underline{\mathcal{C}}$, $X \in \lceil A \rceil^{\mathcal{C}}$ and $a_0 \in A$ define*

$$w_{A,a_0} : \lceil A \rceil^{\mathcal{C}} \to \lceil A - a_0 \rceil^{\mathcal{C}}, \quad X \mapsto X - a_0, \tag{1.4}$$

*and notice that, since it is $G$-equivariant, it induces a function*

$$\underline{w}_{A,a_0} : \lceil A \rceil^{\mathcal{C}}/G \to \lceil A - a_0 \rceil^{\mathcal{C}}/G. \tag{1.5}$$

A straightforward check of the definitions shows that, when $w_{A,a_0}$ is injective (resp. surjective) then $\underline{w}_{A,a_0}$ also is.

**Lemma 1.7.2.** *Let $\mathfrak{S}$ be translative.*

*(a) If $x_0 \in X \in \mathcal{C}$ with $\mathrm{rk}(X) = \mathrm{rk}(X - x_0) + 1$, then $Y \cup g(x_0) \in \mathcal{C}$ for all $g \in G$ and all $Y \in \mathcal{C}$ with $\underline{Y} = \underline{X - x_0}$.*

*(b) If $a_0 \in A \in \underline{\mathcal{C}}$ with $\underline{\mathrm{rk}}(A) = \underline{\mathrm{rk}}(A - a_0) + 1$, then $w_{A,a_0}$ is surjective and, for any choice of $x_0 \in a_0$, a right inverse of $w_{A,a_0}$ is given by*

$$\widehat{w}_{A,a_0} : \lceil A - a_0 \rceil^{\mathcal{C}} \to \lceil A \rceil^{\mathcal{C}}, \; Y \mapsto Y \cup x_0. \tag{1.6}$$

*In particular, $m_{\mathfrak{S}}(A) \geq m_{\mathfrak{S}}(A - a_0)$.*

*(c) If $a_0 \in A \in \underline{\mathcal{C}}$ with $\underline{\mathrm{rk}}(A) = \underline{\mathrm{rk}}(A - a_0)$, then $w_{A,a_0}$ is injective and thus $m_{\mathfrak{S}}(A) \leq m_{\mathfrak{S}}(A - a_0)$.*

*Proof.*

(a) Let $X, x_0$ be as in the claim. For all $g \in G$ consider the central set $g(X)$ of rank $\mathrm{rk}(g(X)) = \mathrm{rk}(X) > \mathrm{rk}(X - x_0)$. By (CR2) there is some $y \in g(X) - (X - x_0)$ with $y \cup (X - x_0) \in \mathcal{C}$ and $\mathrm{rk}(y \cup (X - x_0)) = \mathrm{rk}(X)$. This $y$ must be $g(x_0)$ because every other element $y' \in g(X) - (X \cup g(x_0))$ is of the form $y' = g(x')$ ($\notin X$) for some $x' \in X$, thus $y' \cup (X - x_0) \in \mathcal{C}$ would imply $\{x', g(x')\} \in \mathcal{C}$ which, since by construction $x' \neq g(x')$, is forbidden by the fact that the action is translative. Thus $(X - x_0) \cup g(x_0) \in \mathcal{C}$ for all $g \in G$.

Now consider any $Y$ with $\underline{Y} = \underline{X - x_0}$ and notice that with Lemma 1.6.1 we have

$$\mathrm{rk}((X - x_0) \cup g(x_0)) > \mathrm{rk}(X - x_0) = \mathrm{rk}(Y),$$

and thus by (CR2) there must be $x \in (X - x_0) \cup g(x_0)$ with $Y \cup x \in \mathcal{C}$ and $\mathrm{rk}(Y \cup x) = \mathrm{rk}(Y) + 1$. Since $Y$ consists of translates of elements of $X$, as above the fact that the action is translative enforces $x = g(x_0)$.

Towards (b) and (c), choose any $X \in \lceil A \rceil^{\mathcal{C}}$ and let $x_0 \in X$ be a representative of $a_0$. Then by Definition 1.6.2 and since the action is translative, $\mathrm{rk}(X - x_0) = \mathrm{rk}(X)$ if and only if $\underline{\mathrm{rk}}(A - a_0) = \underline{\mathrm{rk}}(A)$.

(b) Suppose $\underline{\mathrm{rk}}(A - a_0) = \underline{\mathrm{rk}}(A) - 1$. Part (a) ensures that the function $\widehat{w}_{A,a_0}$ is well-defined. Clearly, it is injective and $w_{A,a_0} \circ \widehat{w}_{A,a_0} = \mathrm{id}$. In particular, $w_{A,a_0}$ is surjective.

(c) Suppose now $\underline{\mathrm{rk}}(A - a_0) = \underline{\mathrm{rk}}(A)$ and consider $X_1, X_2 \in \lceil A \rceil^{\mathcal{C}}$. Since the action is translative the sets $X_1 \cap a_0$ and $X_2 \cap a_0$ consist both of a single element, say $x_{0,1}$ and $x_{0,2}$ respectively. If moreover $X_1$, $X_2$ map to the same $Y = X_1 - a_0 = X_2 - a_0$, then $Y \cup x_{0,1}$ and $Y \cup x_{0,2}$ are both central and of the same rank, equal to the rank of $Y$. By (CR1) then $Y \cup \{x_{0,1}, x_{0,2}\} \in \mathcal{C}$, thus $\{x_{0,1}, x_{0,2}\} \in \mathcal{C}$ and since the action is translative we must have $x_{0,1} = x_{0,2}$, hence $X_1 = X_2$.

□

### 1.7.1 Labeling orbits

As was already pointed out, the purpose of this section is to provide the groundwork for proving that the objects introduced in section 1.7.2 are well-defined. The reader wishing to acquire a general view of our setup without delving into technicalities may skip this section with no harm.

Our main task here will be to specify canonical representatives for orbits supported on a molecule, in order for Equation (1.6) to induce a well-defined function between sets of orbits. Again, we consider throughout a $G$-semimatroid $\mathfrak{S}$ defined by an action on $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$, and we assume translativity.

**Definition 1.7.3.** *Given a molecule $(R, F, T)$ of $\mathcal{S}_{\mathfrak{S}}$ fix a numbering of the set $F = \{f_1, ..., f_k\}$. If we consider the elements of $2^{[k]}$ as ordered zero-one-tuples we obtain a total order $\prec$ on $2^{[k]}$ by the lexicographic order on the tuples. Notice that, then, $I \subseteq I'$ implies $I \preceq I'$.*

*We choose representatives $X_R^{(1)}, \ldots, X_R^{(k_R)}$ of the orbits in $\lceil R \rceil^{\mathcal{C}}/G$ and extend $\prec$ to an order on the index set $\{(i, I) \mid i = 1, \ldots, k_R,\ I \in 2^{[k]}\}$ via*

$$(i, I) \prec (i', I') \Leftrightarrow \begin{cases} i < i', \\ \text{or } i = i' \text{ and } I \prec I'. \end{cases} \tag{1.7}$$

*Moreover, choose and fix an element $x_i \in f_i$ for all $i = 1, ..., k$. For all $I \in 2^{[k]}$ set $F_I = \{f_i \mid i \in I\}$ and for all $F' \subseteq F$ define $X_{F'} = \{x_f \mid f \in F'\}$.*

We now can recursively define an ordered partition of $\lceil R \cup F \rceil^{\mathcal{C}}/G$ as follows.

**Definition 1.7.4.** *Set* $\mathscr{Y}^{(1,\varnothing)} := \{G(X_R^{(1)} \cup X_F)\}$*, and for each* $(i,I) \succ (1,\varnothing)$ *let*

$$\mathscr{Y}^{(i,I)} := \left\{ \mathcal{O} \in \frac{\lceil R \cup F \rceil^{\mathcal{C}}}{G} \;\middle|\; \begin{array}{ll} (i) & \mathcal{O} \notin \bigcup_{(j,J)\prec(i,I)} \mathscr{Y}^{(j,J)} \\ (ii) & X_R^{(i)} \cup X_{F \smallsetminus F_I} \subseteq Y \text{ for some } Y \in \mathcal{O} \end{array} \right\}.$$

*Choose an enumeration*

$$\mathscr{Y}^{(i,I)} = \{\mathcal{O}_1, \dots, \mathcal{O}_{h_{(i,I)}}\}$$

*thereby defining the numbers* $h_{(i,I)}$ *(and setting* $h_{(i,I)} = 0$ *if* $\mathscr{Y}^{(i,I)} = \varnothing$*).*

The sets $\mathscr{Y}^{(i,I)}$ indeed partition $\lceil R \cup F \rceil^{\mathcal{C}}/G$ since (i) ensures that they have trivial intersections and (ii) ensures that they exhaust all of $\lceil R \cup F \rceil^{\mathcal{C}}/G$.

**Remark 1.7.5.** *If* $\mathcal{O} \in \mathscr{Y}^{(i,I)}$ *then there exists* $Y \in \mathcal{O}$ *with* $X_{F \smallsetminus F_I} \subseteq Y$*. Moreover, if* $X_{F \smallsetminus F_J} \subseteq Y$ *for some* $Y \in \mathcal{O}$*, then* $J \geq I$*. In particular,* $J \subsetneq I$ *implies* $X_{F \smallsetminus F_J} \nsubseteq Y$ *for all* $Y \in \mathcal{O}$*.*

Now we are ready to define representatives for orbits in $\lceil R \cup F \rceil^{\mathcal{C}}/G$.

**Definition 1.7.6.** *Define then a total ordering* $\lhd$ *on the set*

$$\mathscr{Z}_{R,F} := \{(i,I,j) \mid i = 1, \dots, k_R;\ I \in 2^{[k]};\ j = 1, \dots, h_{(i,I)}\}$$

*by*

$$(i,I,j) \lhd (i',I',j') \Leftrightarrow \begin{cases} (i,I) \prec (i',I') \text{ or} \\ (i,I) = (i',I') \text{ and } j < j'. \end{cases}$$

*For every* $(i,I,j) \in \mathscr{Z}_{R,F}$ *consider the corresponding orbit* $\mathcal{O}_j \in \mathscr{Y}^{(i,I)}$ *and choose a representative* $Y_{R\cup F}^{(i,I,j)}$ *of* $\mathcal{O}_j$ *with*

$$X_R^{(i)} \cup X_{F \smallsetminus F_I} \subseteq Y_{R\cup F}^{(i,I,j)} \in \mathcal{O}_j \tag{1.8}$$

*(such a representative exists by requirement (2) of Definition 1.7.4).*

**Lemma 1.7.7.** $Y_{R\cup F}^{(i,I,j)} \cap X_F = X_{F \smallsetminus F_I}$.

*Proof.* Let $J$ be such that $Y_{R\cup F}^{(i,I,j)} \cap X_F = X_{F \smallsetminus F_J}$. Then $J \subseteq I$ by Equation (1.8). Moreover, if $J \subsetneq I$ then $J \prec I$, a contradiction to Remark 1.7.5. Hence $I = J$ as desired. □

Based on this, we fix representatives for the orbits in $\lceil R \cup F' \rceil^{\mathcal{C}}/G$ for each $F' \subseteq F$.

**Definition 1.7.8.** *Given $F' \subseteq F$, for every $\mathcal{O} \in \lceil R \cup F' \rceil^{\mathcal{C}}/G$ let*

$$z(\mathcal{O}) := \min_{\triangleleft}\{z \in \mathscr{Z}_{R,F} \mid \mathcal{O} \leq GY^{z}_{R\cup F} \text{ in } \mathcal{C}_{\mathfrak{G}}\}$$

*and let $Y^{\mathcal{O}}_{R\cup F'} \in \mathcal{O}$ be the unique representative with*

$$Y^{\mathcal{O}}_{R\cup F'} \subseteq Y^{z(\mathcal{O})}_{R\cup F}.$$

*With these choices, let*

$$\begin{array}{rccc} \widehat{\underline{w}}_{R\cup F, F\smallsetminus F'}: & \lceil R\cup F'\rceil^{\mathcal{C}}/G & \to & \lceil R\cup F\rceil^{\mathcal{C}}/G \\ & \mathcal{O} & \mapsto & G(Y^{\mathcal{O}}_{R\cup F'} \cup X_{F\smallsetminus F'}). \end{array} \tag{1.9}$$

**Lemma 1.7.9.** *Let $\mathfrak{G}$ be translative, let $(R, F, \varnothing)$ be a molecule of the triple $\mathcal{S}_{\mathfrak{G}} = (E_{\mathfrak{G}}, \underline{\mathcal{C}}, \underline{\mathrm{rk}})$ and consider $F'' \subseteq F' \subseteq F$. Then*

*(a) for every $\mathcal{O} \in \lceil R \cup F' \rceil^{\mathcal{C}}/G$*

$$Y^{z(\mathcal{O})}_{R\cup F} = Y^{\mathcal{O}}_{R\cup F'} \cup X_{F\smallsetminus F'};$$

*(b) for every $\mathcal{O} \in \lceil R \cup F'' \rceil^{\mathcal{C}}/G$*

$$Y^{G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\smallsetminus F''})}_{R\cup F'} = Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\smallsetminus F''}.$$

*(c) Furthermore,*

$$\widehat{\underline{w}}_{R\cup F, F\smallsetminus F'} \circ \widehat{\underline{w}}_{R\cup F', F'\smallsetminus F''} = \widehat{\underline{w}}_{R\cup F, F\smallsetminus F''}.$$

*Proof.* In this proof, given any $\mathcal{O} \in \lceil R \cup F' \rceil^{\mathcal{C}}/G$ let us for brevity call $\mathscr{Z}(\mathcal{O})$ the set over which the minimum is taken in Definition 1.7.8 in order to define $z(\mathcal{O})$.

*Part (a).* It is enough to show that $X_{F\smallsetminus F'} \subseteq Y^{z(\mathcal{O})}_{R\cup F}$. In order to prove this, we consider

$$Y' := (Y^{z(\mathcal{O})}_{R\cup F} \smallsetminus X') \cup X_{F\smallsetminus F'}$$

where $X' \in \lceil F \smallsetminus F' \rceil^{\mathcal{C}}$ is defined by $X' \subseteq Y^{z(\mathcal{O})}_{R\cup F}$ (notice that $|X'| = |X_F|$ since $Y^{z(\mathcal{O})}_{R\cup F} \in \lceil R \cup F \rceil^{\mathcal{C}}$ and the action is translative). The set $Y'$ is central by Lemma 1.7.2.(a), because $\mathrm{rk}(Y^{z(\mathcal{O})}_{R\cup F'}) = \mathrm{rk}(Y^{z(\mathcal{O})}_{R\cup F} \smallsetminus X') + |X'|$. Moreover, $GY' \geq \mathcal{O}$ in $\mathcal{C}_{\mathfrak{G}}$ since $Y^{\mathcal{O}}_{R\cup F} \subseteq Y'$.

If $X_{F\smallsetminus F'} \subseteq Y^{z(\mathcal{O})}_{R\cup F}$, then $Y' = Y^{z(\mathcal{O})}_{R\cup F}$. We will prove that if this is not the case, then $z(\mathcal{O}) \neq \min \mathscr{Z}(\mathcal{O})$, reaching a contradition. Suppose then $X_{F\smallsetminus F'} \nsubseteq Y^{z(\mathcal{O})}_{R\cup F}$, and write $z(\mathcal{O}) = (i, I, j)$. By Lemma 1.7.7, we have $I = \{i \mid x_{f_i} \notin Y^{z(\mathcal{O})}_{R\cup F}\}$. Hence,

$$I_{Y'} := \{i \mid x_{f_i} \notin Y'\} = I \cap I_{F'} \subseteq I$$

where $I_{F'} = \{i \mid f_i \in F'\}$, and the last containment is strict (otherwise $Y' = Y^{z(\mathcal{O})}_{R\cup F}$, hence $X_{F\smallsetminus F'} \subseteq Y^{z(\mathcal{O})}_{R\cup F}$, contrary to our assumption). By definition, $I_{Y'} \subsetneq I$ implies $I_{Y'} \lneq I$. Moreover, for $z' = (i, I', j')$ defined by $GY' = \mathcal{O}_{j'} \in \mathscr{Y}^{(i,I')}$ we have in fact by Remark 1.7.5 that $I' \leq I_{Y'}$. Therefore, $I' \leq I_{Y'} \lneq I$. This implies that $z' = (i, I', j') \trianglelefteq (i, I, j) = z(\mathcal{O})$ and $z' \neq z(\mathcal{O})$. Thus, $GY^{z'}_{R\cup F} \in \mathscr{Z}(\mathcal{O})$ but $z'$ strictly precedes $z(\mathcal{O})$, and we reach the annouced contradiction.

*Part (b).* Let $\mathcal{O}$ be as in the claim, and set $\mathcal{U} := G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\smallsetminus F''})$. Then $\mathcal{O} \leq \mathcal{U}$ in $\mathcal{C}_{\mathfrak{S}}$, thus $\mathscr{Z}(\mathcal{O}) \supseteq \mathscr{Z}(\mathcal{U})$ and therefore $z(\mathcal{O}) \trianglelefteq z(\mathcal{U})$. Now, since $Y^{z(\mathcal{O})}_{R\cup F} = Y^{\mathcal{O}}_{R\cup F''} \cup X_{F\smallsetminus F''}$ by part (a), we see that $\mathcal{U} \leq GY^{z(\mathcal{O})}_{R\cup F}$ in $\mathcal{C}_{\mathfrak{S}}$, thus $z(\mathcal{U}) \trianglelefteq z(\mathcal{O})$. In summary, $z(\mathcal{U}) = z(\mathcal{O})$ and, as a subset of $Y^{\mathcal{O}}_{R\cup F''} \cup X_{F\smallsetminus F''}$, we see that $Y^{\mathcal{U}}_{R\cup F'} = Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\smallsetminus F''}$ as claimed.

*Part (c).* For every $\mathcal{O} \in \lceil R \cup F'' \rceil^{\mathcal{C}}/G$ we compute

$$\widehat{\underline{w}}_{R\cup F, F\smallsetminus F'} \circ \widehat{\underline{w}}_{R\cup F', F'\smallsetminus F''}(\mathcal{O}) = \widehat{\underline{w}}_{R\cup F, F\smallsetminus F'}(G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\smallsetminus F''})) =$$
$$G(Y^{G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F'\smallsetminus F''})}_{R\cup F'} \cup X_{F\smallsetminus F'}) = G(Y^{\mathcal{O}}_{R\cup F''} \cup X_{F\smallsetminus F''}) = \widehat{\underline{w}}_{R\cup F, F\smallsetminus F''}(\mathcal{O}),$$

where in the third equality we used (b) and all other equalities hold by definition.

□

**Definition 1.7.10.** *For a molecule $(R, F, T)$ with $F' \subseteq F, T' \subseteq T$ and an orbit $\mathcal{O} \in \lceil R \cup F' \cup T' \rceil^{\mathcal{C}}/G$ the representative $Y^{\mathcal{O}}_{R\cup F'\cup T'} \in \mathcal{O}$ is then*

$$Y^{\mathcal{O}}_{R\cup F'\cup T'} = Y^{\mathcal{O}}_{R\cup F'} \cup Y^{\mathcal{O}}_{T'} = X^{\mathcal{O}}_{R} \cup Y^{\mathcal{O}}_{F'} \cup Y^{\mathcal{O}}_{T'}, \tag{1.10}$$

*where $Y^{\mathcal{O}}_{R\cup F'}$ is given as above and $Y^{\mathcal{O}}_{T'} \in \lceil T' \rceil^{\mathcal{C}}$ is uniquely determined by $X^{\mathcal{O}}_{R}$ since $\underline{w}_{R\cup T', T'}$ is injective by Lemma 1.7.2.(c).*

**Example 1.7.11.** *We go back to our running example (Example 1.1.8), for which we depict in Figure 1.7.11 a piece of the associated periodic arrangement, and consider there the molecule $(\varnothing, F, \varnothing)$, where $F = \{f_a, f_b\}$ is the set of orbits of the orange and green lines.*

*Choose representatives $x_a = a_0$ for the orange lines, $x_b = b_0$ for the green lines and denote their $(0, k)$-translate by $a_k$ (resp. $b_k$).*

*By Definitions 1.7.4 and 1.7.6, we get the following partition of $\lceil F \rceil^{\mathcal{C}}/G$,*

$$\mathscr{Y}^{(1,\varnothing)} = \{\mathcal{O}_0\}, \; \mathscr{Y}^{(1,\{2\})} = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}, \; \mathscr{Y}^{(1,\{1\})} = \mathscr{Y}^{(1,\{1,2\})} = \varnothing,$$

*with representatives*

$$Y^{(1,\varnothing,1)}_F = \{a_0, b_0\}, Y^{(1,\{2\},1)}_F = \{a_0, b_{k_1}\}, Y^{(1,\{2\},2)}_F = \{a_0, b_{k_2}\}, Y^{(1,\{2\},3)}_F = \{a_0, b_{k_3}\},$$

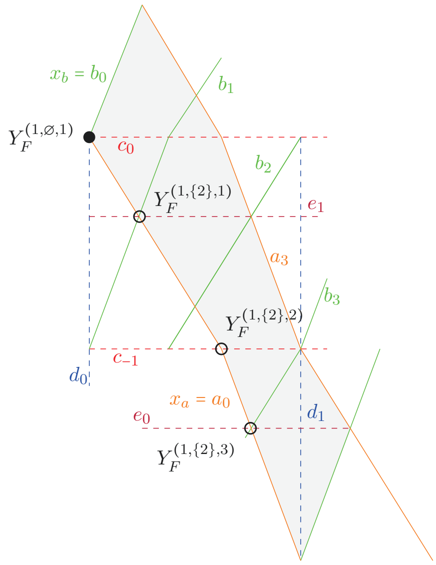



Figure 1.10: An illustration for Example 1.7.11.

*where* $k_1 \neq 0 \mod 4$; $k_2 \neq 0, k_1 \mod 4$; *and* $k_3 \neq 0, k_1, k_2 \mod 4$. *Without loss of generality, one could assume* $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, *and we get the situation depicted in Figure 1.7.11. Moreover, by Definition 1.7.8 we get* $Y_a^{\mathcal{O}_a} = a_0$ *(where* $\lceil f_a \rceil^{\mathcal{C}}/G = \{\mathcal{O}_a\}$*),* $Y_b^{\mathcal{O}_b} = b_0$ *(where* $\lceil f_b \rceil^{\mathcal{C}}/G = \{\mathcal{O}_b\}$*), and* $Y_{\varnothing}^{\varnothing} = \varnothing$ *where* $\lceil \varnothing \rceil^{\mathcal{C}}/G = \{\varnothing\}$.

*Thus,*

$$\widehat{\underline{w}}_{F,F}(\varnothing) = \widehat{\underline{w}}_{F,f_b}(\widehat{\underline{w}}_{f_a,f_a}(\varnothing)) = \widehat{\underline{w}}_{F,f_a}(\widehat{\underline{w}}_{f_b,f_b}(\varnothing)) = G(a_0 b_0) = \mathcal{O}_0.$$

*Notice that an accurate choice of representatives is of the essence. For example, choosing* $Y_a^{\mathcal{O}_a} = a_0$ *and* $Y_b^{\mathcal{O}_b} = b_1$ *as representatives of* $\mathcal{O}_a$*, resp.* $\mathcal{O}_b$*,*

$$\operatorname{im} \widehat{\underline{w}}_{F,f_b} = G(a_0 x_b) = G(a_0 b_0) \neq G(a_0 b_1) = G(x_a b_1) = \operatorname{im} \widehat{\underline{w}}_{F,f_a}.$$

### 1.7.2 Orbit count for molecules

**Definition 1.7.12.** *Given a molecule* $(R, F, T)$ *of a ranked triple, define the following boolean poset*

$$P[R,F,T] := \{(F',T') \mid F' \subseteq F,\ T' \subseteq T\} \text{ with order}$$
$$(F',T') \leq (F'',T'') \Leftrightarrow F' \subseteq F'',\ T' \supseteq T''.$$

*Thus, the maximal element is* $(F, \varnothing)$ *and the minimal element* $(\varnothing, T)$.

Consider now a translative $G$-semimatroid $\mathfrak{S}$ and a molecule $(R, F, T)$ of $\mathcal{S}_{\mathfrak{S}}$. Let $A \in [R, R \cup F \cup T]$, say $A = R \cup F' \cup T'$. Recall that for every $t \in T'$ we have an injective function

$$\underline{w}_{A,t} : \lceil A \rceil^{\mathcal{C}}/G \to \lceil A \smallsetminus t \rceil^{\mathcal{C}}/G$$

by Lemma 1.7.2.(c), and for all $f \in F \smallsetminus F'$ we have the injective function

$$\widehat{\underline{w}}_{R\cup F'\cup f,f} : \lceil R \cup F' \rceil^{\mathcal{C}}/G \to \lceil R \cup F' \cup f \rceil^{\mathcal{C}}/G$$

by Equation (1.9) in Definition 1.7.8.

**Definition 1.7.13.** *Let $\mathfrak{S}$ be a translative $G$–semimatroid and $(R, F, T)$ be a molecule of $\mathcal{S}_{\mathfrak{S}}$. By composing the above functions we obtain, for every $(F', T') \in P[R, F, T]$, an injective function*

$$f^R_{(F',T')} := \widehat{\underline{w}}_{R\cup F,F'} \circ \underline{w}_{R\cup F'\cup T',T'} \tag{1.11}$$

*given by*

$$\begin{array}{rccl} f^R_{(F',T')}: & \lceil R \cup F' \cup T' \rceil^{\mathcal{C}}/G & \to & \lceil R \cup F \rceil^{\mathcal{C}}/G, \\ & \mathcal{O} & \mapsto & G((Y^{\mathcal{O}}_{R\cup F'\cup T'} \smallsetminus \cup T') \cup X_{F\smallsetminus F'}). \end{array}$$

**Remark 1.7.14.** *The functions $f^R_{(F',T')}$ are well-defined by Lemma 1.7.9.(c). Moreover, injectivity implies that $m_{\mathfrak{S}}(A) := |\lceil R \cup F' \cup T' \rceil^{\mathcal{C}}/G| = |\operatorname{im} f^R_{(F',T')}|$.*

**Example 1.7.15.** *In the context of our running example, Example 1.1.8, we have that $(\varnothing, \{a, b\}, \varnothing)$ is a molecule of $\mathcal{S}_{\mathfrak{S}}$. Figure 1.11 depicts the associated poset and maps.*

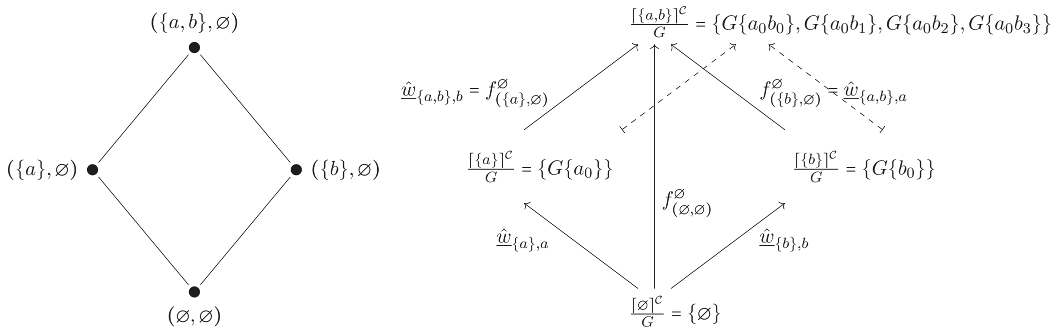



Figure 1.11: The Hasse diagram of the poset $P[\varnothing, \{a, b\}, \varnothing]$ in the context of Example 1.1.8 and, on the right-hand side, the associated diagram of sets and maps.

**Lemma 1.7.16.** *Let $\mathfrak{S}$ be translative and consider a molecule $(R, F, T)$ of $\mathcal{S}_{\mathfrak{S}}$.*

*(a) For $(F', T'), (F'', T'') \in P[R, F, T]$ we have*

$$\begin{aligned} \operatorname{im}(f^R_{(F',T')\wedge(F'',T'')}) &= \operatorname{im}(f^R_{(F'\cap F'',T'\cup T'')}) \\ &= \operatorname{im} f^R_{(F',T')} \cap \operatorname{im} f^R_{(F'',T'')}. \end{aligned}$$

*(b) In particular,*

$$\operatorname{im} f^R_{(F',T')} \subseteq \operatorname{im} f^R_{(F'',T'')} \text{ if } (F',T') \leq (F'',T'').$$

*(c) The function*

$$m_{\mathfrak{S}} : P[R,F,T] \to \mathbb{N}, \quad (F',T') \mapsto m_{\mathfrak{S}}(R \cup F' \cup T')$$

*is (weakly) increasing.*

*Proof.* The function $f^R_{(F',T')}$ is by Definition 1.7.13 a composition of functions of the type exhibited in Equations (1.5) and (1.9). Thus using Lemma 1.7.9.(c) we obtain part (a). Part (b) is an immediate consequence of (a) and by Remark 1.7.14 we conclude (c). □

**Definition 1.7.17.** *Let $(R,F,T)$ be a molecule of $\mathcal{S}_{\mathfrak{S}}$. For every pair $(F',T') \in P[R,F,T]$ define the sets*

$$Z^R(F',T') := \operatorname{im} f^R_{(F',T')}, \quad \overline{Z}^R(F',T') := Z^R(F',T') \smallsetminus \bigcup_{(F'',T'')<(F',T')} Z^R(F'',T''),$$

*and let $n_{\mathfrak{S}}[R](F',T') := |\overline{Z}^R(F',T')|$.*

Then, the following equality holds by Lemma 1.7.16.(a).

$$m_{\mathfrak{S}}(R \cup T' \cup F') = |\operatorname{im} f^R_{(F',T')}| = \sum_{p \leq (F',T')} n_{\mathfrak{S}}[R](p) \tag{1.12}$$

**Lemma 1.7.18.** *If $\mathfrak{S}$ is translative, then for every molecule $(R,F,T)$ in $\mathcal{S}_{\mathfrak{S}}$ we have*

$$\rho(R, R \cup F \cup T) = n_{\mathfrak{S}}[R](F,\varnothing).$$

*Proof.* Let $(R,F,T)$ be a molecule in $\mathcal{S}_{\mathfrak{S}}$ and in this proof let us write $P$ for $P[R,F,T]$. We start by rewriting Definition 1.1.20 as a sum over elements of $P$ as follows.

$$\begin{aligned}\rho(R, R \cup F \cup T) &:= (-1)^{|T|} \sum_{R \subseteq A \subseteq R_1} (-1)^{|R \cup F \cup T| - |A|} m_{\mathfrak{S}}(A) \\ &= \sum_{F' \subseteq F} \sum_{T' \subseteq T} (-1)^{|F \smallsetminus F'| + |T'|} m_{\mathfrak{S}}(R \cup F' \cup T')\end{aligned}$$

Then the poset $P$ has rank function $\operatorname{rk}(F',T') = |F'| + |T \smallsetminus T'|$, and by Möbius inversion from Proposition 0.1.6 (where we call $\mu_P$ the Möbius function of $P$) we can write explicitly the value of $n_{\mathfrak{S}}[R](F,\varnothing)$ from Equation (1.12).

$$
\begin{aligned}
n_{\mathfrak{S}}[R](F,\varnothing) &= \sum_{(F',T')\in P} \mu_P((F',T'),(F,\varnothing)) m_{\mathfrak{S}}(R\cup T'\cup F') \\
&= \sum_{A\in[R,R\cup F\cup T]} (-1)^{|F|+|T|-|F'|-|T\setminus T'|} m_{\mathfrak{S}}(A) \\
&= \sum_{A\in[R,R\cup F\cup T]} (-1)^{|F\setminus F'|+|T'|} m_{\mathfrak{S}}(A) \\
&= \rho(R, R\cup F\cup T)
\end{aligned}
$$

□

Since the function $n_{\mathfrak{S}}[R]$ is - by definition - never negative, as an easy corollary we obtain the following.

**Proposition 1.7.19.** *If $\mathfrak{S}$ is translative, then the pair $(\mathcal{S}_{\mathfrak{S}}, m_{\mathfrak{S}})$ satisfies property (P) of Definition 1.1.22 (and is thus called "pseudo-arithmetic").*

**Definition 1.7.20.** *For a fixed $A \subseteq E_{\mathfrak{S}}$, define the function*

$$\eta_A : \mathcal{C}_{\mathfrak{S}} \to \mathbb{N}, \quad \eta_A(\mathcal{O}) := |\{a \in A \mid a \leq_{\mathcal{P}_{\mathfrak{S}}} \kappa_{\mathfrak{S}}(\mathcal{O})\}|,$$

*where $\mathcal{P}_{\mathfrak{S}}$ is as in Definition 1.3.17.*

**Proposition 1.7.21.** *Let $(R, \varnothing, T)$ be a molecule. Then,*

$$\sum_{L\subseteq T} \rho(R\cup L, R\cup T) x^{|L|} = \sum_{\mathcal{O}\in\lceil R\rceil^{\mathcal{C}}/G} x^{\eta_T(\mathcal{O})}.$$

**Remark 1.7.22.** *Notice that, in terms of the poset $\mathcal{P}_{\mathfrak{S}}$,*

$$\eta_T(\mathcal{O}) = |\{t \in T \mid \kappa_{\mathfrak{S}}(t) \leq_{\mathcal{P}_{\mathfrak{S}}} \kappa_{\mathfrak{S}}(\mathcal{O})\}|.$$

*Thus, in the realizable case we recover the number defined in [20, Section 6].*

*Proof of Proposition 1.7.21.* First notice that, for every $L \subseteq T$, $(R\cup L, \varnothing, T\setminus L)$ is also a molecule, and that by Equation (1.11) we have immediately

$$f^{R}_{(\varnothing, L\cup L')} = f^{R}_{(\varnothing,L)} \circ f^{R\cup L}_{(\varnothing,L')}.$$

Therefore, with Lemma 1.7.16.(b) and Lemma 1.7.18 we can write the following

$$
\begin{aligned}
\rho(R\cup L, R\cup T) &= \left| \lceil R\cup L\rceil^{\mathcal{C}}/G \Big\backslash \bigcup_{t\in T\setminus L} \operatorname{im} f^{R\cup L}_{(\varnothing,\{t\})} \right| \\
&= \left| \operatorname{im} f^{R}_{(\varnothing,L)} \Big\backslash \bigcup_{t\in T\setminus L} \operatorname{im} f^{R}_{(\varnothing,L\cup\{t\})} \right| = |\overline{Z}^{R}(\varnothing, L)|
\end{aligned}
$$

where the second equality follows from injectivity of the functions $f^R$ and $f^{R\cup L}$.

We now prove that, for all $\mathcal{O} \in \lceil R \rceil^{\mathcal{C}}/G$, $\mathcal{O} \in \overline{Z}^R(\varnothing, L)$ implies

$$\{t \in T \mid t \leq_{\mathcal{C}_{\mathfrak{G}}} \kappa_{\mathfrak{G}}(\mathcal{O})\} = L.$$

Let $\mathcal{O} \in \overline{Z}^R(\varnothing, L)$ then for every $t \in T$ we have $\mathcal{O} \in \operatorname{im} f^R_{(\varnothing,t)}$ if and only if there is a representative $X_R$ of $\mathcal{O}$ and some $x_t \in t$ such that $X_R \cup x_t \in \mathcal{C}$. Since we know that $\underline{\mathrm{rk}}(R \cup t) = \underline{\mathrm{rk}}(R)$, the latter is equivalent to saying that $x_t \in \mathrm{cl}_{\mathcal{C}}(X_R)$, i.e., $t \leq \kappa_{\mathfrak{G}}(\mathcal{O})$ in $\mathcal{C}_{\mathfrak{G}}$. Now, by Lemma 1.7.16.(a) we have

$$\operatorname{im} f^R_{(\varnothing,L)} = \bigcap_{t \in L} \operatorname{im} f^R_{(\varnothing,t)}$$

and thus we see that $t \leq \kappa_{\mathfrak{G}}(\mathcal{O})$ if and only if $t \in L$.

In particular, we have that $\eta_T(\mathcal{O}) = |L|$. We can now return to the statement to be proved and write as follows.

$$\begin{aligned}\sum_{L \subseteq T} \rho(R \cup L, R \cup T) x^{|L|} &= \sum_{L \subseteq T} |\overline{Z}^R(\varnothing, L)| x^{|L|} = \sum_{L \subseteq T} \sum_{\mathcal{O} \in \overline{Z}^R(\varnothing, L)} x^{|L|} \\ &= \sum_{\mathcal{O} \in \lceil R \rceil^{\mathcal{C}}/G} x^{\eta_T(\mathcal{O})}\end{aligned}$$

□

## 1.8 Almost arithmetic actions

We now turn to what we call "almost arithmetic" actions. The name is reminiscent of the fact that one additional condition on top of translativity (i.e., normality, see Definition 1.3.9) already ensures that the multiplicity function satisfies "most of" the properties of arithmeticity. This is the content of the main result of this section (Proposition 1.8.5).

First, let us derive some basic properties of normal actions.

**Lemma 1.8.1.** *Let $\mathfrak{G}$ be almost-arithmetic and let $X \in \mathcal{C}$. Then*

- *for all $X' \in \lceil \underline{X} \rceil^{\mathcal{C}}$ we have* $\mathrm{stab}(X) = \mathrm{stab}(X')$,
- *if $x_0 \in X$ and* $\mathrm{rk}(X \setminus x_0) = \mathrm{rk}(X)$*, then* $\mathrm{stab}(X) = \mathrm{stab}(X \setminus x_0)$.

*Proof.* The first item is an immediate consequence of normality. In the second item, the inclusion $\mathrm{stab}(X) \subseteq \mathrm{stab}(X \setminus x_0)$ is evident. For the reverse inclusion, consider $g \in \mathrm{stab}(X \setminus x_0)$. Then we have $gX \cap X \supseteq X \setminus x_0$ and thus $\mathrm{rk}(X) = \mathrm{rk}(gX \cap X)$ which, by (CR1), implies $X \cup g(X) \in \mathcal{C}$ and in particular $\{x_0, g(x_0)\} \in \mathcal{C}$. translativity of the action then ensures $g \in \mathrm{stab}(x_0)$ and thus $g \in \mathrm{stab}(X)$.

□

**Definition 1.8.2.** *For $X_1, \ldots, X_k \in \mathcal{C}$ define*

$$\theta_{X_1,\ldots,X_k} : G \to \prod_{i=1}^{k} \Gamma(X_i), \quad g \mapsto ([g]_{X_1}, \ldots, [g]_{X_k}).$$

Notice that if the action is normal, this map does not depend on the choice of the $X_i$ in $\lceil \underline{X_i} \rceil^{\mathcal{C}}$.

**Lemma 1.8.3.** *Let $\mathfrak{S}$ be almost arithmetic and consider $a_1, \dots, a_k \in E_{\mathfrak{S}}$ and $A \subseteq E_{\mathfrak{S}}$ with $\underline{\mathrm{rk}}(A \cup \{a_1, \dots, a_k\}) = \underline{\mathrm{rk}}(A) + k$. Then*

$$\frac{m_{\mathfrak{S}}(A \cup \{a_1, \dots, a_k\})}{m_{\mathfrak{S}}(A)} = [\Gamma(X) \times \prod_{i=1}^{k} \Gamma(x_i) : \theta_{X, x_1, \dots, x_k}(G)]$$

*where we have chosen some (any) $X \in \lceil A \rceil^{\mathcal{C}}$ and $x_i \in a_i$ for all $i = 1, \dots, k$.*

*Proof.* Let $A$ and $a_1, \dots, a_k$ be as in the statement. Then, since the action is translative, with Lemma 1.7.2.(a) we can identify the two sides of the following equation.

$$\lceil A \cup \{a_1, \dots, a_k\} \rceil^{\mathcal{C}} = \lceil A \rceil^{\mathcal{C}} \times \prod_{i=1}^{k} \lceil a_i \rceil^{\mathcal{C}}$$

For brevity, let us write from now on $A' := A \cup \{a_1, \dots, a_k\}$. Every orbit of the action of $G$ on $\lceil A' \rceil^{\mathcal{C}}$ maps under the projection

$$p_A : \lceil A' \rceil^{\mathcal{C}} \to \lceil A \rceil^{\mathcal{C}},\ Y \mapsto Y \setminus \bigcup_{i=1}^{k} a_i$$

to one of the $m_{\mathfrak{S}}(A)$ orbits of the action on $\lceil A \rceil^{\mathcal{C}}$.

Now choose $X \in \lceil A \rceil^{\mathcal{C}}$ and consider the set of orbits in $\lceil A' \rceil^{\mathcal{C}}$ which project to $GX$, i.e., the orbits of the action of $G$ on

$$p_A^{-1}(GX) = \{(g(X), x_1, \dots, x_k) \mid g \in G,\ \forall i = 1, \dots, k : x_i \in \lceil a_i \rceil^{\mathcal{C}}\}.$$

Recall that for every $a \in E_{\mathfrak{S}}$ and $x \in a$ we have trivially $a = Gx = \lceil a \rceil^{\mathcal{C}}$, and a natural bijection of these with $\Gamma(x)$. In fact, any choice of $x_i \in \lceil a_i \rceil^{\mathcal{C}}$ for $i = 1, \dots, k$ and $X \in \lceil A \rceil^{\mathcal{C}}$ fixes a bijection $p_A^{-1}(GX) \to \Gamma(X) \times \prod_{i=1}^{k} \Gamma(x_i)$, and under this bijection the action of $G$ is the action by composition with elements of the subgroup $\theta_{X, x_1, \dots, x_k}(G)$ defined above.

Therefore we have a bijection

$$p_A^{-1}(GX)/G \to (\Gamma(X) \times \prod_{i=1}^{k} \Gamma(x_i))/\theta_{X, x_1, \dots, x_k}(G)$$

and, by normality, the group on the right hand side does not depend on the choice of $X \in \lceil A \rceil^{\mathcal{C}}$ and $x_i \in a_i$. □

**Lemma 1.8.4.** *The multiplicity function associated to an almost arithmetic $G$-semimatroid $\mathfrak{S}$ satisfies*

$$m_{\mathfrak{S}}(R) m_{\mathfrak{S}}(R \cup F \cup T) = m_{\mathfrak{S}}(R \cup T) m_{\mathfrak{S}}(R \cup F)$$

*for every molecule $(R, F, T)$ of $\mathcal{S}_{\mathfrak{S}}$.*

*Proof.* We choose $X_{R\cup T}\in\lceil R\cup T\rceil^{\mathcal{C}}$ and $X_R\subseteq X_{R\cup T}$ with $X_R\in\lceil R\rceil^{\mathcal{C}}$. Moreover write $F=\{f_1,\dots,f_k\}$ and choose $x_i\in f_i$ for all $i=1,\dots,k$. From Lemma 1.8.3 we obtain the following equalities.

$$\frac{m(R\cup F)}{m(R)}=\Big[\Gamma(X_R)\times\prod_{i=1}^{k}\Gamma(x_i):\theta_{X_R,x_1,\dots,x_k}(G)\Big]$$

$$\frac{m(R\cup T\cup F)}{m(R\cup T)}=\Big[\Gamma(X_{R\cup T})\times\prod_{i=1}^{k}\Gamma(x_i):\theta_{X_{R\cup T},x_1,\dots,x_k}(G)\Big]$$

Since $\underline{\mathrm{rk}}(R\cup T)=\underline{\mathrm{rk}}(R)$, by Lemma 1.8.1 we have $\mathrm{stab}(X_R)=\mathrm{stab}(X_{R\cup T})$, so the right-hand sides are equal. □

**Proposition 1.8.5.** *If $\mathfrak{S}$ is an almost-arithmetic action on a semimatroid, then $m_{\mathfrak{S}}$ satisfies properties (P), (A.1.2) and (A2) with respect to $\mathcal{S}_{\mathfrak{S}}$.*

*Proof.* This follows from Lemma 1.7.19, Lemma 1.8.3 and Lemma 1.8.4. □

We close the section on almost arithmetic actions with a proposition about molecules of the form $(R,F,\varnothing)$, complementing Proposition 1.7.21 above, together with which it will be used later in Section 1.10.

**Definition 1.8.6.** *Let $\mathfrak{S}$ be an almost-arithmetic $G$-semimatroid. Given a molecule $(R,F,\varnothing)$ of $\mathcal{S}_{\mathfrak{S}}$, choose an orbit $\mathcal{O}\in\lceil R\rceil^{\mathcal{C}}/G$ and fix a representative $X_R\in\mathcal{O}$. For every $F'\subseteq F$ let $\mathcal{X}(F')\subseteq\lceil R\cup F'\rceil^{\mathcal{C}}/G$ denote the subset consisting of orbits of the form $GY$ with $X_R\subseteq gY$ for some $g\in G$, i.e.,*

$$\mathcal{X}(F')=(\lceil R\cup F'\rceil^{\mathcal{C}}/G)_{\geq\mathcal{O}}\subseteq\mathcal{C}_{\mathfrak{S}}.$$

*Let $\widetilde{Z}^R_F(F'):=\overline{Z}^R(F',\varnothing)\cap\mathcal{X}(F)$. The sets $\{\widetilde{Z}^R_F(F')\}_{F'\subseteq F}$ partition $\mathcal{X}(F)$. Thus, for every $\mathcal{O}\in\mathcal{X}(F)$ we can define the number*

$$\iota(\mathcal{O}):=|F|-|F'|$$

*where $F'$ is the unique set for which $\mathcal{O}\in\widetilde{Z}^R_F(F')$.*

**Lemma 1.8.7.** *Let $\mathfrak{S}$ be an almost-arithmetic $G$-semimatroid and let $(R,F,\varnothing)$ be a molecule of $\mathcal{S}_{\mathfrak{S}}$. Then for all $F'\subseteq F$ we have*

$$|\widetilde{Z}^R_F(F')|=\frac{\rho(R,R\cup F')}{m_{\mathfrak{S}}(R)},$$

*independently on the choice of the representative $X_R$.*

*Proof.* By construction, $|Z^R(F',\varnothing)\cap\mathcal{X}(F)|=\sum_{(F'',\varnothing)\leq(F',\varnothing)}|\widetilde{Z}^R_F(F'')|$. Hence (following the notation of [95], to which we refer for basics about Möbius transforms), $|\widetilde{Z}^R_F(F')|=(\Psi\mu)(F',\varnothing)$, the evaluation at $(F',\varnothing)$ of the Möbius transform of the function $\Psi:P[R,F',\varnothing]\to\mathbb{Z}$, $(F'',\varnothing)\mapsto|Z^R(F'',\varnothing)\cap\mathcal{X}(F)|$. By Lemma 1.8.3, $\Psi(F'',\varnothing)=m(R\cup F'')/m(R)$ and, by the same computation as in Lemma 1.7.18, the Möbius transform $(\mu*\Psi)$ satisfies $(\mu*\Psi)(F',\varnothing)=\rho(R,R\cup F')/m(R)$. The claim follows. □

**Proposition 1.8.8.** *Let $\mathfrak{S}$ be an almost-arithmetic $G$-semimatroid and $(R, F, \varnothing)$ be a molecule of $\mathcal{S}_{\mathfrak{S}}$. Then*

$$\sum_{F' \subseteq F} \frac{\rho(R, R \cup F')}{m_{\mathfrak{S}}(R)} x^{|F \setminus F'|} = \sum_{\mathcal{O} \in \mathcal{X}(F)} x^{\iota(\mathcal{O})}.$$

*Proof.*

$$\begin{aligned}\sum_{F' \subseteq F} \frac{\rho(R, R \cup F')}{m_{\mathfrak{S}}(R)} x^{|F \setminus F'|} &= \sum_{F' \subseteq F} |\widetilde{Z}_F^R(F')| x^{|F \setminus F'|} \\ &= \sum_{F' \subseteq F} \sum_{\mathcal{O} \in \widetilde{Z}_F^R(F')} x^{|F \setminus F'|} = \sum_{\mathcal{O} \in \mathcal{X}(F)} x^{\iota(\mathcal{O})}\end{aligned}$$

□

## 1.9 Arithmetic actions

In this section we assume that the actions under consideration are arithmetic. This is a much more restrictive assumption than almost-arithmetic, and we will use it in the case when the $G$ is abelian in order to study matroids over rings.

We start off with a general result on arithmetic actions which will allow to state our results.

**Lemma 1.9.1.** *Let $\mathfrak{S}$ be an arithmetic $G$-semimatroid, $A \subseteq E_{\mathfrak{S}}$ and $X, Y \in \lceil A \rceil^{\mathcal{C}}$. Then,*

$$\Gamma^X = \Gamma^Y, \qquad \Gamma(X) = \Gamma(Y), \qquad W(X) = W(Y), \qquad h_X = h_Y.$$

*Proof.* First, in any translative action on a semimatroid, if $X, Y \in \lceil A \rceil^{\mathcal{C}}$ then $X$ and $Y$ contain exactly one element $x_a$ resp. $y_a$ of every orbit in $A$, and $\Gamma^Y \simeq \Gamma^X$. If, moreover, the action is normal, $\mathrm{stab}(x_a) = \mathrm{stab}(y_a)$ and thus $\Gamma^X = \Gamma^Y$.

Furthermore, for every $a \in A$ there is $g_a \in G$ with $x_a = g_a(y_a)$, and there is $\gamma_{YX} \in \Gamma^Y$ with $X = \gamma_{YX}.Y$ (recall Notation 1.3.11). If the action is arithmetic, $W(Y)$ is a group, and in particular multiplication is well-defined. For all $\gamma \in W(Y)$ we must then have $\gamma_{YX}\gamma \in W(Y)$, in particular $(\gamma_{YX}\gamma\gamma_{YX}{}^{-1}).X \in \mathcal{C}$ thus $\gamma_{YX}\gamma\gamma_{YX}{}^{-1} \in W(X)$. We now see that $W(X)$ and $W(Y)$ are isomorphic. If the action is normal, as above $\Gamma^X = \Gamma^Y$ and, since the subgroups $W(Y)$ and $W(X)$ are conjugate, again by normality they are equal. □

Justified by the previous lemma, given $A \in \underline{\mathcal{C}}$, we will choose $X \in \lceil A \rceil^{\mathcal{C}}$ and write

$$\Gamma^A := \Gamma^X, \quad \Gamma(A) := \Gamma(X), \quad W(A) := W(X), \quad h_A := h_X.$$

### 1.9.1 Arithmetic matroids

Theorem C follows easily from the next Lemma which proves that arithmetic actions induce the last of the defining properties of arithmetic matroids which was not fulfilled by almost-arithmetic actions (Example 1.4.3 shows that this difference is non-trivial).

**Definition 1.9.2.** *Let $\mathfrak{S}$ be translative. Given $A \in \underline{\mathcal{C}}$, every $X \in \lceil A \rceil^{\mathcal{C}}$ determines a bijection*

$$b_X : \lceil A \rceil^{\mathcal{C}} \to W(A), \quad \{g_x x \mid x \in X\} \mapsto ([g_x]_x)_{x \in X}$$

*and the action of a $g \in G$ on $\lceil A \rceil^{\mathcal{C}}$ corresponds to diagonal (left) multiplication by $h_A(g)$ in $W(A)$.*

Via $b_X$, for every $a_0 \in A$, the map

$$w_{A,a_0} : \lceil A \rceil^{\mathcal{C}} \to \lceil A \setminus a_0 \rceil^{\mathcal{C}}, \quad X \mapsto X \setminus a_0 \tag{1.13}$$

considered above induces a map $W(A) \to W(A \setminus a_0)$ which, by slight abuse of notation, we also call $w_{A,a_0}$. This is the restriction of the projection $\Gamma^A \to \Gamma^{A \setminus a_0}$. By Lemma 1.7.2, this map is injective whenever $\underline{\mathrm{rk}}(A) = \underline{\mathrm{rk}}(A \setminus a_0)$.

**Lemma 1.9.3.** *Let $\mathfrak{S}$ be a $G$-semimatroid associated to an arithmetic action. Then $m_{\mathfrak{S}}$ satisfies property (A.1.1) of Definition 1.1.22.*

*Proof.* Consider $A \in \underline{\mathcal{C}}$, $a_0 \in A$ such that $\underline{\mathrm{rk}}(A \setminus a_0) = \underline{\mathrm{rk}}(A)$. The injective homomorphism $w_{A,a_0} : W(A) \to W(A \setminus a_0)$ maps $\theta_A(G)$ to $\theta_{A \setminus a_0}(G)$. We have $m_{\mathfrak{S}}(A \setminus a_0) = [W(A \setminus a_0) : \theta_{A \setminus a_0}(G)]$ and

$$m_{\mathfrak{S}}(A) = [W(A) : \theta_A(G)] = [\mathrm{im}(w_{A,a_0}) : \theta_{A \setminus a_0}(G)].$$

Now the claim follows from additivity of the index:

$$m_{\mathfrak{S}}(A \setminus a_0) = [W(A \setminus a_0) : W(A)] m_{\mathfrak{S}}(A).$$

□

As a preparation for the next section, let us here discuss some further aspects of arithmetic actions.

Consider some $A \in \underline{\mathcal{C}}$ and recall from Notation 1.3.11 and Lemma 1.9.1 the map $h_A : G \to W(A)$. Notice that $\mathrm{stab}(A) = \ker h_A$, thus $h_A$ induces an injective $h'_A : \Gamma(A) \to W(A)$ with $\mathrm{im}\, h_A = \mathrm{im}\, h'_A$, and we can write an exact sequence

$$0 \to \Gamma(A) \xrightarrow{h'_A} W(A) \to C(A) \to 0$$

where $C(A) = \mathrm{coker}\, h'_A$ is isomorphic to $W(A)/h_A(G)$.

Given $A \in \underline{\mathcal{C}}$ and $a_0 \in A$, we obtain in particular the commutative square at the left hand side of the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(A) & \xrightarrow{h'_A} & W(A) & \longrightarrow & C(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle i'_{A,a_0}} & & \Big\downarrow{\scriptstyle w_{A,a_0}} & & \Big\downarrow{\scriptstyle c_{A,a_0}} & & \\
0 & \longrightarrow & \Gamma(A \smallsetminus a_0) & \xrightarrow{h'_{A \smallsetminus a_0}} & W(A \smallsetminus a_0) & \longrightarrow & C(A \smallsetminus a_0) & \longrightarrow & 0
\end{array}$$

and thus by functoriality we get a group homomorphism $c_{A,a_0} : C(A) \to C(A \smallsetminus a_0)$ between cokernels.

**Remark 1.9.4.** *The map $b_X$ induces a bijection between $\lceil A \rceil^{\mathcal{C}}/G$ and $C(A)$. The (natural) group structure of $C(A)$ can be seen as additional data that can be extracted from $\mathfrak{S}$. Recent results in the topology of toric arrangements [23, Example 7.3.2] show that this additional data has an algebraic-topological significance. The next section focuses on another point of interest of the finite groups $C(A)$: namely, in the realizable case they appear naturally as torsion subgroups of the associated matroid over $\mathbb{Z}$.*

### 1.9.2 Matroids over rings

Suppose now that $G$ is a finitely generated abelian group and the action arithmetic. Then, we can construct $M_{\mathfrak{S}}$ as in Definition 1.3.14 and ask when this defines a matroid over $\mathbb{Z}$. As discussed in Section 1.4, we do not have non-trivial examples of cases when $M_{\mathfrak{S}}$ is a non-realizable matroid over $\mathbb{Z}$. What we can prove is a characterization in terms of $W(A)$ of when $M_{\mathfrak{S}}$ is a representable matroid over $\mathbb{Z}$. For this, it is enough to stick to the centred case.

**Proposition 1.9.5.** *Let $G$ be a finitely generated abelian group and $\mathfrak{S}$ an arithmetic, centred $G$-semimatroid. Then $M_{\mathfrak{S}}$ is a representable matroid over $\mathbb{Z}$ if and only if the modules $W(A)$ are pure submodules of $\Gamma^A$.*

*Proof.* Let us start by noticing that, since the action is centred, $W(A)$ and $c_{A,a_0}$ are defined for all $A \subseteq E_{\mathfrak{S}}$ and $a_0 \in E_{\mathfrak{S}}$.

If $M_{\mathfrak{S}}$ representable then it arises as in Section 1.2 from a list of primitive integer vectors (i.e., the entries of each vector are mutually coprime). Then, under the canonical isomorphism $\Gamma^A \simeq \mathbb{Z}^A$, the group $W(A)$ maps to the pure subgroup of $\mathbb{Z}^A$ described in Remark 1.2.10.

Suppose now that all $W(A)$ are direct summands of $\Gamma^A$. Then we can form the following diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & C(A) & \longrightarrow & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & W(A) & \longrightarrow & \Gamma^A & \longrightarrow & L(A) & \longrightarrow & 0 \\
 & & \uparrow & & =\uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(A) & \longrightarrow & \Gamma^A & \longrightarrow & J(A) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & K(A) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

and conclude that $C(A)$, which is pure torsion by cardinality reasons, is isomorphic to the kernel $K(A)$ of the map induced between the cokernels $J(A)$ and $L(A)$. This map is surjective (e.g., by the snake lemma its cokernel is 0) and, since $L(A)$ is free by assumption, we conclude that $J(A) \simeq L(A) \oplus C(A) \simeq M_{\mathfrak{G}}(E_{\mathfrak{G}} \setminus A)$.

Now we can show that $M_{\mathfrak{G}}$ is representable, by giving a concrete representation. For $A \subseteq E_{\mathfrak{G}}$ and $a_0 \in E_{\mathfrak{G}}$ let $\mu[A, a_0] : J(Aa_0) \to J(A)$ be the map induced from the universal property of coker in

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(Aa_0) & \longrightarrow & \Gamma^{Aa_0} & \longrightarrow & J(Aa_0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Gamma(A) & \longrightarrow & \Gamma^A & \longrightarrow & J(A) & \longrightarrow & 0
\end{array}
$$

where the left vertical map is induced by the inclusion $\mathrm{stab}(Aa_0) \subseteq \mathrm{stab}(A)$ and the middle one is the standard projection. As above, one easily checks that this is a surjection with cyclic kernel (in fact, by naturality this map corresponds to $c_{A,a_0}$ under the isomorphism $C(A) \simeq J(A)$).

To prove axiom (R) we have now to check that every square corresponding to some $A \subseteq E$, $a_1, a_2 \in E$ is a pushout - but this follows readily from the fact that the cokernel of a (mono)morphism of pushout squares of surjections with cyclic kernels is a pushout square of surjections with cyclic kernels. □

## 1.10 Tutte polynomials of group actions

In this section we study the Tutte polynomial associated to a group action on a semimatroid and, as an application, we extend to the generality of group actions on semimatroids (in particular, beyond the realizable case) two important combinatorial interpretations of Tutte polynomials of toric arrangements.

### 1.10.1 The characteristic polynomial of $\mathcal{P}_{\mathfrak{G}}$

For general cofinite $G$-semimatroids, since the associated action on the semimatroids' geometric semilattice is by rank-preserving maps, the poset $\mathcal{P}_{\mathfrak{G}}$ is ranked (by the rank function $\underline{\mathrm{rk}}$). We can thus define the **characteristic polynomial of $\mathcal{P}_{\mathfrak{G}}$** as

$$\chi_{\mathfrak{G}}(t) := \sum_{p \in \mathcal{P}_{\mathfrak{G}}} \mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0}, p) t^{r - \underline{\mathrm{rk}}(p)},$$

where $r$ is the rank of $\mathcal{S}_{\mathfrak{G}}$ and $\mu_{\mathfrak{G}}$ is the Möbius function of $\mathcal{P}_{\mathfrak{G}}$ (notice that $\mathcal{P}_{\mathfrak{G}}$ has a unique minimal element corresponding to the empty subset of $E_{\mathfrak{G}}$).

**Lemma 1.10.1.** *Let $\mathfrak{G}$ be weakly translative. Then, for every $x \in \mathcal{L}$, the intervals $[\hat{0}, Gx]$ in $\mathcal{P}_{\mathfrak{G}}$ and $[\hat{0}, x]$ in $\mathcal{L}$ are poset-isomorphic. In particular, intervals in $\mathcal{P}_{\mathfrak{G}}$ are geometric lattices.*

*Proof.* For every $q \leq_{\mathcal{P}_{\mathfrak{G}}} p = Gx_p$, by definition there is $x_q \in q$ with $x_q \leq_{\mathcal{L}} x_p$. Any other such $x'_q \in q$ must satisfy $x'_q = gx_q$ for some $g \in G$, thus for every atom $x_a$ of $\mathcal{L}$ with $x_a \leq_{\mathcal{L}} x_q \leq_{\mathcal{L}} x_p$, $gx_a \leq_{\mathcal{L}} x_p$. In particular, for every $s \in x_a$, $\{s, gs\} \in \mathcal{C}$ and by weak translativity $\mathrm{rk}\{s, gs\} = 1$. Thus $gx_a \subseteq \mathrm{cl}\, x_a = x_a$ and, by symmetry, $x_a = gx_a$. This is true for all atoms $x_a$ and hence, because the interval $[\hat{0}, x_p]$ is atomic, we have $x_q = x'_q$.

Therefore the mapping

$$[\hat{0}, p]_{\mathcal{P}_{\mathfrak{G}}} \to [\hat{0}, x_p]_{\mathcal{L}}, \ q \mapsto x_q$$

is well-defined and order preserving. So is clearly its inverse

$$[\hat{0}, x_p]_{\mathcal{L}} \to [\hat{0}, p]_{\mathcal{P}_{\mathfrak{G}}}, \ x \mapsto Gx$$

and thus the two intervals are poset-isomorphic. □

*Proof of Theorem F.* Let us first consider some $p \in \mathcal{P}_{\mathfrak{G}}$ with $p > \hat{0}$. By Hall's theorem [95, Proposition 3.8.5] the value $\mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0}, p)$ is the reduced Euler characteristics of the "open interval" $[\hat{0}, p] \setminus \{\hat{0}, p\}$. By Lemma 1.10.1, the interval $[\hat{0}, p]$ is a geometric lattice with set of atoms, say $A(p)$, and his reduced Euler characteristics can be computed by means of the **atomic complex** [103], a simplicial complex with set of simplices $\Delta_p = \{B \subseteq A(p) \mid \vee B < p\}$. We obtain

$$\mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0}, p) = \sum_{A \in \Delta_p} (-1)^{|A|-1} = \sum_{A \in D_p} (-1)^{|A|},$$

where $D_p := \{A \subseteq A(p) \mid \vee A = p\}$ and the second equality is derived from the boolean identity $\sum_{A \subseteq A(p)} (-1)^{|A|} = 0$. Moreover, with $\widetilde{D}_p := \{\widetilde{A} \subseteq E_{\mathfrak{G}} \mid \mathrm{cl}(\widetilde{A}) = p\}$

an easy computation (using the fact that $\mathcal{S}_{\mathfrak{G}}$ has no loops) shows

$$\begin{aligned}\sum_{\widetilde{A}\in\widetilde{D}_p}(-1)^{|\widetilde{A}|} &= \sum_{A\in D_p}\sum_{\substack{\widetilde{A}=\coprod_{a\in A}X_a\\ \underline{\mathrm{cl}}(X_a)=a}}(-1)^{|\widetilde{A}|}\\ &= \sum_{A\in D_p}\prod_{a\in A}\left[\sum_{\emptyset\neq X_a\subseteq a}(-1)^{|X_a|}\right] = \sum_{A\in D_p}(-1)^{|A|} = \mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0},p).\end{aligned}$$

Notice that the equality $\sum_{\widetilde{A}\in\widetilde{D}_p}(-1)^{|\widetilde{A}|} = \mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0},p)$, which we just proved for $p > \hat{0}$, holds trivially for $p = \hat{0}$. Moreover, $\widetilde{A}\in\widetilde{D}_p$ implies in particular $\underline{\mathrm{rk}}(\widetilde{A}) = \underline{\mathrm{rk}}(p)$. Then,

$$\begin{aligned}\chi_{\mathfrak{G}}(t) &= \sum_{p\in\mathcal{P}_{\mathfrak{G}}}\mu_{\mathcal{P}_{\mathfrak{G}}}(\hat{0},p)t^{r-\underline{\mathrm{rk}}(p)} = \sum_{p\in\mathcal{P}_{\mathfrak{G}}}\sum_{\widetilde{A}\in\widetilde{D}_p}(-1)^{|\widetilde{A}|}t^{r-\underline{\mathrm{rk}}(p)}\\ &= \sum_{\widetilde{A}\in\underline{\mathcal{C}}}(-1)^{|\widetilde{A}|}\sum_{p\in P_{\widetilde{A}}}t^{r-\underline{\mathrm{rk}}(\widetilde{A})}\end{aligned}$$

where $P_{\widetilde{A}} := \{p\in\mathcal{P}_{\mathfrak{G}} \mid \widetilde{A}\in\widetilde{D}_p\} = \lceil\widetilde{A}\rceil^{\mathcal{C}}/G$ for every $\widetilde{A}\in\underline{\mathcal{C}}$ is a set with exactly $m_{\mathfrak{G}}(\widetilde{A})$ elements. Thus,

$$\begin{aligned}\chi_{\mathfrak{G}}(t) &= \sum_{A\in\underline{\mathcal{C}}}(-1)^{|A|}m_{\mathfrak{G}}(A)t^{r-\underline{\mathrm{rk}}(A)}\\ &= (-1)^r\sum_{A\in\underline{\mathcal{C}}}m_{\mathfrak{G}}(A)(-1)^{|A|-\mathrm{rk}(A)}(-t)^{r-\underline{\mathrm{rk}}(A)}\\ &= (-1)^r T_{\mathfrak{G}}(1-t,0)\end{aligned}$$

where, as above, $r$ denotes the rank of $\mathcal{S}_{\mathfrak{G}}$. □

### 1.10.2 The corank-nullity polynomial of $\mathcal{C}_{\mathfrak{G}}$

The corank-nullity polynomial of the poset $\mathcal{C}_{\mathfrak{G}}$ is

$$s(\mathcal{C}_{\mathfrak{G}};u,v) = \sum_{GX\in\mathcal{C}_{\mathfrak{G}}}u^{(r-\mathrm{rk}(X))}v^{(|X|-\mathrm{rk}(X))}.$$

**Proposition 1.10.2.** *If $\mathfrak{G}$ is translative,*

$$T_{\mathfrak{G}}(x,y) = s(\mathcal{C}_{\mathfrak{G}};x-1,y-1).$$

*Proof.* When $\mathfrak{G}$ is translative we have that $|X| = |\underline{X}|$ and (by Corollary 1.6.5) $\mathrm{rk}(X) = \underline{\mathrm{rk}}(\underline{X})$. Then,

$$s(\mathcal{C}_{\mathfrak{G}};u,v) = \sum_{GX\in\mathcal{C}_{\mathfrak{G}}}u^{(r-\mathrm{rk}(X))}v^{(|X|-\mathrm{rk}(X))} = \sum_{A\in\underline{\mathcal{C}}}\sum_{\substack{GX\in\mathcal{C}_{\mathfrak{G}}\\ \underline{X}=A}}u^{(r-\underline{\mathrm{rk}}(A))}v^{(|A|-\underline{\mathrm{rk}}(A))}$$

and the claim follows by setting $u = x-1$ and $v = y-1$. □

### 1.10.3 Activities

We now turn to a generalization and new combinatorial interpretation of the basis-activity decomposition of arithmetic Tutte polynomials as defined in [20]. Consider an almost arithmetic $G$-semimatroid $\mathfrak{S}$ and fix a total ordering of $E_{\mathfrak{S}}$.

**Remark 1.10.3.** *Since we will not need details here, but only the statement of the next lemma, we refer to Ardila [1] for the definition of internal and external activity of bases of a finite semimatroid.*

If $\mathcal{B}_{\mathfrak{S}}$ is the set of bases of the finite semimatroid $\mathcal{S}_{\mathfrak{S}}$, we denote by $I(B)$ and $E(B)$ the sets of internally, resp. externally active elements of any $B \in \mathcal{B}$, and write $R_B := B \setminus I(B)$.

**Lemma 1.10.4** (Proposition 9.11 of [1])**.** *If $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ is a finite semimatroid with set of bases $\mathcal{B}$, then for every basis $B$, $(R_B, I(B), E(B))$ is a molecule, and*

$$\mathcal{C} = \biguplus_{B \in \mathcal{B}} [R_B, B \cup E(B)]$$

We use this decomposition, which generalizes that for matroids proved in [26], in order to rewrite the sum in Equation (1.2) as a sum over all bases.

**Theorem H.** *Let $\mathfrak{S}$ be an almost-arithmetic $G$-semimatroid such that $\mathcal{S}_{\mathfrak{S}}$ is a semimatroid. Then*

$$T_{\mathfrak{S}}(x,y) = \sum_{B \in \mathcal{B}_{\mathfrak{S}}} \left( \sum_{p \in \mathcal{Z}(B)} x^{\iota(p)} \right) \left( \sum_{c \in \lceil R_B \rceil^{\mathcal{C}}/G} y^{\eta_{E(B)}(c)} \right)$$

*where*

$\eta_{E(B)}(c)$ *is the number of $e \in E(B)$ with $e \le \kappa_{\mathfrak{S}}(c)$ in $\mathcal{C}_{\mathfrak{S}}$ (Definition 1.7.20),*

$\mathcal{Z}(B)$ *denotes the set $\mathcal{X}(I(B))$ associated to the molecule $(R_B, I(B), \varnothing)$ in Definition 1.8.6 and, accordingly,*

$\iota(p)$ *is the number defined in Definition 1.8.6.*

*In particular, the theorem holds when $\mathfrak{S}$ is centred, in which case it extends [28, Theorem 6.3] to the non-realizable (and non-arithmetic) case.*

*Proof.* First, using Lemma 1.10.4 we rewrite

$$T_{\mathfrak{S}}(x,y) = \sum_{B \in \mathcal{B}} \sum_{A \in \mu(B)} m_{\mathfrak{S}}(A)(x-1)^{\mathrm{rk}(\mathcal{S}_{\mathfrak{S}}) - \underline{\mathrm{rk}}(A)}(y-1)^{|A| - \underline{\mathrm{rk}}(A)}$$

and then, using Lemma 4.3 of [20] (where only condition (A2) is used) we can rewrite again

$$T_{\mathfrak{S}}(x,y) =$$

$$\sum_{B\in\mathcal{B}}\left(\sum_{F\subseteq I(B)}\frac{\rho(R_B,R_B\cup(I(B)\smallsetminus F))}{m(R_B)}x^{|F|}\right)\left(\sum_{T\subseteq E(B)}\rho(R_B\cup T,R_B\cup E(B))y^{|T|}\right).$$

Where the right-hand side factors are ready to be treated with Proposition 1.7.21 applied to the molecule $(R_B,\varnothing,E(B))$, while the left-hand side factors equal the claimed polynomials by Proposition 1.8.8 applied to the molecule $(R_B,I(B),\varnothing)$. □

### 1.10.4 Tutte-Grothendieck recursion

We have seen (Section 1.3) that the matroid operations of contraction and deletion extend in a natural way to the context of $G$-semimatroids. In this section we study these operations, showing that the Tutte polynomial of a translative action is a Tutte-Grothendieck invariant.

Recall the definitions and notations from Section 1.1.1 and Section 1.3. In the following, given a locally ranked triple $\mathcal{S}$ we will write $\mathcal{C}(\mathcal{S})$ for its "second component" simplicial complex.

**Lemma 1.10.5.** *Let $\mathfrak{S}:G\circlearrowright(S,\mathcal{C},\mathrm{rk})$ be a weakly translative $G$-semimatroid, and $e\in E_{\mathfrak{S}}$. Then,*

*(1) there is a surjection $\phi:\mathcal{C}(\mathcal{S}_{\mathfrak{S}/e})\to\mathcal{C}(\mathcal{S}_{\mathfrak{S}}/e)$ with $\mathrm{rk}_{\mathfrak{S}}(\phi(A)\cup e)-\mathrm{rk}_{\mathfrak{S}}(e)=\mathrm{rk}_{\mathfrak{S}/e}(A)$ which, if the action is translative, also satisfies $|\phi(A)|=|A|$;*

*(2) $\mathcal{P}_{\mathfrak{S}/e}=(\mathcal{P}_{\mathfrak{S}})_{\geq e}$.*

*Moreover,*

*(3) $m_{\mathfrak{S}}(A\cup e)=\sum_{A'\in\phi^{-1}(A)}m_{\mathfrak{S}/e}(A')$.*

*Proof.* Let us choose a fixed representative $x_e\in e$ and write throughout the proof $H:=\mathrm{stab}(x_e)$. In order to prove (1), we start by recalling that $\mathcal{C}_{(\mathfrak{S}/e)}=(\mathcal{C}_{/x_e})/H$ and the natural order on $\mathcal{C}_{\mathfrak{S}}$ (Remark 1.3.3). Define now the following function.

$$\widetilde{\phi}:\mathcal{C}_{\mathfrak{S}/e}\to(\mathcal{C}_{\mathfrak{S}})_{\geq e},\quad H\{x_1,\ldots,x_k\}\mapsto G\{x_1,\ldots,x_k,x_e\}.$$

The function $\widetilde{\phi}$ is a bijection, because the function

$$G\{x_1,\ldots,x_k,gx_e\}\mapsto H\{g^{-1}x_1,\ldots,g^{-1}x_k\}$$

is well-defined and inverse to $\widetilde{\phi}$.

In order to prove (2) we notice that $\widetilde{\phi}$ commutes with the relevant closure operators, i.e.,

$$\widetilde{\phi}\circ\kappa_{\mathfrak{S}/e}=\kappa_{\mathfrak{S}}\circ\widetilde{\phi}.$$

Bijectivity of $\widetilde{\phi}$ implies then that $\mathcal{P}_{\mathfrak{S}/e}=\kappa_{\mathfrak{S}}((\mathcal{C}_{\mathfrak{S}})_{\geq e})$, and the latter is easily seen to equal $(\mathcal{P}_{\mathfrak{S}})_{\geq e}$. Thus, (2) holds.

Consider now the map

$$\phi : \underline{\mathcal{C}_{/x_e}} \to \underline{\mathcal{C}}_{/e}; \{Hx_1, \dots, Hx_k\} \mapsto \{Gx_1, \dots, Gx_k\}$$

and the following diagram

$$\begin{array}{ccc} \mathcal{C}_{\mathfrak{S}/e} & \xrightarrow{\;\widetilde{\phi}\;} & (\mathcal{C}_{\mathfrak{S}})_{\geq e} \\ {\scriptstyle \lfloor\cdot\rfloor}\big\downarrow & & \big\downarrow{\scriptstyle \lfloor\cdot\rfloor\setminus\{e\}} \\ \underline{\mathcal{C}_{/x_e}} & \xrightarrow{\;\phi\;} & \underline{\mathcal{C}}_{/e}. \end{array}$$

Commutativity is evident once we evaluate all maps on a specific argument as follows.

$$\begin{array}{ccc} H\{x_1, \dots, x_k\} & \longmapsto & G\{x_1, \dots, x_k, x_e\} \\ \big\downarrow & & \big\downarrow \\ \{Hx_1, \dots, Hx_k\} & \longmapsto & \{Gx_1, \dots, Gx_k\} \end{array}$$

Now, for every $A \in \underline{\mathcal{C}}_{/e}$ the map $\widetilde{\phi}$ gives a bijection between the $\lfloor\cdot\rfloor \setminus \{e\}$-preimage of $A$ and the $\lfloor\cdot\rfloor$-preimage of $\phi^{-1}(A)$, which proves (3). Claim (1) follows by inspecting the definition of the rank and, for the claim about cardinality, by noticing that if $Hx_1 \neq Hx_2$ and $gx_1 = x_2$ for some $g \in G$, then $\{x_1, gx_1\} \in \mathcal{C}$ and by translativity $x_1 = gx_1 = x_2$, a contradiction. □

**Proposition 1.10.6.** *Let $\mathfrak{S}$ denote a $G$-semimatroid and fix $e \in E_{\mathfrak{S}}$. If $\mathfrak{S}$ is weakly translative– resp. translative, normal, arithmetic –, then so are also $\mathfrak{S}/e$ and $\mathfrak{S}\backslash e$. Moreover, if $\mathfrak{S}$ is weakly translative and cofinite, then $\mathfrak{S}/e$ and $\mathfrak{S}\backslash e$ are also cofinite.*

*Proof.* The treatment of $\mathfrak{S}\backslash e$ is trivial: indeed, the same group acts on a smaller set of elements with the same requirements. We will thus examine the case $\mathfrak{S}/e$. Choose $x_e \in e$ and let $H := \mathrm{stab}(x_e)$.

- $\mathfrak{S}$ *weakly translative.* To check weak translativity for $\mathfrak{S}/e$ consider some $y \in S_{/x_e}$ and suppose $\{y, hy\} \in \mathcal{C}_{/x_e}$ for some $h \in H$. This means by definition that $\{y, hy, x_e\} \in \mathcal{C}$, thus $\{y, hy\} \in \mathcal{C}$ and, by weak translativity of $\mathfrak{S}$, we have $\mathrm{rk}_{\mathcal{C}}(\{y, hy\}) = \mathrm{rk}_{\mathcal{C}}(\{y\})$. Now by (R3) we know

$$\mathrm{rk}_{\mathcal{C}}(\{y\}) + \mathrm{rk}_{\mathcal{C}}(\{y, hy, x_e\}) \leq \mathrm{rk}_{\mathcal{C}}(\{y, x_e\}) + \mathrm{rk}_{\mathcal{C}}(\{y, hy\}).$$

By subtracting $\mathrm{rk}_{\mathcal{C}}(\{y\})$ from both sides we obtain the inequality $\mathrm{rk}_{\mathcal{C}}(\{y, hy, x_e\}) \leq \mathrm{rk}_{\mathcal{C}}(\{y, x_e\})$ and, by (R2), $\mathrm{rk}_{\mathcal{C}}(\{y, hy, x_e\}) = \mathrm{rk}_{\mathcal{C}}(\{y, x_e\})$. We are now left with computing

$$\begin{aligned} \mathrm{rk}_{\mathcal{C}/x_e}(\{y, hy\}) &\overset{\text{def.}}{=} \mathrm{rk}_{\mathcal{C}}(\{y, hy, x_e\}) - \mathrm{rk}_{\mathcal{C}}(\{x_e\}) \\ &= \mathrm{rk}_{\mathcal{C}}(\{y, x_e\}) - \mathrm{rk}_{\mathcal{C}}(\{x_e\}) \overset{\text{def.}}{=} \mathrm{rk}_{\mathcal{C}/x_e}(\{y\}) \end{aligned}$$

- $\mathfrak{S}$ *translative.* As above, consider some $y \in S_{/x_e}$ and suppose $\{y, hy\} \in \mathcal{C}_{/x_e}$ for some $h \in H$. This means that $\{y, hy, x_e\} \in \mathcal{C}$, thus $\{y, hy\} \in \mathcal{C}$ and, by translativity of $\mathfrak{S}$, $hy = y$ as required.

- $\mathfrak{S}$ *normal.* Let $X \in \mathcal{C}_{/x_e}$ then $\operatorname{stab}_H(X) = \operatorname{stab}_G(X) \cap H$ is normal in $G$ because it is the intersection of two normal subgroups. *A fortiori* it is normal in $H$.

- $\mathfrak{S}$ *arithmetic.* Let $X = \{x_1, \dots, x_k\} \in \mathcal{C}_{/x_e}$. For all $i$ there is a natural group homomorphism
$$\omega_i : \Gamma_{/e}(x_i) = H/\operatorname{stab}_H(x_i) \hookrightarrow G/\operatorname{stab}_G(x_i) = \Gamma(x_i)$$
and these induce a natural group homomorphism
$$\omega : \Gamma_{/e}^X \to \Gamma^{X \cup x_e}, \quad (\gamma_1, \dots, \gamma_k) \mapsto (\mathrm{id}, \omega_1(\gamma_1), \dots, \omega_k(\gamma_k)).$$
Now consider $\gamma, \gamma' \in W_{/e}(X)$. Then clearly $\omega(\gamma), \omega(\gamma') \in W(X \cup x_e)$ and, by arithmeticity of $\mathfrak{S}$,
$$\omega(\gamma)\omega(\gamma') = (\mathrm{id}, \omega_1(\gamma_1)\omega_1(\gamma_1'), \dots) = (\mathrm{id}, \omega_1(\gamma_1\gamma_1'), \dots) \in W(X \cup x_e).$$
Now, this means that $\omega(\gamma\gamma').(X \cup x_e) = \gamma\gamma'.X \cup \{x_e\} \in \mathcal{C}$, hence $\gamma\gamma'.X \in \mathcal{C}_{/x_e}$ thus by definition $\gamma\gamma' \in W_{/e}(X)$.

- $\mathfrak{S}$ *(weakly translative and) cofinite.* Cofiniteness of $\mathfrak{S}\backslash e$ is trivial, and that of $\mathfrak{S}/e$ is a consequence of Lemma 1.10.5.(3).

□

We can now state and prove the Tutte-Grothendieck recursion for Tutte polynomials of translative $G$-semimatroids, generalizing the corresponding result of [20] for the arithmetic and centred case.

*Proof of Theorem G.* In this proof for greater clarity we will write $\mathrm{rk}_{\mathfrak{S}}$, resp. $\mathrm{rk}_{\mathfrak{S}/e}$ for the rank functions of $\mathcal{S}_{\mathfrak{S}}$, resp. $\mathcal{S}_{\mathfrak{S}/e}$ (in particular, $\mathrm{rk}_{\mathfrak{S}}$ corresponds to $\underline{\mathrm{rk}}$ in the remainder of this section).

We follow [1, Proposition 8.2], where the analogous results for semimatroids are proved, and start by rewriting the definition.

$$\begin{aligned}
T_{\mathfrak{S}}(x,y) :&= \sum_{A \in \underline{\mathcal{C}}} m_{\mathfrak{S}}(A)(x-1)^{r(\mathcal{S}_{\mathfrak{S}})-\mathrm{rk}_{\mathfrak{S}}(A)}(y-1)^{|A|-\mathrm{rk}_{\mathfrak{S}}(A)} \\
&= \sum_{\underbrace{A \in \underline{\mathcal{C}}, e \notin A}_{A \in \underline{\mathcal{C}}_{\backslash e} = \mathcal{C}(\mathcal{S}_{\mathfrak{S}\backslash e})}} m_{\mathfrak{S}}(A)(x-1)^{r(\mathcal{S}_{\mathfrak{S}})-\mathrm{rk}_{\mathfrak{S}}(A)}(y-1)^{|A|-\mathrm{rk}_{\mathfrak{S}}(A)} \\
&\quad + \sum_{A \cup e \in \underline{\mathcal{C}}} m_{\mathfrak{S}}(A \cup e)(x-1)^{r(\mathcal{S}_{\mathfrak{S}})-\mathrm{rk}_{\mathfrak{S}}(A \cup e)}(y-1)^{|A \cup e|-\mathrm{rk}_{\mathfrak{S}}(A \cup e)}
\end{aligned}$$

The second summand can be rewritten as follows by Lemma 1.10.5.

$$\underbrace{\sum_{A\in\underline{\mathcal{C}}_{/e}}\sum_{A'\in\phi^{-1}(A)}}_{A'\in\mathcal{C}(\mathcal{S}_{\mathfrak{S}/e})} m_{\mathfrak{S}/e}(A')(x-1)^{r(\mathcal{S}_{\mathfrak{S}/e})-\mathrm{rk}_{\mathfrak{S}/e}(A')}(y-1)^{|A'|+1-\mathrm{rk}_{\mathfrak{S}/e}(A')-\mathrm{rk}_{\mathfrak{S}}(e)}$$

If $e$ is neither a loop nor a coloop, by Remark 1.3.25 and Lemma 1.10.5 we have $\mathrm{rk}(\mathcal{S}_{\mathfrak{S}}) = \mathrm{rk}(\mathcal{S}_{\mathfrak{S}\setminus e})$ and $\mathrm{rk}_{\mathfrak{S}}(e) = 1$, thus the two summands are exactly $T_{\mathfrak{S}\setminus e}(x,y)$ and $T_{\mathfrak{S}/e}(x,y)$, respectively. If $e$ is a coloop, $\mathrm{rk}(\mathcal{S}_{\mathfrak{S}}) = \mathrm{rk}(\mathcal{S}_{\mathfrak{S}\setminus e}) - 1$ (and $\mathrm{rk}_{\mathfrak{S}}(e) = 1$) and thus we have $T_{\mathfrak{S}}(x,y) = (x-1)T_{\mathfrak{S}\setminus e}(x,y) + T_{\mathfrak{S}/e}(x,y)$. Finally, when $e$ is a loop we have $\mathrm{rk}_{\mathfrak{S}}(e) = 0$ (but still $\mathrm{rk}(\mathcal{S}_{\mathfrak{S}}) = \mathrm{rk}(\mathcal{S}_{\mathfrak{S}\setminus e})$) and we easily get the claimed identity. □

# Part III

# Sign vector systems and topological representation

# Chapter 2

# Pseudoarrangements

In this chapter we will first give a short introduction to the theory of hyperplane arrangements and toric arrangements. Then we will go over to non-linear arrangements in Euclidean space: arrangements of pseudospheres in $S^d$ and arrangements of pseudolines in $\mathbb{R}^2$ (or $\mathbb{P}^2$). Furthermore, we will present two different attempts [48, 77] to generalize these concepts to a setting of codimension one *pseudosubspaces* for arbitrary dimensions.

Although the "wiggly" pseudoarrangements still maintain a lot of the combinatorial and topological properties of hyperplane arrangements they are quite hard to handle without additional restrictions to make them "nice". For example Edmonds and Mandel used the concept of piecewise linear topology [75] in order to maintain some control on the possible behaviour in arrangements of pseudohemispheres, which were introduced by Folkman and Lawrence [47] as a topological representation of the combinatorial concept of an oriented matroid.

## 2.1 Preliminaries

### 2.1.1 Hyperplane arrangements

In this chapter we introduce the theory of hyperplane arrangements. Basic literature is the book of Orlik and Terao [87] and as a good introduction we recommend the lecture notes of Stanley [96].

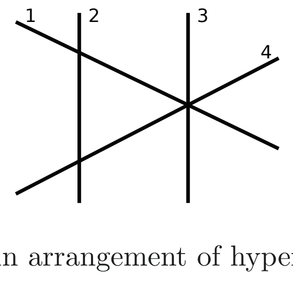



Figure 2.1: An arrangement of hyperplanes in $\mathbb{R}^2$.

**Definition 2.1.1.** *Let $V$ be a $d$-dimensional vector space over a field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$), and let $l \in V^*$ be a linear form, and $b \in \mathbb{K}$. Then an* **affine hyperplane** *is defined as*

$$H = \{v \in V \mid l(v) = b\}$$

*Let $I$ be a countable index set. Then the collection of affine hyperplanes $\mathscr{A} = \{H_i\}_{i \in I}$ is called an* **(affine) hyperplane arrangement**.

In the traditional definition hyperplane arrangements were finite collections. But we will consider **locally finite** arrangements $\mathscr{A} = \{H_i\}_{i \in I}$, that is to say each point in $V$ has a neighbourhood that intersects only a finite number of hyperplanes in $\mathscr{A}$. We call a hyperplane arrangement **central** if the intersection of $\mathscr{A}$ is non-empty, i.e., $\bigcap \mathscr{A} = \bigcap_{i \in I} H_i \neq \emptyset$. In the case when all hyperplanes are linear or equivalently if $0 \in \bigcap \mathscr{A}$, we call the hyperplane arrangement **linear** (the reader should note that our last two notations differ from the ones used in [87]). Furthermore, it is called **real** or **complex** if $V$ is a real or complex vector space.

We are interested in the topology of the complement of the arrangement, which we denote by

$$\mathcal{M}(\mathscr{A}) = V \setminus \bigcup \mathscr{A}.$$

If $\mathscr{A}$ is a real hyperplane arrangement its complement $\mathcal{M}(\mathscr{A}) = \mathbb{R}^d \setminus \bigcup \mathscr{A}$ consists of several contractible connected components, which are called **regions** (or **chambers**) of $\mathscr{A}$. The set of regions will be denoted by $\mathcal{T}(\mathscr{A})$. Later on, we will see their connection to the topes of oriented matroids (see Section 3.1.7).

**Definition 2.1.2.** *Let $\mathscr{A} = \{H_i\}_{i \in I}$ be a hyperplane arrangement in $V$, and let $\mathscr{L}(\mathscr{A})$ be the set of non-empty intersections of hyperplanes in $\mathscr{A}$. Then we define a partial order on $\mathscr{L}(\mathscr{A})$ by reverse inclusion, i.e., $X \leq Y$ if and only if $Y \subseteq X$. Hence, we get the* **intersection poset** *of $\mathscr{A}$ as*

$$\mathscr{L}(\mathscr{A}) = \Big\{ \bigcap_{i \in J} H_i \mid J \subset I \Big\} - \{\emptyset\}$$

*with minimal element $V$ as the "trivial intersection" (thus $\bigcap_I H_i$ where $I = \emptyset$).*

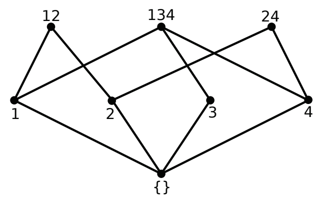



Figure 2.2: Intersection poset of the arrangement in Figure 2.1.

If the hyperplane arrangement is central, we get the center $\cap \mathscr{A}$ as unique maximal element in $\mathscr{L}(\mathscr{A})$. A hyperplane arrangement is **essential** if their exist a subset $\mathscr{A}' \subseteq \mathscr{A}$ such that the intersection $\bigcap \mathscr{A}'$ is a point in $V$.

**Definition 2.1.3.** *Given a real hyperplane arrangement $\mathscr{A}$, we can define the set of (closed)* **faces** *of $\mathscr{A}$ as*

$$\mathcal{F}(\mathscr{A}) := \{\overline{C} \cap X \mid C \in \mathcal{T}(\mathscr{A}), X \in \mathscr{L}(\mathscr{A})\}.$$

*$\mathcal{F}(\mathscr{A}) = \mathcal{F}$ is called the* **face poset** *of $\mathscr{A}$ partially ordered by inclusion.*

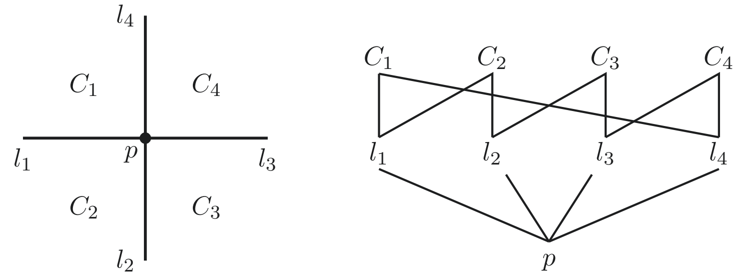



Figure 2.3: A hyperplane arrangement and its face poset.

The face poset $\mathcal{F}(\mathscr{A})$ provides a regular cell decomposition of $\mathbb{R}^d$. Two arrangements of hyperplanes $\mathcal{A}$ and $\mathcal{A}'$ are said to be **combinatorially equivalent** if $\mathcal{F}(\mathcal{A}) \cong \mathcal{F}(\mathcal{A}')$.

A second way to construct the face poset is via the position of vectors with respect to the hyperplanes. This construction is also a way to obtain oriented matroids from hyperplane arrangements (as we will see in Section 3.1.1).

For a real arrangement of hyperplanes $\mathscr{A} = \{H_i\}_{i \in I}$ we have the **positive** and the **negative halfspaces**

$$H_i^+ = \{v \in V \mid l_i(v) > b_i\} \text{ and } H_i^- = \{v \in V \mid l_i(v) < b_i\}.$$

Two vectors $v, w \in V$ are **similarly positioned** with respect to $\mathscr{A}$ if they are on the same side of $H_i$ for all $i \in I$. On the same side of $H_i$ means either both $v$ and $w$ lie in $H_i$, both lie in $H_i^+$ or both lie in $H_i^-$.

Being similarly positioned with respect to $\mathscr{A}$ is obviously an equivalence relation and the equivalences classes are (open) **faces**. This construction is frequently used in literature to define a structure on $V$ obtained by $\mathscr{A}$. In comparison to the first Definition 2.1.3, the faces in this construction are open.

If $F$ is a face in $\mathcal{F}$, then every hyperplane $H \in \mathscr{A}$, which has a non-empty intersection with the interior of $F$, contains $F$. Thus the intersection of all these hyperplanes also contains $F$ and is an affine subspace, which we call the support $\operatorname{supp}(F)$ of $F$. The **dimension of** $F$ is the dimension of its support. Evidently, the dimension of regions is equal to the dimension of $V$. Moreover, we denote the set of the faces of codimension $i$ by $\mathcal{F}_i(\mathscr{A}) = \mathcal{F}_i$.

**Definition 2.1.4.** *A hyperplane arrangement $\mathscr{A}$ in $V = \mathbb{C}^d$ is called* **complexified** *if every hyperplane $H$ in $\mathscr{A}$ is the complexification of a real hyperplane,* *i.e., the defining linear form $l$ lies in $\mathbb{R}^d - \{0\}$ and the defining scalar $b$ is also real, such that*

$$H = \{z = x + iy \in \mathbb{C}^d \mid l(x) + il(y) = b\}.$$

*Let the real part of a complexified hyperplane arrangement be denoted by*

$$\mathscr{A}_{\mathbb{R}} = \{H \cap \mathbb{R}^d \mid H \in \mathscr{A}\} = \{x \in \mathbb{R}^d \mid l_H(x) = b_H, H \in \mathscr{A}\}.$$

### 2.1.2 Toric arrangements

We will transfer the theory of arrangements from vector spaces (and affine spaces) to tori. Now the structure of our spaces is more difficult to handle. In the first case we have simple linear algebra and in the second case we work with the algebraic geometry of tori.

The theory of toric arrangement is a relatively young field, first attempts were made by Lehrer [70] in 1995. Subsequently, progress was made by De Concini and Procesi [32, 33], Ehrenbourg, Readdy and Slone [43], Lawrence [68], Moci [80, 79, 81], Moci and Settepanella [82] and d'Antonio and Delucchi [30, 31]. Let us recall some definitions, which for example can be found in [30] or [31].

**Definition 2.1.5.** *The $d$-dimensional* **complex torus** *is the space $(\mathbb{C}^*)^d$ and the $d$-dimensional* **compact torus** *is $(S^1)^d$, with $S^1$ as the unit circle in $\mathbb{C}$.*

**Definition 2.1.6.** *Let $T = X^d$ be the $d$-dimensional compact or complex torus, thus $X$ is either $S^1$ or $\mathbb{C}^*$. Then the maps $\chi : T \to X$ given by the Laurent monomials over $X$ are the* **characters** *of $T$, this means we have*

$$\chi(x) = x_1^{a_1} \dots x_n^{a_n} \text{ with } a = (a_1, \dots, a_n) \in \mathbb{Z}^d, \text{ for all } x \in T.$$

*The set of all characters of $T$ will be denoted by $\Lambda$. It is a lattice with pointwise multiplication as operation, which is isomorphic to $\mathbb{Z}^d$ via the mapping $a \mapsto x_1^{a_1} \dots x_n^{a_n}$.*

**Definition 2.1.7.** *Given a compact or complex torus $T$ and its set of characters $\Lambda$, then the set*

$$H_{\chi,a} = \{x \in T \mid \chi(x) = a\} \text{ with } \chi \in \Lambda, a \in S^1 \text{ or } a \in \mathbb{C}^*$$

*is a* **hypersurface** *of $T$.*

**Definition 2.1.8.** *Let $A$ be a finite subset of $\Lambda \times \mathbb{C}^*$, a* **(complex) toric arrangement** *$\mathscr{A}$ is the collection of hypersurfaces generated by $A$, i.e.,*

$$\mathscr{A} = \{H_{\chi,a} \mid (\chi, a) \in A\}.$$

*We will also write*

$$\mathscr{A} = \{(\chi, a) \mid \chi \in \Lambda, a \in \mathbb{C}^*\}$$

*and may think of* $\mathscr{A}$ *as the finite collection of the hypersurfaces* $H_{\chi,a}$. *The complement of* $\mathscr{A}$ *is*

$$\mathcal{M}(\mathscr{A}) = (\mathbb{C}^*)^d \smallsetminus \bigcup_{(\chi,a)\in\mathscr{A}} H_{\chi,a}.$$

**Definition 2.1.9.** *If* $A$ *is a finite subset of* $\Lambda \times S^1$ *and* $\Lambda$ *a finitely generated lattice as above, then a* **real toric arrangement** *is given by the collection of hypersurfaces*

$$H^{\mathbb{R}}_{\chi,a} = \{x \in (S^1)^d \mid \chi(x) = a\} \text{ with } (\chi, a) \in A.$$

*As in Definition 2.1.4, when a complex toric arrangement restricts to a real toric arrangement on* $(S^1)^d$, *we call it* **complexified**.

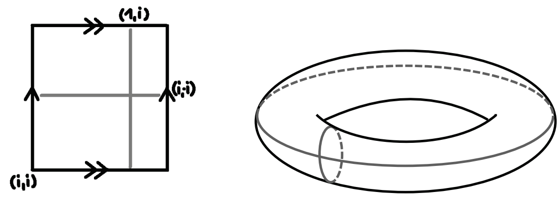



Figure 2.4: The toric arrangement on the 2-dimensional compact torus which is given by the characters $t = 1$ and $s = -i$.

Instead of the former concrete definition of the torus and its lattice, we can also introduce a toric arrangement in a more abstract way, starting with a finitely generated lattice as basic object rather than the "concrete" torus.

**Definition 2.1.10.** *Let* $\Lambda \cong \mathbb{Z}^d$ *be a finitely generated lattice, then we define the corresponding* **complex torus** *to be*

$$T_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C}^*).$$

*Similarly,*

$$T^{\mathbb{R}}_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda; S^1)$$

*is the corresponding* **compact torus**. *For a choice of a basis* $\{\chi_1, ..., \chi_n\}$ *of* $\Lambda$ *we get the isomorphisms*

$$\varphi : T_\Lambda \to (\mathbb{C}^*)^d \text{ with } g \mapsto (g(\chi_1), ..., g(\chi_n)),$$

$$\varphi : T^{\mathbb{R}}_\Lambda \to S^1 \text{ with } g \mapsto (g(\chi_1), ..., g(\chi_n)).$$

**Remark 2.1.11.** *The character lattice of* $T_\Lambda$ *is naturally isomorphic to* $\Lambda$ *(see Remark 12 in [31]), therefore we can identify them in the following.*

Based on this definition, we can construct a toric arrangement as above. In contrast to an affine arrangement, the hypersurfaces in a toric arrangement are not necessarily connected (consider for example the character $t^2 = 1$ of the 2-dimensional compact torus $(S^1)^2$). Even more, the intersection of a finite collection of connected hypersurfaces does not have to be connected in general. Thus we need another combinatorial invariant to study the topology of the complement $\mathcal{M}(\mathscr{A})$ in the toric case corresponding to the intersection poset (see Definition 2.1.2) in the affine case.

**Definition 2.1.12.** *Let $\mathscr{A}$ be a toric arrangement on $T_\Lambda$. Then we consider the set $\mathcal{C}(\mathscr{A})$ of the connected components of non-empty intersections of hypersurfaces in $\mathscr{A}$. The elements in $\mathcal{C}(\mathscr{A})$ are* **layers** *of $\mathscr{A}$, and $\mathcal{C}(\mathscr{A})$, ordered by reverse inclusion, is the* **layer poset** *of $\mathscr{A}$.*

In the same way as the regions and the faces in an affine hyperplane arrangement, we define the toric ones.

**Definition 2.1.13.** *Given a complexified toric arrangement $\mathscr{A}$, let $\mathscr{A}^\mathbb{R}$ denote the arrangement of hypersurfaces on the real torus $T_\Lambda^\mathbb{R}$. Then the* **regions** *of $\mathscr{A}$ are the connected components of $\mathcal{M}(\mathscr{A}^\mathbb{R}) = T_\Lambda^\mathbb{R} \setminus \bigcup H_{\chi,\alpha}^\mathbb{R}$. $\mathcal{T}(\mathscr{A})$ denotes the set of all regions of $\mathscr{A}$. The set of* **faces** *of $\mathscr{A}$ is defined as*

$$\mathcal{F}(\mathscr{A}) := \{\overline{C} \cap X \mid C \in \mathcal{T}(\mathscr{A}), X \in \mathcal{C}(\mathscr{A})\}.$$

*As above, $\mathcal{F}_i$ is the subset of $\mathcal{F}(\mathscr{A})$ containing all faces of codimension $i$.*

The faces of $\mathscr{A}$ are the cells of a cell decomposition as in the affine case - but the cell decomposition does not have to be regular (compare [82]). In comparison to the affine case, the set of faces of a toric arrangement is not a poset but an acyclic category (see 0.4.1). We will refer to $\mathcal{F}(\mathscr{A})$ as **face category** of $\mathscr{A}$.

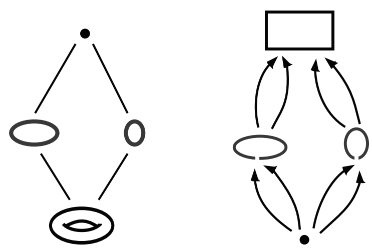


Figure 2.5: Layer poset and face category of the toric arrangement given in Figure 2.4.

**Definition 2.1.14.** *A toric arrangement is called* **essential** *if the layers of maximal codimension are points.*

Unless otherwise stated, our arrangement $\mathscr{A}$ will be essential and complexified from now on. Since there always exists an essentialization for all toric arrangement (see Remark 3.6 in [30]), it is no restriction to consider only essential arrangements.

### Covering space

In this section, we will see the connection between toric arrangements and hyperplane arrangements.

Given a lattice $\Lambda$ of rank $d$, consider the covering map

$$p : \mathbb{C}^d \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C}^*) = T_\Lambda, \qquad g \mapsto \exp \circ g, \tag{2.1}$$

where $\exp : \mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^{2\pi i z}$, is the exponential map. Since we can identify $\mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C})$ with $\mathbb{C}^d$, $p$ is the universal covering map

$$(x_1, ...., x_n) \mapsto (e^{2\pi i x_1}, ..., e^{2\pi i x_n})$$

of the torus $T_\Lambda$. Moreover, we get a restriction of $p$ on the compact torus

$$\mathbb{R}^d \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda; S^1) = T_\Lambda^{\mathbb{R}}. \tag{2.2}$$

Thus we get an associated periodic affine hyperplane arrangement in $\mathbb{C}^d \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C})$ for the toric arrangement $\mathscr{A}$. This hyperplane arrangement is not finite, but locally finite. Besides it is the preimage of $\mathscr{A}$ under $p$ and we will denote it by

$$\mathscr{A}^\uparrow = \{(\chi, z) \in \Lambda \times \mathbb{C} \,|\, (\chi, e^{2\pi i z}) \in \mathscr{A}\}. \tag{2.3}$$

The upwards arrow should indicate that $\mathscr{A}^\uparrow$ is obtained by lifting our original arrangement.

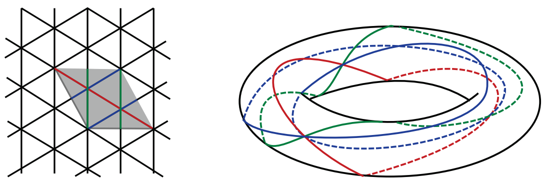


Figure 2.6: A periodic affine arrangement $\mathscr{A}^\uparrow$ in $\mathbb{R}^2$ as the lift of a toric arrangement $\mathscr{A}$ in $(S^1)^2$.

**Remark 2.1.15** (See [31], Remark 17)**.** *If $\mathscr{A}$ is complexified, so is $\mathscr{A}^\uparrow$.*

The lattice $\Lambda$ acts on $\mathbb{C}^d$ and on $\mathbb{R}^d$ as the group of automorphisms of the covering map $p$ of (2.1) above. Consider the map $q : \mathcal{F}(\mathscr{A}^\uparrow) \to \mathcal{F}(\mathscr{A})$ induced by $p$.

**Lemma 2.1.16** (See [30], Lemma 4.8)**.** *Let $\mathscr{A}$ be a complexified toric arrangement. Then the map $q : \mathcal{F}(\mathscr{A}^\uparrow) \to \mathcal{F}(\mathscr{A})$ induces an isomorphism of acylic categories*

$$\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}^\uparrow)/\Lambda.$$

**Lemma 2.1.17** (See Lemma 5.2 in [30])**.** *For the fundamental group $\pi_1(M(\mathscr{A}))$ of a complexified toric arrangement $\mathscr{A}$, we have*

$$\pi_1(M(\mathscr{A})) \simeq \pi_1(M(\mathscr{A}^{\uparrow})) \rtimes \mathbb{Z}^d.$$

## 2.2 Pseudospheres

### 2.2.1 Piecewise linear topology

For a thorough study of piecewise linear topology (or PL topology for short) the reader may be referred to the book [93] by Rushing.

Recall the definition of simplicial complexes, its underlying space and subdivisions of complexes in Section 0.3. For a simplicial complex $\Delta$ with underlying space $||\Delta||$, each point $x \in ||\Delta||$ can be uniquely expressed in terms of barycentric coordinates $\lambda_i$ with respect to the vertices $v_i$ of $\Delta$. These are the real numbers $\lambda_i \geq 0$ such that $x = \sum \lambda_i v_i$, $\sum \lambda_i = 1$, and $\{v_i : \lambda_i > 0\} \in \Delta$. A map $f : ||\Delta|| \to \mathbb{R}^d$ is **linear** if $f(\sum \lambda_i v_i) = \sum \lambda_i f(v_i)$ for each point $x = \sum \lambda_i v_i \in ||\Delta||$.

A **map** $f : \Delta \to \Gamma$ **of simplicial complexes** $\Delta$ and $\Gamma$ is corresponding to the triple $(|f|, ||\Delta||, ||\Gamma||)$ such that $|f| : ||\Delta|| \to ||\Gamma||$ is a continuous map of topological spaces. It is called **simplicial** if every simplex $\sigma \in \Delta$ is mapped to a simplex $\tau_\sigma \in \Gamma$.

**Definition 2.2.1** (See [93])**.** *A map $f : \Delta \to \Gamma$ is **piecewise linear** if there is a simplicial subdivision $\Delta'$ of $\Delta$ such that every simplex $\sigma \in \Delta'$ is mapped linearly into a simplex $\tau_\sigma \in \Gamma$.*

*Two simplicial complexes $\Delta$ and $\Gamma$ are **piecewise linear homeomorphic** or **PL homeomorphic** if there exists a piecewise linear map $f : \Delta \to \Gamma$ which is also a homeomorphism.*

**Definition 2.2.2** (See [12], Definition 4.7.20)**.** *A simplicial complex $\Delta$ is a **PL $d$-ball** if it is PL homeomorphic to a $d$-simplex. It is a **PL $d$-sphere** if it is PL-homeomorphic to the boundary of the $(d+1)$-simplex.*

This definition can be extended for regular cell complexes.

**Lemma 2.2.3** (See [12], Lemma 4.7.25)**.** *A regular cell complex $\Delta$ is a PL $d$-ball if its simplicial subdivision $\Delta_{ord}(\mathcal{F}(\Delta))$ is a PL $d$-ball. Similarly, for PL $d$-spheres.*

The following theorem states some basic technical properties of PL balls and spheres which we will need later on.

**Theorem 2.2.4** (See [12], Theorem 4.7.21)**.**

*(i) The union of two PL $d$-balls , whose intersection is a PL $(d-1)$-ball lying in the boundary of each, is a PL $d$-ball.*

*(ii) The union of two PL d-balls, which intersect along their entire boundaries, is a PL d-sphere.*

*(iii) (Newman's Theorem) The closure of the complement of a PL d-ball embedded in a PL d-sphere is itself a PL d-ball.*

*(iv) The cone over a Pl d-sphere is a PL (d+1)-ball.*

Note that the statements (i) and (iii) with the "PL" removed are false (see [93, p. 69]).

### 2.2.2 Pseudosphere arrangements

Consider an arrangement of linear subspheres $\mathscr{A}' = \{H \cap S^d : H \in \mathscr{A}\}$ where $\mathscr{A}$ is a linear arrangement of hyperplanes. Now the idea behind pseudospheres is that a tame topological deformation that preserves the intersection pattern of the arrangement $\mathscr{A}'$ will also not change the combinatorial type of the arrangement.

Two $(d-1)$-subspheres $S, S'$ of $S^d$ are **equivalent** if $h(S) = S'$ for some homeomorphism $h : S^d \to S^d$.

**Definition 2.2.5** (See [12], Definition 5.1.2)**.** *A $(d-1)$-subsphere $S$ in $S^d$ is a **pseudosphere** in $S^d$ if it satisfies one of the following equivalent conditions.*

*(i) $S$ is equivalent to the equator $S^{d-1} = \{x \in S^d : x_{d+1} = 0\}$.*

*(ii) $S$ is equivalent to some piecewise linearly embedded $(d-1)$-subsphere.*

*(iii) The closure of each connected component of $S^d \setminus S$ is homeomorphic to the d-ball.*

*The two connected components of $S^d \setminus S$ are its **sides** and their closures will be called **pseudohemispheres**.*

The equivalence relation on $(d-1)$-subspheres of $S^d$ defined above has been studied in [93]. The subspheres contained in this equivalence class are the *tame* subspheres, all other codimension 1 subspheres are *wild* (for more information about tame and wild spheres see for example [93]). The reader should note that considering the closure in (iii) is necessary. Although the sides of each pseudosphere in $S^d$ are open $d$-balls, contrary to the intuition this property does not characterize pseudospheres. In [93, p. 68] Rushing gave an example for a wild sphere whose complement consists of two open balls.

The following notion of arrangements of pseudo(hemi)spheres was introduced by Folkman and Lawrence in [47] and much simplified by Edmonds and Mandel in [75].

**Definition 2.2.6** (See [12], Definition 5.1.3)**.** *A finite multiset $\mathcal{A} = (S_e)_{e \in E}$ of pseudospheres in $S^d$ is an **arrangement of pseudospheres** if the following conditions hold:*

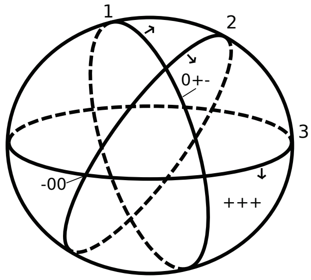



Figure 2.7: A simple example for an arrangement of pseudospheres. (We will make use of the sign vectors later on.)

*(A1)* $S_A = \bigcap_{e \in A} S_e$ *is a sphere, for all* $A \subseteq E$.

*(A2) If* $S_A \nsubseteq S_e$*, for* $A \subseteq E$*,* $e \in E$*, and* $S_e^+$ *and* $S_e^-$ *are the two sides of* $S_e$*, then* $S_A \cap S_e$ *is a pseudosphere in* $S_A$ *with sides* $S_A \cap S_e^+$ *and* $S_A \cap S_e^-$.

*(A3) The intersection of an arbitrary collection of closed sides is either a sphere or a ball.*

In condition (A1), if $S_A \cap S_e = S^{-1} = \varnothing$ is the empty sphere in a zero sphere $S_A \cong S^0$, then the sides of the empty sphere are the two points of $S_A$. The arrangement is called **essential** if $S_E = \bigcap_A S_e = \varnothing$. Furthermore, Edmonds and Mandel [75] proved that (A3) is already implied by (A1) and (A2). Hence, the axiom (A3) is actually redundant but we keep it for the proof of the topological representation theorem in Section 3.1.4 which is following [12, Chapter 4 and 5].

An arrangement $\mathscr{A} = (S_e)_{e \in E}$ in $S^d$ is **centrally symmetric** if $-S_e = S_e$, i.e., each individual pseudosphere is invariant under the antipodal map $x \to -x$ of $S^d$. In the signed case this is equivalent to requiring that $-S_e^+ = S_e^-$ for all $e \in E$.

## 2.3 Pseudolines

Already without the equivalence of arrangements of pseudolines and the combinatorial concept of oriented matroids of rank 3 (see Theorem 3.1.24) the theory of pseudolines was and is an interesting mathematical field on its own. In his paper [71] of 1926, Levi introduced the notion of arrangements of pseudolines and showed, in spite of their resemblance to arrangements of straight lines, they are topologically more general objects. His work was followed among a lot of others by the considerations of Ringel [92] and Grünbaum [57, 56].

There are two different ways to consider *pseudolines*: Either they can be viewed as subsets of Euclidean plane $\mathbb{R}^2$, or as subsets of the real projective plane $\mathbb{P}^2$. The first way of considering them was introduced by Ringel [92] and corresponds mostly to the earlier approaches, see for example [71, 92, 57, 54]. The second definition is used in [12, 53].

For a survey about pseudoline arrangements and a list of references we refer to the article [49] by Goodman.

**Definition 2.3.1** (Definition in $\mathbb{R}^2$ - See [57, 54]). *A **pseudoline** is a simple curve in the plane $\mathbb{R}^2$ which differs only in a bounded segment from a straight line.*

**Definition 2.3.2** (Definition in $\mathbb{R}^2$ - See [57, 54]). *An **arrangement of pseudolines** in $\mathbb{R}^2$ is a finite family of pseudolines such that every two intersect in at most one point, at which they then cross.*

**Remark 2.3.3.** *In this definition compared to our notion of pseudoline arrangement introduced in Example 1.1.8 we do not have to required explicitly that intersection is transversal. This is immediately implied since we consider "just slightly" altered straight lines.*

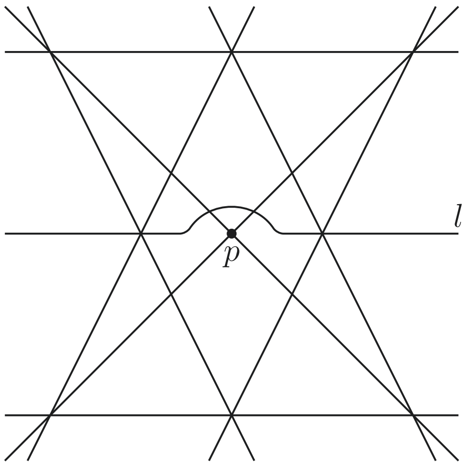



Figure 2.8: The non-Pappus arrangement as an example of a non-stretchable pseudoline arrangement (see [12, p. 16]).

In order to avoid special cases caused by parallel lines one started to look at pseudolines in $\mathbb{P}^2$ instead of in $\mathbb{R}^2$. Recall that a simple closed curve is the image of $S^1$ under an injective continuous map, i.e. a non-self-intersecting continuous loop.

**Definition 2.3.4** (Definition in $\mathbb{P}^2$ - See [12, 53]). *A **pseudoline** is a simple closed curve embedded in $\mathbb{P}^2$ satisfying one of the following equivalent conditions:*

*(1) $L$ is the image of a straight line under a self-homeomorphism of $\mathbb{P}^2$.*

*(2) $\mathbb{P}^2 \smallsetminus L$ is connected.*

**Definition 2.3.5** (Definition in $\mathbb{P}^2$ - See [12, 53])**.** *A collection* $\mathscr{A} = (L_e)_{e\in E}$ *of pseudolines is an* ***arrangement of pseudolines*** *if any pair of pseudolines in* $\mathscr{A}$ *intersects in exactly one point (and necessarily crosses).*

Out of rank considerations Björner at al. require in [12] additionally the condition $\bigcap \mathscr{A} = \varnothing$.

**Definition 2.3.6** (See [12], Section 6.3)**.** *An arrangement* $\mathscr{A} = (L_e)_{e\in E}$ *of pseudolines is called* ***stretchable*** *if it satisfies any one of the following equivalent conditions:*

*(1) Some self-homeomorphism of the projective plane moves all lines* $L_e$ *into straight lines.*

*(2) The cell decomposition of* $\mathbb{P}^2$ *induced by* $\mathscr{A}$ *is combinatorially isomorphic to the cell decomposition induced by some arrangement of straight lines.*

*(3) The oriented matroid corresponding to* $\mathscr{A}$ *is realizable (see Chapter 3).*

Now it is a natural question to ask which arrangements of pseudolines are stretchable and can be straighten to an arrangement of straight lines, and which are not. The following theorem was conjectured by Grünbaum in [56] and proven by Goodman and Pollack in [50].

**Theorem 2.3.7** (See [50], Theorem 1)**.** *Any arrangement of at most eight pseudolines is stretchable.*

The existance of non-stretchable arrangement for $k \geq 9$ was known since [71]. Consider for example the non-Pappus arrangement (see Figure 2.8), trying to straighten all lines of the arrangement forces $l$ to contain $p$. Nevertheless, it is quite hard to decide whether a pseudoline arrangement is stretchable or not. In [78] from 1988, Mnëv showed that it is a NP-hard problem.

Another fundamental tool for working with pseudoline arrangements is the following fact which guarantees that new pseudolines can be added through any not-yet-collinear pair of points, just as in the linear case.

**Lemma 2.3.8** (Levi's Enlargement Lemma, see [71, 56])**.** *If* $\mathscr{A}$ *is an arrangement of pseudolines and* $p, q \in \mathbb{P}^2$ *are two distinct points not on the same member of* $\mathscr{A}$*, there is a pseudoline* $L$ *passing through* $p$ *and* $q$ *such that* $\mathscr{A} \cup \{L\}$ *is an arrangement of pseudolines.*

This fact can not be generalized to higher dimensions. There are examples of arrangements of pseudoplanes in $\mathbb{P}^3$ corresponding to a rank 4 oriented matroid and three given points, such that pseudoplane through these three points can be properly added to the arrangement [51].

**Remark 2.3.9.** *There exist other consructions which are corresponding to pseudoline arrangements, such as wiring diagrams and allowable sequences [52], two-dimensional zonotopal tilings [17, 91] or sweeping [44].*

## 2.4 Pseudohyperplanes

As Levi's Enlargement Lemma 2.3.8 just showed the case of dimension two is special and many of the results about pseudolines cannot be transferred to higher dimensions.

In this section we present two different ideas to generalize the concepts of pseudoline arrangements and pseudosphere arrangements to a more general setting in $\mathbb{R}^d$: The work by Forge and Zaslavsky [48] and by Miller [77]. Other approaches to obtain pseudohyperplanes were made in [18, 40, 39, 106, 89] (an overview of this references is given in the end of this section).

### 2.4.1 Topological hyperplanes

Forge and Zaslavsky (2009) study in [48] the properties of an arrangement of topological hyperplanes in $\mathbb{R}^d$, each topological hyperplane (or topoplane) topologically equivalent to an ordinary straight hyperplane. They analyse when an arrangement of topoplanes corresponds to an arrangement of pseudospheres with one distinguished pseudosphere - an *affine pseudohyperplane arrangement.*

**Definition 2.4.1** (See [48], Section 1)**.** *In a topological space $X$ homeomorphic to $\mathbb{R}^d$, a* ***topological hyperplane*** *or* ***topoplane*** *is a subspace $Y$ such that $(X, Y)$ is homeomorphic to $(\mathbb{R}^d, \mathbb{R}^{d-1})$.*

**Definition 2.4.2** (See [48])**.** *A finite set $\mathcal{H} = \{Y_e\}_{e\in E}$ of topoplanes in $X$ is an* ***arrangement of topoplanes*** *if for every non-empty intersection $F = \bigcap_{A\subseteq E} Y_e$ and every topoplane $Y \in \mathcal{H}$ either $F \subseteq Y$ or $F \cap Y = \varnothing$ or $F \cap Y$ is a topoplane in $Y$.*

As for hyperplane arrangements or pseudosphere arrangements one can define a intersection poset $\mathcal{L} = \{\bigcap \mathcal{S} \mid \mathcal{S} \subseteq \mathcal{H}, \bigcap \mathcal{S} \neq \varnothing\}$ for topoplane arrangement which is a semilattice, but not necessarily geometric. Nevertheless, Forge and Zaslavsky observed the following.

**Proposition 2.4.3** (See [48], Proposition 2)**.** *For an arrangement of topoplanes, each interval in $\mathcal{L}$ is a geometric lattice with rank given by the codimension.*

If every two intersecting topoplanes cross each other, the arrangement is said to be **transsective**. Forge and Zaslavky study if it is possible to transform every arrangement of topoplanes to a transsective arrangement of topoplanes. In dimension two the following holds.

**Theorem 2.4.4** (See [48], Theorem 9)**.** *For any arrangement of topolines, there is a transsective topoline arrangement which has the same faces.*

This theorem cannot be generalized to three or more dimensions, a counterexample in three dimensions can be found in [48, Example 1]. Consider the additional conditions for a topoplane arrangement to be **simple**, i.e. the codimension of every flat equals the number of topoplanes that contain it, and **solid**, i.e. all pairs of topoplanes either do not intersect, cross or they touch without crossing in a topoplane of codimension 2 (for more detail see [48]).

**Theorem 2.4.5** (See [48], Theorem 10). *For a simple, solid arrangement of topoplanes, there is a transsective topoplane arrangement which has the same faces.*

### Connection to projective pseudohyperplanes

Each centrally symmetric pseudosphere arrangement can be identified with a **projective pseudohyperplane arrangement** $\mathcal{P}$ in $\mathbb{P}^d$ by identifying opposite points of $S^d$ [12, 48]. If one removes one pseudohyperplane $H_0 \in \mathcal{P}$ and considers $\mathscr{A} = \{H \setminus H_0 : H \in \mathcal{P}, H \neq H_0\}$ in $X := \mathbb{P}^d \setminus H_0$, we have an **affine pseudohyperplane arrangement** in $\mathbb{R}^d$. A topoplane arrangement is **projectivizable** if it is homeomorphic to an arrangement constructed this way . Furthermore, two topoplanes are called **parallel** if they are disjoint.

**Lemma 2.4.6** (See [48], Lemma 11). *If a topoplane arrangement is projectivizable then it is solid and transsective and parallelism is an equivalence relation on topoplanes.*

We emphasize that in general parallelism does not have to be an equivalence relation in topoplane arrangements. The next theorem will show the importance of this condition.

**Theorem 2.4.7** (See [48], Theorem 13). *A transsective topoline arrangement in $\mathbb{R}^2$ is projectivizable if and only if parallelism in $\mathscr{A}$ is an equivalence relation.*

## 2.4.2 Pseudohyperplanes and smooth manifolds

In [77] from 1987, Miller uses the theory of smooth manifolds and differential topology to generalize Ringel's idea of pseudolines in $\mathbb{R}^2$ (see Definition 2.3.1). Creating pseudohyperplanes in $\mathbb{R}^d$ which correspond to linear hyperplanes outside a bounded region.

Recall that a subset $M \subseteq \mathbb{R}^d$ is a $m$-manifold if each point of $M$ has a neighbourhood in $M$ homeomorphic to an open $m$-ball. A $n$-submanifold of $M$ **locally flat** if for each $x \in N$ there is a neighbourhood $U_x$ of $x$ in $M$ and a homeomorphism $h_x : U_x \to \mathbb{R}^m$ such that $h_x(U_x \cap N) = \mathbb{R}^n$.

**Lemma 2.4.8** (See [77], Lemma 1). *Let $\mathscr{A} = \{M_e : e \in E\}$ be a finite collection of locally flat $(d-1)$-manifolds in $\mathbb{R}^d$, each identical to a hyperplane outside a bounded region. Suppose for $f \in E$ and $A \subseteq E$:*

*(i)* $\bigcap M_A := \bigcap_{e \in A} M_e$ *is empty or homeomorphic to* $\mathbb{R}^k$, $0 \leq k \leq d-1$.

*(ii) Either* $\bigcap M_A \subseteq M_f$ *or* $M_f \cap \bigcap M_A$ *is a locally flat submanifold of* $\bigcap M_A$.

*Then* $\mathscr{A}$ *is an affine pseudohyperplane arrangement.*

A map $f$ is smooth if all its derivatives exist. If $f$ is bijective with smooth inverse $f^{-1}$ it is called a diffeomorphism. A **smooth manifold** is a differentiable manifold such that all its transition maps are smooth. For more detail we refer to [69]. Two submanifolds $X$ and $Y$ of a smooth $d$-manifold $Z$ are **transversal** if for each $x \in X \cap Y$ the tangent spaces of $X$ and $Y$ at $x$ span that of $Z$ at $x$. A collection $\mathscr{A} = \{M_e : e \in E\}$ of smooth $(d-1)$-manifolds in $\mathbb{R}^d$ is **strongly transversal** if for any $A \subseteq E, |A| \leq d$, and any choice of $\{x_e \in M_e : e \in A\}$, the vectors $\{v^e_{x_e} : e \in A\}$ are linearly independent and each $v^e_{x_e}$ is normal to its corresponding tangent space of $M_e$ at $x_e$. In the case, where all $M_e$ are hyperplanes this is the same as being in general position.

**Theorem 2.4.9** (See [77], Main Theorem)**.** *Let* $\mathscr{A} = \{M_e : e \in E\}$ *consist of at least d strongly transversal smooth manifolds in* $\mathbb{R}^d$*, each diffeomorphic to* $\mathbb{R}^{d-1}$*, and each* $M_e$ *identical to a hyperplane outside a bounded region. Then* $\mathscr{A}$ *is an affine arrangement of pseudohyperplanes.*

**Corollary 2.4.10** (See [77], Corollary 1)**.** *Let* $\mathscr{A}$ *be as in Theorem 2.4.9. For* $e \in E$*, set* $M_f$ *to be a translation of* $M_e$ *such that* $M_e \cap M_f = \varnothing$ *then* $\mathscr{A}' = \{M_i : i \in E \cup f\}$ *is an affine arrangement of pseudohyperplanes.*

All in all we obtain arrangements which model affine arrangements of pseudohyperplanes, all of whose pseudohyperplanes are linear outside a bounded region. Björner at al. [12] call these arrangments *Miller arrangements.*

**Definition 2.4.11** (See [12], Exercise 5.11)**.** *Let a* ***Miller arrangement*** *be an arrangement of pseudospheres such that one of the pseudospheres* $S_g$ *is linear and all other pseudospheres* $S_e$ *are linear in an open neighbourhood of* $S_g$.

Unfortunately, these arrangements are not general enough to represent all oriented matroids (see [12, Exercise 5.11]). For $d \geq 4$, there are oriented matroids of rank $d$ which are not representable by a Miller arrangement.

### 2.4.3 Overview of current literature

Already in [105] from 1977, Zaslavsky considers topological dissections of topological spaces by arrangements of affine subspaces of arbitrary dimensions and studies the relation of their number of $i$-dimensional cells and their combinatorial invariants.

Goresky and MacPherson [55, pp. 239,257] compute the cohomology groups of *2-arrangements*, which are arrangements of affine subspaces of codimension 2 in $\mathbb{R}^{2d}$ such that every intersection has even codimension. In [11], Björner studies the properties of *k-arrangements*, which are affine subspace arrangement of $k$-codimensional subspaces such that the codimension of every intersection is a multiple of $k$.

Björner and Ziegler discuss a more general version in [16], the *k-pseudo-arrangements* of codimension $k$ subspheres in $S^{kd-1}$ (and a corresponding notion of *k-matroids*). Where an 1-pseudoarrangement corresponds to an arrangement of pseudospheres and a 2-pseudoarrangement can be considered as an complexified pseudoarrangement.

Hu studies in [59] *arrangements of subspaces and spheres* in $\mathbb{R}^d$, which reappear in the work of Ziegler and Živaljević [108] about diagrams of spaces. Furthermore, arrangements of topological spheres are considered by Pakula in [89].

The groundwork for arrangements of topoplanes where given in [106] by Zaslavsky where he studies *projective topological arrangements* and *Euclidean topological arrangements* which have the additional condition (compared to [48]) that every face has to be a topological cell. Leading to the *topoplane arrangements* by Forge and Zaslavsky [48] as discussed in Subsection 2.4.1.

A different approach to consider more general subsphere arrangements in $S^d$ was made in [19] by Bokowski, Mock and Streinu and in [18] by Bokowski, King, Mock and Streinu via *hyperline sequences*. Considering arrangements of embedded codimension one subspheres in $S^d$ and allowing wild subspheres (see [18, Section 8]).

A generalization of Ringel's definition of pseudolines in $\mathbb{R}^2$ by codimension 1 submanifolds in $\mathbb{R}^d$ is given by Miller [77] (as seen in Subsection 2.4.2). Codimension 1 submanifolds in a smooth manifold were studied again by Deshpande in [40], setting conditions such that the submanifolds in an *arrangement of submanifolds* behave locally like hyperplanes and proving a generalization of Zaslavsky's counting formula [104, Theorem A]. Moreover, Deshpande considered *pseudohyperplanes* in $\mathbb{R}^d$ obtained as the cone over a pseudosphere in $S^{d-1}$ (see [38]).

# Chapter 3

# Oriented matroids - State of the art

Oriented matroids can be thought of as a combinatorial abstraction of many different concepts. Among other things, they arise from point configurations over the reals, from real hyperplane arrangements, from convex polytopes, and from directed graphs. The historical motivation and the different but equivalent ways to axiomatize an oriented matroid emerged from these diverse mathematical theories.

We will start this chapter by giving a geometric intuition for the different aspects of oriented matroids. Followed by an introduction of the main concepts and terminology of oriented matroids from the point of covectors. After this we will be able to consider the Topological Representation Theorem by Folkman and Lawrence [47]. It gives a completely combinatorial description (via oriented matroids) of the geometric concept *"pseudosphere arrangement"*.

In Section 3.1.5, we introduce the concept of affine oriented matroids and explain their connection to non-central hyperplane arrangements. Moreover, the reader will see the axiom system by Karlander [62] which he calls *affine sign vector system*. These systems are in correspondence to affine oriented matroids. For more detail, see also [5].

Subsequently, we will discuss the concept of *conditional oriented matroids* which were introduced by Bandelt, Chepoi and Knauer [3]. The motivation for conditional oriented matroids is to consider sign vectors systems which "look locally like oriented matroids" and can be seen as complexes with oriented matroid as cells.

Furthermore, Section 3.3 familiarises the reader with the theory of zonotopal tilings and the equivalence shown by Bohne in his Dissertation [17]. If $V$ is a vector configuration of $n$ vectors in $\mathbb{R}^d$ then the Bohne-Dress Theorem states that the zonotopal tilings of the zonotope $\mathcal{Z}(V)$ are in one-to-one correspondence with the one-element liftings of the oriented matroid $\mathcal{L}(V)$ (see [107, 17]).

The standard literature for the theory of oriented matroids is the book of

Björner, Las Vergnas, Sturmfels, White and Ziegler [12]. All the omitted proofs in this chapter can be found in the given references.

## 3.1 Oriented matroids

### 3.1.1 Geometric intuition

In Section 0.2, we already had a brief look at the value and importance of matroids and their variety of cryptomorphisms (see Section 0.2). Now we would like to preserve more information. Let us consider the case of a vector configuration in $\mathbb{R}^d$, an oriented set of linear hyperplanes or a directed graph where we would like to keep the information about the orientation.

Suppose $\mathcal{G} = (V, E)$ is a (finite) directed graph with set of vertices $V$ and set of directed edges $E$. Then every edge in cycle $C$ in $\mathcal{G}$ can be either along (+) or against (-) the orientation of an edge in $\mathcal{G}$. In this way, we obtain a sign vector $X \in \{+, -, 0\}^E$ defined by

$$X_e = \begin{cases} + & \text{if } e \in C \text{ along the orientation,} \\ - & \text{if } e \in C \text{ against the orientation,} \\ 0 & \text{otherwise.} \end{cases}$$

By this procedure, we obtain the set $\mathcal{C}$ of (signed) **circuits** of the oriented matroid $\mathcal{M}_\mathcal{G}$ corresponding to $\mathcal{G}$. By a *minimal cut* of $\mathcal{G}$ we mean a partition $V = V_1 \uplus V_2$ of the vertices such that removing the edges between $V_1$ and $V_2$ increases the number of connected components of the graph. For every minimal cut we obtain a (signed) **cocircuit** $X$ of the oriented matroid $\mathcal{M}_\mathcal{G}$ defined as

$$X_e = \begin{cases} + & \text{if the edge } e \text{ is removed and oriented from } V_1 \text{ to } V_2, \\ - & \text{if the edge } e \text{ is removed and oriented from } V_2 \text{ to } V_1, \\ 0 & \text{otherwise.} \end{cases}$$

Oriented matroids arising from a directed graph are called **graphic**.

As a second approach let us consider a vector configuration of $n$ (non-zero) vectors in $\mathbb{R}^d$ or, equivalently, a set of $n$ linear hyperplanes in $\mathbb{R}^d$ as in Figure 3.1. The face poset $\mathcal{F}(\mathscr{A})$ of the hyperplane arrangement $\mathscr{A}$ provides a polyhedral decomposition of $\mathbb{R}^d$ (as seen in Section 2.1.1). The set of **covectors** of the associated oriented matroid is the collection of sign vectors given by the cell decompostion in the following way.

For every hyperplane $H_i$ we have a disjoint union $H_i \uplus H_i^+ \uplus H_i^- = \mathbb{R}^d$ with $H_i^-$ and $H_i^+$ defined as in Section 2.1.1. Then every point $x$ in $\mathbb{R}^d$ is mapped to a sign vector

$$\sigma(x) = (\sigma_1(x), ..., \sigma_n(x)),$$

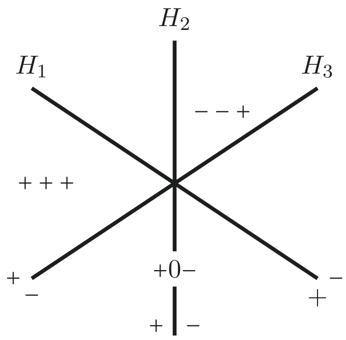



Figure 3.1: An arrangement of hyperplanes with assigned positive and negative halfspaces with three sign vectors given by the cell decomposition of $\mathbb{R}^2$.

where

$$\sigma_i(x) = \begin{cases} + & \text{if } x \in H_i^+, \\ - & \text{if } x \in H_i^-, \\ 0 & \text{if } x \in H_i. \end{cases}$$

**Example 3.1.1.** *The set of covectors of the associated oriented matroid of the arrangement shown in Figure 3.1:*

$$\{000,\ 0{+}{+},\ 0{-}{-},\ {+}0{-},\ {-}0{+},\ {+}{+}0,\ {-}{-}0,\ {+}{+}{+},\ {-}{-}{-},\ {+}{+}{-},\ {-}{-}{+},\ {-}{+}{+},\ {+}{-}{-}\}.$$

Hence, all $x \in \mathbb{R}^d$ with the same sign vector $\sigma(x)$ are contained in the same cell in the polyhedral decomposition given by the arrangement $\mathcal{A} = \{H_1, ..., H_n\}$. If our initial data is given by a set $v_1, ..., v_n$ vectors in $\mathbb{R}^d$, we obtain

$$\sigma(x) = (\text{sign}(v_1^T \cdot x), ..., \text{sign}(v_n^T \cdot x)).$$

Consider the $n$ vectors as columns of a matrix $A$ in $\mathcal{M}_{n,d}(\mathbb{R})$. The oriented matroid given by $A$ is called **realizable** (see also Definition 3.1.8) as it has a geometric realization by an arrangement of linear hyperplanes.

### 3.1.2 Axiomatics

A distinguishing trait of the theory of matroids and oriented matroids is the wealth of cryptomorphisms (for matroid theory see for example the book by Oxley [88] and for oriented matroids the book by Björner at al. [12]). We will use the definition of oriented matroids in terms of covectors. In order to formulate it, let us introduce some notation.

Let $E$ be a finite set and consider sign vectors $X, Y \in \{+, -, 0\}^E$. The **support** of a vector $X$ is $\underline{X} = \{e \in E : X_e \neq 0\}$; its **zero set** is

$$X^0 = E - \underline{X} = \{e \in E : X_e = 0\}.$$

And $-X$ denotes the negative of $X$, that is to say $(-X)_e = -(X_e)$ for all $e \in E$ (sometimes we will also refer to $-X$ as the *opposite*). The **separation set** of $X$ and $Y$ is $S(X,Y) = \{e \in E : X_e = -Y_e \neq 0\}$ and their **composition** $X \circ Y$ is defined by

$$(X \circ Y)_e := \begin{cases} X_e, & \text{if } X_e \neq 0, \\ Y_e, & \text{otherwise.} \end{cases}$$

**Definition 3.1.2** (See [12], Definition 4.1.1)**.** *A set $\mathcal{L} \subseteq \{+,-,0\}^E$ is the set of* ***covectors of an oriented matroid*** *$\mathcal{M}$ if and only if it satisfies:*

*(Sym)* *$X \in \mathcal{L}$ implies $-X \in \mathcal{L}$,*

*(C)* *$X, Y \in \mathcal{L}$ implies $X \circ Y \in \mathcal{L}$,*

*(SE)* *if $X, Y \in \mathcal{L}$ and $e \in S(X,Y)$ then there exists $Z \in \mathcal{L}$ with $Z_e = 0$ and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all $f \notin S(X,Y)$.*

The first property is usually referred to as *symmetry*, the second as *composition* and the third as *strong elimination*. In literature, it is often required in addition to these axioms that the zero vector $0 \in \mathcal{L}$. Although this property is already implied by (Sym) and (SE).

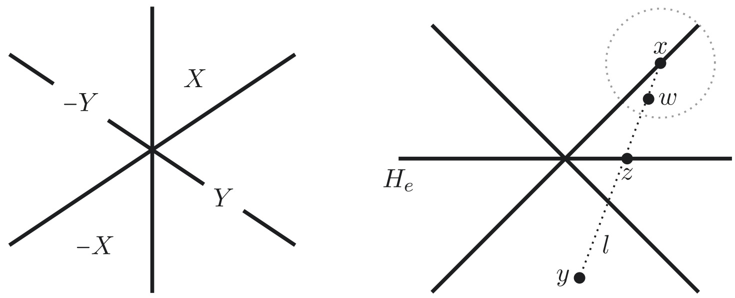



Figure 3.2: Geometric intuition in a hyperplane arrangement for the axiom (Sym) on the left-hand side, for (C) and (SE) on the right-hand side.

In order to gain a geometric intuition for the axioms let us consider an arrangement of linear hyperplanes $\mathscr{A} = \{H_e\}_{e \in E}$. As the arrangement is linear and thus centrally symmetric, the axiom (Sym) is easily seen. For every cell of the induced decomposition its opposite cell, regarding its position to every hyperplane $H_e$ in $\mathscr{A}$, has to be contained in the decomposition as well (see the image on the left-hand side of Figure 3.2). For axioms (C) and (SE) we take a look at the picture on the right-hand side of Figure 3.2. Let $X, Y$ be covectors in $\mathcal{L}$ and $x, y \in \mathbb{R}^2$ points in the corresponding cells of the cell decomposition. If we consider the straight line segment $l$ connecting $x$ and $y$, the composition $X \circ Y$ corresponds to the sign vector of a point $w$ located an $\varepsilon$-step from $x$ in the direction of $y$ (with $\varepsilon$ sufficiently small). Moreover, the axiom (SE) means that, for every separating hyperplane $H_e$, the set $\mathcal{L}$ contains a covector $Z$ which corresponds to the intersection of the line segment $l$ with $H_e$.

The partial order "≤" on $\{+,-,0\}$ (defined by "$0 < -,+$" and $+,-$ incomparable) induces a partial order on $\mathcal{L}$ (see Figure 3.3). Furthermore, the poset $\mathcal{L}$ enlarged by a top element $\hat{1}$ is a graded lattice which we denote by $\hat{\mathcal{L}} = \mathcal{L} \cup \{\hat{1}\}$ (see [12, Theorem 4.1.14] or Theorem 3.1.14). Thus, we are able to speak about the rank rk of covectors, meaning their rank in the lattice $\hat{\mathcal{L}}$. Further, we will refer to $\hat{\mathcal{L}}$ as the **face lattice** of the oriented matroid $\mathcal{M}$.

**Definition 3.1.3.** *The* ***rank*** *of* $\mathcal{M}$ *is the maximal rank in* $\mathcal{L}$.

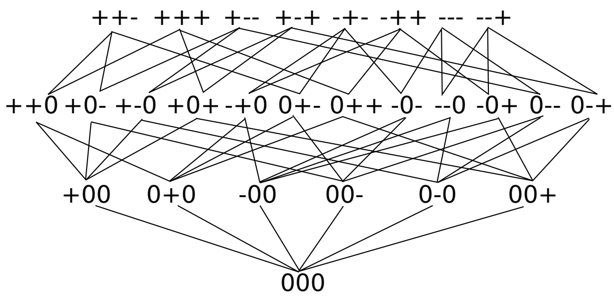



Figure 3.3: (Po)set of covectors of an oriented matroid of rank 3.

Minimal covectors in $\mathcal{L} \setminus \{\hat{0}\}$ are called **cocircuits**, they are covering $\hat{0}$ in $\mathcal{L}$. The set of cocircuits of $\mathcal{L}$ will be denoted by $\mathcal{C}^*$. In Section 3.1.1, we have seen that one natural way to obtain oriented matroids is to consider directed graphs. There, the circuits of the associated graphic oriented matroid correspond to the signed cycles in the graph and the cocircuits to the minimal cuts. The set of covectors can be reconstructed from its cocircuits.

**Proposition 3.1.4** (See [12], Section 3.7)**.** *Let* $\mathcal{L} \subseteq \{+,-,0\}^E$ *be the set of covectors of an oriented matroid. Then* $\mathcal{L}$ *can be reconstructed from its set of cocircuits* $\mathcal{C}^* = \{X \in \mathcal{L} : \mathrm{rk}(X) = 1\} = \mathrm{Min}(\mathcal{L} \setminus \{\hat{0}\})$ *as the set of their compositions, hence*

$$\mathcal{L} = \{X_1 \circ \ldots \circ X_k : X_1, \ldots, X_k \in \mathcal{C}^*\}.$$

The maximal covectors of an oriented matroid are called **topes**. An element $e \in E$ is called a **loop** of $\mathcal{M} = (E, \mathcal{L})$ if $X_e = 0$ for all $X \in \mathcal{L}$. Let $E_0$ denote the set of loops of $\mathcal{M}$. For a tope $T \in \mathcal{L}$ we have $T^0 = E_0$.

**Definition 3.1.5** (See [12], Lemma 4.1.10)**.** *Two distinct elements* $e, f \in E \setminus E_0$ *are* ***parallel*** *if they satisfy one of the following equivalent conditions.*

*(i)* $X_e = X_f$ *for all* $X \in \mathcal{L}$ *or* $X_e = -X_f$ *for all* $X \in \mathcal{L}$.

*(ii) For all $X \in \mathcal{L}$ we have $X_e = 0 \Rightarrow X_f = 0$.*

**Definition 3.1.6.** *An oriented matroid is called **simple** if it has neither loops nor parallel elements.*

**Remark 3.1.7.** *In an arrangement of pseudospheres or hyperplanes topes correspond to the regions of $S^d \setminus \mathscr{A}$ resp. $\mathcal{M}(\mathscr{A}) = \mathbb{R}^d \setminus \mathscr{A}$.*

**Definition 3.1.8.** *An oriented matroid is called **realizable** if it is given by a real valued matrix.*

As we will see later on (by the Topological Representation Theorem 3.1.24), this corresponds in the rank 3 case to the question whether a pseudoline arrangement is stretchable or non-stretchable.

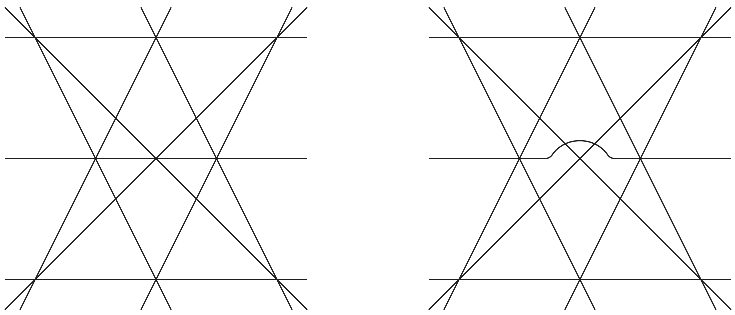


Figure 3.4: The independence structure of the well-known Pappus configuration on the left-hand side gives a realizable matroid. Whereas the matroid given by the non-Pappus configuration on the right-hand side is not realizable.

### 3.1.3 Underlying matroid, minors and duality

The connection of an oriented matroid given by its covectors is provided by the zero map, by which it is possible to show that the zero sets of the covectors are the flats of the underlying matroid.

**Proposition 3.1.9** (See [12], Proposition 4.1.13)**.** *Let $\mathcal{L} \subseteq \{+,-,0\}^E$ be the set of covectors of an oriented matroid.*

*(i) The set $L = \{X^0 : X \in \mathcal{L}\}$ is the collection of flats of the underlying matroid.*

*(ii) The map $z : \mathcal{L} \to L$ is a cover-preserving, order-preserving surjection of $\mathcal{L}$ onto the geometric lattice $L$. It satisfies the algebraic property*

$$(X \circ Y)^0 = X^0 \cap Y^0 = X^0 \vee Y^0.$$

**Remark 3.1.10.** *It is obvious by Proposition 3.1.9 that the rank of an oriented matroid* $\mathcal{M} = (E, \mathcal{L})$ *equals the rank of its underlying matroid* $M$*. Set* $d := \mathrm{rk}(\mathcal{L})$ *then for all* $X \in \mathcal{L}$ *we have* $\mathrm{rk}_M(X^0) = d - \mathrm{rk}_{\mathcal{L}}(X)$*.*

Recall the definition of a minor for an ordinary matroid from Section 0.2. There are natural counterparts for oriented matroids.

**Lemma 3.1.11** (See [12], Lemma 4.1.8)**.** *Let* $\mathcal{L} \subseteq \{+,-,0\}^E$ *be the set of covectors of an oriented matroid and* $A \subseteq E$*.*

*(i) The* ***deletion*** *of* $A$ *given by*

$$\mathcal{L} \backslash A = \{X|_{E \smallsetminus A} : X \in \mathcal{L}\} \subseteq \{+,-,0\}^{E \smallsetminus A}$$

*is an oriented matroid.*

*(ii) The* ***contraction*** *to* $A$ *given by*

$$\mathcal{L}/A = \{X|_{E \smallsetminus A} : X \in \mathcal{L} \text{ with } A \subseteq X^0\} \subseteq \{+,-,0\}^{E \smallsetminus A}$$

*is an oriented matroid.*

**Remark 3.1.12.** *For the rank of the contraction we have*

$$\mathrm{rk}(\mathcal{L}/A) = \mathrm{rk}(\mathcal{L}) - \mathrm{rk}_M(A)$$

*(where* $M$ *is the underlying matroid of* $\mathcal{L}$*).*

By Lemma 3.1.11 oriented matroids are closed under taking deletion and contraction. An oriented matroid obtained by a sequence of deletions and contractions in $\mathcal{L}$ will be denoted as **minor**. We can also speak about **restriction** to $A$, denoted by $\mathcal{L}[A]$, which is the deletion of $E \smallsetminus A$.

## Duality

As it is the case for ordinary matroids, oriented matroids have a natural notion of duality. For the examples in Section 3.1.1, that is to say planar graphs and vector spaces, there exist the notions of planar graph duality or dual spaces in linear algebra. These concepts are generalized by the concept of duality for oriented matroids.

**Proposition 3.1.13** (See [12], Proposition 3.4.1)**.** *Given an oriented matroid* $\mathcal{M}$ *with ground set* $E = \{1, \ldots, n\}$*.*

*(i) The set of cocircuits* $\mathcal{C}^*$ *is the set of circuits of an oriented matroid called the* ***dual*** *of* $\mathcal{M}$ *and denoted by* $\mathcal{M}^*$*.*

*(ii) We have* $(\mathcal{M}^*)^* = \mathcal{M}$*.*

### 3.1.4 Topological Representation Theorem

The Topological Representation Theorem given by Folkman and Lawrence 1978 [47] is an important fact for the significance of oriented matroids. Four years later, the statement was strengthen by Edmonds and Mandel [75] using a proof using PL topology. We obtained the following statement:

*Loop-free oriented matroids of rank $d + 1$ (up to reorientation and isomorphism) are in one-to-one correspondence with essential arrangements of pseudospheres in $S^d$ (up to topological equivalence).*

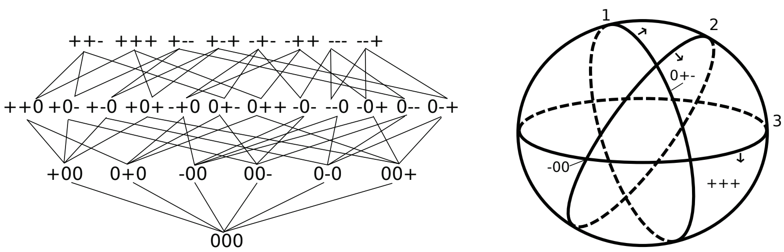



Figure 3.5: Representation of an oriented matroid by a pseudosphere arrangement.

The goal of this subsection is to show the reader a sketch of the proof of this statement. But there are several steps to do before one get to the point to prove the Topological Representation Theorem. We will study the combinatorial, topological and geometric properties of an oriented matroid in order to do so.

#### Step 1: From $\mathcal{L}$ to $\mathscr{A}$

Given the face lattice $\hat{\mathcal{L}}$ of an oriented matroid of rank $\mathrm{rk}(\mathcal{L}) = r$ on a finite set $E$. The goal of this step is to show the following two results:

(i) $\hat{\mathcal{L}}$ is isomorphic to the face lattice $\hat{\mathcal{F}}(\Delta_{\mathcal{L}})$ of some shellable regular cell decomposition of $S^{r-1}$.

(ii) $\mathcal{L}$ gives rise to an arrangement of pseudospheres.

The following result of Folkman and Lawrence [47], Las Vergnas [65] and Edmonds and Mandel [75] states two basic combinatorial properties of the face lattice $\hat{\mathcal{L}}$.

**Theorem 3.1.14** (See [12], Theorem 4.1.14)**.** *Let $\mathcal{L}$ be the set of covectors of an oriented matroid of rank $r$.*

*(i) $\hat{\mathcal{L}} = \mathcal{L} \cup \hat{1}$ is a graded lattice of length $r + 1$.*

*(ii) All intervals of length 2 in $\hat{\mathcal{L}}$ have cardinality 4.*

**Definition 3.1.15.** *A graded poset $P$ is* ***thin*** *if all intervals $[x,y]$ of length 2 have cardinality 4. Furthermore, $P$ is* ***subthin*** *if all intervals $[x,y]$ of length 2 with $y \neq \hat{1}$ have cardinality 4 and all such intervals with $y = \hat{1}$ have cardinality 3 or 4 (at least one with cardinality 3).*

**Remark 3.1.16.** *The property that every interval of length two has exactly four elements is also referred to as* ***diamond property*** *(see [107]).*

Therefore, the Theorem 3.1.14.(ii) states that the face lattice $\hat{\mathcal{L}}$ of an oriented matroid is thin. Furthermore, we also know the following fact about the face lattice. A proposition first shown by Lawrence in [67]. The proof can also be found in [12, Propositon 4.3.2]. For the definition of a recursive coatom ordering see Definition 3.4.6 in the appendix of this chapter.

**Proposition 3.1.17** (Lawrence 1984)**.** *Let $(E,\mathcal{L})$ be an oriented matroid. Then $\hat{\mathcal{L}} = \mathcal{L} \cup \hat{1}$ admits a recursive coatom ordering (given by a particular order on its topes).*

After studying the structural properties of the poset of covectors of an oriented matroid we now want to show their importance for the topological properties of its order complex. Recall the definitions for a regular cell decomposition $\Delta$, the face poset $\mathcal{F}(\Delta)$ and the augmented face poset $\hat{\mathcal{F}}(\Delta) = \mathcal{F}(\Delta) \cup \{\hat{0}, \hat{1}\}$ (see Section 0.3.1).

**Proposition 3.1.18** (See [12], Theorem 4.7.24)**.** *Let $P$ be a graded poset of length $d+2$. Then:*

*(i) $P \cong \hat{\mathcal{F}}(\Delta)$ for some shellable regular cell decomposition $\Delta$ of the d-sphere $\Leftrightarrow$ $P$ is thin and admits a recursive coatom ordering.*

*(ii) $P \cong \hat{\mathcal{F}}(\Delta)$ for some shellable regular cell decomposition $\Delta$ of the d-ball $\Leftrightarrow$ $P$ is subthin and admits a recursive coatom ordering.*

*In both cases $\Delta$ is uniquely determined by $P$ up to cellular homeomorphism, and recursive coatom orderings of $P$ correspond via isomorphism to shellings of $\Delta$.*

By Theorem 3.1.14 and Proposition 3.1.17 the face lattice of an oriented matroid satisfies Proposition 3.1.18.$(i)$. Hence, it corresponds to the face poset of a shellable regular cell decomposition of the $d$-sphere and we obtain immediately the following.

**Theorem 3.1.19** (See [12], Theorem 4.3.3)**.** *Let $(E,\mathcal{L})$ be an oriented matroid of rank r. Then $\hat{\mathcal{L}}$ is isomorphic to the face lattice of a shellable regular cell decomposition $\Delta_{\mathcal{L}}$ of the (r-1)-sphere, unique up to cellular homeomorphism.*

In the next proposition we keep considering the regular cell complex $\Delta_{\mathcal{L}}$, thus the oriented matroid sphere constructed in Theorem 3.1.19. Let us investigate a structure of subcomplexes which is crucial for the proof of the Topological

Representation Theorem 3.1.24. We will use these subcomplexes to obtain an arrangement of pseudospheres starting with an oriented matroid.

The cells of $\Delta_{\mathcal{L}}$ may be identified with the sign vectors of $\mathcal{L} - \{\hat{0}\}$ in a inclusion-preserving manner, thus order ideals in $\mathcal{L}$ can be used to designate subcomplexes of $\Delta_{\mathcal{L}}$.

For each $e \in E, A \subseteq E$, and $i \in \{+,-\}$, let

$$\begin{aligned}\Delta_e^0 &= \{X \in \mathcal{L} : X_e = 0\},\\ \Delta_e^i &= \{X \in \mathcal{L} : X_e \in \{0, i\}\},\\ \Delta_A^0 &= \cap_{e \in A} \Delta_e^0, \text{ and } \Delta_A^i = \cap_{e \in A} \Delta_e^i.\end{aligned} \tag{3.1}$$

We observe that the subcomplexes $\Delta_A^0 \cong \Delta_{\mathcal{L}/A}$ correspond to the matroid contraction.

**Proposition 3.1.20** (See [12], Proposition 4.3.6)**.** *Let $\mathcal{L} \subseteq \{+,-,0\}^E$ be a loop-free oriented matroid of rank $r \geq 2$. The system of subcomplexes (3.1) of the regular cell complex $\Delta_{\mathcal{L}}$ has the following properties:*

*(a)* $\Delta_A^0 = \Delta_{\mathrm{cl}\,A}^0$ *is a shellable* $(r - \mathrm{rk}(A) - 1)$*-sphere, for all $A \subseteq E$.*

*(b) If $e \in E - \mathrm{cl}\,A$, then $\Delta_A^0 \cap \Delta_e^i$, for $i \in \{+,-\}$, are shellable* $(r - \mathrm{rk}(A) - 1)$*-balls, each with boundary $\Delta_{A \cup e}^0$.*

*(c) Every non-empty intersection $\Delta_A^+ \cap \Delta_B^-$, for $A, B \subseteq E$, is a shellable sphere or a shellable ball.*

Comparing this with the Definition 2.2.6 of an arrangement of pseudospheres one obtains immediately the following fact for the proof of the Topological Representation Theorem.

**Corollary 3.1.21.** *For a loop-free oriented matroid $\mathcal{L} \subseteq \{+,-,0\}^E$, the family of subcomplexes* (3.1) *of $\Delta_{\mathcal{L}}$ forms an arrangements of pseudospheres.*

### Step 2: From $\mathscr{A}$ to $\mathcal{L}$

Let $\mathscr{A} = (S_e)_{e \in E}$ be a signed arrangement of pseudospheres in $S^d$ as defined in Definition 2.2.6. For each point in $x \in S^d$ define $\sigma(x) \in \{+,-,0\}^E$ by

$$\sigma(x)_e = \begin{cases} + & \text{if } x \in S_e^+,\\ - & \text{if } x \in S_e^-,\\ 0 & \text{if } x \in S_e.\end{cases}$$

The collection $\sigma(S^d)$ of all such sign vectors serves as an index set of the partition of $S^d$ into "cells" $\sigma^{-1}(X)$, for $X \in \sigma(S^d)$. Set $\Delta(\mathscr{A}) = \{\sigma^{-1}(X) : X \in \sigma(S^d)\}$ the collection of preimages in $S^d$.

**Theorem 3.1.22** (See [12], Theorem 5.1.4)**.** *For a signed arrangement of pseudospheres $\mathscr{A} = (S_e)_{e\in E}$ in $S^d$ the set*

$$\mathcal{L}(\mathscr{A}) = \{\sigma(x) : x \in S^d\} \cup \{0\} \subseteq \{+,-,0\}^E$$

*is the set of covectors of an oriented matroid. If* $\dim S_E = k$*, then the rank of* $\mathcal{L}(\mathscr{A})$ *is* $d-k$*. (In particular, if* $\mathscr{A}$ *is essential then* $\mathrm{rk}(\mathcal{L}(\mathscr{A})) = d+1$*.)*

Two signed arrangements of pseudospheres $\mathscr{A} = (S_e)_{e\in E}$ and $\mathscr{A}' = (S'_e)_{e\in E'}$ in $S^d$ are **topologically equivalent** if there exists a homeomorphism $h : S^d \to S^d$ and a bijection $g : E \to E'$ such that $h(S_e) = S'_{g(e)}$ and $h(S_e^+) = (S'_{g(e)})^+$ for all $e \in E$.

**Theorem 3.1.23** (See [12], Theorem 5.1.6)**.** *Two signed arrangements* $\mathscr{A}$ *and* $\mathscr{A}'$ *in* $S^d$ *are topologically equivalent if and only if* $\mathcal{L}(\mathscr{A}) \cong \mathcal{L}(\mathscr{A}')$*.*

Therefore, the topological equivalence classes of arrangements of pseudospheres are combinatorially determined.

## Step 3: The proof

**Topological Representation Theorem 3.1.24** (See 5.2.1 in [12])**.** *Let $E$ be a finite set and $\mathcal{L} \subseteq \{+,-,0\}^E$ then the following conditions are equivalent:*

*(i)* $\mathcal{L}$ *is the set of covectors of a loop-free oriented matroid of rank* $d+1$*.*

*(ii)* $\mathcal{L} = \mathcal{L}(\mathscr{A})$ *for some signed arrangement* $\mathscr{A} = (S_e)_{e\in E}$ *of pseudospheres in* $S^{d+1+k}$*, such that* $\dim(\bigcap_{e\in E} S_e) = k$*.*

*(iii)* $\mathcal{L} = \mathcal{L}(\mathscr{A})$ *for some signed arrangement* $\mathscr{A}$ *of pseudospheres in* $S^d$*, which is essential and centrally symmetric and whose induced cell complex* $\Delta(\mathscr{A})$ *is shellable.*

*Sketch of the proof of the Topological Representation Theorem 3.1.24.*
The implication (iii)⇒(ii) is trivial, and (ii)⇒(i) is given by Theorem 3.1.22.

For (i)⇒(ii), we know by Theorem 3.1.19 that the face lattice $\hat{\mathcal{L}}$ of a rank $d+1$ oriented matroid is isomorphic to the face lattice $\hat{\mathcal{F}}(\Delta_{\mathcal{L}})$ of a shellable regular cell decomposition $\Delta_{\mathcal{L}}$ of a $d$-sphere. Furthermore, by Corollary 3.1.21 the system of subcomplexes $\mathscr{A} = (\Delta_e^0)_{e\in E}$ from (3.1) forms an arrangement of pseudospheres in $\Delta_{\mathcal{L}} \cong S^d$. The signature of $\mathscr{A}$ is naturally induced by $\mathcal{L}$ such that $\mathcal{L}(\mathscr{A}) = \mathcal{L}$. The construction shows that $\Delta(\mathscr{A}) = \Delta_{\mathcal{L}}$ which is a shellable. The arrangement $\mathscr{A}$ is essential since the cells of $\Delta_{\mathcal{L}}$ are indexed by $\mathcal{L} - \{\hat{0}\}$ and thus, $S_E = \Delta_E^0 = \varnothing$. For the last remaining part that $\mathscr{A}$ is centrally symmetric we refer to the proof in [12]. □

### 3.1.5 Affine oriented matroids

**Definition 3.1.25** (See [12], Definition 4.5.1)**.** *Let $\mathcal{M} = (E, \mathcal{L})$ be an oriented matroid and $g \in E$ no loop. The triple $(E, \mathcal{L}, g)$ is called an* ***affine oriented matroids****. Set*

$$\mathcal{L}^+ = \mathcal{L}_g^+ = \{X \in \mathcal{L} : X_g = +\} \text{ and } \hat{\mathcal{L}}^+ = \mathcal{L}^+ \cup \{0, \hat{1}\},$$

*we call $\mathcal{L}^+$ the* ***affine face poset*** *of $(E, \mathcal{L}, g)$.*

If one considers an oriented matroid as an arrangement of pseudospheres, an affine oriented matroid corresponds to one hemisphere where the pseudosphere $S_g$ corresponding to $g$ is the equator.

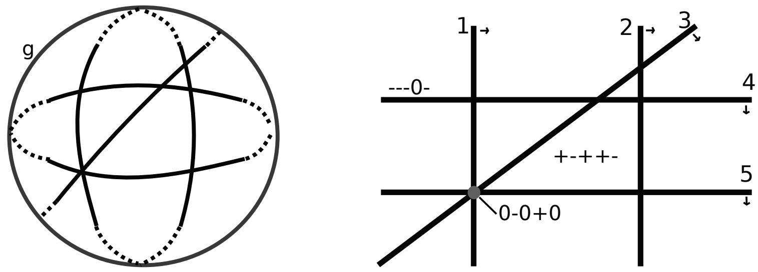



Figure 3.6: The affine oriented matroid $(E, \mathcal{L}, g)$ for an oriented matroid $\mathcal{L}$ and a distinguished element $g \in E$. On the right hand side, the corresponding affine hyperplane arrangement is shown.

In this way a realizable affine oriented matroid can also be obtained from an central hyperplane arrangement by distinguishing one hyperplane $H_g$ and considering the intersection of the arrangement with the translation of $H_g$ satisfying $\alpha_{H_g}(x) = 1$. This procedure is referred to as **deconing**. The other way around, we can also obtain a central arrangement in $\mathbb{R}^{d+1}$ from an affine hyperplane arrangement in $\mathbb{R}^d$ by **coning**, that is to say embedding the affine arrangement $\mathscr{A} = \{H_\alpha\} \subseteq \mathbb{R}^d$ in a translation of $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ – more accurately, in $\{x \in \mathbb{R}^{d+1} : x_{d+1} = 1\} \cong \mathbb{R}^d \subseteq \mathbb{R}^{d+1}$ – and considering the hyperplanes in $\mathbb{R}^{d+1}$ spanned by the $H_\alpha$ and the origin.

**Remark 3.1.26** (Compare Proposition 4.5.3 in [12])**.** *The poset $\mathcal{L}^+$ is pure of length $r-1$ and $\hat{\mathcal{L}}^+$ is a graded lattice of length $r+1$. Particularly, this means that $r_{\mathcal{L}^+}(X) = r_{\mathcal{L}}(X) - 1$ for all $X \in \mathcal{L}^+$ and the rank of the affine oriented matroid $(E, \mathcal{L}, g)$ is $r - 1$.*

**Proposition 3.1.27** (See [12], Proposition 4.5.6)**.** *Let $(E, \mathcal{L}, g)$ be an affine oriented matroid, then each tope in $\mathcal{L}^+$ gives a recursive coatom ordering of $\hat{\mathcal{L}}^+$.*

With this knowlegde in mind, we obtain statements about the topology of the affine oriented matroid. Let the **bounded complex** of $(E, \mathcal{L}, g)$ be given as

$$\mathcal{L}^{++} = \{X \in \mathcal{L}^+ : \mathcal{L}_{\leq X} \subseteq \hat{\mathcal{L}}^+\}.$$

**Theorem 3.1.28** (See [12], Theorem 4.5.7). *Let* $(E, \mathcal{L}, g)$ *be an affine oriented matroid and* $\mathrm{rk}(\mathcal{L}) = r$*. Then:*

*(i) The order complex* $\Delta_{ord}(\mathcal{L}^+)$ *is a shellable* $(r-1)$*–ball.*

*(ii) The bounded complex* $\Delta(\mathcal{L}^{++})$ *is contractible.*

As for an oriented matroid a recursive coatom ordering of $\hat{\mathcal{L}}^+$ is used to show shellability of $\Delta_{ord}(\mathcal{L}^+)$.

### 3.1.6 Affine sign vector systems

In his dissertation [62] in 1992, Johan Karlander gave an axiom system for systems of sign vectors which are corresponding affine oriented matroids. Unfortunately, he never made his research accessible for greater public. Twenty-three years later, Andrea Baum and Yida Zhu revealed and corrected an error in the proof of Karlander's Theorem in [5]. Hence, the main theorem remains valid.

I would like to thank Kolja Knauer for providing me access to [62] and for pointing out [5].

Let $\mathcal{W} \subseteq \{+,-,0\}^E$ be a collection of sign vectors and $X, Y \in \mathcal{W}$ then the vector $X \oplus Y$ is defined as

$$(X \oplus Y)_e = \begin{cases} 0 & \text{if } e \in S(X,Y), \\ (X \circ Y)_e & \text{otherwise.} \end{cases}$$

Furthermore, for $X, Y \in \{+,-,0\}^E$ with $\underline{X} = \underline{Y}$ and $e \in E$ define

$$I_e^=(X,Y) = \{V \in \mathcal{W} | V_e = 0, \forall f \notin S(X,Y) : V_f = X_f\},$$

then the **elimination set** of $X$ and $Y$ is given as

$$I^=(X,Y) = \bigcup_{e \in S(X,Y)} I_e^=(X,Y).$$

Now let $\mathrm{Asym}(\mathcal{W}) := \{X \in \mathcal{W} \mid -X \notin \mathcal{W}\}$ and set

$$\mathcal{P}_{as}^=(\mathcal{W}) = \{X \oplus (-Y) : X, Y \in \mathrm{Asym}(\mathcal{W}), \underline{X} = \underline{Y}, I^=(X,-Y) = I^=(-X,Y) = \varnothing\}$$

to be the **set of parallel vectors** of $\mathcal{W}$. Now we can states Karlander's main theorem (see [62, Theorem 2.1]). For the proof see also [5].

**Theorem 3.1.29** (See [62] and [5]). *A set* $\mathcal{W} \subseteq \{+,-,0\}^E$ *is an affine oriented matroid if and only if* $\mathcal{W}$ *satisfies*

*(FS)* $\mathcal{W} \circ -\mathcal{W} \subseteq \mathcal{W}$,

*(SE*$^=$*)* $X, Y \in \mathcal{W}, \underline{X} = \underline{Y} \implies \forall e \in S(X,Y) : I_e^=(X,Y) \neq \varnothing$,

*($P^{=}_{as}$)* $\mathcal{P}^{=}_{as}(\mathcal{W}) \circ \mathcal{W} \subseteq \mathcal{W}$.

It should be mentioned that [62, 5] additionally require the axiom (C) which is already implied by (FS) (see Remark 3.2.4).

**Definition 3.1.30.** *A set $\mathcal{W} \subseteq \{+,-,0\}^E$ satifying (FS), ($SE^{=}$) and ($P^{=}_{as}$) is called **affine sign vector system**.*

By the Theorem 3.1.29, we obtain a combinatorial characterization for non-central arrangements - which in the realizable case gives a correspondence to affine hyperplane arrangements. But unfortunately, there is still no established theory of *arrangements of pseudohyperplanes* - apart for pseudoline arrangements in dimension two (see Chapter 2). In Chapter 4, we will show that it is possible to loosen the conditions on $\mathcal{P}^{=}_{as}(\mathcal{W})$ and prove a generalization of this axiom system, see Section 4.4.

### 3.1.7 Tope graph

An oriented matroid can also be defined by the structure of its topes and unlike the other ways to define an oriented matroid, the set systems of topes are not a generalization of a concept of unoriented matroids.

In fact, knowing the set of topes $\mathcal{T}$ is enough to deduce the complete knowledge of the oriented matroid. This was first observed by Mandel, but unpublished, see Cordovil [25] and da Silva [27].

**Theorem 3.1.31** (Mandel)**.** *The set $\mathcal{T}$ of topes determines $\mathcal{L}$ via*

$$\mathcal{L} = \{X \in \{+,-,0\}^E : X \circ T \in \mathcal{T} \text{ for all } T \in \mathcal{T}\}.$$

Let us once more consider a hyperplane arrangement in $\mathbb{R}^d$. Here two regions are adjacent if they share a (d-1)-dimensional face. We will transfer this notion to the context of oriented matroids to construct an useful graph and poset structure on $\mathcal{T}$.

Let $\mathcal{L} \subseteq \{+,-,0\}^E$ be the set of covectors of an oriented matroid of rank $r$. The topes are the elements of rank $r$ in $\mathcal{L}$ and we call the elements of rank r-1 **subtopes**. Every subtope is covered by exactly two topes since the face lattice $\hat{\mathcal{L}}$ is thin by Theorem 3.1.14. Hence, it is natural to think of the subtopes as representing edges connecting this these pairs of topes.

**Definition 3.1.32.** *Two topes are **adjacent** if some subtope is covered by both. The adjacent pairs are the edges of the **tope graph** $\mathcal{G}(\mathcal{T})$ and the vertices are the topes.*

By the construction of the tope graph one obtains even more, all information about the topes can be erase, except the relation about adjacency, and we can still reconstruct the oriented matroid.

**Theorem 3.1.33** (Björner, Edelman and Ziegler 1990)**.** *A simple oriented matroid is uniquely determined (up to reorientation) by its unlabeled tope graph.*

## 3.2 Conditional oriented matroids

Conditional oriented matroids were introduced by Bandelt, Chepoi and Knauer in [3], inspired by the theory of lopsided sets (see [66, 4]). A conditional oriented matroid can be viewed as a complex whose cells are oriented matroids (glued together in a lopsided fashion). By this reason, Bandelt, Chepoi and Knauer also give them the alternative name *"complexes of oriented matroids"*.

The geometric intuition for a realizable conditional oriented matroids is a hyperplane arrangement restricted to an open convex set in $\mathbb{R}^d$.

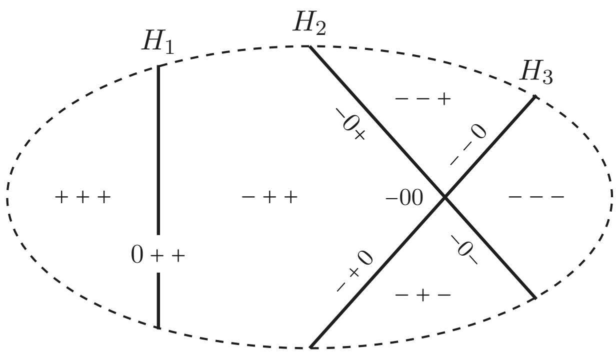



Figure 3.7: An arrangement of lines in $\mathbb{R}^2$ restricted to an open convex set and the corresponding sign vectors.

Let $E$ be a finite (non-empty) ground set and $\mathcal{L} \subseteq \{+,-,0\}^E$ a non-empty set of sign-vectors then the pair $(E,\mathcal{L})$ will be called **system of sign vectors**. As in Section 3.1 the elements of $\mathcal{L}$ are also referred to as covectors.

**Definition 3.2.1** (Basic axioms)**.** *Let $(E,\mathcal{L})$ be a system of sign vectors.*

*(Sym)* $-X \in \mathcal{L}$ *for all* $Y \in \mathcal{L}$.

*(C)* $X \circ Y \in \mathcal{L}$ *for all* $X, Y \in \mathcal{L}$.

*(FS)* $X \circ -Y \in \mathcal{L}$ *for all* $X, Y \in \mathcal{L}$.

*(SE)* *For all* $X, Y \in \mathcal{L}$ *and each* $e \in S(X,Y)$ *there exists* $Z \in \mathcal{L}$ *with* $Z_e = 0$ *and* $Z_f = (X \circ Y)_f = (Y \circ X)_f$ *for all* $f \notin S(X,Y)$.

*(Z)* *The zero sign vector belongs to* $\mathcal{L}$.

Let us explain the geometric motivation for these axioms (see Figure 3.8). Let $X, Y \in \mathcal{L}$ designating subsets of $\mathbb{R}^2$ represented by the points $x, y$ and $l$ be the line through $x$ and $y$. As in Section 3.1.2, by a (sufficiently small) $\varepsilon$-step from $x$ to $y$ we obtain the point $w$ lying in the subset of $\mathbb{R}^2$ designated by $X \circ Y$, which motivates (C). The axiom (FS) is geometrically motivated by an $\varepsilon$-step from $x$ on $l$ in the opposite direction (away from $y$). We obtain the point $u$

contained in the subset designated by $X \circ -Y$. Let $z$ be the point where $l$ crosses the separating hyperplane $H_e$. The subset of $\mathbb{R}^2$ which contains $z$ is designated by a sign vector $Z$ satisfying the conditions of (SE), having $Z_e = 0$ and the same sign as $X$ or $Y$ for each hyperplane which does not separate $x$ and $y$.

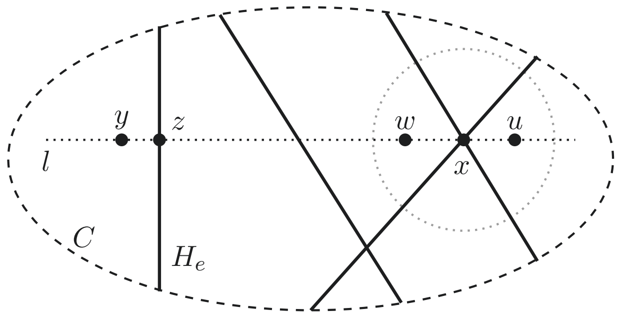



Figure 3.8: Motivating model for the axioms for systems of sign vectors. Compare with [3, Figure 2].

**Definition 3.2.2** (See [3], Definition 1)**.** *A system of sign vectors $(E, \mathcal{L})$ satisfing (FS) and (SE) is called a **conditional oriented matroid**.*

If no confusion can arise we will write $\mathcal{L}$ instead of $(E, \mathcal{L})$.

**Remark 3.2.3.** *A system of sign vectors $(E, \mathcal{L})$ is an oriented matroid if one of the following conditions is satisfied.*

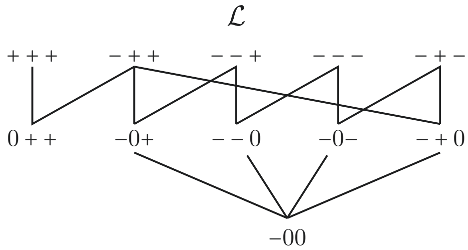
*(i) The pair $(E, \mathcal{L})$ satisfies (Sym), (C) and (SE). (Compare Definition 3.1.2)*

*(ii) The pair $(E, \mathcal{L})$ satisfies (Z), (FS) and (SE). (See [3, Section 2])*

**Remark 3.2.4.** *It should be mentioned that face symmetry (FS) implies composition (C) since for $X, Y \in \mathcal{L}$ we get $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y)$ which lies in $\mathcal{L}$ by (FS).*

**Example 3.2.5.** *Consider an affine hyperplane arrangement and an open convex set $C$ in $\mathbb{R}^d$. Let $E$ be the set of hyperplanes which intersect $C$ and $\mathcal{L}$ be the set of sign vectors which correspond to the faces of the arrangement that intersect with $C$ then the resulting system $(E, \mathcal{L})$ is a conditional oriented matroid.*

*Consider the example in Figure 3.7, we obtain the conditional oriented matroid given by $E = \{1, 2, 3\}$ and the following poset of covectors $\mathcal{L}$.*



### 3.2.1 Minors

**Lemma 3.2.6** (See [3], Lemma 1)**.** *Let $(E,\mathcal{L})$ be a conditional oriented matroid and $A \subseteq E$, then*

*(i) the* ***deletion*** *$(E \setminus A, \mathcal{L}\backslash A)$ of $A$ given by*

$$\mathcal{L}\backslash A = \{X|_{E\setminus A} : X \in \mathcal{L}\} \subseteq \{+,-,0\}^{E\setminus A}$$

*is a conditional oriented matroid.*

*(ii) the* ***contraction*** *$(E \setminus A, \mathcal{L}/A)$ to $A$ given by*

$$\mathcal{L}/A = \{X|_{E\setminus A} : X \in \mathcal{L} \text{ with } A \subseteq X^0\} \subseteq \{+,-,0\}^{E\setminus A}$$

*is a conditional oriented matroid.*

If a system of sign vectors arises by deletions and contractions from another one it is said to be **minor** of it. By Lemma 3.2.6 conditional oriented matroids are closed under taking minors.

**Definition 3.2.7.** *Let $(E,\mathcal{L})$ be a system of sign vectors and $A \subseteq E$.*

*(i) A* ***fibre*** *relative to some $X \in \mathcal{L}$ is defined as*

$$\mathcal{L}_{(\geq X, A)} = \{Y \in \mathcal{L} : Y|_{E\setminus A} = X|_{E\setminus A}\}.$$

*(ii) If $X^0 = A$ then the corresponding fibre $\mathcal{L}_{(\geq X, X^0)}$ is called a* ***conditional face****.*

In [3], the set $\mathcal{L}_{(\geq X, X^0)}$ is just called face. But this term already has a different meaning in the theory of arrangements, so we refer to it as *conditional face.*

**Remark 3.2.8.** *If $(E,\mathcal{L})$ satisfies (C) then the conditional face $\mathcal{L}_{(\geq X, X^0)}$ of $X$ is equal to the upper interval $\mathcal{L}_{\geq X}$. Furthermore, for every $X \in \mathcal{L}$ in a conditional oriented matroid the set $\mathcal{L}_{\geq X}$ corresponds to the deletion $\mathcal{L}\backslash \underline{X}$ which is an oriented matroid by (FS).*

**Lemma 3.2.9** (See [3], Lemma 4)**.** *Let $(E,\mathcal{L})$ be a conditional oriented matroid.*

*(i) All fibres of $(E,\mathcal{L})$ are conditional oriented matroids.*

*(ii) For any $X \in \mathcal{L}$ the restriction $(E-\underline{X}, \mathcal{L}_{\geq X}\backslash \underline{X})$ to $X^0$ is an oriented matroid.*

**Definition 3.2.10** (See [3], Section 1.2 and 11.4)**.** *Let $(E,\mathcal{L})$ be a conditional oriented matroid.*

*(i) If $(E,\mathcal{L})$ corresponds to the intersection of an open convex set with a set of affine hyperplanes (as in Example 3.2.5) it is called* ***realizable.***

*(ii) If the deletion $\mathcal{L}\backslash \underline{X}$ is a realizable oriented matroid for all $X \in \mathcal{L}$ then $(E,\mathcal{L})$ is called* ***locally realizable****.*

**Proposition 3.2.11** (See [3], Propsition 15)**.** *Every realizable conditional oriented matroid is locally realizable.*

### 3.2.2 Decompostion and amalgamation

Now we want to develop a gluing construction such that we can successively glue conditional oriented matroids and obtain conditional oriented matroids again.

Let us fix some notations $E_{\pm} := \{e \in E \mid \{X_e : X \in \mathcal{L}\} \neq \{+\}, \{-\}\}$ and $E_0 = \{e \in E \mid X_e = 0 \text{ for all } X \in \mathcal{L}\}$.

**Definition 3.2.12** (See [3], page 10)**.** *A conditional oriented matroid* $(E, \mathcal{L})$ *is called* ***semisimple*** *if it satisfies*

*(i) for all* $e \in E$ *holds* $\{X_e : X \in \mathcal{L}\} \neq \{0\}$*, i.e.* $E_0 = \varnothing$*,*

*(ii) for all* $e \neq f$ *in* $E_{\pm}$ *there exist* $X, Y \in \mathcal{L}$ *with* $X_e X_f = +$ *and* $Y_e Y_f = -$*.*

Actually, the condition (ii) already implies (i). But we keep both conditions to make it possible to see that this definition originates from loops and parallel elements in an oriented matroid. Although, they are not as strict (for the definition of a simple conditional oriented matroid see [3, p. 10]).

**Remark 3.2.13.** *Actually, Bandelt, Chepoi and Knauer use a stronger condition "for each* $e \in E_{\pm}$ *we have* $\{X_e : X \in \mathcal{L}\} = \{+, -, 0\}$*" instead of (i) in the definition of semisimplicity [3, page 10]. But in the presence of (FS) and (SE) they are equivalent.*

**Definition 3.2.14** (See [3], Section 8)**.** *A system* $(E, \mathcal{L})$ *of sign vectors is an* ***conditional oriented matroid amalgam*** *of two semisimple conditional oriented matroids* $(E, \mathcal{L}')$ *and* $(E, \mathcal{L}'')$ *if the following is satisfied*

*(1)* $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ *with* $\mathcal{L}' \setminus \mathcal{L}'', \mathcal{L}'' \setminus \mathcal{L}', \mathcal{L}' \cap \mathcal{L}'' \neq \varnothing$*;*

*(2)* $(E, \mathcal{L}' \cap \mathcal{L}'')$ *is a semisimple conditional oriented matroid;*

*(3)* $\mathcal{L}' \circ \mathcal{L}'' \subseteq \mathcal{L}'$ *and* $\mathcal{L}'' \circ \mathcal{L}' \subseteq \mathcal{L}''$*;*

*(4) for* $X \in \mathcal{L}' \setminus \mathcal{L}''$ *and* $Y \in \mathcal{L}'' \setminus \mathcal{L}'$ *with* $X^0 = Y^0$ *there exists a shortest path in the graphical hypercube on* $\{+, -\}^{E \setminus X^0}$ *for which all its vertices and barycenters of its edges belong to* $\mathcal{L} \backslash X^0$*.*

**Proposition 3.2.15** (See [3], Proposition 6)**.** *The conditional oriented matroid amalgam of two semisimple conditional oriented matroids is again a semisimple conditional oriented matroid for which every maximal face is a maximal face of at least one of the two constituents.*

**Corollary 3.2.16** (See [3], Corollary 3)**.** *Semisimple conditional oriented matroids are obtained via successive conditional oriented matroid amalgamations from their maximal conditional faces.*

### 3.2.3 Homotopy type

**Lemma 3.2.17** (Gluing Lemma, see Lemma 10.3 in [9]). *Let $\Delta_1, \Delta_2$ be simplicial complexes then*

*(i) if $\Delta_1$ and $\Delta_1 \cap \Delta_2$ are contractible, then $\Delta_1 \cup \Delta_2 \simeq \Delta_2$,*

*(ii) if $\Delta_1$ and $\Delta_2$ are $k$-connected and $\Delta_1 \cap \Delta_2$ is $(k-1)$-connected, then $\Delta_1 \cup \Delta_2$ is $k$-connected,*

*(iii) if $\Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2$ are $k$-connected, then so are $\Delta_1$ and $\Delta_2$.*

**Proposition 3.2.18** (See [3], Proposition 14). *If $(E, \mathcal{L})$ is a conditional oriented matroid, then $\Delta(\mathcal{L})$ is a contractible regular cell complex (and the tope graph of $\mathcal{L}$ is realized by the 1-skeleton of $\Delta(\mathcal{L})$).*

This proposition is proven by using the gluing lemma and the decomposition construction described in [3, Section 8] where a semisimple conditional oriented matroid is constructed by conditional oriented matroid amalgamations.

### 3.2.4 Open conjectures

In the end of their paper Bandelt, Chepoi and Knauer make some conjectures about the connection of conditional oriented matroids and oriented matroids. But there is still a lot of work to do before they are proven.

**Conjecture 3.2.19** (See [3], Section 12). *Every conditional oriented is a fibre of some oriented matroid.*

**Conjecture 3.2.20** (See [3], Section 12). *Every locally realizable conditional oriented matroid is a fibre of a realizable oriented matroid.*

## 3.3 Zonotopal tilings

In 1989, Andreas Dress announced the following theorem: The tilings of a zonotope $Z$ by zonotopes are in bijection with the single-element liftings of the associated oriented matroid $\mathcal{L}(Z)$ of $Z$ [41]. A proof of the theorem was provided in the doctoral dissertation of Jochen Bohne [17] in 1992.

Hence, it yields a straight, euclidean representation also for the non-realizable oriented matroids that have a realizable one-element contraction. In particular, for all oriented matroids of rank at most 3 exists a corresponding zonotopal tiling.

The interested reader is referred to [91], where Richter-Gebert and Ziegler give a nice introduction to the topic and provide a second proof for the theorem.

A zonotope is, to give three equivalent characterization, a affine projection of a cube, a Minkowski sum of line segments and a polytope all of whose faces are centrally symmetric. For more details on zonotopes the reader is referred to [107, Chapter 7].

**Definition 3.3.1.** *Let $V = (v_1, ..., v_n)$ be a vector configuration in $\mathbb{R}^d$ and all $v_i$ non-zero. The* ***zonotope*** *$Z = Z(V)$ of $V$ is Minkowski sum of line segments*

$$Z(V) = \{\sum_{i=1}^{n} \lambda_i v_i : -1 \leq \lambda_i \leq 1\} \subseteq \mathbb{R}^d.$$

**Remark 3.3.2.** *Zonotopes are closely related to oriented matroids. In fact, for every zonotope $Z(V)$ there exists the corresponding hyperplane arrangement $\mathscr{A}_V$ generated by the vector configuration $V$ and this hyperplane arrangement $\mathscr{A}_V$ corresponds naturally to a realizable oriented matroid.*

*More precisely, there is a natural bijection between the non-empty faces of the zonotope $Z(V)$, the faces of the hyperplane arrangement $\mathscr{A}_V$ and the signed covectors of the oriented matroid arising from $V$ (see [107]).*

Intuitively spoken a *zonotopal tiling* of rank $d$ is a $(d-1)$-dimensional polyhedral complex $\mathcal{P}$ such that both the union $|\mathcal{P}|$ and the faces $F \in \mathcal{P}$ are zonotopes. As formal definition we will use the version from Richter-Gebert and Ziegler (which they call *weak zonotopal tiling*) instead of the original version from Bohne.

**Definition 3.3.3** (See [91], Definition 1.4)**.** *Let $Z$ be a zonotope. A* ***zonotopal tiling*** *of $Z$ is a collection of zonotopes $\mathcal{Z} = \{Z_1, ..., Z_m\}$ such that*

*(i)* $\bigcup_{i=1}^{m} Z_i = Z$;

*(ii) if $U$ is a face of $Z_i \in \mathcal{Z}$, then $U \in \mathcal{Z}$;*

*(iii) if $Z_i, Z_j \in \mathcal{Z}$, then the intersection $Z_i \cap Z_j$ is a face of both $Z_i$ and $Z_j$.*

Let $\mathcal{L}$ be an oriented matroid again. Recall that $\mathcal{L}/A$ denotes the contraction of the oriented matroid to $A$ (see Lemma 3.1.11). Now the following set $\mathcal{O}(\mathcal{L}')$ should remind you of the set $\mathcal{L}^+$ from the Defintion 3.1.25 of an affine oriented matroid.

**Definition 3.3.4** (See [91], Definition 1.5)**.** *Let $\mathcal{L} \subseteq \{+,-,0\}^E$ be an oriented matroid. A* ***one-element lifting*** *of $\mathcal{L}$ is an oriented matroid $\hat{\mathcal{L}} \subseteq \{+,-,0\}^{E \cup g}$, such that $\mathcal{L}'/g = \mathcal{L}$ and $g$ is not a loop in $\mathcal{L}'$. We define*

$$\mathcal{O}(\mathcal{L}') = \{X \in \{+,-,0\}^E : (X,+) \in \mathcal{L}'\}.$$

Now we assembled enough substance to state the theorem from Bohne and Dress. For the proof the reader is referred to the Dissertation of Bohne [17] or the paper of Richter-Gebert and Ziegler [91].

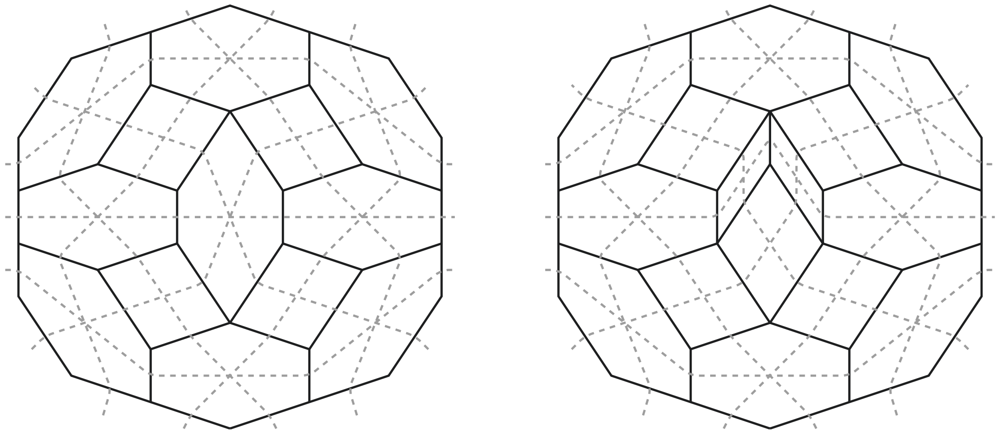


Figure 3.9: The zonotopal tilings corresponding to the Pappus arrangement and the non-Pappus arrangement (see [107, p. 219-220]).

**Bohne-Dress Theorem 3.3.5** (See [17], Theorem 4.1 and 4.2)**.** *Given a vector configuration* $V = (v_1, ..., v_n) \subseteq \mathbb{R}^d$*, let* $\mathcal{Z} = \mathcal{Z}(V)$ *be its zonotope and* $\mathcal{L} = \mathcal{L}(V)$ *its oriented matroid. Then there is a canonical bijection between the zonotopal tilings of* $Z(V)$ *and the one-element liftings* $\mathcal{L}' \subseteq \{+,-,0\}^{[n]\cup g}$ *of the oriented matroid* $\mathcal{L} = \mathcal{L}'/g$*.*

## Bohne's original definition

For later reference, we will give the original definition of a zonotopal tiling by Bohne. This version is slightly stronger as the version from Richter-Gebert and Ziegler in Definition 3.3.3, as it refers explicitly to the vector data of the zonotope. First, for $V = (v_1, ..., v_n) \subseteq \mathbb{R}^d$ and $X \in \{+,-,0\}^E$ set

$$Z_X = Z_X(V) = \left\{ \sum_{e \in E} \mu_e v_e \;\middle|\; \begin{array}{ll} -1 \le \mu_e \le 1 & \text{if} X_e = 0 \\ \mu_e = X_e & \text{if} X_e \neq 0 \end{array} \right\}$$

and for $\lambda \in (\mathbb{R}^d)^*$ define $X^\lambda \in \{+,-,0\}^E$ by $X_e^\lambda = \operatorname{sign}(\lambda(v_e))$ for $e \in E$.

**Definition 3.3.6** (See [17], Definition 1.3)**.** *Given a vector configuration* $V = (v_1, ..., v_n) \subseteq \mathbb{R}^d$*. The set* $\mathcal{O} \subseteq \{+,-,0\}^E$ *is a* ***(strong) zonotopal tiling*** *of* $Z(V)$ *if*

*(i)* $\bigcup_{X \in \mathcal{O}} Z_X = Z(V)$*;*

*(ii) for* $X, Y \in \mathcal{O}$ *with* $Z_X \cap Z_Y \neq \emptyset$ *there exists* $W \in \mathcal{O}$ *with* $Z_W = Z_X \cap Z_Y$*;*

*(iii) for all* $X \in \mathcal{O}$ *we have*

$$\{Y \in \mathcal{O} : Z_Y \subseteq Z_X\} = \{X \circ X^\lambda : \lambda \in (\mathbb{R}^d)^*\}.$$

### 3.3.1 Multiple oriented matroids

Considering the goal of developing a more general axiom system for affine oriented matroids which stays valid and fruitful for an infinite ground set $E$, we have to take a closer look at the Chapters 5 and 6 of Bohne's thesis [17]. There he introduces a generalization of zonotopal tilings and oriented matroids to an infinite set-up. Namely, the concept of a multiple oriented matroid to study the combinatorics of infinite zonotopal tiling, which include infinite periodic arrangement in which we are interested in.

Bohne's concept evolves from the following idea (see also [17, Section 5.1.1]). For every element $e$ in a finite set $E$ we have a possibly infinite collection $\mathcal{A}_e$ of parallel and periodic hyperplanes. Now identify the hyperplane $H_{e,0} \in \mathcal{A}_e$ with 0 and going in one direction the parallel hyperplanes in $\mathcal{A}$ and the subspaces in between will be identified with $1, 2, ...$, in the other direction by $-1, -2, ...$ as in Figure 3.10. In this way, we obtain an identification with a subset of $\mathbb{Z}$.

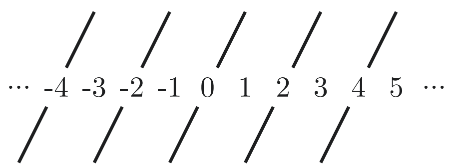



Figure 3.10: Intuition for a multiple oriented matroid (see [17, Section 5.1.1]).

Given a collection $V = (v_1, ..., v_n)$ of vectors in $\mathbb{R}^d$ and $E = \{1, ..., n\}$. For $\mathcal{O} \subseteq \mathbb{Z}^E$ let

$$m_{\mathcal{O}} : E \to \mathbb{N}_0 \cup \infty, e \mapsto \sup\{|X_e| : X \in \mathcal{O}\}$$

be the **multiplicity** of $\mathcal{O}$. The set $\mathcal{O}$ is finite if $m_{\mathcal{O}} \in \mathbb{Z}^E$. Furthermore, for a vector $m \in ((2\mathbb{N} + 1) \cup \infty)^E$ set

$$Z_m(V) = \{\sum_{e \in E} \mu_e v_e : |\mu_e| \leq m_e\}.$$

For $X, Y \in \mathbb{Z}^E$ we define the zonotope

$$Z_X = Z_X(V) = \left\{ \sum_{e \in E} \mu_e v_e \;\middle|\; \begin{array}{ll} |\mu_e - X_e| \leq 1 & \text{if } X_e \equiv 0 \bmod 2 \\ \mu_e = X_e & \text{if } X_e \equiv 1 \bmod 2 \end{array} \right\}$$

and the composition $X \hat{\circ} Y$ by

$$(X \hat{\circ} Y)_e = \begin{cases} X_e + 1, & \text{if } X_e \equiv 0 \bmod 2 \text{ and } Y_e > X_e, \\ X_e - 1, & \text{if } X_e \equiv 0 \bmod 2 \text{ and } Y_e < X_e, \\ X_e, & \text{otherwise.} \end{cases}$$

Moreover, set the vector $X + Y$ to be defined by $(X + Y)_e = X_e + Y_e$ for all $e \in E$. For $m \in \mathbb{Z}^E$ define the map $\varphi_m : \mathbb{Z}^E \to \{+, -, 0\}^E$ by

$$\varphi_m(e) = \text{sign}(X_e - m_e)$$

and for $\mathcal{O} \subseteq \mathbb{Z}^E$ call $\varphi_m(\mathcal{O})$ the $m$-**contraction** of $\mathcal{O}$.

**Definition 3.3.7** (See [17], Definition 5.1)**.** *Given $V = (v_1, ..., v_n) \subseteq \mathbb{R}^d$. Then the collection $\mathcal{O} \subseteq \mathcal{Z}^E$ with multiplicity $m_{\mathcal{O}} \in ((2\mathbb{N}+1) \cup \infty)^E$ is a* ***multiple zonotopal tiling*** *of $Z = Z_{m_{\mathcal{O}}}(V)$ if it satisfies the following conditions.*

*(Z1)* $\bigcup_{X \in \mathcal{O}} Z_X = Z$.

*(Z2) For $X, Y \in \mathcal{O}$ with $Z_X \cap Z_Y \neq \varnothing$ there exists $W \in \mathcal{O}$ with $Z_W = Z_X \cap Z_Y$.*

*(Z3) For all $X \in \mathcal{O}$ we have*

$$\{Y \in \mathcal{O} : Z_Y \subseteq Z_X\} = \{X \hat{\circ} (X + X^\lambda) : \lambda \in (\mathbb{R}^d)^*\}.$$

*(Z4) For all $m \in (2\mathbb{Z})^E$ with $|m_e| < m_{\mathcal{O}}(e)$, for all $e \in E$, the $m$-contraction $\varphi_m(\mathcal{O})$ is a (strong) zonotopal tiling of $Z(V)$.*

**Remark 3.3.8.** *A finite multiple zonotopal tiling satisfies the Definition 3.3.6 (see [17, Theorem 5.4.1]).*

Let the separation set of $X, Y \in \mathcal{O}$ be

$$\hat{S}(X,Y) = \{e \in E : |X_e - Y_e| > 2 \text{ or } (|X_e - Y_e| = 2 \text{ and } X_e \equiv 0 \bmod 2)\}.$$

**Definition 3.3.9** (See [17], Definition 5.2)**.** *Given an oriented matroid $\mathcal{L} \subseteq \{+,-,0\}^E$ without loops. The set $\mathcal{O} \subseteq \mathbb{Z}^E$ with multiplicity $m_{\mathcal{O}} \in ((2\mathbb{N}+1) \cup \infty)^E$ is a* ***multiple oriented matroid*** *with respect to $\mathcal{L}$ if the following is satisfied.*

*(M1) $X, Y \in \mathcal{O}$ implies $X \hat{\circ} Y \in \mathcal{O}$.*

*(M2) If $X, Y \in \mathcal{O}$ and $e \in \hat{S}(X,Y)$ then for $k \in \mathbb{Z}$ with $k \equiv 0 \bmod 2$ and*

$$\min(X_e, Y_e) < k < \max(X_e, Y_e),$$

*there exists $Z \in \mathcal{O}$ such that $Z_e = k$ and*

$$\min((X \hat{\circ} Y)_f, (Y \hat{\circ} X)_f) \leq Z_f \leq \max((X \hat{\circ} Y)_f, (Y \hat{\circ} X)_f)$$

*for all $f \in E$.*

*(M3) For all $X \in \mathcal{L}, Y \in \mathcal{O}$ and $m \in (2\mathbb{N}+1)^E$ with $m_e < m_{\mathcal{O}}(e)$ for all $e \in E$, there exists $Z \in \mathcal{O}$ such that*

$$X_e \cdot Z_e \geq m_e \text{ for all } e \in E \text{ with } X_e \neq 0,$$

$$\text{and } \; Z_e = Y_e \text{ for all } e \in E \text{ with } X_e = 0.$$

*(M4)* *For all* $m \in (2\mathbb{Z})^E$ *with* $|m_e| \leq m_{\mathcal{O}}(e)$ *for all* $e \in E$, *the set*

$$\mathcal{O}_m^+ = \left\{ X \in \{+,-,0\}^{E \cup g} \;\middle|\; \begin{array}{ll} & \text{if } X_g = + \text{ and } X|_E \in \varphi_m(\mathcal{O}) \\ \text{or} & \text{if } X_g = 0 \text{ and } X|_E \in \mathcal{L} \\ \text{or} & \text{if } X_g = - \text{ and } -X|_E \in \varphi_m(\mathcal{O}) \end{array} \right\}$$

*is an oriented matroid on* $E \cup g$.

**Theorem 3.3.10** (See Theorem 6.1 and 6.2 in [17]). *Given a collection* $V = (v_1, ..., v_n)$ *of vectors in* $\mathbb{R}^d$ *and* $E = \{1, ..., n\}$. *A set* $\mathcal{O} \subseteq \mathbb{Z}^E$ *with multiplicity* $m_{\mathcal{O}}$ *is a multiple oriented matroid on* $E$ *if and only if* $\mathcal{O}$ *is a multiple zonotopal tiling of* $Z = Z_{m_{\mathcal{O}}}(V)$.

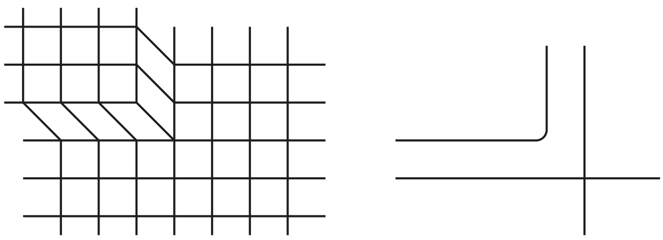


Figure 3.11: Tiling of $\mathbb{R}^2$ which is not a multiple zonotopal tiling (left-hand side), because parallelism is not transitive in its dual line system (right-hand side). See [17, Section 5.1.1].

It should be noted that the tiling of the plane $\mathbb{R}^2$ as in Figure 3.11 is not a multiple zonotopal tiling. In particular, Bohne emphasises in [17, Section 5.1.1] how important it is that parallelism in the dual line system, which arises from the tiling of the space, is transitive (see Figure 3.11). For a multiple zonotopal tiling this is given by condition (Z4).

## 3.4 Appendix: Shellablity

Shellablity is a topological property of cell complexes. Intuitively, it tells us that the cells of a complex can be glued in a well-behaved fashion. Using this fact, it can be shown that shellable complexes have the homotopy type of a wedge of spheres and that their Stanley-Reisner rings admit a combinatorially induced direct sum decomposition.

Considering the order complex of a poset shellablity can be transferred to partially ordered sets. A lot of progress in the theory of shellable posets has been done by Björner and Wachs, for example [13, 14, 15, 8] and [10] can be named. As an introduction to shellable posets we recommend [14].

Now recall the definition of a simplicial complex from Section 0.3 and that a complex is pure if all its maximal cells have the same dimension.

**Definition 3.4.1** (See [14], Definition 2.1). *A simplicial complex $\Delta$ is* ***shellable*** *if all its maximal faces can be arranged in a linear order $F_1, ..., F_l$ in a way such that the subcomplex $(\cup_{i=1}^{k-1}[\varnothing, F_i]) \cap [\varnothing, F_k]$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, ..., l$.*

*Such an ordering of the maximal faces is called a* ***shelling.***

Recall the definition 0.3.5 of an order complex of a poset then a poset is called **shellable** if its order complex is shellable.

**Remark 3.4.2.** *A finite poset $P$ is shellable if and only if the bounded poset $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ is shellable.*

**Proposition 3.4.3** (See [13], Proposition 8.2). *If $P$ is a shellable poset then all intervals of $P$ are shellable.*

An **edge labeling** of a poset $P$ is a map $\lambda : \mathcal{E}(P) \to P'$ where $\mathcal{E}(P) = \{(x,y) \in P \times P : x \lessdot y\}$ is the set of cover relations and $P'$ some poset. For a bounded poset let $\mathcal{M}(P)$ be the set of maximal chains of $P$ and $\mathcal{ME}(P)$ be the set of pairs $(m, x \lessdot y) \in \mathcal{M}(P) \times \mathcal{E}(P)$ consisting of a maximal chain $m$ and an edge $x \lessdot y$ along that chain (i.e., $x, y \in m$). A **chain-edge labeling** of $P$ is a map $\lambda : \mathcal{ME}(P) \to P'$ to some poset $P'$ satisfying:
If two maximal chains $m : \hat{0} = x_0 \lessdot x_1 \lessdot ... \lessdot x_k = \hat{1}$ and $m' : \hat{0} = x'_0 \lessdot x'_1 \lessdot ... \lessdot x'_k = \hat{1}$ coincide along their first $d$ edges, then their labels also coincide along these edges. That is, if $x_i = x'_i$ for $i = 0, 1, ..., d$, then $\lambda(m, x_{i-1} \lessdot x_i) = \lambda(m, x'_{i-1} \lessdot x'_i)$ for $i = 1, ..., d$.

Let $P$ be a poset with $\hat{0}$. If $[x, y]$ is an interval and $r$ an unrefinable chain from $\hat{0}$ to $x$, then the pair $([x,y], r)$ is called a **rooted interval** and denoted by $[x, y]_r$.

**Definition 3.4.4** (See [14], Definition 5.2). *Let $\lambda : \mathcal{ME}(P) \to P'$ be a chain-edge labeling of a bounded poset $P$.*

*(i) $\lambda$ is a CR-labeling (chain rising labeling) if in every rooted interval $[x, y]_r$ of $P$ there is a unique maximal chain $m$ whose label $\lambda_r(m) = (a_1, ..., a_p)$ satisfies $a_1 < ... < a_p$ in $P'$. We call $m$ the rising chain in $[x, y]_r$.*

*If for every rooted interval $[x, y]_r$ the unique rising chain $m$ is lexicographically strictly first, i.e., $\lambda_r(m) < \lambda_r(m')$ for all other maximal chains $m'$ in $[x, y]_r$, then we call $\lambda$ a* ***CL-labeling*** *(chain lexicographic labeling).*

*(ii) A CL-labeling which arises from an edge labeling $\lambda : \mathcal{E} \to P'$ is called an* ***EL-labeling*** *(edge lexicographic labeling).*

*(iii) A bounded poset that admits an EL- or CL-labeling is called* ***EL- or CL-shellable****.*

For the different types of shellablity of posets we have the following relations:

**Proposition 3.4.5** (See [13], Proposition 2.3)**.** *If a bounded poset is EL-shellable it is as well CL-shellable. If a poset is CL-shellable it is shellable.*

For the moment we are still quite closely related to the property of the order complex to be shellable, i.e., that its maximal faces can be ordered in a well-behaved fashion, since the maximal chains of the poset correspond to the maximal faces of the order complex.

The following property is applicable to all abstract posets and can be of use when one wants to show that a poset is the face poset of a cell complex. In fact it is used in the process of the proof of the Topological Representation Theorem 3.1.24. In order to show that the abstractly given covector lattice is the face lattice of a cell decomposition.

**Definition 3.4.6** (See [100], page 383)**.** *A chain-finite poset $P$ with $\hat{0}$ is said to admit a* ***recursive coatom ordering*** *if the length of $P$ is 1, or if $l(P) > 1$ and there is a well ordering $a_1, a_2, \ldots$ of the atoms of $P$ such that:*

*For all $j = 1, 2, \ldots, t$ the upper interval $P_{\geq a_j}$ admits a recursive atom ordering that begins with the minimal elements of $U_j = P_{\geq a_j} \cap \bigcup_{i<j} P_{\geq a_i}$ and $U_j$ is non-empty unless $j = 0$.*

Notice that in the above definition the poset is not required to contain a maximal element $\hat{1}$ contrary to the original definition. This formulation nevertheless is equivalent to the original definition for $P \cup \{\hat{1}\}$.

**Theorem 3.4.7** (See [13], Theorem 3.2)**.** *A bounded poset $P$ admits a recursive atom ordering if and only if $P$ is CL-shellable.*

**Lemma 3.4.8** (See [12], Lemma 4.7.18)**.** *A pure regular cell complex is shellable if and only if its face poset admits a recursive coatom ordering.*

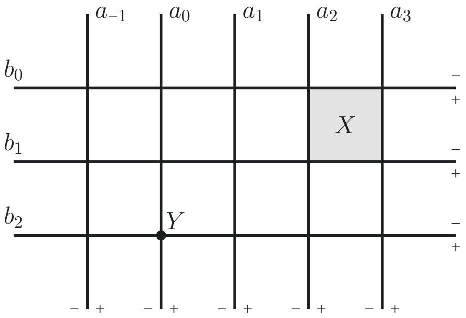
**Remark 3.4.9.** *By [100, Theorem 7.2], every finitary geometric semilattice with a well-order of the atoms which begins with a basic set of atoms (i.e., a maximal independent set of atoms) has a recursive atom ordering and thus, is chain lexicographically shellable. In particular, for this statement the ground set must not be necessaryly finite.*

# Chapter 4

# Oriented semimatroids

## 4.1 Oriented semimatroids

Our primary motivation for developing the concept of an oriented semimatroid is to characterize the face poset of locally finite arrangements in terms of sign vectors (see Figure 4.1). This with the ultimate goal in mind to gain better understanding of the face category of a toric arrangement $\mathscr{A}$ by using the characterization of its lift $\mathscr{A}^{\uparrow}$ (which is an infinite periodic arrangement see Section 2.1.2).




$$X_{a_i} = \begin{cases} + & \text{if } i < 3, \\ - & \text{if } i \geq 3. \end{cases} \qquad X_{b_i} = \begin{cases} + & \text{if } i < 1, \\ - & \text{if } i \geq 1. \end{cases}$$

$$Y_{a_i} = \begin{cases} + & \text{if } i < 0, \\ 0 & \text{if } i = 0, \\ - & \text{if } i > 0. \end{cases} \qquad Y_{b_i} = \begin{cases} + & \text{if } i < 2, \\ 0 & \text{if } i = 2, \\ - & \text{if } i > 2. \end{cases}$$

Figure 4.1: Motivation: Characterization of the cell decomposition given by an infinite periodic arrangement in terms of sign vectors.

A locally finite arrangement of affine hyperplanes restricted to a finite subset with non-empty intersection is a central arrangement for which there exists an associated oriented matroid. Moreover, every finite affine hyperplane arrangement can be seen as the decone of a central arrangement (see Section 3.1.5). So locally, restricted to a finite subset of the ground set $S$, our collection of sign vectors should satisfy the same properties as oriented matroids.

Let $S$ be a countable set and $X, Y \in \{+, -, 0\}^S$. Then the support $\underline{X}$, the zero set $X^0$, the composition $X \circ Y$ and the separation set $S(X, Y)$ are defined as for oriented matroids (see Section 3.1.2).

**Definition 4.1.1** (First Definition)**.** *Let $S$ be a countable set and $\mathcal{O} \subseteq \{+,-,0\}^S$. The pair $(S,\mathcal{O})$ is an* ***oriented semimatroid*** *if it satisfies the following properties.*

*(C)* *If $X,Y \in \mathcal{O}$ then so is $X \circ Y$.*

*(SE)* *For all $X,Y \in \mathcal{O}$ and each $e \in S(X,Y)$, there exists $Z \in \mathcal{O}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all $f \notin S(X,Y)$.*

*(Fin)* *There exists some $m \in \mathbb{N}$ such that all zero sets have at most $m$ elements and all separation sets are finite.*

*(LOM)* *For every $X \in \mathcal{O}$ the restriction $\mathcal{O}[X^0] = \{Y|_{X^0} : Y \in \mathcal{O}\} \subseteq \{+,-,0\}^{X^0}$ is an oriented matroid.*

*(PA)* *Let $X,Y \in \mathcal{O}$ and $A \subseteq X^0$. If the ranks of the oriented matroids $\mathcal{O}[Y^0]$ and $\mathcal{O}[A]$ satisfy $\mathrm{rk}(\mathcal{O}[Y^0]) < \mathrm{rk}(\mathcal{O}[A])$ then there exist $a \in A - Y^0$ and $Z \in \mathcal{O}$ such that $a \cup Y^0 \subseteq Z^0$.*

We can equip the set of sign vectors $\mathcal{O} \subseteq \{+,-,0\}^S$ with the same induced partial order "≤" as an oriented matroid $\mathcal{L}$ (see Section 3.1.2). Since we allow an infinite ground set $S$, the axiom (Fin) is needed in order to restrict our considerations to locally finite structures. If the ground set is infinite (Fin) implies that $0 \notin \mathcal{O}$. By (LOM) we have locally the structure of an oriented matroid. In particular, if a vector $Y \subseteq \{+,-,0\}^{X^0}$ lies in $\mathcal{O}[X^0]$ then so does its negative $-Y$. In the case, when $S$ is finite then $0 \in \mathcal{O}$ implies $\mathcal{O}$ is an oriented matroid.

For $X \in \mathcal{O}$ and $A \subseteq X^0$ the restriction to $A$ is an oriented matroid since $\mathcal{O}[A] = (\mathcal{O}[X^0])[A]$. So we can talk about its rank. The axiom (PA) corresponds to the axiom (G4) of a finitary geometric semilattice (see Definition 1.5.1). Roughly speaking, it gives us some control on parallelism behaviour of elements of $S$. Let us consider the following examples.

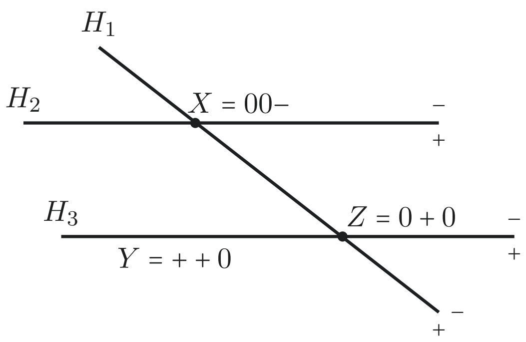



Figure 4.2: Motivation for the axiom (PA).

**Example 4.1.2** (Motivation for (PA))**.** *Consider three hyperplanes* $H_1, H_2, H_3$ *in* $\mathbb{R}^2$*. Suppose* $H_1$ *and* $H_2$ *are not parallel, that is to say they intersect in a point. Then, the third hyperplane* $H_3$ *can not be parallel to both* $H_1$ *and* $H_2$*, thus* $H_3$ *has to cross either* $H_1$ *or* $H_2$*. See Figure 4.2.*
*In the language of sign vectors, this means for* $X = 00- \in \mathcal{O}$ *with* $\mathrm{rk}(\mathcal{O}[X^0]) = 2$ *and* $Y = ++0 \in \mathcal{O}$ *with* $\mathrm{rk}(\mathcal{O}[Y^0]) = 1$ *there exist* $Z \in \mathcal{O}$ *and* $x \in X^0$ *such that* $x \cup Y^0 \subseteq Z^0$*. In our example of Figure 4.2, this is* $Z = 0+0$ *and* $x = 1$*.*

*More general for arbitrary dimensions, let* $H_1, \ldots, H_n$ *be affine hyperplanes in* $\mathbb{R}^d$ *and* $I, I' \subseteq [n]$ *such that* $X := \bigcap_I H_i \neq \varnothing, Y := \bigcap_{I'} H_i \neq \varnothing$ *and* $\mathrm{codim}(Y) < \mathrm{codim}(X)$*. Then, for the linear forms* $l_i \in (\mathbb{R}^d)^*$ *corresponding to the* $H_i$ *and* $L_X := \{l_i : i \in I\}, L_Y := \{l_i : i \in I'\}$ *we have* $\mathrm{rk}_{(\mathbb{R}^d)^*}(L_Y) < \mathrm{rk}_{(\mathbb{R}^d)^*}(L_X)$*, which means there exists* $j \in I \smallsetminus I'$ *with* $l_j \notin \mathrm{span}(L_Y)$*. The hyperplane* $H_j$ *has a non-empty intersection with* $Y$ *(see also [1, Proposition 2.2]).*

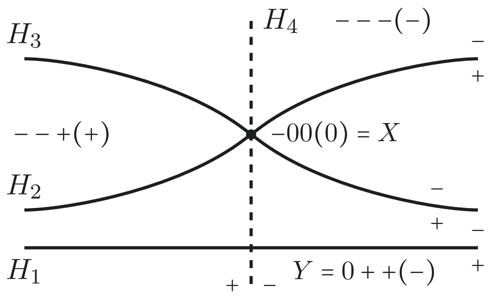



Figure 4.3: Neither the set of sign vectors given by the arrangement of pseudolines $\mathscr{A} = \{H_1, H_2, H_3\}$, nor the set of sign vectors given by the arrangement $\mathscr{A}' = \{H_1, H_2, H_3, H_4\}$ satisfy (PA). Since in both arrangements parallelism is not an equivalence relation.

**Example 4.1.3.** *From the arrangement* $\mathscr{A}$ *of Figure 4.3, we obtain the following set of sign vectors*

$$\mathcal{N}_{\mathscr{A}} = \{-00,\ 0++,\ -0+,\ -0-,\ --0,\ -+0,\ +++,\ -++,\ ---,\ --+.\ -+-\}$$

*on the ground set* $S_{\mathscr{A}} = \{1,2,3\}$ *and from* $\mathscr{A}'$ *of Figure 4.3 we obtain*

$$\begin{aligned}\mathcal{N}_{\mathscr{A}'} = \{ & -000,\ 0++0, \\ & 0+++,\ 0++-,\ +++0,\ ---0,\ -++0,\ -0++,\ -0--,\ --0+,\ -+0-, \\ & ++++,\ +++-,\ -+++,\ -++-,\ ---+,\ ----,\ --++.\ -+--\}\end{aligned}$$

*on the ground set* $S_{\mathscr{A}'} = \{1,2,3,4\}$*.*
*Now let us consider* $X = -00$ *and* $Y = 0++$ *in* $\mathcal{N}_{\mathscr{A}}$*. The set*

$$\mathcal{N}_{\mathscr{A}}[X^0] = \{00,\ 0+,\ 0-,\ +0,\ -0,\ ++,\ --,\ -+,\ +-\}$$

*is an oriented matroid of rank 2 and the set*

$$\mathcal{N}_{\mathscr{A}}[Y^0] = \{0,\ +,\ -\}$$

*is an oriented matroid of rank 1. But there exists no* $Z$ *in* $\mathcal{N}_{\mathscr{A}}$ *with* $\{1,2\} \subseteq Z^0$ *or* $\{1,3\} \subseteq Z^0$*. Hence, the set* $\mathcal{N}_{\mathscr{A}}$ *does not satisfy the axiom (PA).*

*By looking at* $\mathscr{A}'$ *we will see that it is necessary to consider* $A \subseteq X^0$ *for some* $X$ *in the sign vector system, instead of just* $X^0$*. For every* $X, Y \in \mathcal{N}_{\mathscr{A}'}$ *with* $\mathrm{rk}(\mathcal{N}_{\mathscr{A}'}[X^0]) > \mathrm{rk}(\mathcal{N}_{\mathscr{A}'}[Y^0])$ *there exist an element* $x \in X^0 - Y^0$ *and* $Z \in \mathcal{N}_{\mathscr{A}'}$ *such that* $x \cup Y^0 \subseteq Z^0$*, so* $\mathscr{A}'$ *would satisfy the weaker version of (PA). But the weaker version is not closed under deletion since by deleting* $H_4$ *we obtain the arrangement* $\mathscr{A}$*. Considering* $X' = -000$*,* $A = \{2,3\}$ *and* $Y' = 0++-$ *shows that* $\mathcal{N}_{\mathscr{A}'}$ *also does not satisfy the axiom (PA).*

### 4.1.1 Motivation: Infinite affine arrangements

**Theorem 4.1.4.** *Every locally finite hyperplane arrangement* $\mathscr{A} = \{H_i : i \in I\}$ *in* $\mathbb{R}^d$ *with countable index set* $I$ *gives rise to an oriented semimatroid.*

*Proof.* The arrangement $\mathscr{A}$ determines a regular cell decomposition of $\mathbb{R}^d$ (see Section 2.1.1). We will define an indexing set for this cell decomposition of $\mathbb{R}^d$ and show that it satisfies the axioms of an oriented semimatroid.

We have $\mathbb{R}^d = H_i \uplus H_i^+ \uplus H_i^-$ for every hyperplane $H_i \in \mathscr{A}$ where $H_i^+, H_i^-$ denote its positive and negative halfspace. As in Section 3.1.1, we can define a map

$$\sigma : \mathbb{R}^d \to \{+,-,0\}^I,\ x \mapsto (\sigma_i(x))_{i\in I}$$

with

$$\sigma_i(x) = \begin{cases} + & \text{if } x \in H_i^+, \\ - & \text{if } x \in H_i^-, \\ 0 & \text{if } x \in H_i. \end{cases}$$

The collecton $\sigma(\mathbb{R}^d)$ of all such sign vectors serves as an indexing set for a partition of $\mathbb{R}^d$ into "cells" $\sigma^{-1}(X)$, for $X \in \sigma(\mathbb{R}^d)$.

Set $S_{\mathscr{A}} = I$ (sometimes we also use $\mathscr{A}$ as ground set) and

$$\mathcal{O}_{\mathscr{A}} = \{\sigma(x) : x \in \mathbb{R}^d\} \subseteq \{+,-,0\}^I. \tag{4.1}$$

Let us show $(S_{\mathscr{A}}, \mathcal{O}_{\mathscr{A}})$ satisfies the axioms of Definition 4.1.1.

- *The condition (Fin).* The condition (Fin) is given by the local finiteness of the arrangement $\mathscr{A}$. Every point $x \in \mathbb{R}^d$ has a neighbourhood which intersects only finitely many hyperplanes of $\mathscr{A}$ then in particular, the zero set $\sigma(x)^0 = \{i \in I : x \in H_i\}$ of $\sigma(x)$ is finite. Moreover, the line segment between two points $x, y \in \mathbb{R}^d$ intersects only a finite number of hyperplanes in $\mathscr{A}$. This means only for a finite $I' \subseteq I$ the points $x, y$ lie in the opposite halfspaces of $H_i$, $i \in I'$. We have $S(\sigma(x), \sigma(y)) = I'$ is finite.

- *The condition (C).* Let $x, y \in \mathbb{R}^d$. We have to show that there is a point $z \in \mathbb{R}^d$ with $\sigma(z) = \sigma(x) \circ \sigma(y)$. Let $U_x$ be a neighbourhood of $x$ which only intersects the hyperplanes containing $x$. Consider the line segment $l$ from $x$ to $y$ and set $z$ to be a point in $U_x$ obtained by a (sufficiently small) $\varepsilon$-step on $l$ from $x$ to $y$ (see Figure 4.4). By $z \in U_x$ we have $\sigma_i(z) = \sigma_i(x)$ for all $i \in I$ with $x \notin H_i$. Thus $\sigma_i(z) = \sigma_i(x)$ if $\sigma_i(x) \neq 0$.

  Now consider $i \in \{i \in I : \sigma_i(x) = 0\}$. The line segment $l$ is contained in the affine subspace given by the intersection $\bigcap\{H_i \in \mathscr{A} : x, y \in H_i\}$ of all hyperplanes containing $x$ and $y$. So, for all $i$ with $\sigma_i(x) = \sigma_i(y) = 0$ we have $\sigma_i(z) = \sigma_i(y) = 0$. Moreover, if $\sigma_i(x) = 0$ and $\sigma_i(y) \neq 0$ then the line segment $l \setminus \{x\}$ lies in the same halfspaces $H_i^{\sigma_i(y)}$ of $H_i$ as the point $y$. In particular, this means $\sigma_i(z) = \sigma_i(y)$ if $\sigma_i(x) = 0$ and $\sigma_i(y) \neq 0$. All in all, we have $\sigma(z) = \sigma(x) \circ \sigma(y) \in \mathcal{O}_{\mathscr{A}}$. The condition (C) is satisfied.

- *The condition (SE).* Suppose $x, y \in \mathbb{R}^d$ and $i \in S(\sigma(x), \sigma(y))$ then we have $\sigma_i(x) = -\sigma_i(y)$ and $x, y$ lie in opposite halfspaces of $H_i$. Then, the hyperplane $H_i$ lies between $x$ and $y$ and has an intersection with the line segment $l$. Denote this intersection point by $w_i$. For all $j \in I$ such that $H_j$ is not separating $x$ and $y$ the point $w_i \in l \setminus \{x, y\}$ lies in the same halfspace $H_j^{\sigma_j(z)}$ of $H_j$ as the point $z$ from above (with $\sigma(z) = \sigma(x) \circ \sigma(y)$). Thus, for the sign vector $\sigma(w_i) \in \mathcal{O}_{\mathscr{A}}$ we have $\sigma_i(w_i) = 0$ and $\sigma_j(w_i) = (\sigma(x) \circ \sigma(y))_j$ for all $j \notin S(\sigma(x), \sigma(y))$. The condition (SE) is satisfied.

- *The condition (LOM).* For simplification, we use $\mathscr{A}$ as the index set $S_{\mathscr{A}}$ for the remainder of the proof. For all $x \in \mathbb{R}^d$, the set $\mathscr{A}_x := \{H_i \in \mathscr{A} : x \in H_i\}$ is a central hyperplane arrangement. Moreover,

$$\mathcal{O}_{\mathscr{A}}[\sigma(x)^0] = \{\sigma(y)|_{\mathscr{A}_x} : y \in \mathbb{R}^d\}$$

  is the sign vector system corresponding to $\mathscr{A}_x$ and thus, is an oriented matroid (see Section 3.1). Which means (LOM) is satisfied.

- *The condition (PA).* Let $x, y \in \mathbb{R}^d$ and $\mathscr{A}_x, \mathscr{A}_y$ be defined as above. Suppose

$$\operatorname{codim}_{\mathbb{R}^d}(\bigcap \mathscr{A}_y) = \operatorname{rk}(\mathcal{O}_{\mathscr{A}}[\sigma(y)^0]) < \operatorname{rk}(\mathcal{O}_{\mathscr{A}}[\sigma(x)^0]) = \operatorname{codim}_{\mathbb{R}^d}(\bigcap \mathscr{A}_x).$$

  Let $l_i \in (\mathbb{R}^d)^*$ denote the defining linear form of $H_i$ and $L_x = \{l_i : H_i \in \mathscr{A}_x\}$ resp. $L_y = \{l_i : H_i \in \mathscr{A}_y\}$. Then, we have

$$\operatorname{rk}_{(\mathbb{R}^d)^*}(L_y) < \operatorname{rk}_{(\mathbb{R}^d)^*}(L_x)$$

  in the dual space $(\mathbb{R}^d)^*$ (compare with [1, Proof of Proposition 2.2]). This means there exists $l_j \in L_x$ such that $l_j \notin \operatorname{span}_{(\mathbb{R}^d)^*}(L_y)$. For the corresponding hyperplane $H_j \in \mathscr{A}_x$ we have $H_j \cap (\bigcap \mathscr{A}_y) \neq \varnothing$ and $H_j \notin \mathscr{A}_y$. Then, for every $z \in H_j \cap (\bigcap \mathscr{A}_y)$ we have $\sigma(z)|_{\mathscr{A}_y \cup H_j} = 0$ and (PA) is satisfied.

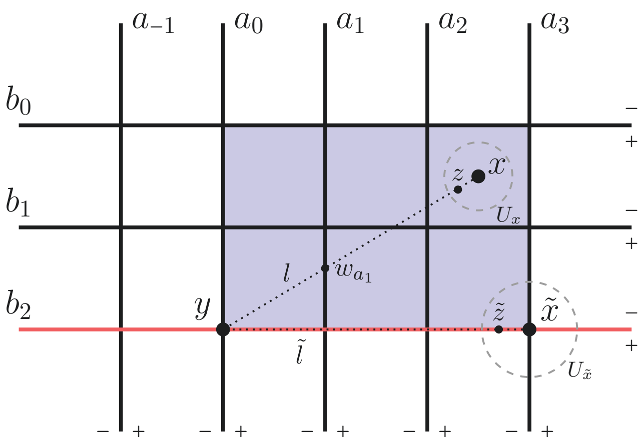



Figure 4.4: A locally finite arrangement in $\mathbb{R}^2$. In blue: The polyhedron bounded by the hyperplanes which do not lie between the points $x$ and $y$. In red: The affine subspace given by the intersection of all hyperplanes containing $\tilde{x}$ and $y$. Moreover, the line segments $l$ (resp. $\tilde{l}$) between $x$ (resp. $\tilde{x}$) and $y$ are drawn.

□

**Definition 4.1.5.** *An oriented semimatroid is called **realizable** if it arises from a locally finite hyperplane arrangement as in Theorem 4.1.4.*

### 4.1.2 Rank function

**Proposition 4.1.6.** *Let $(S, \mathcal{O})$ be an oriented semimatroid. The poset $\mathcal{O}$ is ranked by the function*

$$\mathrm{rk}_{\mathcal{O}} : \mathcal{O} \to \mathbb{N}, X \mapsto d - \mathrm{rk}(\mathcal{O}[X^0]),$$

*where $\mathrm{rk}(\mathcal{O}[X^0])$ is the rank of the oriented matroid $\mathcal{O}[X^0]$ and $d$ is the maximum value of $\mathrm{rk}(\mathcal{O}[X^0])$ over all $X \in \mathcal{O}$.*

**Definition 4.1.7.** *The **rank of an oriented semimatroid** $(S, \mathcal{O})$ is the maximal value of $\mathrm{rk}_{\mathcal{O}}$ on $\mathcal{O}$.*

*Proof of Proposition 4.1.6.* We have to show that every unrefinable chain from a minimal element to a fixed element $X \in \mathcal{O}$ has the same length. We will prove this in two steps. First, we show that $\mathrm{rk}_{\mathcal{O}}$ is consistent with the covering relation in $\mathcal{O}$, i.e. if $Y$ covers $X$ in $\mathcal{O}$ then we have $\mathrm{rk}(\mathcal{O}[X^0]) = \mathrm{rk}(\mathcal{O}[Y^0]) + 1$. Secondly, we show that for all minimal elements in $\mathcal{O}$ the restriction to the zero set has the same rank.

Step 1: *We have* $\mathrm{rk}(\mathcal{O}[X^0]) = \mathrm{rk}(\mathcal{O}[Y^0]) + 1$ *if* $X \lessdot Y$ *in* $\mathcal{O}$.

(i) *Constructing a map from $\mathcal{O}[Y^0]$ to $\mathcal{O}[X^0]$.* Let $X, Y \in \mathcal{O}$ and $Y$ covers $X$, then $Y^0 \subsetneq X^0$ and the map $\psi : \mathcal{O}[Y^0] \to \mathcal{O}[X^0]$ with

$$\psi(Z)_e := \begin{cases} Z_e, & \text{if } e \in Y^0, \\ Y_e, & \text{if } e \in X^0 - Y^0 \end{cases}$$

is an embedding. Let $Z \in \mathcal{O}[Y^0]$ and $\hat{Z} \in \mathcal{O}$ with $\hat{Z}|_{Y^0} = Z$, then the composition $Y \circ \hat{Z}$ satisfies $(Y \circ \hat{Z})|_{X^0} = \psi(Z)$ and $\psi$ is well-defined.

(ii) *The image of an unrefinable chain under $\psi$ is an unrefinable chain.* Let $Y_0 < \ldots < Y_l$ be a maximal chain in $\mathcal{O}[Y^0]$ with length $l = \mathrm{rk}(\mathcal{O}[Y^0])$, then $\psi(Y_0) < \ldots < \psi(Y_l)$ is a chain in $\mathcal{O}[X^0]$. Suppose there exists a covector $W \in \mathcal{O}[X^0]$ with $\psi(Y_i) < W < \psi(Y_{i+1})$ for some $i \in \{0, \ldots, l-1\}$. Thus $W = \hat{W}|_{X^0}$ for some $\hat{W} \in \mathcal{O}$. In $\mathcal{O}[Y^0]$ we have $Y_i < \hat{W}|_{Y^0} < Y_{i+1}$ since $\psi(Y_i)_e = \psi(Y_{i+1})_e$ for all $e \in X^0 - Y^0$. This contradicts our assumption that $Y_0 < \ldots < Y_l$ is maximal. Hence, the chain $\psi(Y_0) < \ldots < \psi(Y_l)$ is maximal between $\psi(Y_0)$ and $\psi(Y_l)$.

(iii) *If $Y_0$ minimal in $\mathcal{O}[Y^0]$ then $\psi(Y_0)$ is an atom in $\mathcal{O}[X^0]$.* For the zero covector in the oriented matroid $\mathcal{O}[X^0]$ we have $0_{\mathcal{O}[X^0]} < \psi(Y_0)$ (strictness holds since $\psi(Y_0)_e = Y_e \neq 0$ for all $e \in X^0 - Y^0$). Suppose there is $Z \in \mathcal{O}[X^0]$ such that $0_{\mathcal{O}[X^0]} < Z < \psi(Y_0)$ and $\hat{Z} \in \mathcal{O}$ with $\hat{Z}|_{X^0} = Z$. Consequently, for all $e \in Y^0$ we have $\hat{Z}_e = 0$ by $Z < \psi(Y_0)$ and $Y_0 = 0_{\mathcal{O}[Y^0]}$ by maximality of the chain. For the composition $X \circ Z$ we get

$$(X \circ Z)_e = \begin{cases} X_e = Y_e, & \text{if } e \notin X^0, \\ 0, & \text{if } e \in Y^0, \\ Z_e < \psi(Y_0)_e = Y_e, & \text{if } e \in X^0 - Y^0. \end{cases}$$

Thereby, we have $X < X \circ Z < Y$ which contradicts the condition that $Y$ covers $X$. Hence, $\psi(Y_0)$ covers $0_{\mathcal{O}[X^0]}$ and is an atom in $\mathcal{O}[X^0]$.

- *Conclusion step 1.* The covector $\psi(Y_l)$ is a maximal covector in $\mathcal{O}[X^0]$ by choice of $Y_l$ and the chain $0_{\mathcal{O}[X^0]} < \psi(Y_0) < \ldots < \psi(Y_l)$ is a maximal chain in $\mathcal{O}[X^0]$ of the length $l + 1 = \mathrm{rk}(\mathcal{O}[Y^0]) + 1$. Hence, by Definition 3.1.3 of the rank of an oriented matroid we have

$$\mathrm{rk}(\mathcal{O}[X^0]) = \mathrm{rk}(\mathcal{O}[Y^0]) + 1 \text{ if } X \lessdot Y \text{ in } \mathcal{O}. \tag{4.2}$$

The step 1 is shown.

Step 2: *For all minimal elements in $\mathcal{O}$ the restriction has the same rank.* Now let $X_0 < \ldots < X_n = X$ and $X'_0 < \ldots < X'_m = X$ be two unrefinable chains in $\mathcal{O}_{\leq X}$ with $X_0, X'_0$ minimal. By (4.2) we have

$$\mathrm{rk}(\mathcal{O}[X^0_{n-i}]) = \mathrm{rk}(\mathcal{O}[X^0]) + i = \mathrm{rk}(\mathcal{O}[(X'_{m-i})^0])$$

for all $i$ for which $X_{n-i}$ or respectively $X'_{m-i}$ exist. This implies

$$\operatorname{rk}(\mathcal{O}[X_0^0]) = \operatorname{rk}(\mathcal{O}[X^0]) + n \quad \text{and} \quad \operatorname{rk}(\mathcal{O}[(X_0')^0]) = \operatorname{rk}(\mathcal{O}[X^0]) + m.$$

Suppose $n < m$ then this would mean $\operatorname{rk}(\mathcal{O}[(X_0')^0]) > \operatorname{rk}(\mathcal{O}[X_0^0])$. By (PA) there exists $Z \in \mathcal{O}$ with $X_0^0 \subseteq Z^0$ and $\operatorname{rk}(\mathcal{O}[Z^0]) = \operatorname{rk}(\mathcal{O}[(X_0')^0])$. Set $k := |S(X_0, Z)|$. But $k = 0$ implies $Z < X_0$ which contradicts the minimality of $X_0$. If $k > 0$ then there exists $Z' \in \mathcal{O}$ with $X_0^0 \subseteq (Z')^0$ and $|S(X_0, Z')| < k$ by (SE). Thus, by induction there exists $Z'' \in \mathcal{O}$ with $X_0^0 \subseteq (Z'')^0$ and $|S(X_0, Z'')| = 0$ which implies again $Z'' < X_0$ and contradicts the minimality of $X_0$. The same reasoning holds for $m < n$, so we get $n = m$.

Thus, for two minimal elements $X_0, X_0' \in \mathcal{O}$ we have

$$\operatorname{rk}(\mathcal{O}[X_0^0]) = \operatorname{rk}(\mathcal{O}[(X_0')^0]) =: d.$$

The value $d$ is maximal by (4.2) and

$$\operatorname{rk}_{\mathcal{O}} : \mathcal{O} \to \mathbb{N}, X \mapsto d - \operatorname{rk}(\mathcal{O}[X^0])$$

is a well-defined rank function on $\mathcal{O}$. □

As in other matroidal structures we can define a notion of loops, parallel elements and simplicity. The interested reader may compare the following with the Definitions 0.2.3, 1.1.3, 3.1.6 of simplicity for matroids, semimatroids, oriented matroids and the Definiton 3.2.12 of semisimplicity for conditional oriented matroids.

**Definition 4.1.8.** *Let $(S, \mathcal{O})$ be an oriented semimatroid then the set of **loops** in $\mathcal{O}$ is defined as*

$$S_0 = \{e \in S : X_e = Y_e \text{ for all } X, Y \in \mathcal{O}\}.$$

*Furthermore, two distinct elements $a, b \in S - S_0$ are **parallel** if $X_a = X_b$ for all $X \in \mathcal{O}$ or $X_a = -X_b$ for all $X \in \mathcal{O}$.*

*When $(S, \mathcal{O})$ has neither loops nor parallel elements, we call it **simple**.*

**Remark 4.1.9.** *A maximal element $X$ in $(\mathcal{O}, <)$ has the zero set $X^0 = S_0$ by condition (C). Hence, we have for maximal elements*

$$\mathcal{O}[X^0] = \{Y|_{S_0} : Y \in \mathcal{O}\} \cong \{0\}$$

*and $\operatorname{rk}_{\mathcal{O}}(X) = d - \operatorname{rk}(\mathcal{O}[X^0]) = d$. Which also means for the rank of the oriented semimatroid $\operatorname{rk}(\mathcal{O}) = d$.*

**Corollary 4.1.10.** *The set of sign vectors of an oriented semimatroid is a pure poset.*

*Proof.* A poset is pure if all maximal chains have the same length. By Proposition 4.1.6, every unrefinable chain from a minimal element in $\mathcal{O}$ to a fixed element $X \in \mathcal{O}$ has the same length $\mathrm{rk}_{\mathcal{O}}(X)$.

Moreover, all maximal elements in the poset $(\mathcal{O}, <)$ have the same rank $d$ by Remark 4.1.9. Then, all maximal chains in $(\mathcal{O}, <)$ must have the same length (namely $d$).

□

**Remark 4.1.11.** *Contrary to an oriented semimatroid, the poset of sign vectors of a conditional oriented matroid is not pure. See Example 4.2.6.*

### 4.1.3 Underlying semimatroid

From now on we will suppose that $(S, \mathcal{O})$ is simple.

**Proposition 4.1.12.** *Let $(S, \mathcal{O})$ be an oriented semimatroid, then the collection $L_{\mathcal{O}} = \{Z^0 \subseteq S : Z \in \mathcal{O}\}$ of zero sets of $\mathcal{O}$ (ordered by inclusion) forms a finitary geometric semilattice. Moreover, the map $\zeta : \mathcal{O} \to L_{\mathcal{O}}$ is a cover-preserving, order-reversing and rank-reversing surjection satisfying $(X \circ Y)^0 = X^0 \wedge Y^0$.*

In order to proof the Proposition 4.1.12 we need the following lemma, the proof of which works exactly as for oriented matroids.

**Lemma 4.1.13** (Compare [12], Lemma 4.1.12)**.** *Suppose $X, Y \in \mathcal{O}$ with $\underline{X} \subseteq \underline{Y}$ and $X \nleq Y$. Then there exists $V \in \mathcal{O}$ such that $V \lessdot Y$ and $V_e = Y_e$ for all $e \notin S(X, Y)$.*

*Proof of Proposition 4.1.12.* Let $X, Y \in \mathcal{O}$, it is a straightforward observation that $(X \circ Y)^0 = X^0 \cap Y^0$ is the maximal set contained in $X^0$ and $Y^0$. Therefore, the set $L_{\mathcal{O}} = \{Z^0 \subseteq S : Z \in \mathcal{O}\}$ ordered by inclusion forms a meet semilattice, where $(X \circ Y)^0$ is the meet of $X^0$ and $Y^0$.

Let us show that $\zeta$ preserves cover relations. Assume $X, Y, Z \in \mathcal{O}$ such that $X < Y$ and $X^0 \supsetneq Z^0 \supsetneq Y^0$. We will show that $Y$ does not cover $X$. Replacing $Z$ by $X \circ Z$ we may assume $X < Z$. If $Z < Y$ we are done. Otherwise, by Lemma 4.1.13 there exists $V \in \mathcal{O}$ with $V < Y$ and $V_e = Y_e$ for all $e \notin S(Z, Y)$. Then $X < V$, since $X_e = Y_e = V_e$ for all $e \in \underline{X}$, and $X \neq V$, since $X_f = 0 \neq Y_f = V_f$ for any $f \in \underline{Y} \smallsetminus \underline{Z}$. Thus, we get $X < V < Y$. Which shows that $\zeta$ is cover-preserving.

Any unrefineable chain in $\mathcal{O}$ gives an unrefineable chain in $L_{\mathcal{O}}$ (in reversed order). By (C) we know $\varnothing \in L_{\mathcal{O}}$, which then is its unique minimal element $\hat{0}_{L_{\mathcal{O}}}$ (for $\mathcal{O}$ not simple we have $\hat{0}_{L_{\mathcal{O}}} = \{e \in S : X_e = 0 \text{ for all } X \in \mathcal{O}\}$). Hence, $L_{\mathcal{O}}$ is ranked with rank function

$$\mathrm{rk}_{L_{\mathcal{O}}}(X^0) = \mathrm{rk}(\mathcal{O}) - \mathrm{rk}_{\mathcal{O}}(X) = \mathrm{rk}(\mathcal{O}[X^0]), \tag{4.3}$$

which is bounded by some $m \in \mathbb{N}$ by (Fin).

- *Axiom (G3).* For every $X \in \mathcal{O}$ the interval $[\hat{0}, X^0] \subseteq L_{\mathcal{O}}$ corresponds to the zero sets in the restriction $\mathcal{O}[X^0]$. Thus, by (LOM) every interval $[\hat{0}, X^0]$ in $L_{\mathcal{O}}$ is a geometric lattice. Moreover, the axiom (Fin) gives a bound for the number of atoms of $[\hat{0}, X^0]$. So (G3) is satisfied.

- *Axiom (G4).* Suppose $X_1, ..., X_n \in \mathcal{O}$ such that $X_1^0, ..., X_n^0$ are atoms in $L_{\mathcal{O}}$ for which the join $X_1^0 \vee ... \vee X_n^0 \in L_{\mathcal{O}}$ exists and $\mathrm{rk}_{L_{\mathcal{O}}}(X_1^0 \vee ... \vee X_n^0) = n$. In particular, this means there exists $W \in \mathcal{O}$ with

$$W^0 = X_1^0 \vee ... \vee X_n^0 \supseteq X_1^0 \cup ... \cup X_n^0.$$

We have $\mathcal{O}[X_1^0 \cup ... \cup X_n^0] = (\mathcal{O}[W^0])[X_1^0 \cup ... \cup X_n^0]$ is an oriented matroid. Let us show that

$$\mathrm{rk}(\mathcal{O}[X_1^0 \cup ... \cup X_n^0]) = n. \tag{4.4}$$

For all $V \in \mathcal{O}$ with $X_1^0 \cup ... \cup X_n^0 \subseteq V^0$ we have $W^0 \subseteq V^0$ and

$$\mathrm{rk}(\mathcal{O}[W^0]) \leq \mathrm{rk}(\mathcal{O}[V^0])$$

by the choice of $W$. In particular, there exists no covector $V' \in \mathcal{O}[W^0]$ with $X_1^0 \cup ... \cup X_n^0 \subseteq (V')^0 \subsetneq W^0$, which means

$$\mathrm{cl}_{M_W}(X_1^0 \cup ... \cup X_n^0) = W^0$$

in the underlying matroid $M_W$ of $\mathcal{O}[W^0]$. Then (4.4) follows by

$$\begin{aligned} \mathrm{rk}(\mathcal{O}[X_1^0 \cup ... \cup X_n^0]) &= \mathrm{rk}_{M_W}(X_1^0 \cup ... \cup X_n^0) = \mathrm{rk}_{M_W}(W^0) \\ &= \mathrm{rk}(\mathcal{O}[W^0]) = \mathrm{rk}_{L_{\mathcal{O}}}(W^0) = n. \end{aligned} \tag{4.5}$$

Now consider $Y \in \mathcal{O}$ with $\mathrm{rk}_{L_{\mathcal{O}}}(Y^0) = \mathrm{rk}(\mathcal{O}[Y^0]) < n$ then by (PA) there exists $x \in (X_1^0 \cup ... \cup X_n^0) - Y^0$ and $Z \in \mathcal{O}$ such that $x \cup Y^0 \subseteq Z^0$. By simplicity of $\mathcal{O}$, we get $\{x\} = X_i^0$ for some $i$. Which means $X_i^0 \nsubseteq Y^0$ and their join $X_i^0 \vee Y^0 (\subseteq Z^0)$ in $L_{\mathcal{O}}$ exists. So the axiom (G4) is proven.

□

**Remark 4.1.14.** *The proof of Proposition 4.1.12 works as well without the assumption of simplicity. Then, the argumentation in the last part of the proof just has to be adapted to the case* $\mathrm{cl}_{M_W}(\{x\}) = X_i^0$ *for some* $i$.

By Proposition 4.1.12 and Theorem E we get immediately.

**Corollary 4.1.15.** *An oriented semimatroid* $(S, \mathcal{O})$ *has an underlying finitary semimatroid* $\mathcal{S}_{\mathcal{O}} = (S, \mathcal{C}_{\mathcal{O}}, \mathrm{rk}_{\mathcal{C}_{\mathcal{O}}})$ *with the subsets of the zero sets of* $\mathcal{O}$ *as central sets, i.e.*

$$\mathcal{C}_{\mathcal{O}} = \{A \subseteq S \mid A \subseteq X^0 \textit{ for some } X \in \mathcal{O}\},$$

*and the rank function*

$$\mathrm{rk}_{\mathcal{C}_{\mathcal{O}}} : \mathcal{C}_{\mathcal{O}} \to \mathbb{N}, A \mapsto \mathrm{rk}_{L_{\mathcal{O}}}(\bigvee A)$$

*where* $\bigvee A$ *is the join of* $A$ *in* $L_{\mathcal{O}}$ *and* $\mathrm{rk}_{L_{\mathcal{O}}}$ *as in (4.3).*

**Remark 4.1.16.** *For $A \in \mathcal{C}_\mathcal{O}$, the join of its elements $\bigvee A$ in $L_\mathcal{O}$ is the smallest zero set of $\mathcal{O}$ that contains $A$.*

**Remark 4.1.17.** *For every $X \in \mathcal{O}$ we have*

$$\mathrm{rk}_{\mathcal{C}_\mathcal{O}}(X^0) = \mathrm{rk}_{L_\mathcal{O}}(X^0) = \mathrm{rk}(\mathcal{O}) - \mathrm{rk}_\mathcal{O}(X) = \mathrm{rk}(\mathcal{O}[X^0]).$$

*Furthermore, for $A \in \mathcal{C}_\mathcal{O}$ and $W \in \mathcal{O}$ with $W^0 = \bigvee A$ we have*

$$\mathrm{rk}_{\mathcal{C}_\mathcal{O}}(A) = \mathrm{rk}_{L_\mathcal{O}}(\bigvee A) = \mathrm{rk}(\mathcal{O}) - \mathrm{rk}_\mathcal{O}(W) = \mathrm{rk}(\mathcal{O}[W^0]) = \mathrm{rk}(\mathcal{O}[A]),$$

*the last equality follows as in (4.5).*

### 4.1.4 Minors

**Proposition 4.1.18.** *Let $(S, \mathcal{O})$ be an oriented semimatroid.*

*(a) For $A \subseteq S$ with $A \subseteq X^0$ for some $X \in \mathcal{O}$, set*

$$\mathcal{O}/A = \{Z|_{S-A} : Z \in \mathcal{O} \text{ with } A \subseteq Z^0\}.$$

*The **contraction** $(S - A, \mathcal{O}/A)$ **to** $A$ is an oriented semimatroid.*

*(b) For $A \subseteq S$. The **deletion** $(S - A, \mathcal{O}\backslash A)$ **of** $A$ given by*

$$\mathcal{O}\backslash A = \{X|_{S-A} : X \in \mathcal{O}\} \subseteq \{+, -, 0\}^{S-A}$$

*is an oriented semimatroid.*

*Proof.* (a) *Contraction.* The properties (C), (SE) and (Fin) are transmitted directly from $\mathcal{O}$ to $\mathcal{O}/A$. Furthermore, for $\tilde{Z} \in \mathcal{O}$, $A \subseteq \tilde{Z}^0$ and $Z := \tilde{Z}|_{S-A} \in \mathcal{O}/A$ we have

$$(\mathcal{O}/A)[Z^0] = (\mathcal{O}[\tilde{Z}^0])/A, \tag{4.6}$$

which is an oriented matroid by Lemma 3.1.11. So, the property (LOM) is satisfied.

For proving (PA), let $Z, Y \in \mathcal{O}/A$ and $B \subseteq Z^0$. Suppose

$$\mathrm{rk}((\mathcal{O}/A)[Y^0]) < \mathrm{rk}((\mathcal{O}/A)[B]). \tag{4.7}$$

There exist $\tilde{Y}, \tilde{Z} \in \mathcal{O}$ with $Z = \tilde{Z}|_{S-A}$, $\tilde{Z}|_A = 0$ and $Y = \tilde{Y}|_{S-A}$, $\tilde{Y}|_A = 0$. For the disjoint union $A \uplus B \subseteq \tilde{Z}^0$ we have $\mathcal{O}[A \uplus B] = (\mathcal{O}[\tilde{Z}^0])[A \uplus B]$, which is an oriented matroid, and

$$\mathcal{O}[A \uplus B]/A = (\mathcal{O}/A)[B]. \tag{4.8}$$

For the remaining part of the proof, let us fix some notations and make an observation. The underlying matroids of $\mathcal{O}[\tilde{Y}^0]$, $\mathcal{O}[A \uplus B]$ and $\mathcal{O}[A]$

are denoted by $M_Y$, $M_{A\cup B}$ and $M_A$ respectively. Using $A \subseteq Y^0$, $A \subseteq A \uplus B$ and the fact

$$M_Y[A] = M_A = M_{A\cup B}[A],$$

we get for the rank of $A$ in the different matroids

$$\mathrm{rk}_{M_Y}(A) = \mathrm{rk}_{M_A}(A) = \mathrm{rk}_{M_{A\uplus B}}(A). \tag{4.9}$$

The equalities in (4.9) are implied since all considered matroids are obtained by deletions (for the rank of a deletion of a matroid see Definition 0.2.5). We can conclude

$$\begin{aligned}
\mathrm{rk}(\mathcal{O}[\tilde{Y}^0]) &= \mathrm{rk}((\mathcal{O}[\tilde{Y}^0])/A) + \mathrm{rk}_{M_Y}(A)\\
&\overset{(4.6)}{=} \mathrm{rk}((\mathcal{O}/A)[Y^0]) + \mathrm{rk}_{M_Y}(A)\\
&\overset{(4.7,4.9)}{<} \mathrm{rk}((\mathcal{O}/A)[B]) + \mathrm{rk}_{M_{A\cup B}}(A)\\
&\overset{(4.8)}{=} \mathrm{rk}(\mathcal{O}[A \uplus B]/A) + \mathrm{rk}_{M_{A\cup B}}(A)\\
&= \mathrm{rk}(\mathcal{O}[A \uplus B])
\end{aligned}$$

using the properties of the rank of the contraction of a matroid for the first and the last equality (see Definition 0.2.5 and Remark 3.1.10).

To recapitulate we have $\tilde{Z}, \tilde{Y} \in \mathcal{O}$ such that $A \uplus B \subseteq \tilde{Z}^0$ and $\mathrm{rk}(\mathcal{O}[\tilde{Y}^0]) < \mathrm{rk}(\mathcal{O}[A \uplus B])$. By the axiom (PA) for $\mathcal{O}$ there exist

$$b \in (A \uplus B) - \tilde{Y}^0 \subseteq S - A$$

and $\tilde{W} \in \mathcal{O}$ such that $A \subseteq \tilde{Y^0} \cup b \subseteq \tilde{W}^0$. The restriction $\tilde{W}|_{S-A} =: W \in \mathcal{O}/A$ satisfies (PA) in $\mathcal{O}/A$. This means $\mathcal{O}/A$ is an oriented semimatroid and (a) is proven.

(b) *Deletion.* As for the contraction, the axioms (C), (Fin) and (SE) are directly transmitted to the deletion $\mathcal{O}\backslash A$. For $\tilde{Z} \in \mathcal{O}$ with $Z := \tilde{Z}|_{S-A} \in \mathcal{O}\backslash A$ we have

$$(\mathcal{O}\backslash A)[Z^0] = (\mathcal{O}[\tilde{Z}^0])\backslash A,$$

which is an oriented matroid by Lemma 3.1.11. The deletion satisfies (LOM).

For (PA), let $X, Y \in \mathcal{O}\backslash A$ and $B \subseteq X^0$ such that

$$\mathrm{rk}((\mathcal{O}\backslash A)[Y^0]) < \mathrm{rk}((\mathcal{O}\backslash A)[B]). \tag{4.10}$$

For later reference, we make the following observation

$$B \subseteq X^0 \subseteq E - A \quad \text{and} \quad Y^0 \subseteq E - A. \tag{4.11}$$

Let $\tilde{Y}, \tilde{X} \in \mathcal{O}$ with $X = \tilde{X}|_{S-A}$ and $Y = \tilde{Y}|_{S-A}$. By $Y^0 \subseteq \tilde{Y}^0$, we get $Y^0 \in \mathcal{C}_\mathcal{O}$ for the underlying finitary semimatroid $\mathcal{S}_\mathcal{O}$ of $(S, \mathcal{O})$ as in Corollary 4.1.15.

Then, there exists some $W \in \mathcal{O}$ with $W^0 = \bigvee Y^0$, i.e. whose zero set is the join of $Y^0$ in $L_{\mathcal{O}}$. Thus, we have $\tilde{X}, W \in \mathcal{O}$, $B \subseteq \tilde{X}^0$ and

$$\begin{aligned} \mathrm{rk}(\mathcal{O}[W^0]) &= \mathrm{rk}(\mathcal{O}[Y^0]) \\ &\overset{(4.11)}{=} \mathrm{rk}((\mathcal{O}\backslash A)[Y^0]) \\ &\overset{(4.10)}{<} \mathrm{rk}((\mathcal{O}\backslash A)[B]) \\ &\overset{(4.11)}{=} \mathrm{rk}(\mathcal{O}[B]), \end{aligned}$$

where the first equality follows by Remark 4.1.17. By the condition (PA) in $(S, \mathcal{O})$, there exist $b \in B - W^0$ and $\tilde{Z} \in \mathcal{O}$ such that $b \cup Y^0 \subseteq b \cup W^0 \subseteq \tilde{Z}^0$. By $b \cup Y^0 \subseteq E - A$ we have for $Z := \tilde{Z}|_{E-A} \in \mathcal{O}\backslash A$ that $b \cup Y^0 \subseteq Z^0$. Hence, $(S - A, \mathcal{O}\backslash A)$ satisfies (PA) and is an oriented semimatroid.

□

Therefore, oriented semimatroid are closed under taking deletion and contraction. An oriented semimatroid obtained by a sequence of deletions and contractions in $\mathcal{O}$ is called a **minor** of $\mathcal{O}$. When we say **restriction** to $A$, we mean the deletion of $S - A$.

## 4.2 Relationship to existing concepts

### 4.2.1 Relationship to conditional oriented matroids

**Definition 4.2.1** (Second Definition)**.** *Let $S$ be countable and $\mathcal{O} \subseteq \{+,-,0\}^S$. The pair $(S, \mathcal{O})$ is an **oriented semimatroid** if it satisfies the following properties.*

*(FS) If $X, Y \in \mathcal{O}$ then so is $X \circ -Y$.*

*(SE) For all $X, Y \in \mathcal{O}$ and each $e \in S(X,Y)$, there exists $Z \in \mathcal{O}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all $f \notin S(X,Y)$.*

*(Fin) The zero set $X^0$ and the separation set $S(X,Y)$ are finite for all $X, Y \in \mathcal{O}$.*

*(PA) Let $X, Y \in \mathcal{O}$ and $A \subseteq X^0$. If the ranks of the oriented matroids $\mathcal{O}[Y^0]$ and $\mathcal{O}[A]$ satisfy $\mathrm{rk}(\mathcal{O}[Y^0]) < \mathrm{rk}(\mathcal{O}[A])$ then there exist $a \in A - Y^0$ and $Z \in \mathcal{O}$ such that $a \cup Y^0 \subseteq Z^0$.*

**Lemma 4.2.2.** *Let $S$ be a countable set and $\mathcal{O} \subseteq \{+,-,0\}^S$. Then the following is equivalent:*

*(i) $(S, \mathcal{O})$ is an oriented semimatroid;*

*(ii) $(S, \mathcal{O})$ satisfies (C), (SE), (Fin), (LOM) and (PA); (Definition 4.1.1)*

*(iii)* $(S,\mathcal{O})$ *satisfies (FS), (SE), (Fin) and (PA). (Definition 4.2.1)*

*Proof of the lemma 4.2.2.* For proving the equivalence of Definition 4.1.1 and Definition 4.2.1 we have to show that the conditions (C) and (LOM) imply (FS) and on the other hand, that the condition (FS) together with (SE) implies (C) and (LOM).

- *(C) and (LOM) imply (FS).* Assume the pair $(S,\mathcal{O})$ satisfies the conditions of Definition 4.1.1. For $X,Y \in \mathcal{O}$ we have $Y|_{X^0} \in \mathcal{O}[X^0]$. Then, $\mathcal{O}[X^0]$ also contains its negative $-(Y|_{X^0})$ since it is an oriented matroid by (LOM). Thus, there is $Z \in \mathcal{O}$ with $Z|_{X^0} = -Y|_{X^0}$. By (C) we get $X \circ (-Y) = X \circ Z \in \mathcal{O}$ and $(S,\mathcal{O})$ satisfies the axiom (FS).

  Hence, if $(S,\mathcal{O})$ satisfies Definition 4.1.1 it also satisfies Definition 4.2.1.

- *(FS) together with (SE) implies (C) and (LOM).*

  (i) A pair $(S,\mathcal{O})$ satisfying (FS) satisfies (C) by Remark 3.2.4.

  (ii) Suppose the pair $(S,\mathcal{O})$ satisfies (FS) and (SE). By Remark 3.2.3 we have to show that for all $X \in \mathcal{O}$ the restriction $\mathcal{O}[X^0] = \{Y|_{X^0} : Y \in \mathcal{O}\}$ satisfies the conditions (Z), (FS) and (SE). Let $X \in \mathcal{O}$. The set $\mathcal{O}[X^0]$ satisfies (FS) and (SE) since this means considering the conditions (FS) and (SE) of $(S,\mathcal{O})$ restricted to a subset $X^0 \subseteq S$ of the ground set. Moreover, $X|_{X^0} = 0_{\mathcal{O}[X^0]} \in \mathcal{O}[X^0]$ which means (Z) is satisfied.

  Thus, if $(S,\mathcal{O})$ satisfies Definition 4.2.1 it also satisfies Definition 4.1.1.

Therefore, the two definitions are equivalent and the lemma is proven. □

Comparing Lemma 4.2.2 with the Definition 3.2.2 of a conditional oriented matroid we get immediately the following.

**Corollary 4.2.3.** *Every finite oriented semimatroid is a conditional oriented matroid.*

**Corollary 4.2.4.** *If* $(S,\mathcal{O})$ *is a finite oriented semimatroid then its order complex* $\Delta(\mathcal{O})$ *is a contractible regular cell complex.*

*Proof.* The proof follows immediately by Corollary 4.2.3 and Proposition 3.2.18. □

**Corollary 4.2.5.** *If* $(S,\mathcal{O})$ *is a finite oriented semimatroid,* $X \in \mathcal{O}$ *and* $A \subseteq S$ *then the fibre*

$$\mathcal{O}_{(\geq X,A)} = \{Y \in \mathcal{O} : Y|_{E\setminus A} = X|_{E\setminus A}\}$$

*is a conditional oriented matroid.*

*Proof.* The proof follows immediately by Corollary 4.2.3 and Lemma 3.2.9. □

On the contrary, not every conditional oriented matroid is an oriented semimatroid. Consider the following example.

**Example 4.2.6.** *Let us consider the sign vector system* $(\{1,2,3\},\mathcal{L})$ *given in the Figure 4.5. The set* $\mathcal{L}$ *is equal to* $\mathcal{N}_{\mathscr{A}}$ *of Example 4.1.3. Hence, the intersection of the open convex set with an affine hyperplane arrangement on the left-hand side of Figure 4.5 and the arrangement of pseudolines* $\mathscr{A}$ *shown in Figure 4.3 determine equivalent stratifications and are combinatorially equivalent.*

*By Definition 3.2.10, a realizable conditional oriented matroid corresponds to a hyperplane arrangement restricted to an open convex set in* $\mathbb{R}^d$*, and we trust the reader will be able to check that (FS) and (SE) are satisfied in* $(\{1,2,3\},\mathcal{L})$ *(see Definition 3.2.2). But, as we have seen in Example 4.1.3, the pair* $(\{1,2,3\},\mathcal{L})$ *is not an oriented semimatroid since it does not satisfy (PA).*

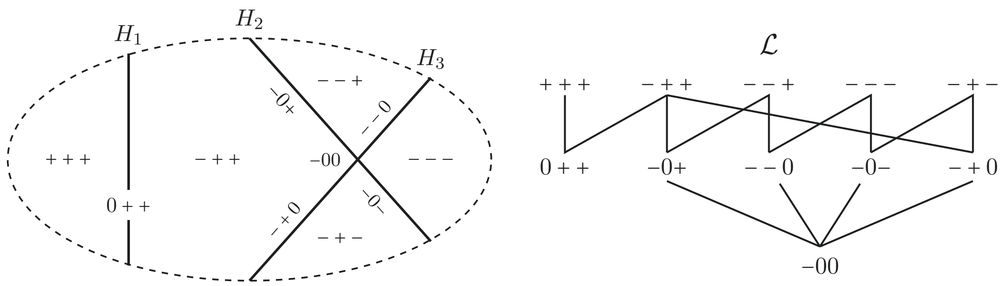



Figure 4.5: A realizable conditional oriented matroid which is not an oriented semimatroid. Its poset of sign vectors $\mathcal{L}$ on the right-hand side and the corresponding intersection of an open convex set with a hyperplane arrangement on the left-hand side.

**Conjecture 4.2.7.** *The Definition 3.2.2 of a conditional oriented matroid can be loosened to an infinite ground set* $E$.

### 4.2.2 Relationship to affine oriented matroids

**Theorem 4.2.8.** *Let* $(E,\mathcal{L},g)$ *be an affine oriented matroid and the set* $\mathcal{L}^+ = \{X \in \mathcal{L} : X_g = +\}$ *its affine face poset. The pair* $(E-g,\mathcal{L}^+)$ *is an oriented semimatroid.*

*Proof.* Given an affine oriented matroid $(E,\mathcal{L},g)$. Clearly, the affine face poset $\mathcal{L}^+$ satisfies (Fin). Moreover, for $X,Y \in \mathcal{L}^+$ we have $X \circ Y \in \mathcal{L}$ with $(X\circ Y)_g = +$, so $\mathcal{L}^+ \circ \mathcal{L}^+ \subseteq \mathcal{L}^+$ and (C) is satisfied.

Consider $X,Y \in \mathcal{L}^+ \subseteq \mathcal{L}$ and $e \in S_{\mathcal{L}^+}(X,Y) = S_{\mathcal{L}}(X,Y)$. Since $\mathcal{L}$ satisfies (SE) there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X\circ Y)_f = (Y\circ X)_f$ for all $f \notin S_{\mathcal{L}}(X,Y)$. By $X_g = Y_g = +$ we have $Z_g = (X\circ Y)_g = +$ and $Z \in \mathcal{L}^+$. So, $\mathcal{L}^+$ satisfies (SE).

Now consider the restriction $\mathcal{L}^+[X^0] = \{Z|_{X^0} : Z \in \mathcal{L}^+\}$ for $X \in \mathcal{L}^+$. For every $Y \in \mathcal{L}$ we have $X \circ Y \in \mathcal{L}^+$ and $Y|_{X^0} = (X \circ Y)|_{X^0}$, which means $\mathcal{L}^+[X^0] = \mathcal{L}[X^0]$. Then $\mathcal{L}^+[X^0]$ is an oriented matroid by Proposition 3.1.11. The condition (LOM) is satisfied.

For proving (PA), let $X, Y \in \mathcal{L}^+$ and $A \subseteq X^0$. As shown above, the restrictions $\mathcal{L}^+[Y^0]$ and $\mathcal{L}^+[A] = (\mathcal{L}^+[X^0])[A]$ are oriented matroids. Suppose we have for their ranks $\mathrm{rk}(\mathcal{L}^+[Y^0]) < \mathrm{rk}(\mathcal{L}^+[A])$. We know the rank of an oriented matroid and its of its underlying matroid are the same (see Remark 3.1.10) and for the rank in the deletion $M \backslash D$ in a matroid $M = (E, \mathrm{rk})$ we have $\mathrm{rk}_{M\backslash D}(F) = \mathrm{rk}(F)$ for all $F \subseteq E - D$ (see Definition 0.2.5).

Let $M$ be the underlying matroid of $\mathcal{L}$. Then, by using $\mathcal{L}^+[Y^0] = \mathcal{L}[Y^0]$ and $(\mathcal{L}^+[X^0])[A] = (\mathcal{L}[X^0])[A]$ we get

$$\begin{aligned}\mathrm{rk}_M(Y^0) &= \mathrm{rk}(M[Y^0]) = \mathrm{rk}(\mathcal{L}[Y^0]) = \mathrm{rk}(\mathcal{L}^+[Y^0]) \\ &< \mathrm{rk}(\mathcal{L}^+[A]) = \mathrm{rk}(\mathcal{L}[A]) = \mathrm{rk}(M[A]) = \mathrm{rk}_M(A).\end{aligned}$$

Thus, we can extend $Y^0$ by an element $a \in A - Y^0$ such that its rank in $M$ increases $\mathrm{rk}_M(Y^0 \cup a) = \mathrm{rk}_M(Y^0) + 1$. If $g \notin \mathrm{cl}_M(Y^0 \cup a)$, let $Z \in \mathcal{L}$ such that $Z^0 = \mathrm{cl}_M(Y^0 \cup a)$ which exists by Proposition 3.1.9. Either $Z$ or $-Z$ lies in $\mathcal{L}^+$, satisfying the requirements of (PA).

Otherwise, we have $g \in \mathrm{cl}_M(Y^0 \cup a)$. Thus, an according covector $Z \in \mathcal{L}$ with $Z^0 = \mathrm{cl}_M(Y^0 \cup a)$ will not lie in $\mathcal{L}^+$. We will show that there exists $b \in A$ with $\mathrm{rk}_M(Y^0 \cup b) = \mathrm{rk}_M(Y^0) + 1$ and $g \notin \mathrm{cl}_M(Y^0 \cup b)$. Recall the fourth closure axiom for the matroid $M$ from [88, Lemma 1.4.3].

(CL4) If $B \subseteq E$ and $x, y \in E$, then $y \in \mathrm{cl}_M(B \cup x) - \mathrm{cl}_M(B)$ implies $x \in \mathrm{cl}(B \cup y)$

First, assume for all $b \in A$ we have $\mathrm{rk}_M(Y^0 \cup a \cup b) = \mathrm{rk}_M(Y^0 \cup a)$. Then, $A \subseteq \mathrm{cl}_M(Y^0 \cup a)$ and $\mathrm{rk}_M(A) = \mathrm{rk}_M(Y^0 \cup a)$. Hence,

$$\mathrm{cl}_M(Y^0 \cup a) = \mathrm{cl}_M(A) \subseteq \mathrm{cl}_M(X^0) = X^0$$

by the matroid version of Proposition 4.6.2.(e). Which contradicts the assumption that $g \notin X^0$. So, there exists $b \in A$ with $\mathrm{rk}_M(Y^0 \cup a \cup b) = \mathrm{rk}_M(Y^0 \cup a) + 1$. Since $b \notin \mathrm{cl}_M(Y^0 \cup a) = \mathrm{cl}_M(Y^0 \cup g)$ we have $g \notin \mathrm{cl}_M(Y^0 \cup b)$ by (CL4) in $M$. By Proposition 3.1.9 there is a covector $W \in \mathcal{L}$ with $W^0 = \mathrm{cl}_M(Y^0 \cup b)$ which is as required (or $-W$). Hence, $\mathcal{L}^+$ satisfies the axiom (PA) and the pair $(E - g, \mathcal{L}^+)$ is an oriented semimatroid. □

But without additional restrictions, finite oriented semimatroids are more general than affine oriented matroids. Let us consider the following example.

**Example 4.2.9.** *Let us consider the finite oriented semimatroid given by the arrangement of pseudolines shown in Figure 4.6. We have $S = \{1, 2, 3\}$ and*

$$\mathcal{O} = \{0{+}{+},\ {+}0{+},\ {+}{+}0,\ {+}{+}{+},\ {+}{+}{-},\ {+}{-}{+},\ {-}{+}{+}\}$$

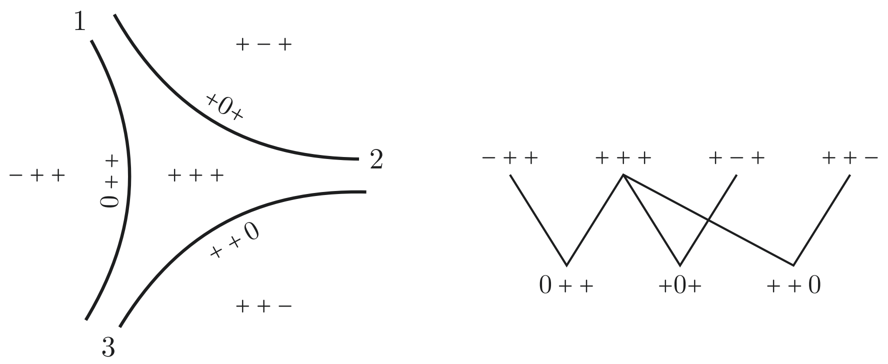



Figure 4.6: A finite oriented semimatroid which is not an affine oriented matroid. On the left-hand side, the corresponding set of pseudolines in $\mathbb{R}^2$ is shown and on the right-hand side, the poset of sign vectors.

*as set of sign vectors in* $\{+,-,0\}^S$. *It is an easy exercise to check that* $(S,\mathcal{O})$ *satisfies the axioms of Definition 4.1.1 for an oriented semimatroid.*

*We will show that* $(S,\mathcal{O})$ *is not an affine oriented matroid. Recall that* $(S,\mathcal{O})$ *is an affine oriented matroid if and only if it satisfies the axioms (FS), (SE*$^=$*) and (P*$^=_{as}$*), see Theorem 3.1.29. The axiom (P*$^=_{as}$*) would require the following*

$$\mathcal{P}^=_{as}(\mathcal{O}) \circ \mathcal{O} \subseteq \mathcal{O},$$

*where*

$$\mathcal{P}^=_{as}(\mathcal{O}) := \{X \oplus (-Y) \mid X, Y \in \mathrm{Asym}(\mathcal{O}), \underline{X} = \underline{Y}, I^=(X,-Y) = I^=(-X,Y) = \varnothing\}$$

*with*

$$\mathrm{Asym}(\mathcal{O}) := \{X \in \mathcal{O} \mid -X \notin \mathcal{O}\},$$
$$I^=_e(X,Y) := \{Z \in \mathcal{O} \mid Z_e = 0, \forall f \notin S(X,Y) : Z_f = X_f\},$$
$$I^=(X,Y) = \bigcup_{e \in S(X,Y)} I^=_e(X,Y).$$

*Consider* $X = +-+$ *and* $Y = ++-$ *in* $\mathcal{O}$. *We have* $\underline{X} = \underline{Y}$, *and* $-X = -+- \notin \mathcal{O}$ *as well as* $-Y = --+ \notin \mathcal{O}$, *so* $X, Y \in \mathrm{Asym}(\mathcal{O})$. *Moreover,*

$$I^=(X,-Y) = \bigcup_{e \in S(X,-Y)} I^=_e(X,-Y) = I^=_1(X,-Y) = \varnothing$$

*and*

$$I^=(-X,Y) = \bigcup_{e \in S(-X,Y)} I^=_e(-X,Y) = I^=_1(-X,Y) = \varnothing.$$

*Hence, we have* $X \oplus (-Y) = (+-+) \oplus (--+) = 0-+ \in \mathcal{P}^=_{as}(\mathcal{O})$. *But for* $Z = 0++ \in \mathcal{O}$ *we have*

$$(X \oplus (-Y)) \circ Z = (0-+) \circ (0++) = 0-+ = X \oplus (-Y) \notin \mathcal{O}.$$

*Therefore, the pair* $(S,\mathcal{O})$ *does not satisfy (P*$^=_{as}$*) and is not an affine oriented matroid.*

### 4.2.3 Relationship to multiple oriented matroids

Bohne proves a correspondence between multiple oriented matroids and multiple zonotopal tilings in [17], see Theorem 3.3.10. So with his multiple oriented matroids, he already gives a combinatorial description of a certain kind of infinite arrangements in $\mathbb{R}^d$ – the ones which arise as the dual line system of a multiple zonotopal tiling. In particular, they include periodic hyperplane arrangements in $\mathbb{R}^d$ which are given by a finite set of affine hyperplanes and translates thereof, as described in Section 1.2.

Multiple oriented matroids are given as a set $\mathcal{O} \subseteq \mathbb{Z}^E$ where $E$ is finite, compared to an oriented semimatroid $\mathcal{O} \subseteq \{+,-,0\}^S$ with $S$ countable. But the axioms (M1-M4) of Definition 3.3.9 have similar meaning. The axiom (M1) corresponds to (C), the axiom (M2) to (SE) and (M4) corresponds to (LOM). The axiom (M3) is a condition on the behaviour going to infinity and its symmetry. Notably, in the case when the multiplicity $m_{\mathcal{O}}(e)$ is finite for some $e \in E$, i.e. the parallel class of $e$ is finite, the axiom (M3) implies that there exist $X, X' \in \mathcal{O}$ with $X_e = m_{\mathcal{O}}(e)$ and $X'_e = -m_{\mathcal{O}}(e)$ (see [17, Remark 5.2.1]).

Nevertheless, oriented semimatroids are more general than multiple oriented matroids. For example the oriented semimatroid

$$(S, \mathcal{O}) = (\{1,2,3\}, \{0++, +0+, ++0, +++, ++-, +-+, -++\})$$

shown in Figure 4.6 is no multiple oriented matroid. The ground set $S$ can not be ordered in a way such that $(S, \mathcal{O})$ satisfies the structure of a multiple oriented matroid (see in Figure 3.10).

## 4.3 Importance of parallelism

In Lemma 2.4.6 and Theorem 2.4.7 by Forge and Zaslavsky [48] for topoplane arrangements, we already got a glimpse of the relevance for arrangements to have parallelism as an equivalence relation. In general, parallelism is not transitive in topoplane arrangements (see Section 2.4.1).

In a same way, Bohne restricts his considerations to multiple zonotopal tilings where parallelism in the corresponding dual line system is transitive (see Section 3.3.1). The transitivity is given by the condition (Z4) of Definition 3.3.7.

Recall the realizable conditional oriented matroid in Example 4.2.6. From the intersection of the open convex set with a set of affine hyperplanes we obtain a collection of subsets of these hyperplanes, each corresponding to an $e \in E$. It is easily seen that for the subsets parallelism is not transitive and accordingly no equivalence relation (see Figure 4.5).

**Definition 4.3.1.** *Two elements $a, b \in S$ of an oriented semimatroid $(S, \mathcal{O})$ are **affine parallel** if $Z_a = 0$ implies that $Z_b \neq 0$ for all $Z \in \mathcal{O}$.*

**Remark 4.3.2.** *Affine parallelism is an equivalence relation since it is obviously reflexive and symmetric and transitivity follows by (PA).*

In the case of an oriented semimatroid $(S_{\mathscr{A}}, \mathcal{O}_{\mathscr{A}})$ given by a locally finite hyperplane arrangement $\mathscr{A}$, affine parallelism of two elements $a, b \in S_{\mathscr{A}}$ means that the hyperplanes $H_a$ and $H_b$ have no intersection and only coincide at infinity.

Two distinct elements $a, b \in E - g$ are **parallel in an affine oriented matroid** $(E, \mathcal{L}, g)$ if they are parallel (i.e. $\{a, b\}$ is a circuit) in the contraction $\mathcal{L}/g$ (see [12, Definition 10.5.1]).

**Lemma 4.3.3.** *Two elements $a, b \in E - g$ are parallel in the affine oriented matroid $(E, \mathcal{L}, g)$ then they are either parallel or affine parallel in the oriented semimatroid $(E - g, \mathcal{L}^+)$.*

*Proof.* Let $a, b \in E - g$ be parallel in $\mathcal{L}/g = \{X|_{E-g} : X \in \mathcal{L}, g \in X^0\}$, that is to say $a, b \notin E_0 = \{e \in E : X_e = 0 \text{ for all } X \in \mathcal{L}\}$ and $X_a = X_b$ for all $X \in \mathcal{L}/g$ or $X_a = -X_b$ for all $X \in \mathcal{L}/g$. In the underlying matroid $M/g$ of $\mathcal{L}/g$ we have

$$\mathrm{rk}_{M/g}(a) = \mathrm{rk}_{M/g}(b) = \mathrm{rk}_{M/g}(a, b) = 1.$$

Consider the following two cases.

(i) Suppose $\mathrm{rk}_M(a, b) = 1$ in the underlying matroid $M$ of $\mathcal{L}$. Then we have $\mathrm{rk}_M(a, b) = \mathrm{rk}_M(a) = \mathrm{rk}_M(b) = 1$ and $a, b$ are parallel in $M$. Moreover, $\mathrm{cl}_M(a) = \mathrm{cl}_M(b)$ by the matroid version of Proposition 4.6.2.(e). Hence, for all $X \in \mathcal{L}$ we have $X_a = 0$ if and only if $X_b = 0$, which means $a, b$ are parallel elements in $\mathcal{L}$ and satisfy

$$X_e = X_f \text{ for all } X \in \mathcal{L} \text{ or } X_e = -X_f \text{ for all } X \in \mathcal{L}^+$$

by Definition 3.1.5. This means $a, b$ are parallel in the oriented semimatroid $\mathcal{L}^+$.

(ii) Otherwise, if $\mathrm{rk}_M(a, b) = 2$ in the underlying matroid $M$ of $\mathcal{L}$. Since $a, b$ are parallel in $\mathcal{L}/g$ we have $\mathrm{rk}_M(a, b) = 2 = \mathrm{rk}_{M/g}(a, b) + \mathrm{rk}_M(g) = \mathrm{rk}_M(a, b, g)$ and $g \in \mathrm{cl}_M(a, b)$ by Proposition 4.6.2.(e). This means for all $X \in \mathcal{L}$ with $a, b \in X^0$ we have $g \in X^0$ as well, then $X \notin \mathcal{L}^+$. Therefore, for all $X \in \mathcal{L}^+$ with $X_a = 0$ we have $X_b \neq 0$ and $a, b$ are affine parallel in the oriented smimatroid $\mathcal{L}^+$.

By (i) and (ii) we also characterized when $a, b$ are parallel and when affine parallel in the oriented semimatroid $(E - g, \mathcal{L}^+)$. □

## 4.4 Generalization of affine sign vector systems

We prove a generalization of affine sign vector systems as defined in Section 3.1.6. We will show that it is possible to loosen the conditions for the elimination set $I(X, Y)$ and the set of parallel vectors $\mathcal{P}(\mathcal{W})$ so that the axiom system from Theorem 3.1.29 for an affine oriented matroid is still valid.

This section is based on joint work with Kolja Knauer and Emanuele Delucchi [35].

Let $(E, \mathcal{W})$ be a system of sign vectors, $X, Y \in \{+,-,0\}^E$ and $e \in E$. Define

$$\begin{aligned} I_e^{=}(X,Y) &:= \{Z \in \mathcal{W} \mid Z_e = 0, \forall f \notin S(X,Y) : Z_f = X_f\}, \text{ and} \\ I_e(X,Y) &:= \{Z \in \mathcal{W} \mid Z_e = 0, \forall f \notin S(X,Y) : Z_f = (X \circ Y)_f\}. \end{aligned}$$

As in Section 3.1.6, set

$$I^{=}(X,Y) = \bigcup_{e \in S(X,Y)} I_e^{=}(X,Y) \text{ and } I(X,Y) = \bigcup_{e \in S(X,Y)} I_e(X,Y).$$

Recall $\mathrm{Asym}(\mathcal{W}) := \{X \in \mathcal{W} \mid -X \notin \mathcal{W}\}$ and $X \oplus Y$ is defined as

$$(X \oplus Y)_e = \begin{cases} 0 & \text{if } e \in S(X,Y) \\ (X \circ Y)_e & \text{otherwise.} \end{cases}$$

Using these terms we define:

$$\begin{aligned} \mathcal{P}_{as}^{=}(\mathcal{W}) &:= \{X \oplus (-Y) \mid X, Y \in \mathrm{Asym}(\mathcal{W}), \underline{X} = \underline{Y}, I^{=}(X,-Y) = I^{=}(-X,Y) = \varnothing\}, \\ \mathcal{P}(\mathcal{W}) &:= \{X \oplus (-Y) \mid X, Y \in \mathcal{W}, I(X,-Y) = I(-X,Y) = \varnothing\}. \end{aligned}$$

Recall the axiom system from Theorem 3.1.29 for an affine oriented matroid.

**Definition 4.4.1** (See [62, 5])**.** *Let $E$ be finite and $\mathcal{W} \subseteq \{+,-,0\}^E$ then the pair $(E, \mathcal{W})$ is an affine oriented matroid iff*

*(FS)* $\mathcal{W} \circ (-\mathcal{W}) \subseteq \mathcal{W}$,

*(SE$^=$)* $X, Y \in \mathcal{W}, \underline{X} = \underline{Y} \implies \forall e \in S(X,Y) : I_e^{=}(X,Y) \neq \varnothing$,

*($P_{as}^=$)* $\mathcal{P}_{as}^{=}(\mathcal{W}) \circ \mathcal{W} \subseteq \mathcal{W}$.

We propose the following seemingly simpler and stronger version.

**Proposition 4.4.2.** *Let $E$ be finite and $\mathcal{W} \subseteq \{+,-,0\}^E$ then the pair $(E, \mathcal{W})$ is an affine oriented matroid iff*

*(FS)* $\mathcal{W} \circ (-\mathcal{W}) \subseteq \mathcal{W}$,

*(SE)* $X, Y \in \mathcal{W} \implies \forall e \in S(X,Y) : I_e(X,Y) \neq \varnothing$,

*(P)* $\mathcal{P}(\mathcal{W}) \circ \mathcal{W} \subseteq \mathcal{W}$.

*Proof.* In the remaining proof we will make use of some straightforward observations. Let $X, Y \in \{+,-,0\}^E$, then we have

1. $S(X,Y) = S(X \circ Y, Y \circ X)$,
2. $\underline{X \circ Y} = \underline{Y \circ X}$,

3. if $\underline{X} = \underline{Y}$, then $X_f = (X \circ Y)_f$,
4. $X \oplus Y = (X \circ Y) \oplus (Y \circ X)$,
5. $I^{=}(X,Y) = I(X,Y)$ for $X, Y$ with $\underline{X} = \underline{Y}$.

- *Part I: (SE$^{=}$) $\Longleftrightarrow$ (SE).* Clearly, (SE) implies (SE$^{=}$) by 5. Conversely, we will show that together with (FS) the axiom (SE$^{=}$) implies (SE). Indeed, for $X, Y \in \mathcal{W}$ we have $S(X,Y) = S(X \circ Y, Y \circ X)$ by 1 and as well as $(X \circ Y)_f = ((X \circ Y) \circ (Y \circ X))_f$ by 2 and 3. This implies

$$I_e(X,Y) = I_e^{=}(X \circ Y, Y \circ X) = I_e(X \circ Y, Y \circ X).$$

Thus, if $e \in S(X \circ Y, Y \circ X)$ with $I_e^{=}(X \circ Y, Y \circ X) \neq \varnothing$ then also $e \in S(X,Y)$ and $I_e(X,Y) \neq \varnothing$ which means (SE$^{=}$) implies (SE).

- *Part II: (P$^{=}_{as}$) $\Longleftrightarrow$ (P).* By 5, we can write $\mathcal{P}^{=}_{as}(\mathcal{W})$ as

$$\{X \oplus (-Y) \mid X, Y \in \operatorname{Asym}(\mathcal{W}), \underline{X} = \underline{Y}, I(X,-Y) = I(-X,Y) = \varnothing\},$$

which gives $\mathcal{P}^{=}_{as}(\mathcal{W}) \subseteq \mathcal{P}(\mathcal{W})$ and thus, (P) implies (P$^{=}_{as}$). To conclude we show (P$^{=}_{as}$) implies (P). First observe that one can drop the asymmetry condition. For $X, Y, -Y \in \mathcal{W}$ we have $I(X,-Y) \neq \varnothing$ by (SE), unless $S(X,-Y) = \varnothing$. If $S(-X,Y) = \varnothing$ and $\underline{X} = \underline{Y}$ we have $X = -Y$, which implies $X \oplus (-Y) = X \in \mathcal{W}$ and $X \oplus (-Y) \circ \mathcal{W} \subseteq \mathcal{W}$ by (FS). The same argument works for $X, -X, Y \in \mathcal{W}$. Hence, (P$^{=}_{as}$) implies

$$\{X \oplus (-Y) \mid X, Y \in \mathcal{W}, \underline{X} = \underline{Y}, I(X,-Y) = I(-X,Y) = \varnothing\} \circ \mathcal{W} \subseteq \mathcal{W}.$$

We proceed by showing that

$$\{X \oplus (-Y) \mid X, Y \in \mathcal{W}, \underline{X} = \underline{Y}, I(X,-Y) = I(-X,Y) = \varnothing\} \supseteq \mathcal{P}(\mathcal{W}).$$

Let $X \oplus (-Y) \in \mathcal{P}(\mathcal{W})$ and consider the vectors $X \circ (-Y), Y \circ (-X)$. We have $-(Y \circ (-X)) = -Y \circ X$ and

$$\underline{X \circ (-Y)} \overset{2}{=} \underline{-Y \circ X} = \underline{-(Y \circ (-X))} = \underline{Y \circ (-X)}, \tag{4.12}$$

where the last equality follows from $\underline{Z} = \underline{-Z}$. By 1 we get

$$S(X,-Y) = S(X \circ (-Y), -Y \circ X), \; S(-X,Y) = S(-X \circ Y, Y \circ (-X)).$$

Which implies

$$I(X,-Y) = I(X \circ (-Y), -Y \circ X), \; I(-X,Y) = I(-X \circ Y, Y \circ (-X)). \tag{4.13}$$

Finally, 4 gives $X \oplus (-Y) = (X \circ (-Y)) \oplus (-Y \circ X)$ and together with (4.12) and (4.13) we obtain that $X \oplus (-Y)$ is contained in the set on the left-hand side. This concludes the proof of (P$^{=}_{as}$) $\Longrightarrow$ (P).

□

### 4.4.1 Going forward: Locally finite affine oriented matroids

Given the characterization of an affine oriented matroid in terms of affine sign vector systems $(E, \mathcal{W})$, one naturally wonders if it is possible to generalize the concept to an infinite ground set. As a starting point for future work we suggest the following.

**Definition 4.4.3.** *Let $S$ be a countable set and $\mathcal{W} \subseteq \{+,-,0\}^S$. The pair $(S,\mathcal{W})$ is a **locally finite affine oriented matroid** iff*

*($C^\infty$)* $\bigcirc^\infty \mathcal{W} \subseteq \mathcal{W}$,

*(Fin)* $\forall X, Y \in \mathcal{W} : |S(X,Y)| < \infty$ *and* $|\bigcap_{X\in\mathcal{W}} X^0| < \infty$,

*(FS)* $\mathcal{W} \circ (-\mathcal{W}) \subseteq \mathcal{W}$,

*($SE^=$)* $X, Y \in \mathcal{W}, \underline{X} = \underline{Y} \implies \forall e \in S(X,Y) : I_e^=(X,Y) \neq \varnothing$,

*($P_{as}^=$)* $\mathcal{P}_{\mathrm{as}}^=(\mathcal{W}) \circ \mathcal{W} \subseteq \mathcal{W}$.

**Proposition 4.4.4.** *In a locally finite affine oriented matroid $(S,\mathcal{W})$ all zero sets are finite.*

*Proof.* Suppose there is $X \in \mathcal{W}$ with $X^0 = e_1, \ldots = \infty$, then by (Fin) for all $e_i \in X^0$ except possibly a finite subset there exists $Y^i \in \mathcal{W}$ with $Y^i_{e_i} \neq 0$. Now let $Y = Y_1 \circ Y_2 \circ \ldots$ be their composition. By ($\mathrm{C}^\infty$) we have $Y \in \mathcal{W}$. Then $X \circ (-Y) \in \mathcal{W}$ by (FS) and $S(X \circ Y, X \circ (-Y)) = \infty$. This contradicts the assumption in (Fin). □

**Problem 1.** *Is the poset $(\mathcal{W}, <)$ of a locally finite affine oriented matroid a ranked meet semilattice?*

**Problem 2.** *Determine minors and duality in a locally finite affine oriented matroid.*

## 4.5 Topology of oriented semimatroids

Our geometric intuition of oriented semimatroids of rank $d$ is that they correspond to essential arrangements in $\mathbb{R}^d$. A desirable property of an oriented matroid of rank $d$ would be that its order complex is homeomorphic to $\mathbb{R}^d$. But unfortunately, this is not in general the case.

**Example 4.5.1.** *Consider again the oriented semimatroid $(S,\mathcal{O})$ from Example 4.2.9. By looking at its poset $(\mathcal{O},<)$ of sign vectors in Figure 4.6 one sees immediately that the rank of $(S,\mathcal{O})$ is one. But the order complex of $(S,\mathcal{O})$ is not homeomorphic to $\mathbb{R}$ (see Figure 1.7).*

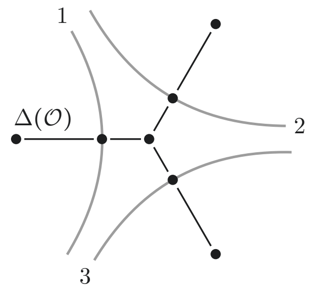



Figure 4.7: Example of an oriented semimatroid $\mathcal{O}$ of rank 1 whose (geometric realization of the) order complex $\Delta(\mathcal{O})$ is not homeomorphic to $\mathbb{R}$.

Nevertheless, studying oriented semimatroid is interesting since they characterize the cell decomposition of $\mathbb{R}^d$ given by a locally finite hyperplane arrangement (see Theorem 4.1.4). Furthermore, we know for a finite oriented semimatroid which arises from an affine oriented matroid (see Theorem 4.2.8) that its order complex is a shellable $d$-ball (see Theorem 3.1.5).

## 4.6 Appendix: Closure in finitary semimatroids

Recall the Definition 1.1.9 of the closure and flats of a finitary semimatroid $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk}_{\mathcal{C}})$.

**Proposition 4.6.1** (See Proposition 2.4 in [1])**.** *The closure operator of a finitary semimatroid satisfies the following properties, for all $X, Y \in \mathcal{C}$ and $x, y \in S$.*

*(CLR1) The closure* $\mathrm{cl}(X)$ *is a central set and* $\mathrm{rk}_{\mathcal{C}}(\mathrm{cl}(X)) = \mathrm{rk}_{\mathcal{C}}(X)$.

*(CL1) The set $X$ is a subset of* $\mathrm{cl}(X)$.

*(CL2) If $X \subseteq Y$ then* $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.

*(CL3)* $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.

*(CL4) If $X \cup x \in \mathcal{C}$ and $y \in$* $\mathrm{cl}(X \cup x) - \mathrm{cl}(X)$, *then $X \cup y \in \mathcal{C}$ and $x \in$* $\mathrm{cl}(X \cup y)$.

The proof of Proposition 4.6.1 follows by the same reasoning as in [1]. Most of the properties valid for ordinary matroid are also satisfied in the case of finitary semimatroids. This also holds for the following.

**Proposition 4.6.2** (Compare [88], p. 31)**.** *Let $\mathcal{C}$ be a finitary semimatroid with rank function* $\mathrm{rk}_{\mathcal{C}}$ *and closure operator* $\mathrm{cl}$. *If $X, Y$ are central sets of $\mathcal{C}$ then the following is satisfied:*

*(a) If $X \subseteq$* $\mathrm{cl}(Y)$ *and* $\mathrm{cl}(Y) \subseteq \mathrm{cl}(X)$, *then* $\mathrm{cl}(X) = \mathrm{cl}(Y)$.

*(b) If* $Y \subseteq \mathrm{cl}(X)$*, then* $X \cup Y \in \mathcal{C}$ *and* $\mathrm{cl}(X \cup Y) = \mathrm{cl}(X)$.

*(c) The intersection of all flats containing* $X$ *equals* $cl(X)$.

*(d) If* $X \cup Y \in \mathcal{C}$*, then*
$\mathrm{rk}_{\mathcal{C}}(X \cup Y) = \mathrm{rk}_{\mathcal{C}}(X \cup \mathrm{cl}(Y)) = \mathrm{rk}_{\mathcal{C}}(\mathrm{cl}(X) \cup \mathrm{cl}(Y)) = \mathrm{rk}_{\mathcal{C}}(\mathrm{cl}(X \cup Y))$.

*(e) If* $X \subseteq Y$ *and* $\mathrm{rk}_{\mathcal{C}}(X) = \mathrm{rk}_{\mathcal{C}}(Y)$*, then* $cl(X) = cl(Y)$.

*Proof.* The property (a) follows immidiately by (CL2) and (CL3). Now let $X, Y$ be central sets with $Y \subseteq \mathrm{cl}(X)$, then since $X$ and $Y$ are subsets of $\mathrm{cl}(X)$ so is their union. Therefore, the union $X \cup Y$ is also central by the definition of a simplicial complex and thus it equals $\mathrm{cl}(X)$ by (CL2) and (CL3). Hence, (b) is satisfied.

For $X \in \mathcal{C}$, let $B$ denote the intersection of all flats containing $X$. Clearly, the set $B$ is a subset of the closure of $X$. On the other hand say $A$ is a flat containing $X$, this implies by (CL2) that $\mathrm{cl}(X) \subseteq \mathrm{cl}(A) = A$. Thus $\mathrm{cl}(X) \subseteq B$ and (c) follows. Assume $X, Y$ and $X \cup Y$ are central sets. By (CLR1), the closure $\mathrm{cl}(X \cup Y)$ is a central set and $\mathrm{rk}_{\mathcal{C}}(X \cup Y) = \mathrm{rk}_{\mathcal{C}}(\mathrm{cl}(X \cup Y))$. Furthermore, we have the following sequence of sets:

$$X \cup Y \subseteq X \cup \mathrm{cl}(Y) \subseteq \mathrm{cl}(X) \cup \mathrm{cl}(Y) \subseteq \mathrm{cl}(X \cup Y),$$

where the last containment follows by (CL2). All elements of this sequence are central since $\mathrm{cl}(X \cup Y)$ is and thus (d) is satisfied by (R2).

Suppose $X, Y$ are central with $X \subseteq Y$ and $\mathrm{rk}_{\mathcal{C}}(X) = \mathrm{rk}_{\mathcal{C}}(Y)$. By (CL2), we have $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$. Now let $y \in \mathrm{cl}(Y)$, then $Y \cup y \in \mathcal{C}$ (and thus $X \cup y$ as well). Hence,

$$\mathrm{rk}_{\mathcal{C}}(X \cup y) \leq \mathrm{rk}_{\mathcal{C}}(Y \cup y) = \mathrm{rk}_{\mathcal{C}}(Y) = \mathrm{rk}_{\mathcal{C}}(X) \leq \mathrm{rk}_{\mathcal{C}}(X \cup y)$$

by (R2) and $y \in \mathrm{cl}(X)$. So, we have $\mathrm{cl}(X) = \mathrm{cl}(Y)$ and (e) is satisfied. □

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