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Mathematics of Generalized Versions of the Melitz, Krugman, and Armington Models with Detailed Derivations

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We provide detailed textbook style mathematical derivations of an extended version of the heterogenous firms model of Melitz (2003), as well as the Armington (1969) and Krugman (1980) models. Our model of heterogeneous firms extends the model of Melitz (2003) by allowing multiple sectors, intermediates, heterogeneous regions based on data, labor-leisure choice, initial heterogeneous tariffs, multiple factors of production, the possibility of sector-specific inputs and trade imbalances based on data, and we incorporate global and unilateral tariff policy shocks. Although the models in this paper are extensions in numerous directions of the Melitz trade model of heterogeneous firms, the pedagogical approach in this paper should substantially facilitate the accessibility of the applied heterogenous-firms model of international trade. Balistreri and Tarr (2022) apply these models to GTAP data where they assess the relative welfare impacts in the Armington, Krugman, and Melitz style models of trade cost reductions in eighteen model variants. This paper documents the equations of those models, and we hope it will be a clear roadmap for understanding and constructing modern multi-sector, multi-region international trade models that must be fitted to data.

JEL codes: F12, F13, C65, C68, D58.

Keywords: Heterogeneous Firms; Welfare Gains from the New Trade Theory; Intermediates; Labor-Leisure Choice; Tariff Shocks.

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1. Introduction

1.1 Motivation

This paper serves two purposes: (i) to document the extended versions of the Melitz, Krugman, and Armington models of Balistreri and Tarr (2022) by providing the mathematics of the computer code; and (ii) to provide detailed, pedagogical mathematical derivations of those models to facilitate understanding.

The seminal contribution of Melitz (2003) has contributed to the substantial expansion of both theoretical and empirical investigations of the impacts of heterogeneous firms in international trade. For applied policy modeling, however, it is necessary to extend the Melitz model in numerous directions. Our models with heterogeneous firms allow numerous policy relevant extensions of Melitz (2003), including: multiple sectors, intermediates with sector use shares based on data, heterogeneous regions based on data, labor-leisure choice calibrated to estimates of labor supply elasticities, tariff data based on actual tariffs (as well as iceberg trade costs), multiple factors of production, sector-specific inputs and trade imbalances based on the data.

Our approach to the model with heterogeneous firms is in the style of Melitz, for example, variables like firm productivity, prices and quantities are continuous, there is a fixed cost of entry that must be paid prior to knowing the firm's productivity, there is a fixed cost of selling in any market, and the free-entry condition is the expected value of profits are zero in the steady state with a death rate of firms. The mathematical derivations are especially detailed regarding the extended version of the heterogenous-firms model of Melitz (2003), but this paper also provides detailed derivations of the Armington (1969) and Krugman (1980) models. Comments we have received on earlier drafts have indicated that even though our version of the heterogeneous-firms model is considerably more general than Melitz (2003), our detailed derivations have made that model more accessible. We hope this will be a clear roadmap for understanding the theory of modern multi-sector, multi-region international trade models that must be fitted to data.

Conceptually, we define a full policy model which includes all data, model variants and policy instruments. Then many of the various model variations may be thought of as special cases of the general model. For example, the general model allows labor-leisure choice and sector-specific factors, where the share of primary factors that are sector-specific may range from zero to one. Then the model without sector-specific factors is a special case of the general model where the share of primary factors that are sector-specific is zero; and the model with no labor-leisure choice is a special case of the general model with perfectly inelastic supply of labor.

The mathematics we lay out shows there is a subset of the equations that contains equations common to the Armington, Krugman, and Melitz models. In these common equations, the prices of goods are interpreted as either Armington or Dixit-Stiglitz price indices, with associated quantities available for absorption in the economy. Beyond the equations that are common across all models, there are some equations that are specific to the Krugman-style model and a slightly larger set that are specific to the Melitz style model.

The remainder of this introduction is structured to provide additional context for the models and their importance in contemporary analysis. In section 1.2 we summarize the key results of Balistreri and Tarr (2022). Our intent is to highlight the importance of structure for welfare analysis of international trade and motivate the reader to learn the details of our generalized version of the heterogeneous-firms model. We survey the literature on the mathematics of the three model structures we consider in section 1.3. In section 1.4, we explain that our generalized heterogeneous-firms model allows us to explain not only the typical situation, where a minority of firms export and most firms only sell in their home market, but also situations where some firms export without selling in their home market. In section 1.5 we outline the body of the paper in terms of sections and purpose.

1.2 What Model Features Indicate Different Welfare Gains Across the Structures?

To demonstrate the value of the generalized approach adopted, we begin with a discussion of what model features distinguish the welfare gains between models based on Melitz (2003), Krugman (1980), and Armington (1969). The well-known paper of Arkolakis, Costinot, and Rodríguez-Clare (2012) found that in a single-sector stylized model, in response to a global reduction in iceberg trade costs, the welfare gains in the Armington, Krugman, and Melitz models are equal, provided the trade responses are equalized based on a gravity estimate.[1] Costinot and Rodríguez-Clare (2014) showed that with intermediates in the model, the monopolistic competition models yield larger gains than the Armington model.[2] Balistreri and Tarr (2022) produce a comprehensive welfare comparison between Armington, Krugman and Melitz style models based on 18 model variants. They assess which model features and policy instruments are important to distinguish the welfare gains from changes in trade costs in Melitz, Krugman or Armington style models. In Table 1 we summarize the aggregate results of Balistreri and Tarr (2022).

Define A, K, and M as the global welfare gains from a reduction of trade costs in the Armington, Krugman, and Melitz models, respectively. Results are shown in columns 2-4 for holding trade responses equal across the three model structures

[1] Stylized model features are: one sector; one primary factor; no intermediates; no labor-leisure choice; no initial tariffs; iceberg trade cost policy shocks; and multiple regions with balanced trade in all regions.

[2] See Balistreri and Tarr (2021) for a review of the literature on the relative welfare gains in the Armington, Krugman, and Melitz models.

Table 1. Relative welfare gains measured in global *equivalent variation*: Armington gains indexed at one.

		Relative Welfare Gains (trade responses equalized)			Krugman Share of Melitz gains above Armington	Relative Welfare Gains (trade responses NOT equalized[a])		
	New or Known Result	Armington	Krugman	Melitz		Armington	Krugman	Melitz
	1	2	3	4	5	6	7	8
I Global 10% reduction in iceberg trade costs		σ^A adjusted	σ^K adjusted	$\sigma^M = 5.0$ $a = 4.58$	$S = \frac{K-A}{M-A}$	$\sigma^A = 5.58$	$\sigma^K = 5.58$	$\sigma^M = 5.0$ $a = 4.58$
A.One-Sector Model			K/A	M/A			K/A	M/A
1.stylized model features[b]	Known	1	1	1	1	1	1	1
2.stylized model with trade imbalances[c]	New	1	1	1	1	1	1	1
3.stylized with labor-leisure choice[d]	New	1	1.06	1.07	0.87	1	1.06	1.07
4.stylized with an intermediate good	Known	1	1.35	1.43	0.82	1	1.35	1.43
5.stylized with an intermediate good and labor-leisure choice	New	1	1.61	1.79	0.78	1	1.61	1.79
B.Four-Sector Model (one primary factor, no labor-leisure choice)								
6.with one aggregate intermediate good	Known	1	1.38	1.47	0.80	1	1.40	1.49
7.with Cobb-Douglas demand for 4 intermediate goods	New	1	1.55	1.72	0.76	1	1.59	1.76
8.with CES demand for 4 intermediates ($\sigma^T = 0.5$)	New	1	1.20	1.24	0.84	1	1.22	1.26
C.Four-Sector Model (includes intermediates (σ^T=0.5) and tariff data)								
9.with 1 mobile primary factor	New	1	1.18	1.22	0.83	1	1.21	1.25
10.with 3 mobile primary factors	New	1	1.18	1.21	0.83	1	1.21	1.24
11.with 3 primary factors with one of them (capital) 20% sector-specific	New	1	1.16	1.20	0.83	1	1.18	1.21
12.with 3 mobile primary factors and labor-leisure choice	New	1	1.26	1.32	0.82	1	1.28	1.34
D.Policy Model[e]								
13.Policy model except no labor-leisure choice	New	1	1.12	1.13	0.88	1	1.15	1.16
14.Policy model	New	1	1.17	1.20	0.87	1	1.20	1.23
II Tariff Changed: Movement to Global Free Trade								
15.Policy model except no labor-leisure choice	New	1	1.66	1.69	0.95	1	2.55	2.63
16.Policy model starting from uniform tariffs no labor-leisure choice	New	1	4.06	4.67	0.83	1	5.21	5.76
17.Policy model	New	1	1.97	2.08	0.90	1	3.08	3.26
III Uniltaral Increases in All Tariffs to 25%								
18. Sign of the welfare change in the policy model	New	all ten regions positive	six out of ten regions negative	seven out of ten regions negative		all ten regions positive	six out of ten regions negative	seven out of ten regions negative

a Central elasticities equilibrate trade responses in the one-sector models, but not the multi-sector models.
b Stylized model features are: one sector, one primary factor, no intermediates, no labor-leisure choice, no initial tariffs, iceberg trade cost policy shocks, and multiple regions with balanced trade in all regions.
c All additional models contain data-based trade imbalances.
d This extends Balistreri, Hillberry, and Rutherford (2010) and Arkolakis and Esposito (2014) result to holding the trade responses equal and to a comparison of Krugman to Armington.
e Policy Model includes nine sectors, labor-leisure choice, one sector-specific and two mobile primary factors, initial data-based tariffs and trade balances, CES demand for intermediates with data-based shares and elasticity of substitution of 0.5, and 10 heterogeneous regions. The monopolistic competition models contain four Armington and five monopolistically competitive sectors.
Source: Balistreri and Tarr (2022).

and in columns 6-8 for not holding trade responses equal.[3] In all model variants beyond the stylized one-sector model of Arkolakis et al. (2012), the *global* welfare *gains* from the *global reduction* in iceberg or tariff costs, the Melitz structure produces the largest welfare gains and the Armington model produces the least welfare gains.[4] In these 15 model variants, the ratios of welfare impacts of columns 2-4 are bound as follows: $(1.07)A \leq M \leq (4.67)A$ and $(1.06)A \leq K \leq (4.06)A$. The Armington model understates the gravity-consistent welfare gains in the monopolistic competition models by between six percent and 367 percent depending on the model variant.

The most important data and model features that increase the gains under monopolistic competition relative to the Armington structure are labor-leisure choice and intermediate inputs with intermediate shares based on input-output accounts. Further, labor-leisure choice interacts synergistically with intermediates. The marginal impact of labor-leisure choice on the relative gains in the monopolistic competition models compared to Armington is more than three times larger in the presence of intermediates. Models that exclude intermediates and labor-leisure choice will find welfare results in the Melitz and Krugman models much closer to Armington.

When evaluating unilateral tariff changes, market structure is especially important. For each of the the model regions separately, Balistreri and Tarr (2022) evaluate unilateral tariff increases to 25 percent of all tariffs below 25 percent. In the Armington model, they estimate welfare gains for all regions due to terms-of-trade gains. On the contrary, in the Melitz or Krugman models they estimate welfare losses for most of regions because the product variety and productivity effects of the monopolistic competition models work in the opposite direction of the terms-of-trade effects. These results show that the monopolistic competition models typically lower the implied optimal tariff compared with the Armington model and move the policy conclusion away from protectionism. The analysis of Caliendo and Feenstra (2022) and Balistreri and Markusen (2009) provide additional intuition and evidence consistent with this result.

Other that the first two stylized models of Table 1, we have $M > K > A$. For these models, define the parameter: $0 < S = \frac{K-A}{M-A} < 1$. The closer S is to one, the closer are the results of the Krugman model to the Melitz model. Table 1, column 5 shows that, beyond stylized models, $0.76 \leq S \leq 0.95$ indicating that the estimated welfare gains of the Krugman model are much closer to the estimated welfare gains of the Melitz model rather than the Armington model. This suggests that the variety effect (which is present in both monopolistic competition models)

[3] See Balistreri and Tarr (2022) for a discussion of how they address the issue of whether trade responses should be held equal and for a more comprehensive review of the welfare comparison literature.

[4] One exception is when we introduce a non-zero trade balance in an otherwise stylized one-sector model consistent with Arkolakis, Costinot, and Rodríguez-Clare (2012).

is quantitatively more important than the productivity gains from the selection effect of the Melitz model in differentiating the welfare gains of the monopolistic competition models from Armington.

On the other hand, Balistreri and Tarr (2022) find some model features that result in little or no difference in impacts between the market structures, such as multiple primary factors of production versus labor only. In models with multiple sectors, they often find cases of "reversed welfare rankings" where, for individual regions, the estimated welfare gains from the reduction of trade costs are largest in the Armington model and smallest in the Melitz model. Parameters for the terms-of-trade and a comprehensive variety measure are developed that explain these reversed welfare ranking results.

1.3 Literature on the Mathematics of the Melitz, Krugman, and Armington Models

The mathematical details of Melitz (2003) are explained well in Redding (2010b) and Redding (2010a); and Donaldson (2016) provides a pedagogical derivation of the mathematics of the Melitz model. The basic model of Melitz must be extended, however, in numerous dimensions to be an appropriate model for policy analysis.[5]

Balistreri and Rutherford (2013) provide equations that are close to our list of equations in section 2. Balistreri and Rutherford (2013) is an important complement to this paper as it also explains calibration issues and provides computer code for the three classes of models we consider.[6] Balistreri and Rutherford (2013) do not, however, derive the equations of the models.

The paper that is closest to ours in providing details of the mathematics of a generalized heterogeneous-firms model is Akgul, Villoria, and Hertel (2016). Akgul, Villoria, and Hertel (2016) provide computer code for how to implement a heterogeneous-firms model in GTAP. They also do an excellent job of explaining the economic intuition for the equations of the heterogeneous-firms model. We differ from Akgul, Villoria, and Hertel (2016) in that we assume that firms use a composite input that is a linearly homogeneous function of all primary and intermediate inputs used in variable costs. For fixed costs we assume that fixed costs use the same composite input, whereas Akgul, Villoria, and Hertel (2016) and Jafari and Britz (2018) assume that fixed costs use only primary inputs. In our approach, there are variety gains but no rationalization gains in the Krugman model. With different cost structures for fixed and variable costs, the welfare gains

[5] Costinot and Rodriguez-Clare (2013) provide a mathematical appendix for their heterogeneous-firms model using the "exact-hat" approach. The exact-hat approach has a long tradition among CGE modelers. Johansen (1960) style models are built specifically around proportional-change formulations (Dixon et al., 1982), and this is the standard approach for models that utilize the GEMPACK software (Harrison and Pearson, 1996).

[6] The computer code consistent with our equations, which is used in the analysis of Balistreri and Tarr (2022), is available for download at the following URL: https://www2.econ.iastate.edu/faculty/balistreri/avkvm/model_dist_econinq.zip.

in monopolistic competition models potentially also include rationalization gains (or losses). We illustrate in the case of the Krugman model in equation (3.28) below.

As noted by Dixon, Jerie, and Rimmer (2018, appendix 7.3) and Bekkers and Francois (2018), in Akgul, Villoria, and Hertel (2016) the price of the representative firm to all markets does not differ by market based on the bilateral representative firm productivities. This simplification precludes bilateral selection.[7] In our approach, the representative firm's price depends on bilateral export cutoff productivities as in Melitz.

Dixon, Jerie, and Rimmer (2018) provide the most comprehensive treatment of the economics, mathematics, calibration and computer code for implementing Armington, Krugman, and Melitz models in GEMPACK. Our approach differs from theirs in a few respects as follows.

First, we use continuous variables in the model of heterogeneous firms (not in the Armington or Krugman models), which is consistent with, and can be related more easily to, the theoretical literature of heterogeneous firms. By contrast, Dixon, Jerie, and Rimmer (2018, p.11) primarily employ a discrete variable approach. All theoretical contributions in international trade of which we are aware employ the continuous approach including Melitz (2003), Arkolakis, Costinot, and Rodríguez-Clare (2012), Chaney (2008), Costinot and Rodríguez-Clare (2014), Melitz and Redding (2015), Feenstra (2010), Demidova and Rodríguez-Clare (2009), and Caliendo et al. (2020).

Second, we allow households to have a labor-leisure choice. Balistreri and Tarr (2022) show that including a labor-leisure choice indicates substantionally larger welfare gains in the heterogeneous-firms model relative to the Armington model, especially when intermediates are included.

Finally, as in Melitz (2003), we employ a general probability function when we solve for the firm's profit maximizing price and quantity and the resulting firm-level revenue. Similarly, our derivations of industry aggregates in terms of the representative firm generalizes (namely the price and quantity indices, aggregate revenue and profits) hold for any probability distribution in which the representative firms' productivities are well-defined. This is of practical importance for researchers who wish to introduce a probability distribution other than the Pareto. For example, Fernandes et al. (2019) employ a log-normal distribution in place of a Pareto.[8] Our derivations for the industry aggregates apply to the log-normal distribution as well, so there is no need for an independent deriva-

[7] See Dixon, Jerie, and Rimmer (2018, appendix 7.3) for a numerical assessment of the impact of a lack of bilateral selection.

[8] Fernandes et al. (2019) show that if the Pareto distribution is replaced by the log-normal distribution in a heterogeneous-firms model, the trade responses (the intensive margins in particular) more closely match the data, but the welfare results in the one-sector Melitz model are not significantly changed by the log-normal distribution.

tion of the form of the industry aggregates. One could follow our derivations up to the point of solving for the zero-profit productivity cutoffs and introduce the alternate probability distribution at that point. Notwithstanding the paper of Fernandes et al. (2019), to date, all large-scale models of heterogeneous firms, that we know, employ the Pareto distribution.

1.4 Where Firms Export

It is well known that among the set of firms in a country, exporting firms are a significant minority—most firms only sell in their domestic market. Our model is consistent with these data as iceberg costs, tariffs, and usually additional fixed costs of exporting, all lead to a higher zero-profit productivity cutoff for exporting than for sales in the home market. There is also evidence, however, that there are firms that export without selling in their home market. In section 4, we show analytically that our model with heterogeneous regions also explains situations where some firms only export without selling in their home market. This could happen if there are large export markets relative to the home market, relatively weak home market preferences for the product of a sector, or lower fixed costs of exporting to some region relative to home market fixed costs. In his one-sector model with two heterogeneous regions, Feenstra (2010, p.13) also shows the possibility of firms exporting without selling in their home market. Our result generalizes the Feenstra result to an arbitrary number of heterogeneous regions and arbitrary number of sectors.

1.5 Plan of the paper

In section 2, we list the sets, variables, parameters, and instruments that are the basis of the computer code for the models used in Balistreri and Tarr (2022). The equations of the three models, showing which are common to all three models, are conveniently grouped in Table 2 that appears at the end of section 2. A listing of the equilibrium conditions presented as complementary slack condtions, as they are coded for computation, is provided in Appendix B. Detailed derivations of the equations are in section 3, where it is necessary to introduce numerous variables that are not in the computer code. The more verbose notation is used to clearly illustrate the dual versus primal representations of technologies and preferences. Section 4 contains a proof of the cutoff productivities determining where firms export (bilateral selection). Finally we conclude the paper in section 5.

2. Equations of the Computer Code

In this section, we list the variables, parameters, sets, instruments, and equations of the model that are the basis for the computer code of Balistreri and Tarr (2022). This allows the reader to see exactly which equations and variables are required to program the model without extraneous variables and equations possibly confusing the reader regarding what is required for the solution of the models.

We use the definitions of these variables in section 3 where we provide details of the derivations of the equations. In Appendix B, we provide the equations as arranged as complementary slack conditions (which is how the model is coded) with some explanation of their role in the model, but without the derivations in section 3. In table 2, we group the equations of the model into four categories: (i) equations common to the three market structures models; (ii) those specific to the Armington model; (iii) those specific to the Krugman model; and (iv) those specific to the Melitz model. Since the variables listed in this section (section 2) are limited to those necessary to solve the models, the reader will notice that there are many variables in section 3 that are necessary for the derivations that are not present in section 2. This is especially true in the case of the Melitz model, as that model is solved for equilibrium conditions in terms of the *representative* firms. For the derivations of section 3, however, it is necessary to introduce notation for the full model.

2.1 Sets and indices defined

Let R be the set of all regions indexed by r or s. I is the set of all goods and services indexed by i or j, with $K \subseteq I$ as the subset of Krugman sectors and $M \subseteq I$ as the subset of Melitz sectors. We reserve the index $k \in K$ for Krugman sectors and the index $m \in M$ for Melitz sectors. F is the set of primary factors indexed by f, with $\tilde{F} \subset F$ as the subset of sector-specific factors.

2.2 Variables defined

- D_r Indices of "full" consumption in region r, equal to one in the benchmark equilibrium.[9]
- Q_{ir} Indices for composite-good supply for good or service i in region r, equal to one in the benchmark equilibrium; in Armington mode, this is an index of the Armington aggregate good; in the monopolistic competition models, it is the index on the Dixit-Stiglitz aggregate.
- Y_{ir} Indices of composite *inputs* in sector i of region r that are equal to one in the benchmark equilibrium. All primary factors and intermediates in sector i of region r are combined into a single composite input with a quantity index of Y_{ir}. Under monopolistic competition, this composite input is used for both variable costs and fixed costs. Under Armington the price of this composite input is marginal cost which is the price of industry i's output. Intermediate inputs of good j that are a component of the composite input for sector i are either Armington on Dixit-Stiglitz aggregates.[10]
- e_r Unit expenditure indices (true cost-of-living index) in region r.

[9] With labor-leisure choice this includes the imputed value of leisure.
[10] In Section 3.2.2 we denote the input quantity of Armington or Dixit-Stiglitz intermediate good j into sector i composite input for region r as X_{jir}.

- c_{ir} Domestic price of the composite input used in sector i of region r. For monopolistically competitive firms it is the price of composite inputs used for both fixed and variable costs in region r.
- P_{ir} Price of goods and services (Dixit-Stiglitz price in monopolistic competition) in sector i of region r.
- w_{fr} Price of mobile primary factor f in region r (e.g., w_{Lr} is the wage rate of mobile labor).
- $\tilde{w}_{fir}$ Price of sector-specific primary factor f in sector i in region r. [11]
- n_{ir} Number of active firms for $i \in K$ in region r, and the number of firms that enter for $i \in M$.
- N_{mrs} Number of Melitz firms in sector $m \in M$ of region r selling in region s.
- p_{krs} Gross firm-level price of Krugman firms in sector $k \in K$ from region r selling in region s; includes firm markup, tariffs, and iceberg costs.
- q_{krs} Firm-level quantity of Krugman firms in sector $k \in K$ from region r selling in region s.
- $\tilde{p}_{mrs}$ Gross firm-level price of Melitz representative firms in sector $m \in M$ from region r selling in region s; includes firm markup, tariffs, and iceberg costs.
- $\tilde{q}_{mrs}$ Firm-level quantity of Melitz representative firms in sector $m \in M$ from region r selling in region s.
- $\tilde{\varphi}_{mrs}$ Firm-level productivity of Melitz representative firms in sector $m \in M$ from region r selling in region s.
- $\mathcal{I}_r$ Nominal income of region r (measured in units of the numeraire). With labor-leisure choice, this includes the imputed value of leisure.

2.3 Instruments defined

- τ_{irs} Iceberg trade costs, number of units ≥ 1 of sector i from region r needed to export one unit to region s.
- t_{irs} Tariff rates in sector i on imports into region s from region r.

2.4 Parameters defined

- $d0_r$ Benchmark value of full consumption in region r.[12]
- $q0_{ir}$ Benchmark value of composite domestic and imported supply in sector i in region r.
- $y0_{ir}$ Benchmark value of sector i gross output in region r.
- α_{jir} Benchmark coefficient of intermediate input j in gross output of sector i in region r.

[11] The tilde in the definition of a specific primary factor is used to distinguish the specific primary factor price from the mobile primary factor price. It is unrelated to the tilde in the Melitz equations, where the tilde is related to Melitz representative firms.
[12] Includes the imputed value of leisure if there is a labor-leisure choice.

α_{Wir} Benchmark coefficient of aggregate (total) primary inputs in gross output of sector i in region r.
β_{fir} Benchmark share of primary factor f in the value-added of sector i in region r.
η_r^L Preference weight on leisure in utility.
η_r^C Preference weight on consumption of goods and services in utility.
θ_{ir} Benchmark share of i in total consumption of goods and services of region r.
σ^A Elasticity of substitution in Armington sectors between goods from different regions.[13]
σ^K Elasticity of substitution in Krugman sectors.[14]
σ^M Elasticity of substitution in Melitz sectors.[15]
λ_{irs}^A Preference weights in Armington aggregation of good i regional varieties from region r in region s.
λ_{krs}^K Preference weights in Krugman aggregation of firm varieties of good $k \in K$ from region r in region s.
λ_{mrs}^M Preference weights in Melitz aggregation of firm varieties of good $m \in M$ from region r in region s.
σ^L Elasticity of substitution between leisure and consumption of goods and services.
σ^T Elasticity of substitution between intermediates and value added.
f_{kr}^K Fixed cost (in composite input units) of Krugman firms in sector $k \in K$ from region r.
f_{mrs}^M Fixed cost (in composite input units) of Melitz firms in sector $m \in M$ from region r supplying to region s.
f_{mr}^E Sunk entry cost (in composite input units) of Melitz firms in sector $m \in M$ from region r.
a Pareto distribution shape parameter.
b Pareto distribution lower support.
δ Annual probability of firm death in Melitz model.

[13] In the model as formulated, we assume that all elasticities are the same across sectors. While this is not standard practice in the policy literature, it was assumed in Balistreri and Tarr (2022) so our analysis was comparable with the structural-comparison literature (Costinot and Rodríguez-Clare, 2014; Arkolakis, Costinot, and Rodríguez-Clare, 2012) based on gravity. With a single elasticity it was relatively transparent to scale the Armington and Krugman elasticity across sectors to match trade responses across the structures, which was a central condition for our research question. All of the analysis in this paper generalizes to the relatively minor computer-code enhancement to include sector-specific elasticities.
[14] See footnote 13.
[15] See footnote 13.

$\overline{F}_{fr}$ Endowment of mobile factor f in region r. (For $f =$ labor, without labor-leisure choice the observed labor supply is the endowment. With labor-leisure choice this is the *time endowment*.)

$\overline{SF}_{fir}$ Endowment of specific factor f in sector i of region r.

$\overline{BOP}_r$ Benchmark capital account surplus.

2.5 Equations

In table 2 we enumerate the set of equations that are the basis for the computer code of Balistreri and Tarr (2022). In that analysis we allow models to include either a mixed market structure or a strictly unique market structure. Associated with each equation is a variable, which corresponds to the complementary-slack relationships outlined in Appendix B. For exposition we show the equilibrium conditions in table 2 as equalities, which are consistent with our computed interior solutions. As coded, however, the equilibrium conditions are the weak inequalities shown in Appendix B.

Table 2. Equations for the Armington, Krugman, and Melitz Models

Description (link to equation in section 3)	**Algebraic expression**	**Associated variable and set inclusions**	**Dimensions** (set name indicates number of elements in set)
1. Equations Common to the Armington, Krugman, and Melitz Models			
AKM.1 Unit expenditure function (3.1)	$e_r = \left[\eta_r^L w_{Lr}^{1-\sigma^L} + \eta_r^C \left(\prod_{i\in I} P_{ir}^{\theta_{ir}}\right)^{1-\sigma^L}\right]^{1/(1-\sigma^L)}$	D_r $\forall\, r \in R$	R
AKM.2 Unit cost function under CES intermediates (3.5)	$c_{ir} = \Big[\sum_{j\in J} \alpha_{jir} P_{jr}^{1-\sigma^T} + \alpha_{Wir} \left(\prod_{f\in(F\setminus\tilde{F})} (w_{fr})^{\beta_{fir}} \prod_{f\in\tilde{F}} (\tilde{w}_{fir})^{\beta_{fir}}\right)^{1-\sigma^T}\Big]^{1/(1-\sigma^T)}$ Note: For alternative Cobb-Douglas and single-composite intermediate-input formulations see equations (3.3) and (B.4).	Y_{ir} $\forall\, i \in I$ and $r \in R$	$I \times R$
AKM.3 Market clearance for domestic use (3.81)	$q0_{ir} Q_{ir} = d0_r D_r \frac{\partial e_r}{\partial P_{ir}} + \sum_{j\in I} y0_{jr} Y_{jr} \frac{\partial c_{jr}}{\partial P_{ir}}$	P_{ir} $\forall\, i \in I$ and $r \in R$	$I \times R$
AKM.4 Market clearance for sector-specific primary factors (3.86)	$\overline{SF}_{fir} = y0_{ir} Y_{ir} \frac{\partial c_{ir}}{\partial \tilde{w}_{fir}}$	$\tilde{w}_{fir}$ $\forall\, f \in \tilde{F}$, $i \in I$, and $r \in R$	$\tilde{F} \times I \times R$
AKM.5 Market clearance for mobile primary factors (3.87)	$\overline{F}_{fr} = \sum_{i\in I} y0_{ir} Y_{ir} \frac{\partial c_{ir}}{\partial w_{fr}} + d0_r D_r \frac{\partial e_r}{\partial w_{fr}}$	w_{fr} $\forall\, f \in F \setminus \tilde{F}$ and $r \in R$	$(F - \tilde{F}) \times R$
AKM.6 Real Consumption and Welfare (3.95) or (3.104)	$d0_r D_r = \frac{\mathcal{I}_r}{e_r}$	e_r $\forall\, r \in R$	R
AKM.7 Income (3.88) Note: full income with labor-leisure choice.	$\mathcal{I}_r = \sum_{f\in(F\setminus\tilde{F})} w_{fr} \overline{F}_{fr} + \sum_{f\in\tilde{F}} \sum_{i\in I} \tilde{w}_{fir} \overline{SF}_{fir}$ $+ \sum_{i\in(I\setminus(K\cup M))} \sum_{s\in R} t_{isr} c_{is} \tau_{isr} q0_{ir} Q_{ir} \frac{\partial P_{ir}}{\partial[(1+t_{isr})\tau_{isr} c_{is}]}$ $+ \sum_{k\in K} \sum_{s\in R} \frac{t_{ksr} n_{ks} p_{ksr} q_{ksr}}{1+t_{ksr}}$ $+ \sum_{m\in M} \sum_{s\in R} \frac{t_{msr} N_{msr} \tilde{p}_{msr} \tilde{q}_{msr}}{1+t_{msr}}$ $+ e_{\mathrm{USA}} \overline{BOP}_r$	$\mathcal{I}_r$ $\forall\, r \in R$	R
Number of equations and variables common to the three models: $R \times \left[3 + F - \tilde{F} + I \times (2 + \tilde{F})\right]$			

Table 2. Continued: Equations for the Armington, Krugman, and Melitz Models

Description (link to equation in section 3)	Algebraic expression	Associated variable and set inclusions	Dimensions (set name indicates number of elements in set)
2. Armington Specific Equations			
A.8 Price index Armington sectors (3.7)	$P_{is} = \left[\sum_{r\in R} \lambda^A_{irs}[(1+t_{irs})\tau_{irs}c_{ir}]^{1-\sigma^A}\right]^{1/(1-\sigma^A)}$	Q_{is} $\forall s \in R$ and $i \in I \setminus (K \cup M)$	$R \times (I-K-M)$
A.9 Market clearance composite input (3.82)	$y0_{ir}Y_{ir} = \sum_{s\in R} \tau_{irs}q0_{is}Q_{is}\frac{\partial P_{is}}{\partial[(1+t_{irs})\tau_{irs}c_{ir}]}$	c_{ir} $\forall r \in R$ and $i \in I \setminus (K \cup M)$	$R \times (I-K-M)$
Number of Armington specific equations and variables: $2R \times (I-K-M)$			
3. Krugman Specific Equations			
K.8 Dixit-Stiglitz price index (3.22)	$P_{ks} = \left[\sum_{r\in R} \lambda^K_{krs}n_{kr}p^{1-\sigma^K}_{krs}\right]^{1/(1-\sigma^K)}$	Q_{ks} $\forall s \in R$ and $k \in K$	$R \times K$
K.9 Market clearance for composite input (3.83)	$y0_{kr}Y_{kr} = n_{kr}\left(f^K_{kr} + \sum_{s\in R}\tau_{krs}q_{krs}\right)$	c_{kr} $\forall r \in R$ and $k \in K$	$R \times K$
K.10 Firm-level demand (3.18)	$q_{krs} = \lambda^K_{krs}q0_{ks}Q_{ks}\left(\frac{P_{ks}}{p_{krs}}\right)^{\sigma^K}$	p_{krs} $\forall r \in R$, $s \in R$, and $k \in K$	$R^2 \times K$
K.11 Firm-level pricing (3.17)	$p_{krs} = \frac{(1+t_{krs})\tau_{krs}c_{kr}}{1-1/\sigma^K}$	q_{krs} $\forall r \in R$, $s \in R$, and $k \in K$	$R^2 \times K$
K.12 Zero-profit (free-entry) (3.20)	$f^K_{kr}c_{kr} = \sum_{s\in R}\frac{p_{krs}q_{krs}}{\sigma^K(1+t_{krs})}$	n_{kr} $\forall r \in R$ and $k \in K$	$R \times K$
Number of Krugman specific equations and variables: $K \times \left(3R + 2R^2\right)$			
4. Melitz Specific Equations			
M.8 Dixit-Stiglitz price index (3.54)	$P_{ms} = \left[\sum_{r\in R} \lambda^M_{mrs}N_{mrs}\tilde{p}^{1-\sigma^M}_{mrs}\right]^{1/(1-\sigma^M)}$	Q_{ms} $\forall s \in R$ and $m \in M$	$R \times M$
M.9 Market clearance for composite input (3.85)	$y0_{mr}Y_{mr} = \delta f^E_{mr}n_{mr} + \sum_{s\in R} N_{mrs}\left(f^M_{mrs} + \frac{\tau_{mrs}\tilde{q}_{mrs}}{\tilde{\varphi}_{mrs}}\right)$	c_{mr} $\forall r \in R$ and $m \in M$	$R \times M$
M.10 Firm-level demand (3.55)	$\tilde{q}_{mrs} = \lambda^M_{mrs}q0_{ms}Q_{ms}\left(\frac{P_{ms}}{\tilde{p}_{mrs}}\right)^{\sigma^M}$	$\tilde{p}_{mrs}$ $\forall r \in R$, $s \in R$, and $m \in M$	$R^2 \times M$
M.11 Firm-level pricing (3.52)	$\tilde{p}_{mrs} = \frac{(1+t_{mrs})\tau_{mrs}c_{mr}}{\tilde{\varphi}(1-1/\sigma^M)}$	$\tilde{q}_{krs}$ $\forall r \in R$, $s \in R$, and $k \in K$	$R^2 \times K$
M.12 Bilateral selection: *zero-profit productivity cutoff* (3.72)	$f^M_{mrs}c_{mr} = \frac{\tilde{p}_{krs}\tilde{q}_{krs}}{(1+t_{krs})}\frac{a+1-\sigma^M}{a\sigma^M}$	N_{mrs} $\forall r \in R$, $s \in R$, and $m \in M$	$R^2 \times M$
M.13 Zero expected profit: *free-entry* (3.77)	$\delta f^E_{mr}c_{mr} = \sum_{s\in R}\left(\frac{N_{mrs}}{n_{mr}}\right)\frac{\tilde{p}_{mrs}\tilde{q}_{mrs}(\sigma^M-1)}{(1+t_{mrs})a\sigma^M}$	n_{mr} $\forall r \in R$ and $m \in M$	$R \times M$
M.14 Representative-firm productivity (3.79)	$\tilde{\varphi}_{mrs} = b\left(\frac{N_{mrs}}{n_{mr}}\right)^{-1/a}\left(\frac{a+1-\sigma^M}{a}\right)^{1/(1-\sigma^M)}$	$\tilde{\varphi}_{mrs}$ $\forall r \in R$, $s \in R$, and $m \in M$	$R^2 \times M$
Number of Melitz specific equations and variables: $M \times \left(3R + 4R^2\right)$			

The number of equations must equal the number of variables. Only relative prices are determined, however, so we choose a numeraire in which all prices and nominal income are expressed. Thus, the overall dimensions are reduced by one market-clearance condition and one price. The market for the numeraire good is ensured to clear by Walras's Law, the values of excess demand across the general equilibrium must sum to zero. As indicated we can run the model with either mixed market structures or strictly a unique market structure. The overall dimensions of the models, inclusive of the numeraire, when we adopt a unique market structure are as follows (where the set name indicates the number of elements):

- **Armington ($K = M = \emptyset$)** model dimensions:
$$R \times \left[3 + F - \tilde{F} + I \times \left(4 + \tilde{F}\right)\right].$$
- **Krugman ($K = I$)** model dimensions:
$$\left(2R^2 \times I\right) + R \times \left[3 + F - \tilde{F} + I \times \left(5 + \tilde{F}\right)\right].$$
- **Melitz ($M = I$)** model dimensions:
$$\left(4R^2 \times I\right) + R \times \left[3 + F - \tilde{F} + I \times \left(5 + \tilde{F}\right)\right].$$

When the model is run with a mixture of market structures the overall dimensions are dependent on the mix, but can be calculated from table 2.

3. Equations Explained

We proceed in this section to explain and derive the model equations. In subsection 3.1 we consider top-level consumer preferences over leisure and consumption of goods and services. Subsection 3.2 considers the technology of the *Composite Input*, which is used for both fixed and marginal cost. It is important to read this subsection in order to understand the derivations that follow. Subsections 3.3, 3.4, and 3.5 derive the equations that are specific to the Armington, Krugman, and Melitz models. The set of market clearance conditions are covered in subsection 3.6, and income balance and the numeraire are covered in subsection 3.7. Subsection 3.8 covers a set of details related to the regional expenditure function, and we derive our formula for calculating Hicksian equivalent variation.

3.1 Preferences

The technology and preference equations of the computer code are described in the dual. For intuition, we also show the primal equations in sections 3.1 and 3.2. With the option to include a labor-leisure choice, the unit expenditure function in region r is:

$$e_r = \left[\eta_r^L w_{\text{L}r}^{1-\sigma^L} + \eta_r^C \left(\prod_{i \in I} P_{ir}^{\theta_{ir}}\right)^{1-\sigma^L}\right]^{1/(1-\sigma^L)} \quad \forall\, r \in R. \qquad (3.1)$$

Equation (3.1), which is equation AKM.1 of table 2, is dual to the following utility function defined over leisure and consumption of goods and services:

$$U_r = d0_r D_r = \left[\eta_r^{L\,\frac{1}{\sigma^L}} l_r^{\frac{\sigma^L-1}{\sigma^L}} + \eta_r^{C\,\frac{1}{\sigma^L}} \left(\frac{1}{\theta_r} \prod\nolimits_{i \in I} C_{ir}^{\theta_{ir}} \right)^{\frac{\sigma^L-1}{\sigma^L}} \right]^{\frac{\sigma^L}{\sigma^L-1}} \quad \forall\, r \in R, \tag{3.2}$$

where l_r is leisure, C_{ir} is consumption of the i-th composite good or service in region r and $\theta_r = \prod_i \theta_{ir}^{\theta_{ir}}$ is the Cobb-Douglas scaling parameter used to assure consistency between the primal and the form of the expenditure function. P_{ir} is the price of a unit of the composite commodity of the i-th sector given by the price indices shown below in sections 3.3, 3.4, and 3.5, which depend on the market structure of the sector, and w_{Lr} is the wage rate in region r. The consumer maximizes utility, subject to her income constraint given by AKM.7. Equation (3.1) follows from substitution of the optimum values of consumption and leisure into the utility function and using the duality identities to solve for the unit expenditure function.[16] Details of the derivation are in section 3.8 below.

3.2 Technology of the Composite Input

We next explain equation AKM.2 of the model. As is common in the literature, we assume that the inputs required for both fixed and marginal costs are identical, and the costs of these inputs may be represented by a function that is a linearly homogeneous, quasi-concave composite function of all inputs.[17] We al-

[16] The assumptions of Cobb-Douglas demand for aggregate goods and services and CES demand for leisure versus aggregate goods and services are comparable to the established literature, including Costinot and Rodríguez-Clare (2014), Arkolakis, Costinot, and Rodríguez-Clare (2012), Balistreri, Hillberry, and Rutherford (2010), and Balistreri, Hillberry, and Rutherford (2011). While we adopt the standard linearly homogeneous (constant-returns to scale) CES form for preferences over leisure and the consumption aggregate, it is not as restrictive as one might think. Empirical labor-supply responses are usually summarized in terms of the uncompensated and compensated elasticities, where the elasticity with respect to money income is implied through the Slutsky relationship. Adjusting the time endowment along with the elasticity of substitution allows one to calibrate to these observables. Ballard (2000) presents the theory and formulae for calibrating the CES form to these empirical observables. Ballard specifically argues that the time endowment is open to interpretation, and so it should be adjusted to indicate the appropriate compensated labor supply elasticity. Thus, although CES imposes unitary elasticities with respect to "full income," the function can be calibrated to an observable income elasticity. The CES form, of course, still imposes the strong (and unrealistic) assumption that leisure is separable from consumption of goods and services.

[17] Studies that adopt the assumption that inputs required for both fixed and marginal costs are identical include, for example, Helpman and Krugman (1985, p.12), Costinot and Rodríguez-Clare (2013, equation 7), Balistreri and Rutherford (2013), Dixon, Jerie, and Rimmer (2019), and Markusen, Rutherford, and Tarr (2005, equations 5 and 6). See also the discussion in the text regarding the alternate approach of Akgul, Villoria, and Hertel (2016) and Jafari and Britz (2018).

ways assume that value-added inputs combine in a Cobb-Douglas nest, but we employ multiple treatments of intermediates. We show the case of CES demand for intermediates in table 2 but employ multiple representations of intermediate demand in the applied model.

3.2.1 Cobb-Douglas

With Cobb-Douglas demand for intermediates, the production technology for the composite input in the dual is given by:

$$c_{ir} = \prod_{j \in I} (P_{jr})^{\alpha_{jir}} \prod_{f \in F \setminus \tilde{F}} (w_{fr})^{\alpha_{Wir}\beta_{fir}} \prod_{f \in \tilde{F}} (\tilde{w}_{fir})^{\alpha_{Wir}\beta_{fir}}, \tag{3.3}$$

where $\alpha_{jir} \geq 0$ and $\beta_{fir} \geq 0$; and constant returns to scale requires $\alpha_{Wir} = (1 - \sum_{j \in I} \alpha_{jir})$ and $\sum_{f \in F} \beta_{fir} = 1$. The unit cost function of (3.3) is dual to the Cobb-Douglas production function:

$$y0_{ir} Y_{ir} = \Phi_{ir} \prod_{j \in I} (X_{jir})^{\alpha_{jir}} \prod_{f \in F \setminus \tilde{F}} (x_{fir})^{\alpha_{Wir}\beta_{fir}} \prod_{f \in \tilde{F}} (\tilde{x}_{fir})^{\alpha_{Wir}\beta_{fir}}, \tag{3.4}$$

where in sector i of region r, $y0_{ir}Y_{ir}$ is the quantity of composite inputs, in the Armington model this is simply output of sector i. We denote X_{jir} as intermediate use of the aggregate good Q_{jr} (either Armington or Dixit-Stiglitz) from sector j used as intermediates in sector i of region r. Inputs of mobile factor $f \in F \setminus \tilde{F}$ by sector i in region r are denoted x_{fir}, and inputs of specific factor $f \in \tilde{F}$ by sector i in region r are denoted $\tilde{x}_{fir}$. The parameter Φ_{ir} is the Cobb-Douglas scaling parameter used to assure unit consistency between the primal and the form of the cost function. We define the composite of inputs $y0_{ir}Y_{ir}$, where $y0_{ir}$ is the benchmark value of *gross output* and Y_{ir} are the endogenous variables that take the value of one in the benchmark. This formulation makes it transparent that the endogenous variables for which we solve directly indicate proportional changes. Minimization of the cost of acquiring one unit of the composite input subject to the production function in (3.4) yields the unit cost function (3.3). If there are no intermediates, then the α_{jir} parameters are all zero and we only have the Cobb-Douglas nest of primary factors.

3.2.2 Constant Elasticity of Substitution (CES)

If we assume that intermediates and a value-added composite substitute with an elasticity of substitution $\sigma^T \neq 1$, we have the unit cost function of (3.5) which is equation AKM.2 in table 2:

$$c_{ir} = \left[\sum_{j \in J} \alpha_{jir} P_{jr}^{1-\sigma^T} + \alpha_{Wir} \left(\prod_{f \in (F \setminus \tilde{F})} (w_{fr})^{\beta_{fir}} \prod_{f \in \tilde{F}} (\tilde{w}_{fir})^{\beta_{fir}} \right)^{1-\sigma^T} \right]^{1/(1-\sigma^T)}, \tag{3.5}$$

where $\alpha_{jir} \geq 0$ and $\beta_{fir} \geq 0$; and constant returns to scale in the primary factors nest requires $\sum_{f \in F} \beta_{fir} = 1$. The unit cost function of (3.5) is dual to the CES production function with the value-added composite as one of the inputs:

$$\begin{aligned} y0_{ir} Y_{ir} = \Bigg[& \sum_{j \in J} \alpha_{jir}^{1/\sigma^T} X_{jir}^{(\sigma^T-1)/\sigma^T} \\ & + \alpha_{Wir}^{1/\sigma^T} \left(\frac{1}{\beta_{ir}} \prod_{f \in (F \setminus \tilde{F})} (x_{fr})^{\beta_{fir}} \prod_{f \in \tilde{F}} (\tilde{x}_{fir})^{\beta_{fir}} \right)^{(\sigma^T-1)/\sigma^T} \Bigg]^{\sigma^T/(\sigma^T-1)}, \end{aligned} \tag{3.6}$$

where we introduce $\beta_{ir} = \prod_f \beta_{fir}^{\beta_{fir}}$ as the Cobb-Douglas scaling parameter used to assure consistency between the primal and the form of cost function. Minimization of the cost of acquiring one unit of the composite input subject to the production function in (3.6) yields the unit cost function (3.5). If there are no intermediates, then the α_{jir} parameters are all zero and we only have the Cobb-Douglas nest of primary factors.

3.3 Armington Prices and Quantities[18]

In the Armington model, the price of good i in its home market, say region r, is its marginal cost, which is c_{ir}. The delivered price to region s of good or service i includes the iceberg and tariff costs, i.e., $p_{irs} = (1 + t_{irs})\tau_{irs}c_{ir}$. In the Armington case, the price of a good or service is given by the unit cost of the constant returns to scale CES aggregation of prices of varieties from different regions. This gives us model equation A.8 in table 2, which we rewrite here as (3.7):

$$P_{is} = \left[\sum_{r \in R} \lambda_{irs}^{A} [(1 + t_{irs})\tau_{irs}c_{ir}]^{1-\sigma^A} \right]^{1/(1-\sigma^A)}. \tag{3.7}$$

We note that the Armington price P_{is} is the price of the composite good or service i in region s available for all uses, which is for consumption or intermediates in our model. By defining the Armington aggregate price based on absorption shares (explained below in this subsection), we have a unique Armington price of goods

[18] We discuss the Dixit-Stiglitz price aggregates in the Krugman and Melitz models in sections 3.4 and 3.5.

or services in sector i of region s for all uses. This facilitates exposition and is consistent with standard gravity formulations (e.g., Anderson and Van Wincoop, 2003; Balistreri and Hillberry, 2007; and Bernard et al., 2003). It is also consistent with the standard monopolistic-competition structure where there is a single price index representing a CES composite of different varieties (as discussed in sections 3.4 and 3.5).

The Armington price is dual to a good that is defined by the Armington aggregator function (3.8).

$$q0_{is}Q_{is} = \left[\sum_{r\in R} (\lambda^A_{irs})^{1/\sigma^A} Q_{irs}^{(\sigma^A-1)/\sigma^A} \right]^{\sigma^A/(\sigma^A-1)}, \tag{3.8}$$

where Q_{irs} is the quantity of good i supplied by region r to region s. The Armington good in any sector i is an aggregate of the goods supplied from different regions of the model and expresses the total quantity of an aggregate Armington good that is available for consumption and intermediate use. The quantities supplied from the different regions are not consumed by the representative agent or used as intermediates independently. They are demanded for consumption or intermediates only as a component of the Armington aggregate. Given the prices P_{is}, consumers and firms optimize the purchases of the aggregate Armington good based on their expenditure and cost functions, which in the initial equilibrium determine the parameters λ^A_{irs} of the Armington aggregate.

We define the Armington composite quantity as $q0_{is}Q_{is}$, where $q0_{is}$ is the benchmark value of composite Armington output of sector i in region s and Q_{is} is the associated endogenous variable that takes the value of one in the benchmark. Then Q_{is} indicates the proportional change in the Armington composite good or service in the counterfactual equilibrium.

It is useful at this point to mention the procedure for calibrating the Armington aggregation. To obtain CES preference weights, the λ in (3.7) and (3.8) we use a set of observed values from the benchmark equilibrium. The discussion largely follows the presentation of calibration under the *Calibrated-share Form* of CES functions in Appendix A.[19] Denote by the superscript "0" the value of the bilateral prices in the benchmark equilibrium, $p^0_{irs} = (1 + t^0_{irs})\tau^0_{irs}c^0_{ir}$. In addition, consider the assumed benchmark price of the Armington composite, P^0_{is}, which might take on a value of one by our choice of units. Notice, however, that the bilateral prices are rampant with potentially different bilateral distortions and transport margins. So, although we might chose units such that the $c^0_{ir} = 1$ in the benchmark, the prices faced by agents in a given region s will be inclusive of idiosyncratic distortions and not be one. Let us introduce the parameter ν_{irs} as the value share

[19] The calibrated-share form was first discussed by Rutherford (1995). Our Appendix A is a set of relatively concise notes directly based on Rutherford (1995).

of region s expenditures on good or service i sourced from region r. This is directly observable in the accounts as the gross (of distortion) expenditures on good or service i sourced from r by region s divided by the value of total absorption of good or service i in region s, where $\sum_r \nu_{irs} = 1$ by definition. Now consider rewriting equation (3.7) in the calibrated-share form as presented in Appendix A:

$$P_{is} = P_{is}^0 \left[\sum_{r \in R} \nu_{irs} \left(\frac{(1 + t_{irs}) \tau_{irs} c_{ir}}{(1 + t_{irs}^0) \tau_{irs}^0 c_{ir}^0} \right)^{1-\sigma^A} \right]^{1/(1-\sigma^A)} . \tag{3.9}$$

Translating the added data into the CES coefficient λ_{irs} yields the following calibration formula:

$$\lambda_{irs}^A = \nu_{irs} \left[\frac{P_{is}^0}{(1 + t_{irs}^0) \tau_{irs}^0 c_{ir}^0} \right]^{1-\sigma^A} ,$$

where everything on the right-hand side is observed in the benchmark equilibrium, except σ^A which must be assumed.

3.4 Krugman Specific Equations

Total costs for Krugman firms consist of a fixed cost and constant marginal costs of total output. As discussed above, we assume that the costs of all inputs may be represented by a composite cost function (3.3) or (3.5). We further assume that the inputs required for both fixed and marginal costs are identical. If the firm produces any output, it incurs a fixed cost of f_{kr}^K units of the composite input. Firms in region r do not face an additional fixed cost of exporting to market $s \neq r$. They do, however, face iceberg trade costs $\tau_{krs} \geq 1$. All firms in a given region have identical costs, so we may represent total costs TC_{kr} of all firms in sector k of region r by a representative firm's total cost function:

$$TC_{kr} = c_{kr} \left(f_{kr}^K + \sum_{s \in R} \tau_{krs} q_{krs} \right) , \tag{3.10}$$

where the delivered quantity to a consumer in regon s is q_{krs} and the marginal cost is simply c_{kr}.

We want to derive the demand for a variety of an individual firm in sector k of region r and sold in region s. Each firm chooses to produce a unique (yet symmetric) variety.[20] Let ω_{kr} index a variety produced by an individual firm in

[20] If two firms both produce the same variety (with non-zero output), they would share the demand for that variety. Given the internal scale economies this would indicate higher average cost relative to symmetric varieties produced by single firms. Only one firm

sector k of region r and define the following:

$p_s(\omega_{kr})$	=	the price of variety ω_{kr} in region s;
$q_s(\omega_{kr})$	=	the quantity of variety ω_{kr} in region s;
$\pi_s(\omega_{kr})$	=	the operating profit of variety ω_{kr} in region s (excludes the fixed costs of operation); and
V_{krs}	=	the set of varieties from sector k sold in region s by firms from region r. (We subsequently show that an active firm sells its variety in all markets, so the index s is redundant).

3.4.1 Firm price, quantity, and profit

We first write the Dixit-Stiglitz price index so that the individual firm varieties appear and the prices are not necessarily equal. We allow regional preference weights, but not firm level preference weights, and discuss calibration of the weights below. The price index is given by:

$$P_{ks} = \left[\sum_{r \in R} \sum_{\omega_{kr} \in V_{krs}} \lambda^K_{krs} p_s(\omega_{kr})^{1-\sigma^K} \right]^{1/(1-\sigma^K)} . \tag{3.11}$$

The Dixit-Stiglitz price index is dual to the following technology for aggregating firm quantities:

$$q0_{ks} Q_{ks} = \left[\sum_{r \in R} \sum_{\omega_{kr} \in V_{krs}} (\lambda^K_{krs})^{1/\sigma^K} q_s(\omega_{kr})^{(\sigma^K-1)/\sigma^K} \right]^{(\sigma^K-1)/\sigma^K} , \tag{3.12}$$

where we define the Dixit-Stiglitz composite quantity as $q0_{ks}Q_{ks}$. As in the Armington model, $q0_{ks}$ is the benchmark value of good or service i in region s (available for absorption), and Q_{ks} is the associated endogenous variable that takes the value of one in the benchmark. A similar definition will apply in the Melitz model.

We seek the demand function for an individual variety, ω'_{kr} in region s. Since P_{ks} is the minimum cost of acquiring one unit of good or service k in region s, and our technology is linearly homogeneous, the cost function for all varieties of good k in region s is $q0_{ks}Q_{ks}P_{ks}$. From Shephard's Lemma, the conditional demand function for an individual variety is $q0_{ks}Q_{ks}\frac{\partial P_{ks}}{\partial p_s(\omega'_{kr})}$.

Define Z_{ks} as the term inside the brackets of equation (3.11) so that $P_{ks} = (Z_{ks})^{1/(1-\sigma^K)}$. Then

producing a given variety would be able to survive in the proposed equilibrium.

$$
\begin{aligned}
\frac{\partial P_{ks}}{\partial p_s(\omega'_{kr})} &= \frac{1-\sigma^K}{1-\sigma^K} Z_{ks}^{\frac{1}{1-\sigma^K}-1} \left(\lambda^K_{krs} p_s(\omega'_{kr})^{-\sigma^K}\right) \\
&= P_{ks} Z_{ks}^{-1} \left(\lambda^K_{krs} p_s(\omega'_{kr})^{-\sigma^K}\right) \\
&= \lambda^K_{krs} \left(\frac{P_{ks}}{p_s(\omega'_{kr})}\right)^{\sigma^K},
\end{aligned}
$$

where in the last simplification we use $Z_{ks}^{-1} = P_{ks}^{\sigma^K-1}$. The demand for the individual variety ω'_{kr} is

$$
q_s(\omega'_{kr}) = \lambda^K_{krs} q0_{ks} Q_{ks} \left(\frac{P_{ks}}{p_s(\omega'_{kr})}\right)^{\sigma^K}.
$$

Since choice of the variety ω'_{kr} was arbitrary, we may write the demand for any variety as

$$
q_s(\omega_{kr}) = \lambda^K_{krs} q0_{ks} Q_{ks} \left(\frac{P_{ks}}{p_s(\omega_{kr})}\right)^{\sigma^K}. \tag{3.13}
$$

We define prices inclusive of tariffs, but the firm does not receive the tariff revenue, so firm-level revenue is $p_s(\omega_{kr})q_s(\omega_{kr})/(1+t_{krs})$. The quantity delivered to the consumer is net of the iceberg costs, so the firm must produce $\tau_{krs}q_s(\omega_{kr})$ units for $q_s(\omega_{kr})$ to be delivered to the customer.[21] Since the firm does not incur an additional fixed cost of exporting in the Krugman model, we have the following definition of operating profit for the individual firm in sector k of region r on sales in region s:

$$
\pi_s(\omega_{kr}) = \frac{p_s(\omega_{kr})q_s(\omega_{kr})}{(1+t_{krs})} - \tau_{krs}c_{kr}q_s(\omega_{kr}), \tag{3.14}
$$

which is firm revenue less the variable cost of delivered product. The first-order condition for profit maximization yields:

$$
\frac{\partial \pi_s(\omega_{kr})}{\partial q_s(\omega_{kr})} = \frac{p_s(\omega_{kr})}{(1+t_{krs})} + \frac{q_s(\omega_{kr})}{(1+t_{krs})}\frac{dp_s(\omega_{kr})}{dq_s(\omega_{kr})} - \tau_{krs}c_{kr} = 0. \tag{3.15}
$$

[21] First introduced by Samuelson (1954) and independent of its realism, the international trade literature commonly uses "iceberg" costs to model barriers to trade and transport costs. The formulation is convenient (especially in stylized analytic models) because the transport margin is paid in units of the good being traded, so there is no need to track a new price or good related to transport services.

Define the "aggregates" of the demand for a variety on the right side of (3.13) as $\Gamma_{krs} \equiv \lambda^K_{krs} q0_{ks} Q_{ks} (P_{ks})^{\sigma^K}$, such that demand for a variety is given by

$$q_s(\omega_{kr}) = \Gamma_{krs} p_s(\omega_{kr})^{-\sigma^K}. \tag{3.16}$$

The Krugman (and Melitz) model assumes large-group monopolistic competition, in which firms assume they do not impact aggregates. This implies the firm assumes that its price does not impact Γ_{krs}. Using the Inverse Function Theorem for the derivative of the demand function, for the middle term in (3.15) we have:

$$\begin{aligned}
\frac{q_s(\omega_{kr})}{(1+t_{krs})} \frac{dp_s(\omega_{kr})}{dq_s(\omega_{kr})} &= \frac{1}{(1+t_{krs})} \frac{q_s(\omega_{kr})}{\frac{dq_s(\omega_{kr})}{dp_s(\omega_{kr})}} \\
&= \frac{\Gamma_{krs} p_s(\omega_{kr})^{-\sigma^K}}{(1+t_{krs})\Gamma_{krs}\left(-\sigma^K p_s(\omega_{kr})^{(-\sigma^K-1)}\right)} \\
&= -\frac{p_s(\omega_{kr})}{\sigma^K(1+t_{krs})}.
\end{aligned}$$

Substituting this term back into (3.15) and rearranging we have the price that maximizes profit for the individual (small) firm in sector k of region r selling in region s

$$\begin{aligned}
p_s(\omega_{kr}) &= \frac{(1+t_{krs})\tau_{krs} c_{krs}}{1-1/\sigma^K} \\
&= \frac{\sigma^K}{\sigma^K-1}(1+t_{krs})\tau_{krs} c_{kr} \\
p_{krs} &= \frac{\sigma^K}{\sigma^K-1}(1+t_{krs})\tau_{krs} c_{kr}.
\end{aligned} \tag{3.17}$$

This indicates the standard markup above marginal cost for firms under monopolistic competition. Firms charge a constant markup above marginal costs, where the markup increases as the Dixit-Stiglitz elasticity decreases toward one. Since costs are identical for all firms in a given sector and region, all firms in sector k from region r apply the same markup over costs in any market (differing by market depending on tariffs and iceberg costs); then we may drop the firm or variety index from the final equality of price equation (3.17) and substitute p_{krs} for $p_s(\omega_{kr})$. This establishes equation K.11 in table 2. Substitute p_{krs} for $p_s(\omega_{kr})$ in the equation for demand for a variety (3.13). We see that all firms from sector k in region r produce the same quantity of sales in region s. Thus, we also drop the variety index regarding price, quantity, and profits. In particular, all firms in sector k of region r selling in region s face the same demand curve, i.e., $q_s(\omega_{kr}) = q_{krs}$, and we write the demand for any variety ω_{kr} in region s as:

$$q_{krs} = \lambda^K_{krs} q0_{ks} Q_{ks} \left(\frac{P_{ks}}{p_{krs}} \right)^{\sigma^K}. \tag{3.18}$$

This establishes model equation K.10 in table 2.

3.4.2 Free Entry Condition

Free entry leads to zero profits, so accumulated operating profits across all markets just cover fixed costs. Rearrange equation (3.17), so we have $\tau_{krs} c_{kr} = \frac{p_{krs}(1-1/\sigma^K)}{(1+t_{krs})}$. Substitute this into the the equation for operating profits (3.14) of any firm from region r on sales to a particular region s such that profits are maximized gives us the following:

$$\begin{aligned} \pi_s(\omega_{kr}) &= \frac{p_{krs} q_{krs}}{(1+t_{krs})} - \tau_{krs} c_{kr} q_s(\omega_{kr}) \\ &= \frac{p_{krs} q_{krs}}{(1+t_{krs})} - \frac{p_{krs}(1-1/\sigma^K) q_{krs}}{(1+t_{krs})} \\ &= \frac{p_{krs} q_{krs}}{\sigma^K (1+t_{krs})}. \end{aligned} \tag{3.19}$$

From (3.19) we see that any firm that is active in its home market will sell in all regions s since operating profits are positive in all regions. That is, the index s in the set V_{krs} is redundant. Setting the fixed entry cost equal to operating profits summed over all regions s yields the zero-profit condition, which is model equation K.12 in table 2:

$$f^K_{kr} c_{kr} = \sum_{s \in R} \frac{p_{krs} q_{krs}}{\sigma^K (1+t_{krs})}. \tag{3.20}$$

Associated with the zero-profit condition is n_{kr}, the number of firms active in sector k of region r.[22] Since there is a one-to-one correspondence between firms and varieties, n_{kr} is the number of varieties in the set V_{krs}.

The remaining Krugman specific model conditions are arrived at as follows. Use equation (3.17) to obtain

[22] Despite the fact that n_{kr} does not appear explicitly in equations (3.20), it has an intuitive association related to the complementary-slack relationship of equation (B.10) in appendix B. Consider that firm entry into the market is an activity (with intensity measured by n_{kr}). The marginal cost of establishing a firm is the left-hand side of (3.20) and the marginal benefit is the operating profits or the right-hand side. Intuitively, the activity (entry) will intensify up to the point of zero profits. Thus, in terms of the complementary slack condition (B.10) we have that $MC(n) - MB(n) \geq 0$, $n \geq 0$, and $n(MC(n) - MB(n)) = 0$, where $MC(n)$ and $MB(n)$ are marginal cost and marginal benefit of entry, and n is the intensity of entry.

$$\sum_{\omega_{kr} \in V_{krs}} p_s(\omega_{kr})^{1-\sigma^K} = \sum_{\omega_{kr} \in V_{krs}} p_{krs}^{1-\sigma^K} = n_{kr} p_{krs}^{1-\sigma^K}. \tag{3.21}$$

Then substitute (3.21) into the Dixit-Stiglitz disaggregated price index, (3.11) to obtain model equation K.8 in table 2

$$P_{ks} = \left[\sum_{r \in R} \lambda_{krs}^K n_{kr} p_{krs}^{1-\sigma^K} \right]^{1/(1-\sigma^K)}. \tag{3.22}$$

Similarly, substitute q_{krs} for $q_s(\omega_{kr})$ into the Dixit-Stiglitz disaggregated quantity index (3.12) and use

$$\sum_{\omega_{kr} \in V_{ksr}} q_{krs}^{(\sigma^K-1)/\sigma^K} = n_{kr} q_{krs}^{(\sigma^K-1)/\sigma^K}$$

to obtain the Dixit-Stiglitz aggregate quantity for this model

$$q0_{ks} Q_{ks} = \left[\sum_{r \in R} (\lambda_{krs}^K)^{1/\sigma^K} n_{kr} q_{krs}^{(\sigma^K-1)/\sigma^K} \right]^{\sigma^K/(\sigma^K-1)}. \tag{3.23}$$

3.4.3 Constant Output per Firm

It helps with interpretation of results to show that output per firm is constant in this model. Substitute for price in (3.20) to get:

$$\sum_{s \in R} \frac{\frac{\sigma^K}{\sigma^K-1}(1+t_{krs})\tau_{krs} c_{kr} q_{krs}}{\sigma^K(1+t_{krs})} = f_{kr}^K c_{kr} \quad ; \tag{3.24}$$

rearranging we have

$$q_{kr} = \sum_{s \in R} \tau_{krs} q_{krs} = f_{kr}^K(\sigma^K - 1). \tag{3.25}$$

Since $\tau_{krs} q_{krs}$ is the amount of output firms of sector k in region r produce for sales in region s, the left-hand side of (3.25) is total firm output. Since the right-hand side of (3.25) is constant, output per firm is constant, i.e., there are no rationalization effects in this model and results differ from Armington only due to the Dixit-Stiglitz variety externality.

The absence of rationalization gain, however, depends on our assumption that fixed and variable costs share the same cost structure. Some authors assume that variable costs may be represented by a linearly homogeneous composite function of *all* inputs, but fixed costs only use *primary inputs*, (similarly treated as a composite input). Let the unit cost functions for variable and fixed costs may be represented by c_{kr} and c^f_{kr}, respectively. In this case, equation (3.25) becomes

$$q_{kr} = \sum_{s \in R} \tau_{krs} q_{krs} = f^K_{kr}(\sigma^K - 1)\frac{c^f_{kr}}{c_{kr}}. \tag{3.26}$$

The percentage change in output per firm is equal to the percentage change in the ratio of the unit cost functions, and rationalization gains are possible.[23] The problem for an applied modeler, however, is that characterization of these different cost structures is not possible based only on the data in the social accounts. As noted by Hertel and Swaminathan (1996, p.32), without additional industry level data, any assumptions about these cost structures could be criticized. It is incumbent on the modeler to make any assumption transparent.

3.4.4 The Dixit-Stiglitz externality in the dual

Consider, again, Z_{ks} as the term inside the brackets of the Dixit-Stiglitz price index in equilibrium, equation (3.22), so that $P_{ks} = Z_{ks}^{1/(1-\sigma^K)}$. Partially differentiate with respect to $n_{kr'}$ for some $r' \in R$. Given that $\sigma^K > 1$, we have

$$\begin{aligned}
\frac{\partial P_{ks}}{\partial n_{kr'}} &= \frac{1}{1-\sigma^K} Z_{ks}^{\frac{1}{1-\sigma^K}-1}\left(\lambda_{krs} p_{krs}^{1-\sigma^K}\right) \\
&= \frac{1}{1-\sigma^K} P_{ks} Z_{ks}^{-1}\left(\lambda_{krs} p_{krs}^{1-\sigma^K}\right) \\
&= \frac{1}{1-\sigma^K} \lambda_{krs} P_{ks}^{\sigma^K} p_{krs}^{1-\sigma^K} < 0, \quad \therefore \\
\frac{\partial P_{ks}}{\partial n_{kr'}} &< 0.
\end{aligned} \tag{3.27}$$

Since r' was chosen arbitrarily, (3.27) holds for all $r \in R$. This shows that the cost of a unit of utility in region s, declines (increases) as the number of varieties from any region r increases (decreases).

[23] Akgul, Villoria, and Hertel (2016) estimate gains in the Melitz model for the world that are 9-10 times larger than the estimated gains in their Armington model. This is a larger ratio of Melitz to Armington gains than has been found in the literature to date, including by Balistreri and Tarr (2022) in any of their 18 model variants. A reduction in trade costs will have a first-order impact on the price of intermediates and a second-order effect on the price of primary inputs. While the first-order, second-order observation is not definitive, a new equilibrium may result where intermediate prices fall by more than primary input prices; then c^f_{kr}/c_{kr} will rise and there will be rationalization gains. The results of Akgul, Villoria, and Hertel (2016, table 4) show large scale effects in their firm heterogeneity model, suggesting that their cost structure assumption is a component in explaining the large relative welfare gains in the Melitz model compared to Armington.

3.4.5 The Dixit-Stiglitz externality in the primal

In equilibrium of the Krugman model, the Dixit-Stiglitz quantity index in region s of goods in sector k may be written in the form of equation (3.23). Note that equation (3.23) may be rewritten as a Dixit-Stiglitz aggregate of Dixit-Stiglitz sub-aggregates of goods from the regions r. That is,

$$\begin{aligned} q0_{ks}Q_{ks} &= \left[\sum_{r\in R}(\lambda^K_{krs})^{1/\sigma^K} n_{kr} q_{krs}^{(\sigma^K-1)/\sigma^K}\right]^{\sigma^K/(\sigma^K-1)} \\ &= \left[\sum_{r\in R} DS_{ksr}^{(\sigma^K-1)/\sigma^K}\right]^{\sigma^K/(\sigma^K-1)}, \end{aligned} \tag{3.28}$$

where

$$DS_{ksr} \equiv \left[(\lambda^K_{krs})^{1/\sigma^K} n_{kr} q_{krs}^{(\sigma^K-1)/\sigma^K}\right]^{\sigma^K/(\sigma^K-1)}.$$

Now consider the impact of an increase in the number of varieties on one of the Dixit-Stiglitz sub-aggregates.

$$\begin{aligned} DS_{ksr} &= \left[(\lambda^K_{krs})^{1/\sigma^K} n_{kr} q_{krs}^{(\sigma^K-1)/\sigma^K}\right]^{\sigma^K/(\sigma^K-1)} \\ &= (\lambda^K_{krs})^{1/(\sigma^K-1)} n_{kr}^{\sigma^K/(\sigma^K-1)} q_{krs} \\ &= (\lambda^K_{krs})^{1/(\sigma^K-1)} n_{kr}^{1/(\sigma^K-1)} n_{kr} q_{krs}. \end{aligned} \tag{3.29}$$

The cost to users in region s of all varieties of goods in sector k from region r is $n_{kr}p_{krs}q_{krs}$ which increases in proportion to the number of varieties. But, since $1/(\sigma^K - 1) > 0$, the value to consumers or effective supply to firms increases more than proportionately with the number of firms. This means that for a given expenditure on commodity k from region r, region s gets more utility if there is an increase in varieties from r.

3.4.6 Preference weights in the the Dixit-Stiglitz Price Equation of the Krugman model

Using our definition of the value share, ν_{irs}, from the discussion around equation (3.9), the calibrated-share form (see Appendix A for a definition of the calibrated-share form) of the Dixit-Stiglitz price index is

$$P_{ks} = P^0_{ks}\left[\sum_{r\in R}\nu_{krs}\frac{n_{kr}}{n^0_{kr}}\left(\frac{p_{krs}}{p^0_{krs}}\right)^{1-\sigma^K}\right]^{1/(1-\sigma^K)}. \tag{3.30}$$

where the superscript "0" denotes the value of the variable in the benchmark equilibrium. Without loss of generallity, we take $n^0_{kr} = 1$ and define:[24]

$$\lambda^K_{krs} \equiv \nu_{krs}(P^0_{ks}/p^0)^{1-\sigma^K}.$$

Substituting λ^K_{krs} into the calibrated-share form of the CES price aggregator, we obtain (3.22), which is equation K.8 in table 2.

3.5 Melitz Specific Equations

3.5.1 Firm Basics

Since we desire to calibrate our model to real data, we generalize the model of Melitz (2003). Among our extensions are that we allow for asymmetry among the regions of the world, for multiple sectors, intermediates based on real data, the possibility of alternate elasticities of substitution in the cost function of the firm, labor-leisure choice, multiple factors of production, both mobile and specific factors of production, and initial tariffs in addition to iceberg trade costs. As in Melitz (2003), we assume there is a continuum of firms each choosing to produce a unique variety. All Melitz firms in sector m of region r incur a sunk entry cost (equal to $c_{mr} f^E_{mr}$) of entering the market prior to knowing its productivity. All these firms also incur a fixed cost of selling in any market ($c_{mr} f^M_{mrs}$). Unlike in Melitz (2003), we do **not** assume that $r \neq s \Rightarrow f^M_{mrs} > f^M_{mrr}$. This latter condition assumes that if a firm chooses to export, it incurs a larger fixed cost of serving any market other than its home market. With homogeneous regions, as in Melitz (2003), this latter condition is required to produce the result that all active firms sell in the home market, but only a fraction of them export. We show in section 4, however, that with heterogeneous regions, it is possible that some firms export without selling in the home market. This could occur, for example, if the size of the home market is small relative to the export market or if the preferences of home market consumers for the product the firm produces is low compared with preferences of consumers in some foreign markets. In our calibration of the model, we allow the data to tell us the relationship between home market and foreign market fixed costs. If, for example, we find in the calibration that a larger share of firms export to a given market than sell in the home market we would calculate the cutoff productivities such that it takes a lower cutoff productivity to export to that market than sell in the home market. This is not a likely situation, but possible under our formulation.

After entering the market, the firm receives a productivity level φ_{mr} as a random draw from a probability density function (PDF) $g(\varphi_{mr})$. There is a one-to-one

[24] We have shown that the welfare results are independent of the initial value of the number of Krugman firms. Welfare only depends on the proportional changes in n_{kr} not its benchmark scale.

correspondence between the firm's productivity and its variety. We will thus use the productivity draw φ_{mr} to also index the unique variety produced by the firm drawing that productivity level. In the first three subsections, we allow $g(\varphi_{mr})$ to be a general PDF of a continuous random variable that takes positive values, subject to the convergence of the integral for the productivity of the representative firm. In section 3.5.4, in order to solve the model for the cutoff productivities and other parameters, we restrict $g(\varphi_{mr})$ to the Pareto probability distribution (untruncated above) with shape parameter a and lower bound b.

For a firm in sector m of region r producing a variety with productivity φ_{mr} and selling in region s, define:

$p_s(\varphi_{mr}) =$ the firm's gross price in region s inclusive of tariffs and iceberg costs;
$q_s(\varphi_{mr}) =$ the firm's quantity of sales in region s;
$\pi_s(\varphi_{mr}) =$ the firm's profits in region s; and
$r_S(\varphi_{mrs}) =$ the firm's revenue in region s.

Define V_{mrs} as the set of productivities of firms in sector m of region r that sell in region s. The sets of productivity values will have different minimum values depending on the export market s. Define $TC_{mr}(\varphi_{mr})$ as the total operating costs of the firm in sector m of region r. Total operating costs for active Melitz firms with productivity φ_{mr}, include the (constant) marginal costs of output and the fixed cost of serving any market in which they are active, but they exclude the sunk entry costs:

$$TC_{mr}(\varphi_{mr}) = c_{mr}\left(\sum_{s\in R} f^M_{mrs} + \frac{1}{\varphi_{mr}}\sum_{s\in R}\tau_{mrs}q_{mrs}(\varphi_{mr})\right), \tag{3.31}$$

where the excluded sunk entry costs are equal to $c_{mr}f^E_{mr}$.[25]

3.5.2 Firm-level Prices, Quantities, Profits, and Revenue

There is a continuum of demand functions. To obtain the demand for the variety of an individual Melitz firm in sector m of region r selling in region s, we write the Dixit-Stiglitz price index as P_{ms}:

$$P_{ms} = \left[\sum_{r\in R}\lambda^M_{mrs}\int_{\varphi_{mr}\in V_{mrs}} p_s(\varphi_{mr})^{1-\sigma^M} d\varphi_{mr}\right]^{1/(1-\sigma^M)}. \tag{3.32}$$

The Dixit-Stiglitz price index is dual to the following technology for aggregating firm quantities:

[25] As in the Krugman model, we assume that the inputs required for all fixed costs and marginal costs are identical, and the costs of these inputs may be represented by a function that is a linearly homogeneous, composite function of all inputs. In the heterogeneous-firms case, this assumption also applies to the sunk entry costs.

$$q0_{ms}Q_{ms} = \left[\sum_{r\in R}(\lambda^M_{mrs})^{1/\sigma^M}\int_{\varphi_{mr}\in V_{mrs}} q_s(\varphi_{mr})^{(\sigma^M-1)/\sigma^M} d\varphi_{mr}\right]^{(\sigma^M-1)/\sigma^M} . \tag{3.33}$$

We want to derive the demand for an individual variety. Denote an individual variety in sector m of region r as φ'_{mr}. Analogous to the Krugman case, we use the linearly homogeneous property of the CES and the fact that Dixit-Stiglitz price in (3.32) is the minimum cost of acquiring one unit of the Dixit-Stiglitz good to write the conditional cost function as $q0_{ms}Q_{ms}P_{ms}$. We then apply Shephard's Lemma to the conditional cost function to get that the demand function of an individual variety is $q0_{ms}Q_{ms}\frac{\partial P_{ms}}{\partial p_s(\varphi'_{mr})}$.

Define Z_{ms} as the term inside the brackets of equation (3.32) so that $P_{ms} = (Z_{ms})^{1/(1-\sigma^M)}$. Then

$$\begin{aligned}\frac{\partial P_{ms}}{\partial p_s(\varphi'_{mr})} &= \frac{\lambda^M_{mrs}(1-\sigma^M)}{1-\sigma^M} Z_{ms}^{\frac{1}{1-\sigma^M}-1} p_s(\varphi'_{mr})^{-\sigma^M} \\ &= \lambda^M_{mrs} P_{ms} Z^{-1}_{ms} p_s(\varphi'_{mr})^{-\sigma^K} \\ &= \lambda^M_{mrs}\left(\frac{P_{ms}}{p_s(\varphi'_{mr})}\right)^{\sigma^M},\end{aligned}$$

where we used the property that:

$$\frac{\partial\left(\int_{\varphi_{mr}\in V_{mrs}} p_s(\varphi_{mr})^{1-\sigma^M} d\varphi_{mr}\right)}{\partial p_s(\varphi'_{mr})} = (1-\sigma^M)p_s(\varphi'_{mr})^{-\sigma^M}. \tag{3.34}$$

In order to derive the demand for an individual variety with a continuum of varieties, authors implicitly or explicitly use the property in equation (3.34).[26]

Given (3.34) the demand for a variety is

$$q_s(\varphi'_{mr}) = \lambda^M_{mrs} q0_{ms} Q_{ms}\left(\frac{P_{ms}}{p_s(\varphi'_{mr})}\right)^{\sigma^M} .$$

Since choice of the variety φ'_{mr} was arbitrary, we may write the demand for any variety as

$$q_s(\varphi_{mr}) = \lambda^M_{mrs} q0_{ms} Q_{ms}\left(\frac{P_{ms}}{p_s(\varphi_{mr})}\right)^{\sigma^M} . \tag{3.35}$$

[26] While clearly intuitive, the mathematical derivation of (3.34) is elusive, at least from our perspective. We provide Appendix C as a review of what we found on the topic.

To simplify the notation define $\Gamma_{mrs} \equiv \lambda^M_{mrs} q0_{ms} Q_{ms} (P_{ms})^{\sigma^M}$. Then

$$q_s(\varphi_{mr}) = \Gamma_{mrs} p_s(\varphi_{mr})^{-\sigma^M}. \tag{3.36}$$

Profit for the individual firm in sector m of region r selling in region s is:

$$\pi_s(\varphi_{mr}) = \frac{p_s(\varphi_{mr}) q_s(\omega_{mr})}{(1 + t_{mrs})} - \frac{\tau_{mrs} c_{mr}}{\varphi_{mr}} q_s(\varphi_{mr}) - c_{mr} f^M_{mrs}. \tag{3.37}$$

Profit maximization analogous to the Krugman case above yields the optimal price for the individual firm:

$$p_s(\varphi_{mr}) = \frac{(1 + t_{mrs})\tau_{mrs} c_{mr}}{\varphi_{mr}(1 - 1/\sigma^M)}. \tag{3.38}$$

We make two observations about prices of firms in sector m of region r selling in region s. We see that more productive firms charge lower prices, i.e., $\partial p_s(\varphi_{mr})/\partial \varphi_{mr} < 0$ and, from the demand curve (3.35), sell larger quantities. Second, in the home market iceberg costs are unity and tariffs are zero. Thus, the firm's export prices to market s differ from its home market price by a term that is the product of its iceberg and tariff costs:

$$p_r(\varphi_{mr}) = \frac{c_{mr}}{\varphi_{mr}(1 - 1/\sigma^M)} \quad \text{and} \quad p_s(\varphi_{mr}) = (1 + t_{mrs})\tau_{mrs} p_r(\varphi_{mr}). \tag{3.39}$$

We now derive several expressions that will allow us to calculate the bilateral cutoff productivities for exporting to any market. From equation (3.38), the ratio of the prices of two firms from sector m in region r active in market s, is inversely related to the ratio of their productivities.

$$\frac{p_s(\varphi''_{mr})}{p_s(\varphi'_{mr})} = \frac{\frac{(1+t_{mrs})\tau_{mrs} c_{mr}}{\varphi''_{mr}(1-1/\sigma^M)}}{\frac{(1+t_{mrs})\tau_{mrs} c_{mr}}{\varphi'_{mr}(1-1/\sigma^M)}} = \frac{\varphi'_{mr}}{\varphi''_{mr}}. \tag{3.40}$$

Revenue of the firm in sector m from region r with productivity φ_{mr} on sales in region s is:

$$\begin{aligned} r_s(\varphi_{mr}) &= \frac{p_s(\varphi_{mr}) q_s(\varphi_{mr})}{(1 + t_{mrs})} \\ &= \frac{p_s(\varphi_{mr}) \left(\Gamma_{mrs} p_s(\varphi_{mr})^{-\sigma^M} \right)}{(1 + t_{mrs})} \\ &= \frac{\Gamma_{mrs} p_s(\varphi_{mr})^{1-\sigma^M}}{(1 + t_{mrs})}. \end{aligned} \tag{3.41}$$

We note that $\partial r_s(\varphi_{mr})/\partial\varphi_{mr}$ is positive, so revenue on exports to market s increases with the firm's productivity.

The ratio of the revenue of any two firms in sector m of region r on their sales in region s may be expressed as a function of the ratio of their productivities and the Dixit-Stiglitz elasticity. To simplify notation, define

$$\Lambda_{mrs} \equiv \frac{(1+t_{mrs})\tau_{mrs}c_{mr}}{(1-1/\sigma^M)} \quad \text{, so} \quad p_s(\varphi_{mr}) = \frac{\Lambda_{mrs}}{\varphi_{mr}}.$$

Then

$$\frac{r_s(\varphi''_{mr})}{r_s(\varphi'_{mr})} = \left(\frac{\frac{\Lambda_{mrs}}{\varphi''_{mr}}}{\frac{\Lambda_{mrs}}{\varphi'_{mr}}}\right)^{1-\sigma^M} = \left(\frac{\varphi''_{mr}}{\varphi'_{mr}}\right)^{\sigma^M-1}. \tag{3.42}$$

Now rearranging (3.38) as

$$\frac{p_s(\varphi_{mr})(1-1/\sigma^M)}{(1+t_{mrs})} = \frac{\tau_{mrs}c_{mr}}{\varphi_{mr}},$$

and substituting the right-hand side of this expression out of equation (3.37) and add the fixed costs of selling in region s to both sides to obtain the operating profit of the firm in market s. Operating profits in market s are:

$$\begin{aligned}\pi_s(\varphi_{mr}) + c_{mr}f^M_{mrs} &= \frac{p_s(\varphi_{mr})q_s(\varphi_{mr})}{(1+t_{mrs})} - \frac{(1-1/\sigma^M)}{(1+t_{mrs})}p_s(\varphi_{mr})q_s(\varphi_{mr}) \\ &= \frac{p_s(\varphi_{mr})q_s(\varphi_{mr})}{(1+t_{mrs})\sigma^M} \\ &= \frac{r_s(\varphi_{mr})}{\sigma^M}.\end{aligned}$$

The firm's profits on sales in market s are

$$\pi_s(\varphi_{mr}) = \frac{r_s(\varphi_{mr})}{\sigma^M} - c_{mr}f^M_{mrs}. \tag{3.43}$$

Define φ^*_{mrs} as the productivity cutoff for firms in sector m of region r selling in region s. The productivity draw of firms does not depend on the destination market. Due to regional heterogeneity, however, the zero-profit productivity cutoff depends on the parameters of the export market. Therefore, the productivity value that yields zero profit may be represented as:

$$\pi_s(\varphi^*_{mrs}) = 0 \quad \Leftrightarrow \quad \frac{r_s(\varphi^*_{mrs})}{\sigma^M} = c_{mr}f^M_{mrs}. \tag{3.44}$$

From (3.42) take ${\varphi'}_{mr} = \varphi^*_{mrs}$, and we have

$$\begin{aligned} r_s(\varphi_{mr}) &= r_s(\varphi^*_{mrs}) \left(\frac{\varphi_{mr}}{\varphi^*_{mrs}}\right)^{\sigma^M-1} \\ &= \sigma^M c_{mr} f^M_{mrs} \left(\frac{\varphi_{mr}}{\varphi^*_{mrs}}\right)^{\sigma^M-1}. \end{aligned} \tag{3.45}$$

Substitute for $r_s(\varphi_{mr})$ from (3.45) into (3.43) for profits. We have the profits of an operating firm as a function of their relative productivity

$$\pi_s(\varphi_{mr}) = \frac{\sigma^M c_{mr} f^M_{mrs} \left(\frac{\varphi_{mr}}{\varphi^*_{mrs}}\right)^{\sigma^M-1}}{\sigma^M} - c_{mr} f^M_{mrs} = c_{mr} f^M_{mrs} \left[\left(\frac{\varphi_{mr}}{\varphi^*_{mrs}}\right)^{\sigma^M-1} - 1\right].$$

This gives us the conditional profits in market s for firm with productivity φ_{mr}:

$$\pi_s(\varphi_{mr}) = \begin{cases} c_{mr} f^M_{mrs} \left[\left(\frac{\varphi_{mr}}{\varphi^*_{mrs}}\right)^{\sigma^M-1} - 1\right] > 0 & \text{iff} \quad \varphi_{mr} > \varphi^*_{mrs} \\ 0 & \text{otherwise.} \end{cases} \tag{3.46}$$

For firms with productivity values below the zero-profit cutoff, the firm will not sell in market s. That is, $q_s(\varphi_{mr}) = 0$ if $\varphi_{mr} < \varphi^*_{mrs}$.

3.5.3 Aggregate and Representative Firm Variables

In this subsection we derive the aggregate price and quantity indices, industry revenue and profits for each sector and region of the model in terms of the representative firm. After entering the market, the firm receives a productivity level $\varphi_{mr} > 0$ as a random draw from a continuous probability density function $g(\varphi_{mr})$. Even though there is a continuum of heterogeneous firms operating on any bilateral link in which firms are active, the key to solving the model is to describe the equilibrium in terms of a single firm on each active bilateral link.[27] That is, the distribution of firms on a bilateral link is not part of the market equilibrium conditions. The market equilibrium on a particular bilateral link is *represented* as a function of a single firm. The model is equivalent to a model in which all firms selling on that bilateral link have the same productivity as that single firm we call the representative firm.

For the definition of the aggregate variables and the bilateral representative firms, we may retain generality of the PDF defined as a continuous random variable that takes only positive values, subject to the constraint that the integral in equation (3.50) below converges. The other integrals below also converge if (3.50) converges. Define the continuous probability density function (PDF) of the posi-

[27] By bilateral "link" we refer to a bilateral export-import trading relationship.

tive productivities faced by firms in sector m of region r as $g(\varphi_{mr})$, and let $G(\varphi_{mr})$ be the cumulative distribution function for this PDF. Given that firms from sector m of region r are not active in a market s for $\varphi_{mr} < \varphi^*_{mrs}$, $G(\varphi^*_{mrs})$ is the probability that the firm is not active in market s. Then the probability that the firm is active in market s is:

$$1 - G(\varphi^*_{mrs}) = \int_{\varphi^*_{mrs}}^{\infty} g(\varphi_{mr}) d\varphi_{mr}. \tag{3.47}$$

Let y be any real number with $y \geq \varphi^*_{mrs}$. We seek $G(y|y \geq \varphi^*_{mrs})$, the conditional distribution function for the random variable φ_{mr} conditional on $\varphi_{mr} \geq \varphi^*_{mrs}$. For notation, let C be a subset of the set of possible outcomes of a random experiment, and $\mathsf{Pr}[C]$ be the probability of the event C (or the probability measure of the set C). Define the sets:

$$A = \{\varphi_{mr} | \varphi_{mr} \leq y\} \text{ and } B = \{\varphi_{mr} | \varphi_{mr} \geq \varphi^*_{mrs}\}.$$

We have $A \cap B = \{\varphi_{mr} | \varphi^*_{mrs} \leq \varphi_{mr} \leq y\}$. Note that for any $y \geq \varphi^*_{mrs}$, the conditional probability of A given B (denoted $\mathsf{Pr}[A|B]$), is equal to the value of the conditional cumulative distribution function. i.e.,

$$G(y|y \geq \varphi^*_{mrs}) = \mathsf{Pr}[A|B] = \frac{\mathsf{Pr}[A \cap B]}{\mathsf{Pr}[B]}.$$

We have

$$\begin{aligned} G(y|y \geq \varphi^*_{mrs}) &= \frac{\mathsf{Pr}[A \cap B]}{\mathsf{Pr}[B]} \\ &= \frac{\int_{\varphi^*_{mrs}}^{\infty} g(\varphi) d\varphi}{1 - G(\varphi^*_{mrs})} \\ &= \int_{\varphi^*_{mrs}}^{\infty} \frac{g(\varphi_{mr})}{1 - G(\varphi^*_{mrs})} d\varphi. \end{aligned} \tag{3.48}$$

Then the conditional PDF of productivities, conditional on the firm of sector m in region r being active in market s, is (at each point of continuity of $g(\varphi_{mr})$):

$$g(\varphi_{mr} | \varphi_{mr} \geq \varphi^*_{mrs}) = \begin{cases} \frac{g(\varphi_{mr}}{1 - G(\varphi^*_{mrs})} & \text{if} \varphi_{mr} \geq \varphi^*_{mrs} \\ 0 & \text{otherwise.} \end{cases} \tag{3.49}$$

Representative Firm Productivity: Define the productivity of our representative firm from sector m in region r that is active in export market s as $\tilde{\varphi}_{mrs}$ where

$$\tilde{\varphi}_{mrs} = \left[\int_{\varphi^*_{mrs}}^{\infty} \varphi_{mr}^{\sigma^M - 1} \frac{g(\varphi_{mr})}{1 - G(\varphi^*_{mrs})} d\varphi_{mr} \right]^{1/(\sigma^M - 1)}. \tag{3.50}$$

The productivity of our representative firm is indexed by the destination market since the zero-profit productivity value cutoff, φ^*_{mrs}, depends on the destination market. We may rewrite (3.50) as:

$$\tilde{\varphi}_{mrs}^{\sigma^M - 1} = \int_{\varphi^*_{mrs}}^{\infty} \varphi_{mr}^{\sigma^M - 1} \frac{g(\varphi_{mr})}{1 - G(\varphi^*_{mrs})} d\varphi_{mr}. \tag{3.51}$$

We make repeated use of (3.51). In this form, it is evident that $\tilde{\varphi}_{mrs}^{\sigma^M - 1}$ is equal to $\mathsf{E}[\varphi_{mrs}^{\sigma^M - 1} | \varphi_{mr} \geq \varphi^*_{mrs}]$ the mathematical expectation of $\varphi_{mrs}^{\sigma^M - 1}$ conditional on $\varphi_{mr} \geq \varphi^*_{mrs}$.[28] Substitute the particular productivity value for $\tilde{\varphi}_{mrs}$ for φ_{mr} into (3.38), the optimal price equation of the firm gives us

$$p_s(\tilde{\varphi}_{mrs}) = \tilde{p}_{mrs} = \frac{(1 + t_{mrs})\tau_{mrs} c_{mr}}{\tilde{\varphi}_{mrs}(1 - 1/\sigma^M)} = \frac{\Lambda_{mrs}}{\tilde{\varphi}_{mrs}}. \tag{3.52}$$

which is equation M.11 of table 2, where we use the notation $\tilde{p}_{mrs} \equiv p_s(\tilde{\varphi}_{mrs})$.[29] Similarly, for quantity, revenue and profits, define $\tilde{q}_{mrs} \equiv q_s(\tilde{\varphi}_{mrs})$; $\tilde{r}_{mrs} \equiv r_s(\tilde{\varphi}_{mrs})$; and $\tilde{\pi}_{mrs} \equiv \pi_s(\tilde{\varphi}_{mrs})$.

The Dixit-Stiglitz Price Index in Terms of the Representative Firms Prices: Using our conditional PDF, rewrite the Dixit-Stiglitz price index of (3.32) as follows:

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[28] Melitz (2003, footnote 9) interprets this productivity as a weighted harmonic mean, weighted by relative productivities. We do not judge the economic interpretation of this productivity value as important. What is important about it is that it allows us to solve the model in the sense that we may derive industry variables in terms of it. Dixon, Jerie, and Rimmer (2018, section2.2), however, provides an economic interpretation of this productivity.

[29] As Dixon, Jerie, and Rimmer (2018, p.180-181) noted first, Akgul, Villoria, and Hertel (2016, equation 12) make assumptions that reduce the generality of the heterogeneous-firms model. That is, they assume (in our notation) that

$$\tilde{p}_{mrs} = \tilde{p}_{mr} = \frac{c_{mr}}{\tilde{\varphi}_{mr}(1 - 1/\sigma^M)} \quad \forall s \in R,$$

where they define $\tilde{\varphi}_{mr}$ as the average productivity of firms in sector m of region r. That is, the price of the representative firm in sector m of region r does not differ by destination market based on the representative firm productivity. This is unlike in Melitz and in our equation (3.52), where the representative firm and its price differs by destination market.

$$P_{ms} = \left[\sum_{r \in R} \lambda^M_{mrs} \int^{\infty}_{\varphi^*_{mrs}} N_{mrs} p_s(\varphi_{mr})^{1-\sigma^M} \frac{g(\varphi_{mr})}{1 - G(\varphi^*_{mrs})} d\varphi_{mr} \right]^{1/(1-\sigma^M)}, \tag{3.53}$$

where N_{mrs} is the mass of firms in sector m of region r that sell in region s. We use (3.38), (3.51), and the definition above that $p_s(\varphi_{mr}) = \Lambda_{mrs}/\varphi_{mr}$ to simplify the value of the integral in equation (3.53). The simplification is as follows:

$$\begin{aligned}
\int^{\infty}_{\varphi^*_{mrs}} N_{mrs} p_s(\varphi_{mr})^{1-\sigma^M} \frac{g(\varphi_{mr})}{1 - G(\varphi^*_{mrs})} d\varphi_{mr} &= N_{mrs} \Lambda^{1-\sigma^M}_{mrs} \int^{\infty}_{\varphi^*_{mrs}} \varphi^{\sigma^M - 1}_{mr} \frac{g(\varphi_{mr})}{1 - G(\varphi^*_{mrs})} d\varphi_{mr} \\
&= N_{mrs} \Lambda^{1-\sigma^M}_{mrs} \tilde{\varphi}^{\sigma^M - 1}_{mrs} \\
&= N_{mrs} p(\tilde{\varphi}_{mrs})^{1-\sigma^M} \\
&= N_{mrs} \tilde{p}^{1-\sigma^M}_{mrs}.
\end{aligned}$$

Substituting $N_{mrs} \tilde{p}^{1-\sigma^M}_{mrs}$ for the integral in (3.53) to rewrite the Dixit-Stiglitz price index as

$$P_{ms} = \left[\sum_{r \in R} \lambda^M_{mrs} N_{mrs} \tilde{p}^{1-\sigma^M}_{mrs} \right]^{1/(1-\sigma^M)}, \tag{3.54}$$

which is equation M.8 of table 2. Equation (3.54) represents the industry aggregate price index in terms of the representative firm alone. We do not need to employ the distribution of productivities to characterize the industry price index. Since $\partial P_{ms}/\partial N_{mrs} < 0$, the Dixit-Stiglitz price index for good m in region s declines in the mass of varieties for any region r. This is analogous to the Dixit-Stiglitz externality indicated for the Krugman model of homogeneous varieties (see sections 3.4.4 and 3.4.5 for a more detailed discussion).

The Dixit-Stiglitz Quantity Index in Terms of Representative Firm Quantities: In (3.35), choose $\varphi_{mr} = \tilde{\varphi}_{mrs}$; then the demand for the representative variety is:

$$\tilde{q}_{mrs} = q_s(\tilde{\varphi}_{mrs}) = \lambda^M_{mrs} q0_{ms} Q_{ms} \left(\frac{P_{ms}}{\tilde{p}_{mrs}} \right)^{\sigma^M} = \Gamma_{mrs} \tilde{p}^{-\sigma^M}_{mrs}. \tag{3.55}$$

which is equation M.10 in table 2.

Analogous to the Dixit-Stiglitz price index, we may use our conditional PDF to rewrite the Dixit-Stiglitz technology of equation (3.33) as:

$$q0_{ms}Q_{ms} = \left[\sum_{r\in R}(\lambda^M_{mrs})^{1/\sigma^M}\int_{\varphi^*_{mrs}}^{\infty} N_{mrs}q_{mrs}^{(\sigma^M-1)/\sigma^M}\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr}\right]^{(\sigma^M-1)/\sigma^M}. \tag{3.56}$$

From equation (3.36) we have $q_s(\varphi_{mr}) = \Gamma_{mrs}q_s(\varphi_{mr})^{-\sigma^M}$, and from equation (3.40) we have $p_s(\varphi''_{mr}) = p_s(\varphi'_{mr})\varphi'_{mr}/\varphi''_{mr}$. Choose $\tilde{\varphi}_{mrs} = \varphi'_{mr}$ and $\varphi_{mr} = \varphi''_{mr}$ in equation (3.40), then we have

$$p_s(\varphi_{mr}) = p_s(\tilde{\varphi}_{mrs})\frac{\tilde{\varphi}_{mrs}}{\varphi_{mr}}. \tag{3.57}$$

Substituting from equation (3.57) into (3.36) to obtain:

$$q_s(\varphi_{mr}) = q_s(\tilde{\varphi}_{mrs})\left(\frac{\tilde{\varphi}_{mrs}}{\varphi_{mr}}\right)^{-\sigma^M}. \tag{3.58}$$

Substituting from equation (3.58) into the integral of equation (3.56) we have

$$\begin{aligned}
&\int_{\varphi^*_{mrs}}^{\infty} N_{mrs}q_s(\varphi_{mr})^{(\sigma^M-1)/\sigma^M}\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr} \\
&= N_{mrs}\int_{\varphi^*_{mrs}}^{\infty}\left[q_s(\tilde{\varphi}_{mrs})^{(\sigma^M-1)/\sigma^M}\left(\frac{\tilde{\varphi}_{mrs}}{\varphi_{mr}}\right)^{1-\sigma^M}\right]\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr} \\
&= N_{mrs}q_s(\tilde{\varphi}_{mrs})^{(\sigma^M-1)/\sigma^M}\tilde{\varphi}_{mrs}^{1-\sigma^M}\int_{\varphi^*_{mrs}}^{\infty}\varphi_{mr}^{\sigma^m-1}\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr} \\
&= N_{mrs}q_s(\tilde{\varphi}_{mrs})^{(\sigma^M-1)/\sigma^M}.
\end{aligned} \tag{3.59}$$

Substituting from (3.59) for the integral in (3.56), the Dixit-Stiglitz aggregate quantity may be written as a function of representative firm quantities:

$$q0_{ms}Q_{ms} = \left[\sum_{s\in R}(\lambda^M_{mrs})^{1/\sigma^M}N_{mrs}q_s(\tilde{\varphi}_{mrs})^{\frac{\sigma^M-1}{\sigma^M}}\right]^{\frac{\sigma^M}{\sigma^M-1}}. \tag{3.60}$$

Aggregate Industry Revenue in Terms of Representative Firm Revenues: Using (3.41), the the expected revenue for an individual firm in sector m of region r on sales in region s, conditional on the firm being active in region s, is:

$$\begin{aligned}
\mathsf{E}[r_s(\varphi_{mr})] &= \mathsf{E}\left[\frac{p_s(\varphi_{mr})q_s(\varphi_{mr})}{1+t_{mrs}}\right] \\
&= \frac{\Gamma_{mrs}}{1+t_{mrs}}\int_{\varphi^*_{mrs}}^{\infty} p_s(\varphi_{mr})^{1-\sigma^M}\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr} \\
&= \frac{\Gamma_{mrs}\Lambda_{mrs}^{1-\sigma^M}}{1+t_{mrs}}\int_{\varphi^*_{mrs}}^{\infty} \varphi_{mr}^{\sigma^M-1}\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr} \\
&= \frac{\Gamma_{mrs}\Lambda_{mrs}^{1-\sigma^M}}{1+t_{mrs}}\tilde{\varphi}_{mrs}^{\sigma^M-1} \\
&= \frac{\Gamma_{mrs}}{1+t_{mrs}}p_s(\tilde{\varphi}_{mrs})^{1-\sigma^M} \\
&= r_s(\tilde{\varphi}_{mrs}) \\
&= \tilde{r}_{mrs}.
\end{aligned} \tag{3.61}$$

For aggregate revenue, R_{mrs}, for all firms in sector m of region r on sales in region s, we must aggregate over the mass of firms. This is:

$$R_{mrs} = \int_{\varphi^*_{mrs}}^{\infty} N_{mrs}r_s(\varphi_{mr})\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr} = N_{mrs}r_s(\tilde{\varphi}_{mrs}). \tag{3.62}$$

For aggregate revenue for all firms in sector m of region r on total sales to all regions, R_{mr}, we must aggregate over all regions. This is:

$$R_{mr} = \sum_{s\in R} N_{mrs}r_s(\tilde{\varphi}_{mrs}). \tag{3.63}$$

Aggregate Sector Profits in Terms of Representative Firm Revenues: For aggregate profits, we use (3.43). The conditional expected profits for an individual firm in sector m of region r on sales in region s are:

$$\begin{aligned}
\mathsf{E}[\pi_s(\varphi_{mr})] &= \int_{\varphi^*_{mrs}}^{\infty}\left[\frac{r_s(\varphi_{mr})}{\sigma^M} - c_{mr}f^M_{mrs}\right]\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr} \\
&= \frac{r_s(\tilde{\varphi}_{mrs})}{\sigma^M} - c_{mr}f^M_{mrs} \\
&= \pi_s(\tilde{\varphi}_{mrs}) \\
&= \tilde{\pi}_{mrs}.
\end{aligned}$$

For aggregate profits for all firms, Π_{mrs}, in sector m of region r on sales in region s, we must aggregate over the mass of firms:

$$\Pi_{mrs} = \int_{\varphi^*_{mrs}}^{\infty} N_{mrs}\pi_s(\varphi_{mr})\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})}d\varphi_{mr} = N_{mrs}\pi_s(\tilde{\varphi}_{mrs}). \tag{3.64}$$

For aggregate profits for all firms in sector m of region r on total sales to all regions, Π_{mr}, we must aggregate over all regions. This is:

$$\Pi_{mr} = \sum_{s \in R} N_{mrs} \pi_s(\tilde{\varphi}_{mrs}). \tag{3.65}$$

Summarizing these four aggregate relationships, we have:

$$(3.54) \quad P_{ms} = \left[\sum_{r \in R} \lambda^M_{mrs} N_{mrs} \tilde{p}^{1-\sigma^M}_{mrs} \right]^{1/(1-\sigma^M)} \quad ;$$

$$(3.60) \quad q0_{ms} Q_{ms} = \left[\sum_{s \in R} (\lambda^M_{mrs})^{1/\sigma^M} N_{mrs} q_s(\tilde{\varphi}_{mrs})^{\frac{\sigma^M - 1}{\sigma^M}} \right]^{\frac{\sigma^M}{\sigma^M - 1}} \quad ;$$

$$(3.63) \quad R_{mr} = \sum_{s \in R} N_{mrs} r_s(\tilde{\varphi}_{mrs}) \quad ;$$

$$(3.65) \quad \Pi_{mr} = \sum_{s \in R} N_{mrs} \pi_s(\tilde{\varphi}_{mrs}).$$

These four equations allow us to represent the industry aggregates in terms of the representative firms alone. We do not need to employ the distribution of productivities to characterize the industry aggregates.

3.5.4 The Pareto Distribution

The free-entry equilibrium condition, the zero-profit productivity cutoffs and the representative firm productivity level depend on the probability density function (PDF). We assume that productivity is a random variable with a Pareto PDF that is untruncated above. A Pareto distributed random variable X has strictly positive constants a and b with the PDF

$$g(x) = \begin{cases} \frac{ab^a}{x^{a+1}} & x \geq b, \\ 0 & \text{otherwise.} \end{cases}$$

The cumulative distribution function (CDF) for the Pareto distribution is

$$G(x) = \begin{cases} 1 - \left(\frac{b}{x}\right)^a & x \geq b, \\ 0 & \text{otherwise.} \end{cases}$$

Taking $x = \varphi^*_{mrs}$ in the CDF for a Melitz firm m in region r and using equation (3.49), the conditional Pareto PDF (conditional on the productivity exceeding φ^*_{mrs}) is

$$\frac{g(\varphi_{mr})}{1-G(\varphi^*_{mrs})} = \begin{cases} \frac{ab^a}{\varphi_{mr}^{a+1}}\left(\frac{\varphi^*_{mrs}}{b}\right)^a = \frac{a(\varphi^*_{mrs})^a}{\varphi_{mr}^{a+1}} & \varphi_{mr} \geq \varphi^*_{mrs}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.66}$$

3.5.5 Selection or Zero-profit cutoff

We first solve for zero-profit productivity cutoff for selection into each market in terms of the representative firm productivity. Take $\tilde{\varphi}_{mrs} = \varphi_{mr}$ in (3.45) to get

$$r_s(\tilde{\varphi}_{mrs}) = \sigma^M c_{mr} f^M_{mrs}\left(\frac{\tilde{\varphi}_{mrs}}{\varphi^*_{mrs}}\right)^{\sigma^M-1}. \tag{3.67}$$

Given the conditional Pareto PDF, we may derive the ratio of the productivity of the representative firm to the zero-profit productivity cutoff. The productivity of our representative firm is:

$$\begin{aligned}\tilde{\varphi}_{mrs} &= \left[\int_{\varphi^*_{mrs}}^{\infty}\varphi_{mr}^{\sigma^m-1}\frac{a(\varphi^*_{mrs})^a}{\varphi_{mr}^{a+1}}d\varphi_{mr}\right]^{\frac{1}{\sigma^M-1}} \\ &= \left[a(\varphi^*_{mrs})^a\int_{\varphi^*_{mrs}}^{\infty}\varphi_{mr}^{(\sigma^M-a-2)}d\varphi_{mr}\right]^{\frac{1}{\sigma^M-1}} \\ &= \varphi^*_{mrs}\left[\frac{a+1-\sigma^M}{a}\right]^{\frac{1}{1-\sigma^M}},\end{aligned} \tag{3.68}$$

where we assume that $\sigma^M - a - 1 < 0$ or equivalently $a > \sigma_M - 1$ for the integral to converge.[30] Then, we have

$$\left(\frac{\tilde{\varphi}_{mrs}}{\varphi^*_{mrs}}\right)^{\sigma^M-1} = \left[\frac{a+1-\sigma^M}{a}\right]^{-1}. \tag{3.69}$$

Substituting (3.69) into (3.67) to yield:

[30] Taking the integral $\int_{\varphi^*_{mrs}}^{\infty}\varphi_{mr}^{(\sigma^M-a-2)}d\varphi_{mr} = \frac{\varphi_{mr}^{(\sigma^M-a-1)}}{\sigma^M-a-1}\Big|_{\varphi^*_{mrs}}^{\infty} = \left[\lim_{\varphi_{mr}\to\infty}\left(\frac{\varphi_{mr}^{(\sigma^M-a-1)}}{\sigma^M-a-1}\right)\right] - \left(\frac{(\varphi^*_{mrs})^{\sigma^M-a-1}}{\sigma^M-a-1}\right) = \frac{(\varphi^*_{mrs})^{\sigma^M-a-1}}{a+1-\sigma^M}$, where we assume that $\sigma^M - a - 1 < 0$ for the limit term to converge, in which case it equals zero.

$$r_s(\tilde{\varphi}_{mrs}) = \sigma^M c_{mr} f_{mrs}^M \left[\frac{a+1-\sigma^M}{a}\right]^{-1} \quad \text{or}$$
$$c_{mr} f_{mrs}^M = r_s(\tilde{\varphi}_{mrs}) \left[\frac{a+1-\sigma^M}{a\sigma^M}\right]. \tag{3.70}$$

From (3.55), we have that $\tilde{q}_{mrs} = \Gamma_{mrs}\tilde{p}_{mrs}^{-\sigma^M}$, then

$$\frac{\tilde{p}_{mrs}\tilde{q}_{mrs}}{1+t_{mrs}} = \frac{\Gamma_{mrs}\tilde{p}_{mrs}^{1-\sigma^M}}{1+t_{mrs}} = r_s(\tilde{\varphi}_{mrs}) = \tilde{r}_{mrs}, \tag{3.71}$$

where we use (3.61). So we may replace representative revenue in (3.70) with product of representative price and quantity over tariff revenue, yielding (3.72):

$$c_{mr} f_{mrs}^M = \frac{\tilde{p}_{mrs}\tilde{q}_{mrs}}{1+t_{mrs}} \left[\frac{a+1-\sigma^M}{a\sigma^M}\right]. \tag{3.72}$$

Equation (3.72) is the zero-profit productivity cutoff condition in terms of the representative firm's profits, which is equation M.12 in table 2.

3.5.6 Free Entry

We use the various aggregates derived above for the firm's expected profits, revenue, price and quantity. The conditional expected profits for an individual firm in sector m of region r on sales in region s are:

$$\begin{aligned} \mathsf{E}[\pi_s(\varphi_{mr})] &= \frac{\mathsf{E}[r_s(\varphi_{mr})]}{\sigma^M} - c_{mr} f_{mrs}^M \\ &= \frac{\tilde{p}_{rms}\tilde{q}_{rms}}{\sigma^M(1+t_{rms})} - c_{mr} f_{mrs}^M \end{aligned} \tag{3.73}$$

conditional on $\varphi_{mr} > \varphi_{rms}^*$. Following Melitz (2003), we assume that these profits are steady-state profits. But there is an annual probability of death of the firm δ where $\delta \in (0,1)$, so that $1-\delta$ is the probability of survival. The conditional expected profits into the indefinite future of a potential entrant into sector m of region r on sales to region s are:

$$\begin{aligned} \sum_{t=0}^{\infty}(1-\delta)^t \mathsf{E}[\pi_s(\varphi_{mr})] &= \frac{\mathsf{E}[\pi_s(\varphi_{mr})]}{\delta} \\ &= \begin{cases} \frac{1}{\delta}\left[\frac{\tilde{p}_{rms}\tilde{q}_{rms}}{\sigma^M(1+t_{rms})} - c_{mr} f_{mrs}^M\right] & \varphi_{mr} \geq \varphi_{mrs}^*, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.74}$$

A potential entrant will realize these profits with probability $[1 - G(\varphi^*_{mrs})]$. The free-entry equilibrium condition is that the conditional expected profits into the indefinite future of a potential entrant summed over all markets just equals the sunk costs of entering the market:

$$\frac{1}{\delta} \sum_{s \in R} \left([1 - G(\varphi^*_{mrs})] \left[\frac{\tilde{p}_{rms} \tilde{q}_{rms}}{\sigma^M (1 + t_{rms})} - c_{mr} f^M_{mrs} \right] \right) = c_{mr} f^E_{mr}. \tag{3.75}$$

Since in the steady state equilibrium $[1 - G(\varphi^*_{mrs})]$ is equal to the share of entering firms in sector m of region r that are active in market s, we have:

$$[1 - G(\varphi^*_{mrs})] = \frac{N_{mrs}}{n_{mr}}.$$

Then we rewrite (3.75) as

$$\frac{1}{\delta} \sum_{s \in R} \left(\frac{N_{mrs}}{n_{mr}} \left[\frac{\tilde{p}_{rms} \tilde{q}_{rms}}{\sigma^M (1 + t_{rms})} - c_{mr} f^M_{mrs} \right] \right) = c_{mr} f^E_{mr}. \tag{3.76}$$

From (3.72) we can substitute $c_{mr} f^M_{mrs}$ out of (3.76) and simplify. This gives us equation M.13 in table 2, the free entry condition of the model equations:

$$\delta f^E_{mr} c_{mr} = \sum_{s \in R} \left(\frac{N_{mrs}}{n_{mr}} \right) \frac{\tilde{p}_{mrs} \tilde{q}_{mrs} (\sigma^M - 1)}{(1 + t_{mrs}) a \sigma^M}. \tag{3.77}$$

The free entry equilibrium condition is that the sunk costs of entering the market just equals the conditional expected steady-state profits into the indefinite future of a potential entrant summed over all active markets.

3.5.7 Representative Firm Productivity Level

The Pareto (CDF) is, again,

$$G(x) = \begin{cases} 1 - \left(\frac{b}{x}\right)^a & x \geq b, \\ 0 & \text{otherwise.} \end{cases}$$

The probability that the firm's productivity is greater than or equal to φ^*_{mrs} is $(b/\varphi^*_{mrs})^a$, which in equilibrium equals the share of entering firms that are active in market s. Thus, we may solve for φ^*_{mrs} from

$$\left(\frac{b}{\varphi^*_{mrs}} \right)^a = \frac{N_{mrs}}{n_{mr}} \quad \Leftrightarrow \quad \varphi^*_{mrs} = b \left(\frac{N_{mrs}}{n_{mr}} \right)^{-1/a}. \tag{3.78}$$

Using (3.69), and substituting from (3.78) for φ^*_{mrs} we have

$$\tilde{\varphi}_{mrs} = b \left(\frac{N_{mrs}}{n_{mr}} \right)^{-1/a} \left(\frac{a + 1 - \sigma^M}{a} \right)^{1/(1-\sigma^M)}. \tag{3.79}$$

This is equation M.14 from table 2 of the model equations.

3.5.8 Preference Weights in the Dixit-Stiglitz Price Equation of the Melitz Model

We may relate the preference weights in (3.54), the Dixit-Stiglitz price index of the model, to value shares of absorption in the benchmark data ν_{mrs}.[31] In the Melitz case, the calibrated-share form (see Appendix A for a definition of the calibrated-share form) of the Dixit-Stiglitz price index is

$$P_{ms} = P^0_{ms} \left[\sum_{r \in R} \nu_{mrs} \frac{N_{mrs}}{N^0_{mrs}} \left(\frac{\tilde{p}_{mrs}}{\tilde{p}^0_{mrs}} \right)^{1-\sigma^M} \right]^{1/(1-\sigma^M)}. \tag{3.80}$$

where the superscript "0" denotes the value of the variable in the benchmark equilibrium. Without loss of generallity, we take $N^0_{mrs} = 1$ and define:

$$\lambda^M_{mrs} \equiv \nu_{mrs} (P^0_{ms} / \tilde{p}^0_{mrs})^{1-\sigma^M}.$$

Substituting λ^M_{mrs} into the calibrated-share form of the CES price aggregator, we obtain (3.54), which is equation M.8 in table 2.

3.6 Market Clearance Conditions

3.6.1 Supply and Demand Balance for Domestic Use

The total quantity supplied of good or service i in region r is $q0_{ir} Q_{ir}$. In equilibrium, this must equal the sum of consumer and intermediate demand. Since we have linearly homogeneous preferences, the compensated demand function for final goods is obtained by applying Shepard's Lemma to the unit expenditure function and multiplying by total final demand in the economy. This gives us the first term on the right-hand side of (3.81). For demand for good i as an input in sector j of region r, we apply Shepard's Lemma to the unit cost function for inputs in sector j and multiply by total demand for intermediates in sector j. Sum over all sectors j to arrive at the second term on the right-hand side of (3.81). The market clearance conditions of supply and demand are given by

[31] The value share, ν_{irs}, is defined for the Armington case and applied in equation (3.9). The definition is the same in the Melitz case. We simply replace the sector index i with $m \in M$ for a Melitz sector.

$$q0_{ir}Q_{ir} = d0_r D_r \frac{\partial e_r}{\partial P_{ir}} + \sum_{j \in I} y0_{jr} Y_{jr} \frac{\partial c_{jr}}{\partial P_{ir}}. \tag{3.81}$$

Equation (3.81) is equation AKM.3 of table 2. In (3.81), the price depends on the market structure of the sector. The price index P_{ir} is defined by A.8, K.8, or M.8 of table 2, depending on whether the market structure is Armington, Krugman, or Melitz.

3.6.2 Market Clearance for Composite Input i of Region r under Armington

Under Armington, the unit cost of good i in region s is given by (3.7):

$$P_{is} = \left[\sum_{r \in R} \lambda^A_{irs} [(1+t_{irs})\tau_{irs} c_{ir}]^{1-\sigma^A} \right]^{1/(1-\sigma^A)} \quad \forall s \in R \text{ and } i \in I \setminus (K \cup M),$$

and the associated price of good i from region r sold in region s is $(1+t_{irs})\tau_{irs}c_{ir}$. Applying Shephard's Lemma to the unit cost function and scaling by total demand for good i in region s, gives us the demand for good i from region r in region s:

$$q0_{is}Q_{is} \frac{\partial P_{is}}{\partial [(1+t_{irs})\tau_{irs} c_{ir}]}.$$

Since under Armington the use of the composite input in sector i is equal to output of the sector (see section 3.2) this is also the derived demand in region s for the composite input i from region r. Summing over all regions s for good i from region r gives us total demand for composite input i from region r:

$$\sum_{s \in R} q0_{is}Q_{is} \frac{\partial P_{is}}{\partial [(1+t_{irs})\tau_{irs} c_{ir}]}.$$

Firms in region r, must ship τ_{irs} units of good i to region s for one unit of good i to arrive in region s. Incorporating the additional composite input required to cover the iceberg costs gives us our market clearance condition for composite inputs used in Armington goods:

$$y0_{ir}Y_{ir} = \sum_{s \in R} \tau_{irs} q0_{is}Q_{is} \frac{\partial P_{is}}{\partial [(1+t_{irs})\tau_{irs} c_{ir}]} \quad \forall r \in R \text{ and } i \in I \setminus (K \cup M). \tag{3.82}$$

This is equation A.9 of table 2. It is important to point out that under Armington the *composite input*, which trades at a market price of c_{ir} $\forall i \in I \setminus (K \cup M)$, derives its demand from the Armington aggregation activities across the destination s markets. This is the right-hand side of (3.82). For Krugman and Melitz goods, however, the composite input, which trades at a price c_{ir} $\forall i \in (K \cup M)$, is used as the single *input* to Krugman or Melitz firms. Under monopolistic competition demand for the composite input comes from summing across the individual firms' input use as indicated in the subsections 3.6.3 and 3.6.4 that follow immediately.[32]

3.6.3 Market Clearance for Composite Input k in Region r in all uses

For the monopolistic competition models, we must account for the use of the composite input for both variable and fixed costs. Given the cost function in (3.10), use of the composite input by Krugman firms for its variable costs for its sales to all markets is equal to $\sum_{s\in R}\tau_{krs}q_{krs}$. Krugman firms also incur a fixed cost of operating which is also defined in units of the composite input. Then $f^K_{kr}+\sum_{s\in R}\tau_{krs}q_{krs}$ is the total use of the composite input by a single firm in sector k of region r. Supply and demand balance for the total use of the composite input in a Krugman sector in region r is:

$$y0_{kr}Y_{kr}=n_{kr}\left(f^K_{kr}+\sum_{s\in R}\tau_{krs}q_{krs}\right)\quad\forall r\in R\ \text{ and }\ k\in K. \tag{3.83}$$

This is equation K.9 of table 2.

3.6.4 Market Clearance for Composite Input m in Region r in all uses

We first want to find the amount of the composite input required for variable costs to produce the output of all firms in sector m of region r on their sales in region s. We use the notation developed above:

$$q_s(\varphi_{mr})=\Gamma_{mrs}p_s(\varphi_{mr})^{-\sigma^M}=\Gamma_{mrs}\Lambda^{-\sigma^M}\varphi_{mr}^{\sigma^M},$$

where $\Gamma_{mrs}\equiv\lambda^M_{mrs}q0_{ms}Q_{ms}(P_{ms})^{\sigma^M}$ and $\Lambda_{mrs}\equiv\frac{(1+t_{mrs})\tau_{mrs}c_{mr}}{(1-1/\sigma^M)}$. The amount of the composite input required for variable costs by a firm in sector m of region r with productivity φ_{mr} for their sales in region s is $\frac{\tau_{mrs}q_s(\varphi_{mr})}{\varphi_{mr}}$. Aggregating over all firms in sector m of region r on their sales in region s, we get

[32] Shephard's Lemma does apply to the Dixit-Stigliz price indexes under monopolistic competition to derive firm-level demand as presented in sections 3.4 and 3.5, but this is not demand for the composite input directly. Demand for the composite input comes from summing across the individual firms' input use.

$$
\begin{aligned}
N_{mrs}\mathsf{E}\left[\frac{\tau_{mrs}q_s(\varphi_{mr})}{\varphi_{mr}}\right] &= N_{mrs}\tau_{mrs}\Gamma_{mrs}\Lambda_{mrs}^{-\sigma^M}\int_{\varphi^*_{mrs}}^{\infty}\frac{\varphi_{mr}^{\sigma^M}}{\varphi_{mr}}\frac{g(\varphi_{mr}}{1-G(\varphi^*_{mrs})}d\varphi_{mr} \\
&= N_{mrs}\tau_{mrs}\Gamma_{mrs}\Lambda_{mrs}^{-\sigma^M}\tilde{\varphi}_{mrs}^{\sigma^M-1} \\
&= N_{mrs}\tau_{mrs}\Gamma_{mrs}p_s(\tilde{\varphi}_{mrs})^{-\sigma^M}\frac{1}{\tilde{\varphi}_{mrs}} \\
&= N_{mrs}\frac{\tau_{mrs}q_s(\tilde{\varphi}_{mrs})}{\tilde{\varphi}_{mrs}}. \qquad (3.84)
\end{aligned}
$$

In addition, we must account for the use of the composite input to cover the fixed costs of operation and sunk entry costs. Summing over all markets, gives us the Melitz market clearance condition for the composite input in sector m:

$$y0_{mr}Y_{mr} = \delta f^E_{mr}n_{mr} + \sum_{s\in R}N_{mrs}\left(f^M_{mrs} + \frac{\tau_{mrs}\tilde{q}_{mrs}}{\tilde{\varphi}_{mrs}}\right) \quad \forall r\in R \text{ and } m\in M. \qquad (3.85)$$

This is equation M.9 of table 2.

3.6.5 Market Clearance for primary factors

The demand for a sector-specific factor of production f used in sector i of region r is obtained by applying Shephard's Lemma to the unit cost function of sector i of region r. The unit cost function is defined by either (3.3) or (3.5). Given linearly homogeneous technologies, we scale the demand by total use of the composite input to obtain the total demand for the specific factor.

$$\overline{SF}_{fir} = y0_{ir}Y_{ir}\frac{\partial c_{ir}}{\partial\tilde{w}_{fir}}. \qquad (3.86)$$

This is equation AKM.4 of table 2.

For primary mobile factors of production, we account for demand across all sectors $i\in I$ and $r\in R$:

$$\overline{F}_{fr} = \sum_{i\in I}y0_{ir}Y_{ir}\frac{\partial c_{ir}}{\partial w_{fr}} + d0_rD_r\frac{\partial e_r}{\partial w_{fr}}. \qquad (3.87)$$

Leisure demand is given by the final term on the right-hand side, which is non-zero only if the factor is labor and we have chosen a non-zero elasticity of labor supply. This yields equation AKM.5 of table 2.

3.7 Income Balance and the Numeraire

The model is based on relative prices (the model is homogeneous of degree zero in prices). We define the numeraire as the price of a unit of utility in the

United States, i.e.,

$$e_{\text{USA}} \equiv 1.$$

All prices are relative to this numeraire. Nominal income and any added nominal reports are understood to be measure in units of the numeraire.

For income balance we must have nominal income equal to expenditures of the representative agent in region r. Income equals the value of factor endowments plus any tariff revenue plus the net value of any capital account surplus in the benchmark equilibrium, where we hold the net capital account surplus constant in any counterfactual.

$$\begin{aligned}\mathcal{I}_r = & \sum_{f \in (F \setminus \tilde{F})} w_{fr} \overline{F}_{fr} \\ & + \sum_{f \in \tilde{F}} \sum_{i \in I} \tilde{w}_{fir} \overline{SF}_{fir} \\ & + \sum_{i \in (I \setminus (K \cup M))} \sum_{s \in R} t_{isr} c_{is} \tau_{isr} q0_{ir} Q_{ir} \frac{\partial P_{ir}}{\partial [(1 + t_{isr}) \tau_{isr} c_{is}]} \\ & + \sum_{k \in K} \sum_{s \in R} \frac{t_{ksr} n_{ks} p_{ksr} q_{ksr}}{1 + t_{ksr}} \\ & + \sum_{m \in M} \sum_{s \in R} \frac{t_{msr} N_{msr} \tilde{p}_{msr} \tilde{q}_{msr}}{1 + t_{msr}} \\ & + e_{\text{USA}} \overline{BOP}_r. \end{aligned} \tag{3.88}$$

This is equation AKM.7 of table 2.

The first two terms on the right-hand side of (3.88) are the values of mobile factor endowments and sector-specific factor endowments, respectively. With no labor-leisure choice, the initial total labor supply is the endowment. With labor-leisure choice, the endowment of the representative agent is the total time endowment; then income is "full" income, as it includes the imputed value of leisure. The third term is the value of tariff revenue collected in all Armington sectors. The quantity demanded in region r of imports of good or service $i \in (I \setminus (K \cup M))$ from region s is

$$q0_{ir} Q_{ir} \frac{\partial P_{ir}}{\partial [(1 + t_{isr}) \tau_{isr} c_{is}]}.$$

For each unit of good i from region s that arrives in region r, exporters charge for the melt, which is $\tau_{isr} \geq 1$ units for each unit that arrives. The tariff in region r is assessed on the value $c_{is} \tau_{isr}$ per unit.

The fourth term is tariff revenue collected in Krugman sectors. The customs value of imports from all Krugman firms in sector k of region s into region r equals

$n_{ks} p_{ksr} q_{ksr} / (1 + t_{ksr})$. We then apply the tariff rate on these imports and aggregate over all regions and Krugman sectors.

The fifth term is tariff revenue collected in Melitz sectors. Using equations (3.61), (3.62) and (3.71), the customs value of imports from all Melitz firms in sector m of region s into region r equals:

$$N_{msr} \mathsf{E} \left[\frac{p_r(\varphi_{ms}) q_r(\varphi_{ms})}{1 + t_{msr}} \right] = N_{msr} r_r(\tilde{\varphi}_{mrs}) = N_{msr} \left[\frac{\tilde{p}_{mrs} \tilde{q}_{mrs}}{1 + t_{msr}} \right]. \tag{3.89}$$

To obtain the fifth term in (3.88), we apply the tariff rate to the value in (3.89) and sum over all Melitz sectors and regions.

The final term allows for non-zero trade balances based on the data. A trade deficit augments the income available to the representative agent to spend on goods and services. We hold the trade deficit constant in any counterfactual to avoid providing a permanent free lunch that would distort the welfare analysis.

This model is designed to contribute to the literature on the relative welfare gains of market structure. To maintain comparability with the key papers in this literature, in particular Arkolakis, Costinot, and Rodríguez-Clare (2012) and Costinot and Rodríguez-Clare (2014), we subsume all investment and government demand in consumption. Therefore, we do not have an accounting of investment or government demand in the model. All final demand is included in the expeniture function for the representative agent.

3.8 The Consumers' Expenditure Function, Indirect Utility, and Hicksian Equivalent Variation

In this section, we derive the regional consumer's expenditure function and the formula for Hicksian equivalent variation with and without labor-leisure choice.

3.8.1 Expenditure function with no labor-leisure choice

We first derive the expenditure function for the case of no labor-leisure choice. The goal is to show that equation (3.1) is consistent with consumer optimization over goods and services when $\eta_r^L = 0$. We can do this from the familiar primal utility maximization problem and then use the duality identities to specify the expenditure function. Start with the primal objective

$$U_r = d0_r D_r = \mathcal{H}_r \left[\frac{1}{\theta_r} \Pi_{i \in I} C_{ir}^{\theta_{ir}} \right], \tag{3.90}$$

where to simplify notation in the absence of a labor-leisure choice we define $\mathcal{H}_r \equiv (\eta_r^C)^{(1/(\sigma^L - 1))}$ which is a monotonic increasing transformation of our utility function that allows us to choose appropriate units associated with utility. Maximization of $U_r = d0_r D_r$ subject to the budget, $\mathcal{I}_r = \sum_{i \in I} P_{ir} C_{ir}$, yields the optimum consumption values:

$$C_{ir}^{*} = \frac{\theta_{ir}\mathcal{I}_r}{P_{ir}}. \tag{3.91}$$

Indirect utility in region r (where $\boldsymbol{p}_r$ is the vector of prices) is

$$\begin{aligned} V_r(\boldsymbol{p}_r, \mathcal{I}_r) &= \mathcal{H}_r \left[\frac{1}{\theta_r} \prod_{i \in I} \left(\frac{\theta_{ir}\mathcal{I}_r}{P_{ir}} \right)^{\theta_{ir}} \right] \\ &= \mathcal{H}_r \quad \left(\frac{\mathcal{I}_r}{\prod_{i \in I} P_{ir}^{\theta_{ir}}} \right), \end{aligned} \tag{3.92}$$

where the final term uses the fact that $\prod_i \mathcal{I}_r^{\theta_{ir}} = \mathcal{I}_r$ and the definition of $\theta_r = \prod_i \theta_{ir}^{\theta_{ir}}$. Now use one of the fundamental duality identities (Varian, 1992, p.106) to find the expenditure function. We use $V_r(\boldsymbol{p}_r, E_r(\boldsymbol{p}_r, U_r)) \equiv U_r$; then we replace $\mathcal{I}_r$ in (3.92) with the expenditure function as follows:

$$\begin{aligned} V_r(\boldsymbol{p}_r, E_r(\boldsymbol{p}_r, U_r)) &= U_r = d0_r D_r \\ \mathcal{H}_r \left(\frac{E_r(\boldsymbol{p}_r, U_r)}{\prod_{i \in I} P_{ir}^{\theta_{ir}}} \right) &= U_r \\ E_r(\boldsymbol{p}_r, U_r) &= \frac{U_r}{\mathcal{H}_r} \prod_{i \in I} P_{ir}^{\theta_{ir}}. \end{aligned} \tag{3.93}$$

The expenditure function is linear in utility (a result of linearly-homogeneous preferences), so we can specify the unit expenditure function

$$\begin{aligned} e_r(\boldsymbol{p}_r) &= \frac{1}{\mathcal{H}_r} \prod_{i \in I} P_{ir}^{\theta_{ir}} \\ &= \eta_r^{C\,\frac{1}{1-\sigma^L}} \prod_{i \in I} P_{ir}^{\theta_{ir}} \\ &= \left[\eta_r^C \left(\prod_{i \in I} P_{ir}^{\theta_{ir}} \right)^{1-\sigma^L} \right]^{\frac{1}{1-\sigma^L}}. \end{aligned} \tag{3.94}$$

Equation (3.94) is the right-hand side of equation (3.1) under the special restriction that $\eta_r^L = 0$. Thus we have shown that (3.1) is a proper unit expenditure function that embeds the optimizing behavior of consumers. We could have arrived at exactly the same unit expenditure function by minimizing expenditures and substituting the compensated demand functions back into the objective. In an equilibrium we will have income equal to expenditures and given our derivation of the unit expenditure function in (3.94), with $e_r = e_r(\boldsymbol{p}_r)$ from (3.1), we can rearrange (3.93) to directly derive equation AKM.6 of table 2 in terms of the model variables. That is, we already have

$$U_r = \mathcal{H}_r \frac{E_r(\boldsymbol{p}_r, U_r)}{\Pi_{i\in I} P_{ir}^{\theta_{ir}}},$$

which yields the following in terms of the model variables:

$$d0_r D_r = \frac{\mathcal{I}_r}{e_r}. \tag{3.95}$$

For completeness the compensated demand functions $[h_{ir}(\boldsymbol{p}_r, U_r)]$ in the special case of no labor-leisure choice are given by

$$h_{ir}(\boldsymbol{p}_r, U_r) = \frac{\theta_{ir}}{\mathcal{H}_r} \frac{\Pi_{j\in I} P_{jr}^{\theta_{jr}}}{P_{ir}}. \tag{3.96}$$

3.8.2 Expenditure function with a labor-leisure choice

With a labor-leisure choice the consumer's optimization problem is more complex. We take advantage of the fact that the Cobb-Douglas subutility function on non-leisure goods is separable and linearly homogeneous. This allows us to apply two-stage budgeting where we define the price index ($\mathcal{P}_r^C$) and quantity of subutility ($\mathcal{C}_r$) as follows:

$$\mathcal{P}_r^C \equiv \Pi_{i\in I} P_{ir}^{\theta_{ir}}$$
$$\mathcal{C}_r \equiv \frac{1}{\theta_r} \Pi_{i\in I} C_{ir}^{\theta_{ir}}.$$

By the preceeding derivations in section 3.8.1 we can show that $\mathcal{P}_r^C$ is an ideal price index

$$\mathcal{P}_r^C = \mathcal{P}_r(\boldsymbol{p}) = \min_{\{C_{ir} \forall i\in I\}} \left[\sum_{i\in I} P_{ir} C_{ir} \quad \text{s.t.} \quad 1 = \frac{1}{\theta_r} \Pi_{i\in I} C_{ir}^{\theta_{ir}} \right],$$

where the only deviation from section 3.8.1 is that we make the monotonic transformation such that the coefficient $\mathcal{H}_r$ is eliminated in the derivation of the unit expenditure index. We now substitute the subutility quantity ($\mathcal{C}_r$) into equation (3.2) to give us the primal objective in leisure and the composite of goods consumption:

$$U_r = d0_r D_r = \left[\eta_r^{L\,\frac{1}{\sigma^L}} l_r^{\frac{\sigma^L - 1}{\sigma^L}} + \eta_r^{C\,\frac{1}{\sigma^L}} \mathcal{C}_r^{\frac{\sigma^L - 1}{\sigma^L}} \right]^{\frac{\sigma^L}{\sigma^L - 1}}. \tag{3.97}$$

Let the total time endowment in region r be denoted $\overline{L}_r = l_r + L_r$ where L_r is labor supply. Income from labor is $w_{\mathrm{L}r}L_r$ and denote non-labor income $\mathcal{I}_r^{NL}$. Income from labor services and non-labor income will equal expenditures on goods and services (excluding leisure):

$$w_{\mathrm{L}r}L + \mathcal{I}_r^{NL} = \mathcal{P}_r^C \mathcal{C}_r.$$

Taking the imputed value of time as part of *full income*, we define full income as

$$\mathcal{I}_r = w_{\mathrm{L}r} l_r + \mathcal{P}_r^C \mathcal{C}_r. \tag{3.98}$$

We maximize (3.97) with respect to l and C subject to expenditures on leisure and commodities being equal to full income (equation 3.98). This is the usual CES optimization, yielding:

$$l_r^* = \eta_r^L \frac{\mathcal{I}_r}{\mathcal{P}_r} \left(\frac{\mathcal{P}_r}{w_{\mathrm{L}r}} \right)^{\sigma^L} \quad \text{and} \quad C_r^* = \eta_r^C \frac{\mathcal{I}_r}{\mathcal{P}_r} \left(\frac{\mathcal{P}_r}{\mathcal{P}_r^C} \right)^{\sigma^L}, \tag{3.99}$$

where the new price index $\mathcal{P}_r$ over full consumption is defined as follows:

$$\begin{aligned} \mathcal{P}_r &\equiv \left[\eta_r^L w_{\mathrm{L}r}^{1-\sigma^L} + \eta_r^C \mathcal{P}_r^{C\,1-\sigma^L} \right]^{\frac{1}{1-\sigma^L}} \\ &= \left[\eta_r^L w_{\mathrm{L}r}^{1-\sigma^L} + \eta_r^C \left(\prod_{i \in I} P_{ir}^{\theta_{ir}} \right)^{1-\sigma^L} \right]^{\frac{1}{1-\sigma^L}}. \end{aligned} \tag{3.100}$$

Substitute the optimal values of l_r and $\mathcal{C}_r$ into (3.97) gives us the indirect utility function:

$$
\begin{aligned}
V_r(\boldsymbol{p}_r, \mathcal{I}_r) &= \left[\eta_r^{L\,\frac{1}{\sigma^L}} \left(\eta_r^L \frac{\mathcal{I}_r}{\mathcal{P}_r} \left(\frac{\mathcal{P}_r}{w_{\mathrm{L}r}} \right)^{\sigma^L} \right)^{\frac{\sigma^L-1}{\sigma^L}} + \eta_r^{C\,\frac{1}{\sigma^L}} \left(\eta_r^C \frac{\mathcal{I}_r}{\mathcal{P}_r} \left(\frac{\mathcal{P}_r}{\mathcal{P}_r^C} \right)^{\sigma^L} \right)^{\frac{\sigma^L-1}{\sigma^L}} \right]^{\frac{\sigma^L}{\sigma^L-1}} \\
&= \frac{\mathcal{I}_r}{\mathcal{P}_r} \left[\eta_r^L \left(\left(\frac{\mathcal{P}_r}{w_{\mathrm{L}r}} \right)^{\sigma^L} \right)^{\frac{\sigma^L-1}{\sigma^L}} + \eta_r^C \left(\left(\frac{\mathcal{P}_r}{\mathcal{P}_r^C} \right)^{\sigma^L} \right)^{\frac{\sigma^L-1}{\sigma^L}} \right]^{\frac{\sigma^L}{\sigma^L-1}} \\
&= \frac{\mathcal{I}_r \mathcal{P}_r^{\sigma^L}}{\mathcal{P}_r} \left[\eta_r^L w_{\mathrm{L}r}^{1-\sigma^L} + \eta_r^C \mathcal{P}_r^{C\,1-\sigma^L} \right]^{\frac{\sigma^L}{\sigma^L-1}} \\
&= \frac{\mathcal{I}_r \mathcal{P}_r^{\sigma^L}}{\mathcal{P}_r} \mathcal{P}_r^{-\sigma^L} \\
&= \frac{\mathcal{I}_r}{\mathcal{P}_r}. \qquad (3.101)
\end{aligned}
$$

Now using the duality identity that $V_r(\boldsymbol{p}_r, E_r(\boldsymbol{p}_r, U_r)) \equiv U_r$ we derive the expenditure function. Replacing income in indirect utility with the expenditure function, we have:

$$
\begin{aligned}
V_r(\boldsymbol{p}_r, E_r(\boldsymbol{p}_r, U_r)) &= U_r = d0_r D_r \\
\frac{E_r(\boldsymbol{p}_r, U_r)}{\mathcal{P}_r} &= U_r \\
E_r(\boldsymbol{p}_r, U_r) &= U_r \mathcal{P}_r. \qquad (3.102)
\end{aligned}
$$

The expenditure function is linear in utility (a result of linearly-homogeneous preferences), so we can specify the unit expenditure function $e_r(\boldsymbol{p}_r) \equiv E_r(\boldsymbol{p}_r, 1) = E_r(\boldsymbol{p}_r, U_r)/U_r$ to give us equation (3.1) in its general form to include a labor-leisure choice, which we reproduce here as (3.103):

$$
\begin{aligned}
e_r(\boldsymbol{p}_r) &= \mathcal{P}_r \\
&= \left[\eta_r^L w_{\mathrm{L}r}^{1-\sigma^L} + \eta_r^C \mathcal{P}_r^{C\,1-\sigma^L} \right]^{\frac{1}{1-\sigma^L}} \\
&= \left[\eta_r^L w_{\mathrm{L}r}^{1-\sigma^L} + \eta_r^C \left(\prod_{i \in I} P_{ir}^{\theta_{ir}} \right)^{1-\sigma^L} \right]^{1/(1-\sigma^L)}. \qquad (3.103)
\end{aligned}
$$

Just as in the case with no labor-leisure choice we can derive AKM.6 in table 2 by noting that equilibrium income equals expenditures and our derivation of the unit expenditure function, where $e_r = e_r(\boldsymbol{p}_r)$ from (3.1). From 3.102 we have

$$U_r = \frac{E_r(\boldsymbol{p}_r, U_r)}{\mathcal{P}_r},$$

which yields the following in terms of the model variables:

$$d0_r D_r = \frac{\mathcal{I}_r}{e_r}. \tag{3.104}$$

3.8.3 Money Metric Indirect Utility and Equivalent Variation

In this section we outline our specific choice of units for utility and its relationship with our welfare or *Equivalent Variation* reports. In sections 3.8.1 and 3.8.2 we derive equation AKM.6 in table 2, which is the preceding equation (3.104):

$$d0_r D_r = \frac{\mathcal{I}_r}{e_r}.$$

where $e_r = e_r(\boldsymbol{p}_r)$ is the minimum expenditure required to obtain one unit of utility (true-cost-of-living index), $d0_r$ is the initial value of aggregate final demand, D_r is a variable that takes the value of one in the initial equilibrium.

There is an important nuance regarding cardinalization in AKM.6 or (3.104). In section 3.8.2 we show that $U_r \equiv V_r(\boldsymbol{p}_r, E_r(\boldsymbol{p}_r, U_r)) = \mathcal{I}_r / e_r(\boldsymbol{p}_r)$. This expression may be derived with utility as an ordinal indicator. Our utility function, however, is cardinal since we define utility equal to final demand: $U_r = d0_r D_r$. To see this note that in the benchmark, since $D_r = 1$, we must have $U_r = d0_r = \mathcal{I}_r$. Final demand and income are fixed at their observed levels in the benchmark, so this equality will fail to hold with a positive monotonic transformation of utility. That is, for equation AKM.6 or (3.104) to be consistent with the data, we must choose units of utility such that the initial value of the unit expenditure function is equal to one: $e_r^0 = e_r(\boldsymbol{p}_r^0) = 1$.[33] We show below that this choice of units yields a simple and convenient model report of equivalent variation, because at a benchmark with $e_r^0 = e_r(\boldsymbol{p}_r^0) = 1$ indirect utility is measured in a money metric. That is,

$$V_r(\boldsymbol{p}_r^0, \mathcal{I}_r^0) = \mathcal{I}_r^0 = d0_r D_r = d0_r,$$

where by definition $D_r = 1$ at the benchmark and so drops out of the final term.

[33] The level of utility equals the value of full consumption under the appropriate calibration of η_r^L and η_r^C such that $e_r(\boldsymbol{p}_r) = 1$ at the benchmark. Let ν_r^L be the benchmark value share of leisure in full income for region r, $w_{\mathrm{L}r}^0$ be the benchmark wage, and $\mathcal{P}_r^{C0}$ be the benchmark value of $\mathcal{P}_r^C$, then setting $\eta_r^L = \nu_r^L \left(w_{\mathrm{L}r}^0\right)^{\sigma^L - 1}$ and $\eta_r^C = (1 - \nu_r^L) \left(\mathcal{P}_r^{C0}\right)^{\sigma^L - 1}$ gives us a scaling of units such that utility equals the value of full consumption and $e_r(\boldsymbol{p}_r) = 1$ at the benchmark.

Now consider the definition of the **money metric indirect utility function** as presented by Varian (1992, p.110). In our notation this is

$$\mu_r(\boldsymbol{p}_r; \boldsymbol{p}'_r,\ \mathcal{I}'_r) \equiv E_r(\boldsymbol{p}_r, V_r(\boldsymbol{p}'_r, \mathcal{I}'_r)).$$

Varian (1992) explains that $\mu_r(\boldsymbol{p}_r; \boldsymbol{p}'_r,\ \mathcal{I}'_r)$ indicates how much money one would need at prices $\boldsymbol{p}_r$ to be as well off when faced with a different price vector $\boldsymbol{p}'_r$ and income level $\mathcal{I}'_r$. Let us take the initial price vector as the basis for evaluation $\boldsymbol{p}_r = \boldsymbol{p}^0_r$. Thus, under our choice of units and linear homogeneity, money metric indirect utility is given by

$$\mu_r(\boldsymbol{p}^0_r; \boldsymbol{p}'_r,\ \mathcal{I}'_r) = e_r(\boldsymbol{p}^0_r) V_r(\boldsymbol{p}'_r, \mathcal{I}'_r) = \frac{\mathcal{I}'_r}{e(\boldsymbol{p}'_r)} = d0_r D_r.$$

So any numeric solution of $d0_r D_r$ at an equilibrium with price vector $\boldsymbol{p}'_r$ and income level $\mathcal{I}'_r$ gives us money metric indirect utility evaluated at initial prices, $\boldsymbol{p}^0_r$. At this point we can drop the prime notation, because the initial price vector is distinguished with a "0" superscript.

Now consider the textbook (Varian, 1992, p.161) definition of Hicksian *Equivalent Variation*:

$$EV_r \equiv \mu_r(\boldsymbol{p}^0_r; \boldsymbol{p}_r,\ \mathcal{I}_r) - \mu_r(\boldsymbol{p}^0_r; \boldsymbol{p}^0_r,\ \mathcal{I}^0_r).$$

It follows that at a numeric solution we calculate equivalent variation as

$$EV_r = d0_r D_r - d0_r = d0_r (D_r - 1), \tag{3.105}$$

or equivalently using equation (3.104) we express the same report in terms of the change in real income

$$EV_r = \frac{\mathcal{I}_r}{e_r} - \mathcal{I}^0_r. \tag{3.106}$$

In the case that we have a labor-leisure choice it is important to note that income measured by $\mathcal{I}_r$ includes the imputed value of leisure. $\mathcal{I}_r$ is full income, which will exceed measured income as $w_{\mathrm{L}r} L + \mathcal{I}^{NL}_r$ in the accounts. This does not change the theory behind our measure of EV_r. With a labor-leisure choice, the ratio of EV to benchmark measured income will not be equal to D_r. To report EV as a proportion of measured benchmark income one needs to use the full report of $\frac{EV_r}{w_{\mathrm{L}r} L + \mathcal{I}^{NL}_r}$ to account for the difference between $\mathcal{I}^0_r$ and $w_{\mathrm{L}r} L + \mathcal{I}^{NL}_r$. Of course, the ratio of EV to benchmark *full income* is still equal to D_r.

4. Market Productivity Cutoffs: Impact of Market Size and Preferences

For firms in any region r of our model, we derive the condition for the ratio of the zero-profit productivity cutoff in any of its export markets s to the zero-profit productivity cutoff condition in its home market:

$$\frac{\varphi^*_{mrs}}{\varphi^*_{mrr}}.$$

We show that iceberg costs and tariffs lead to a higher zero-profit productivity cutoff for exporting than for sales in the home market. Further, if there are higher fixed costs of exporting compared to fixed costs of selling in the home market, then relative fixed costs as well as tariffs and iceberg costs all lead to the Melitz (2003) result (under homogeneous regions) that exporting firms are a proper subset of firms that sell in the home market. This supports the stylized fact that exporting firms are a minority of total firms in any region. With heterogeneous regions, however, we could find that some firms find it profitable to export to some markets without serving the home market. For example, Jakubiak et al. (2006) surveyed 510 Ukrainian firms that export to the European Union. They report that about ten percent of these firms report that they do not sell in Ukraine. This could happen if there are large export markets relative to the home market or relatively weak home market preferences for the product of a sector. Feenstra (2010, equation 19) derives a similar condition for a one-sector model with two heterogeneous regions. We extend Feenstra's result to an arbitrary finite number of sectors and regions.

From equation (3.44) we have $r_s(\varphi^*_{mrs}) = \sigma^M c_{mr} f^M_{mrs}$ and $r_r(\varphi^*_{mrr}) = \sigma^M c_{mr} f^M_{mrr}$. Then, the ratio of the revenues at the zero-profit productivity cutoffs is:

$$\frac{r_s(\varphi^*_{mrs})}{r_r(\varphi^*_{mrr})} = \frac{f^M_{mrs}}{f^M_{mrr}}. \tag{4.1}$$

Consider the ratio of the revenue of firms from region r with productivities φ^*_{mrs} and φ^*_{mrr} on sales in their home market r. Equation (3.42) holds for all destination markets $s \in R$, in particular, for the home market. In (3.42) substitute φ^*_{mrs} for ${\varphi''}_{mr}$ and φ^*_{mrr} for ${\varphi'}_{mr}$. We have that

$$\frac{r_r(\varphi^*_{mrs})}{r_r(\varphi^*_{mrr})} = \left(\frac{\varphi^*_{mrs}}{\varphi^*_{mrr}}\right)^{\sigma^M-1}. \tag{4.2}$$

We want to express the ratio of the cutoff productivities in terms of parameters. Using equation (3.41), and recalling that $t_{mrr} = 0$ and $\tau_{mrr} = 1$, we have

$$
\begin{aligned}
r_s(\varphi_{mr}) &= \frac{\Gamma_{mrs}}{1+t_{mrs}} p_s(\varphi_{mr})^{1-\sigma^M} \\
&= \frac{\Gamma_{mrs}}{1+t_{mrs}} \left[(1+t_{mrs})\tau_{mrs} p_r(\varphi_{mr})\right]^{1-\sigma^M} \\
&= \frac{\Gamma_{mrs}}{\Gamma_{mrr}} \left[(1+t_{mrs})\tau_{mrs}\right]^{1-\sigma^M} \frac{\Gamma_{mrr} p_r(\varphi_{mr})^{1-\sigma^M}}{1+t_{mrs}} \\
&= \frac{\Gamma_{mrs}}{\Gamma_{mrr}} (1+t_{mrs})^{-\sigma^M} \tau_{mrs}^{1-\sigma^M} r_r(\varphi_{mr}) \\
&= \Psi_{mrs} r_r(\varphi_{mr}), \qquad (4.3)
\end{aligned}
$$

where to simplify notation we define

$$\Psi_{mrs} \equiv \frac{\Gamma_{mrs}}{\Gamma_{mrr}} (1+t_{mrs})^{-\sigma^M} \tau_{mrs}^{1-\sigma^M}.$$

Divide both sides of (4.3) by Ψ_{mrs} to obtain

$$r_r(\varphi_{mr}) = \frac{r_s(\varphi_{mr})}{\Psi_{mrs}} \quad \forall \ \varphi_{mr},$$

and, in particular,

$$r_r(\varphi^*_{mrs}) = \frac{r_s(\varphi^*_{mrs})}{\Psi_{mrs}}.$$

Substitute $\frac{r_s(\varphi^*_{mrs})}{\Psi_{mrs}}$ for $r_r(\varphi^*_{mrs})$ in (4.2) and use (4.1) to get

$$\frac{r_s(\varphi^*_{mrs})}{r_r(\varphi^*_{mrr})} = \left(\frac{\varphi^*_{mrs}}{\varphi^*_{mrr}}\right)^{\sigma^M-1} \Psi_{mrs} = \frac{f^M_{mrs}}{f^M_{mrr}}. \qquad (4.4)$$

Rearranging, we have

$$\frac{\varphi^*_{mrs}}{\varphi^*_{mrr}} = \left(\frac{f^M_{mrs}}{\Psi_{mrs} f^M_{mrr}}\right)^{\frac{1}{\sigma^M-1}} = \left(\frac{f^M_{mrs}}{f^M_{mrr}}\right)^{\frac{1}{\sigma^M-1}} (1+t_{mrs})^{\frac{\sigma^M}{\sigma^M-1}} \tau_{mrs} \left(\frac{\Gamma_{mrr}}{\Gamma_{mrs}}\right)^{\frac{1}{\sigma^M-1}}. \qquad (4.5)$$

The iceberg costs and the tariffs lead to a higher zero-profit productivity cutoff for exporting than for sales on the home market. If the fixed costs of exporting are higher than the fixed costs of serving the home market, these would also lead to a smaller share of firms in a sector of a region exporting. Thus, these parameters on the right-hand side of (4.5) are consistent with the stylized fact that exporting firms are a significant minority of total firms of a region and exporting firms are larger. On the other hand, anything that contributes to

$$\frac{\Gamma_{mrr}}{\Gamma_{mrs}}$$

being less than one could lead to some firms exporting without serving their domestic market. With the definition of Γ_{mrs} the ratio is

$$\frac{\Gamma_{mrr}}{\Gamma_{mrs}} = \frac{\lambda_{mrr}^{M} q0_{mr} Q_{mr} P_{mr}^{\sigma^M}}{\lambda_{mrs}^{M} q0_{ms} Q_{ms} P_{ms}^{\sigma^M}}. \tag{4.6}$$

Notably, for firms in small home markets, there may be sufficiently large export markets such that the zero-profit productivity cutoff for exports is lower than the home market zero-profit productivity cutoff; this would result in some firms exporting without selling in their home market.

5. Conclusion

While the work of Melitz (2003) is seminal, it is necessary to extend that model from its simplifying assumptions to make it suitable for applied general equilibrium policy modeling. The first objective of this paper is to extend the Melitz (2003) model in numerous directions such that it may be used in an applied general equilibrium policy context and to provide detailed pedagogical derivations to substantially facilitate the accessibility of the generalized model. Our model of heterogeneous firms extends the model of Melitz (2003) by allowing multiple sectors, intermediates with shares based on data from the input-output tables, heterogeneous regions based on data, labor-leisure choice, initial heterogeneous tariffs as well as iceberg trade costs, multiple factors of production, the possibility of sector-specific inputs and trade balances based on data. In addition to shocks to iceberg trade costs considered by Melitz, we incorporate global and unilateral tariff policy shocks. Redding (2010b), Redding (2010a), and Donaldson (2016) provide good pedagogical derivations of the basic Melitz model, but not the extensions mentioned above.

Although there are papers in the literature that provide mathematical derivations of many of the extensions of the heterogeneous-firms models we consider (e.g., Costinot and Rodríguez-Clare, 2014; Akgul, Villoria, and Hertel, 2016; and Dixon, Jerie, and Rimmer, 2018) our approach is more detailed or pedagogical and, in some cases, depending on the alternate approach in the literature, more general (e.g., labor-leisure choice) or closer to the theoretical literature begun by Melitz (2003). We have provided detailed textbook style mathematical derivations of an extended version of the heterogenous firms model of Melitz (2003), as well as the Armington (1969) and Krugman (1980) models. Comments we have received on earlier drafts of this paper have indicated that, although our version of the heterogeneous-firms model is considerably more general than Melitz (2003), our detailed derivations have made the heterogeneous-firms model of international trade more accessible.

The second objective of the paper is to document the models of Balistreri and Tarr (2022). Balistreri and Tarr apply these models to GTAP data where they assess the relative welfare impacts in the Armington, Krugman, and Melitz style models of trade cost reductions in eighteen model variants. This paper documents the equations of those models and provides a summary of the key results.

Conceptually, there is a full policy model which includes all data, model variants and policy instruments. Then many of the various model variations that are considered in Balistreri and Tarr (2022) may be thought of as special cases. For example, the general model allows labor-leisure choice and sector-specific factors, where the share of primary factors that are sector-specific may range from zero to one. Then the model without sector-specific factors is a special case of the general model where all primary factors of production are mobile; and the model with no labor-leisure choice is a special case of the general model with a variable elasticity of labor supply.

The mathematics we lay out in table 2 shows that there is a set of equations that is common to the Armington, Krugman and Melitz models. The difference in these common equations in that set is that the interpretation or mathematics of the industry prices and quantities are the Armington aggregates or the Dixit-Stiglitz aggregates. Beyond industry prices and quantities, there are some equations that are specific to the Krugman style model and a slightly larger set of equations that are specific to the Melitz style model. We hope this paper will be a clear roadmap for understanding and constructing modern multi-sector, multi-region international trade models that must be fitted to data.

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Appendix A. Constant Elasticity of Substitution Calibrated-share Form

This appendix outlines the basic calibrated-share form of constant-elasticity-of-substitution (CES) functions. The presentation is a condensed version of the pedagogical treatment by Rutherford (1995), which is intended to help students simplify empirical calibration procedures. Other authors have developed an analogous CES form for analytic work, calling it the *normalized CES* function (Growiec, 2013). We will focus on a linearly-homogeneous CES production technology. Of course, the treatment ports directly into a representation of homothetic preferences, where cost functions are recast as expenditure functions. The treatment in this appendix is generalized, and the notation is apart from the rest of the paper.

A textbook presentation of the CES technology for producing y with a vector of inputs, $\boldsymbol{x}$, will take on a form analogous to

$$y(\boldsymbol{x}) = \left(\sum_i \alpha_i x_i^{\rho}\right)^{1/\rho}. \tag{A.1}$$

The cost function, which conveniently embeds the firms' optimization over inputs, is given by

$$C(\boldsymbol{r}, y) = y\left(\sum_i \alpha_i^{\sigma} r_i^{1-\sigma}\right)^{1/(1-\sigma)}, \tag{A.2}$$

where $\sigma = 1/(1-\rho)$ indicates the *constant* elasticity of substitution and the $r_i \in \boldsymbol{r}$ are input prices. In the context of an empirical calibration of the α_i to a *benchmark* equilibrium this form is inconvenient. The formulae that transform the data into the parameters are non-linear and inherently dependent on our choice of scaling—or units choice—for the observed variables. One thing we can leverage, however, is the fact that linear homogeneity allows us to factor out any positive scalar, ϕ, such that:

$$y(\boldsymbol{x}) = \phi\left(\sum_i \beta_i x_i^{\rho}\right)^{1/\rho}, \tag{A.3}$$

where $\beta_i = \alpha_i \phi^{-\rho}$. A common simplification is to choose ϕ such that $\sum_i \beta_i = 1$ making it clear that the coefficients represent input weights. Note that these weights are still dependent on the units we use to measure inputs. Further, the new parameter ϕ translates the non-linear term into the specific units with which we choose to measure output y.

Consider an alternative ϕ that equals the observed quantity of output at an established *benchmark* equilibrium, y^0. The β_i will take on a unique and convenient relationship with the input data under this rescaling, and the non-linear term will

Table A.1. Typical calibration data

Observed Value	Reference Quantity	Reference Price
$vom = p^0 y^0$	y^0	p^0
$vfm_i = r_i^0 x_i^0 \quad \forall i$	$x_i^0 \quad \forall i$	$r_i^0 \quad \forall i$

be equal to one at the benchmark. To see this consider table A.1, which includes typical data as observed in the social accounts. The convention of labeling gross value of output as *vom* and factor inputs as vfm_i is derived from the GTAP database notation. In table A.1 prices and quantities embellished with a superscript 0 are benchmark data where the balance between price versus quantity is a result of choice of units. With these data we can also calculate a key summary parameter—the benchmark value share of inputs:

$$\theta_i \equiv \frac{r_i^0 x_i^0}{\sum_j r_j^0 x_j^0} = \frac{r_i^0 x_i^0}{p^0 y^0},$$

where the last equality holds for micro-consistent data, in which revenues are disbursed as input payments ($p^0 y^0 \equiv \sum_j r_j^0 x_j^0$). Note that the new share parameter, θ_i, is unitless, and thus independent of the scale in which we chose to measure inputs. We can, of course, use the conditional input demands evaluated at the benchmark (with zero profits) to directly calibrate the α_i in terms of the data and the added value share θ_i. Conditional input demand is given by

$$x_i(\boldsymbol{r}, Y) = \frac{\partial C(\boldsymbol{r}, Y)}{\partial r_i} = y \left(\frac{\alpha_i c(\boldsymbol{r})}{r_i} \right)^{\sigma} \tag{A.4}$$

where $c(\boldsymbol{r})$ is the unit cost function which has a benchmark equilibrium value of p^0. Evaluating equation (A.4) at the benchmark equilibrium and inverting to solve for α_i we have

$$\alpha_i = \frac{r_i^0 x_i^0}{p^0 y^0} \left(\frac{y^0}{x_i^0} \right)^{\rho} = \theta_i \left(\frac{y^0}{x_i^0} \right)^{\rho} \tag{A.5}$$

Now let us define $\beta_i \equiv \theta_i (x_i^0)^{-\rho} = \alpha_i \phi^{-\rho}$ where $\phi \equiv y^0$. Substituting the β_i and ϕ out of equation (A.3) using their definitions gives us what Rutherford (1995) calls the *calibrated share form*:

$$y(\boldsymbol{x}) = y^0 \left[\sum_i \theta_i \left(\frac{x_i}{x_i^0} \right)^{\rho} \right]^{1/\rho}. \tag{A.6}$$

The corresponding cost function (in calibrated share form) would be

$$C(\boldsymbol{r}, y) = p^0 y \left[\sum_i \theta_i \left(\frac{r_i}{r_i^0} \right)^{(1-\sigma)} \right]^{1/(1-\sigma)}, \tag{A.7}$$

where we take advantage of the fact that

$$\alpha_i^{\sigma} = \theta_i^{\sigma} \left(\frac{y^0}{x_i^0} \right)^{\sigma-1} = \theta_i \left(\frac{p^0}{r_i^0} \right)^{1-\sigma}.$$

As an extension to this basic presentation consider benchmark distortions in the form of input taxes. The reference prices and calibrated value shares need to take into account cost minimization is on a gross-of-tax basis. Let the gross-of-tax input price faced by producers be given by $(1 + t_i)r_i$. The reference price would include the benchmark markup: $(1 + t_i^0)r_i^0$, and the value share would be given by

$$\theta_i \equiv \frac{(1 + t_i^0) r_i^0 x_i^0}{p^0 y^0}.$$

The cost function extended for input taxes would be

$$C(\boldsymbol{r}, y) = p^0 y \left[\sum_i \theta_i \left(\frac{(1 + t_i) r_i}{(1 + t_i^0) r_i^0} \right)^{(1-\sigma)} \right]^{1/(1-\sigma)}. \tag{A.8}$$

Linear homogeneity indicates that all of the information about the technology can be captured in the *unit* cost function:

$$c(\boldsymbol{r}) = p^0 \left[\sum_i \theta_i \left(\frac{(1 + t_i) r_i}{(1 + t_i^0) r_i^0} \right)^{(1-\sigma)} \right]^{1/(1-\sigma)}. \tag{A.9}$$

Input demand using the calibrated-share conventions is found by applying Shephard's lemma. Conditional input demand for good i equals the output level times the partial derivative of the unit cost function with respect to the gross-of-tax input price of good i:

$$
\begin{aligned}
x_i(\boldsymbol{r}, y) &= y \frac{\partial c(\boldsymbol{r})}{\partial[(1+t_i)r_i]} \\
&= y p^0 \frac{1-\sigma}{1-\sigma} \left(\frac{c(\boldsymbol{r})}{p^0} \right)^{\sigma} \frac{\theta_i}{[(1+t_i^0)r_i^0]^{1-\sigma}} [(1+t_i)r_i]^{-\sigma} \\
&= y p^0 \theta_i [(1+t_i^0)r_i^0]^{-1} \left(\frac{(1+t_i^0)r_i^0}{(1+t_i)r_i} \frac{c(\boldsymbol{r})}{p^0} \right)^{\sigma} \\
&= y p^0 \frac{(1+t_i^0)r_i^0 x_i^0}{p^0 y^0} [(1+t_i^0)r_i^0]^{-1} \left(\frac{(1+t_i^0)r_i^0}{(1+t_i)r_i} \frac{c(\boldsymbol{r})}{p^0} \right)^{\sigma} \\
&= x_i^0 \frac{y}{y^0} \left(\frac{(1+t_i^0)r_i^0}{(1+t_i)r_i} \frac{c(\boldsymbol{r})}{p^0} \right)^{\sigma}.
\end{aligned}
\tag{A.10}
$$

In application we often arrange the unit cost function as an equilibrium condition, such that price equals marginal cost (zero unit profits):

$$
p = p^0 \left[\sum_i \theta_i \left(\frac{(1+t_i)r_i}{(1+t_i^0)r_i^0} \right)^{(1-\sigma)} \right]^{1/(1-\sigma)}. \tag{A.11}
$$

We can also use this to show the popular *exact-hat* form. Define $\hat{p}$ as the proportional change in p between the benchmark and counterfactual equilibrium. We divide through by p^0 (which might be one by our choice of units) to get

$$
\hat{p} = \left[\sum_i \theta_i \left(\frac{(1+t_i)r_i}{(1+t_i^0)r_i^0} \right)^{(1-\sigma)} \right]^{1/(1-\sigma)}. \tag{A.12}
$$

The convenience of the calibrated-share form of the CES technology is notable in that we can go directly from the accounts to the functions as represented in the computer code simply by calibrating (calculating) the θ_i (and assuming a value for σ). This avoids the tedious non-linear and scale dependent formulae for the α_i. Credit for insights gleaned from this appendix should be attributed to Rutherford (1995). Blame for mistakes and added confusion lies with the authors.

Appendix B. Equations as Implemented in the Computer Code

In this appendix we itemize the equations of the computer code as applied by Balistreri and Tarr (2022). The model is formulated as a mixed complementarity problem in the Mathiesen-Rutherford tradition (Rutherford, 1999). The equilibrium conditions are coded as a set of complementary-slack conditions of the form:

$$f(\boldsymbol{x}) \geq 0; \quad x \geq 0; \quad \text{and} \quad x f(\boldsymbol{x}) = 0,$$

where $\boldsymbol{x}$ is a vector of endogenous variables and x is a particular positive variable associated with the function $f(\boldsymbol{x})$. So, for example, we would have market clearance as excess supply is greater than or equal to zero which is complementary to a positive price. Excess supply is consistent only with an equilibrium that has a zero price for the associated good. If the price is positive then excess supply must be zero. To simplify the presentation we use the $\perp$ symbol to indicate the complementarity. So we present

$$f(\boldsymbol{x}) \geq 0 \perp x \geq 0.$$

In section B.1 we present the equations that represent preferences and transformation technologies. These can generally be thought of as *zero-profit* conditions for constant-returns-to-scale activity, where the marginal cost of the activity is given by its cost or expenditure function and the marginal benefit is given by the price of its output. The equations specific to the Krugman and Melitz models are presented in sections B.2 and B.3. Market clearance equations are presented in section B.4, and finally the income balance equations are presented in section B.5.

B.1 Technologies and preferences

With the option to include a labor-leisure choice, the unit expenditure function is given by:

$$\left[\eta_r^L w_{\text{L}r}^{1-\sigma^L} + \eta_r^C \left(\prod_{i \in I} P_{ir}^{\theta_{ir}}\right)^{1-\sigma^L}\right]^{1/(1-\sigma^L)} - e_r \geq 0 \quad \perp \quad D_r \geq 0. \tag{B.1}$$

If $\eta_r^L = 0$ labor supply is perfectly inelastic; then the unit expenditure function reduces to only the Cobb-Douglas preference nest over goods and services. Equilibrium condition (B.1) is consistent with AKM.1 in table 2 and equation (3.1) in the text.

The production technology for the composite input in the dual is given by one of the following formulations. We always assume that value-added inputs combine in a Cobb-Douglas nest, but we employ multiple treatments of intermediates.

If we assume that intermediates and a value-added composite substitute with an elasticity of substitution $\sigma^T \neq 1$ we have the cost function of (B.2),

$$\left[\sum_{j\in J} \alpha_{jir} P_{jr}^{1-\sigma^T} + \alpha_{wir}\left(\prod_{f\notin\tilde{F}}(w_{fr})^{\beta_{fir}}\prod_{f\in\tilde{F}}(\tilde{w}_{fir})^{\beta_{fir}}\right)^{1-\sigma^T}\right]^{\frac{1}{1-\sigma^T}} - c_{ir} \geq 0 \perp Y_{ir} \geq 0, \tag{B.2}$$

where $\alpha_{jir} \geq 0$, $\beta_{fir} \geq 0$, and $\sum_{f\in F}\beta_{fir} = 1$. This equilibrium condition is consistent with AKM.2 in table 2 or equation (3.5) in the text. If there are no intermediates, then the α_{jir} parameters are all zero, $\alpha_{wir} = 1$ and we only have the Cobb-Douglas nest of primary factors. If $\sigma^T = 1$, we have the cost function of Cobb-Douglas technology in (B.3):

$$\left[\prod_{j\in J}(P_{jr})^{\alpha_{jir}}\prod_{f\notin\tilde{F}}(w_{fr})^{\beta_{fir}(1-\sum_j\alpha_{jir})}\prod_{f\in\tilde{F}}(\tilde{w}_{fir})^{\beta_{fir}(1-\sum_j\alpha_{jir})}\right] - c_{ir} \geq 0 \quad \perp \quad Y_{ir} \geq 0. \tag{B.3}$$

This equilibrium condition is consistent with equation (3.3) in the text.

Two important numerical papers in this field, Balistreri, Hillberry, and Rutherford (2011) and Costinot and Rodríguez-Clare (2014), assume a single-composite intermediate in each sector of each region. Then, for example, steel is used in the same proportion in automobile production as in the accounting sector. While apparently unrealistic, Balistreri and Tarr (2022) assess the implications of this assumption for the relative gains from trade in the Armington, Krugman and Melitz models. Under the assumption of a single composite intermediate in each sector and region, the price of intermediates is the unit expenditure function, and the cost function is shown by (B.4):

$$\left[(e_r)^{\sum_j\alpha_{jir}}\prod_{f\notin\tilde{F}}(w_{fr})^{\beta_{fir}(1-\sum_j\alpha_{jir})}\prod_{f\in\tilde{F}}(\tilde{w}_{fir})^{\beta_{fir}(1-\sum_j\alpha_{jir})}\right] - c_{ir} \geq 0 \quad \perp \quad Y_{ir} \geq 0, \tag{B.4}$$

where e_r is given by equilibrium condition (B.1). In the case when we do not have any intermediates, the α share parameters in the above equations would all be zero.

The next set of equations indicate aggregation of varieties consumed or used as intermediates in region s. For the Armington structure, we have the condition for optimal supply of the composite good or service (Q_{is}) available for absorption:

$$\left[\sum_{r\in R}\lambda_{irs}^A[(1+t_{irs})\tau_{irs}c_{ir}]^{1-\sigma^A}\right]^{1/(1-\sigma^A)} - P_{is} \geq 0 \quad \perp \quad Q_{is} \geq 0, \tag{B.5}$$

This equilibrium condition is consistent with A.8 in table 2 and equation (3.7) in the text. If we assume a monopolistic competition structure, the dual price is either a Dixit-Stiglitz aggregation of Krugman firm-level varieties, as:

$$\left[\sum_{r\in R}\lambda^K_{krs}n_{kr}p_{krs}^{1-\sigma^K}\right]^{1/(1-\sigma^K)} - P_{ks} \geq 0 \quad \perp \quad Q_{ks} \geq 0; \tag{B.6}$$

or (as we show in section 3.5) an aggregation of Melitz representative firm varieties as:

$$\left[\sum_{r\in R}\lambda^M_{mrs}N_{mrs}\tilde{p}_{mrs}^{1-\sigma^M}\right]^{1/(1-\sigma^M)} - P_{ms} \geq 0 \quad \perp \quad Q_{ms} \geq 0. \tag{B.7}$$

Equilibrium condition (B.6) corresponds to K.8 in table 2 and equation (3.22) in the text; and (B.7) corresponds to M.8 in table 2 and equation (3.54) in the text.

B.2 Krugman specific equilibrium conditions

Excess supply for a variety produced by an individual firm in region r and sold in region s must be greater than or equal to zero, with the associated variable being the firm-level price:

$$q_{krs} - \lambda^K_{krs}q0_{ks}Q_{ks}\left(\frac{P_{ks}}{p_{krs}}\right)^{\sigma^K} \geq 0 \quad \perp \quad p_{krs} \geq 0. \tag{B.8}$$

This equilibrium condition corresponds to K.10 in table 2 and equation (3.18) in the text. Faced with demand for its variety, given in (B.8), a firm supplying anything will supply a quantity that equates marginal cost to marginal revenue:

$$\tau_{krs}c_{kr} - \frac{(1-1/\sigma_K)p_{krs}}{1+t_{krs}} \geq 0 \quad \perp \quad q_{krs} \geq 0, \tag{B.9}$$

where (B.9) reorients the standard markup condition in K.11 and equation (3.17) into a (marginal cost less marginal revenue) complementary-slack condition associated with firm-level supply. Free entry leads to zero profits, so firms enter until the cost of establishing a firm equals the accumulated quasi-rents that firm earns across all markets:

$$f^K_{kr}c_{kr} - \sum_{s\in R}\frac{p_{krs}q_{krs}}{\sigma_K(1+t_{krs})} \geq 0 \quad \perp \quad n_{kr} \geq 0 \tag{B.10}$$

This equilibrium condition is consistent with K.12 and equation (3.20) in the text.

B.3 Melitz specific equations

The Melitz model posits an infinite number of firms corresponding to a continuous distribution of firm productivities. Nonetheless, the Melitz *equilibrium* is defined by a single firm (variety) on each bilateral trade link. That is, the distribution of firm productivities does not appear in the market equilibrium. The market equilibrium on a particular bilateral link is equivalent to a model in which all firms selling on that bilateral link have the same productivity as that single firm. For that reason, we call the firm with the productivity in the market equilibrium the "representative" firm. This should not be confused with the representative firm in a Krugman model, where all firms are actually identical. For details of the derivation starting from the distribution of all firms, see section 3.5.

Excess supply for a variety produced by the representative firm of r selling in region s must be greater than or equal to zero, with the associated variable being the price of the representative variety:

$$\tilde{q}_{mrs} - \lambda^M_{mrs} q0_{ms} Q_{ms} \left(\frac{P_{ms}}{\tilde{p}_{mrs}}\right)^{\sigma^M} \geq 0 \quad \perp \quad \tilde{p}_{mrs} \geq 0. \tag{B.11}$$

This equilibrium condition corresponds to M.10 in table 2 and equation (3.55) in the text.

Similar to (B.9) we have a complementary-slack condition associated with the quantity supplied by the representative firm:

$$\frac{\tau_{mrs} c_{mr}}{\tilde{\varphi}_{mrs}} - \frac{\tilde{p}_{mrs}(1 - 1/\sigma_M)}{(1 + t_{mrs})} \geq 0 \quad \perp \quad \tilde{q}_{mrs} \geq 0. \tag{B.12}$$

Again this condition reorients the standard markup condition in M.11 and equation (3.52) into a complementary-slack condition where the first term is marginal cost and the second term is marginal revenue.

Next we specify the condition that determines selection into each bilateral market. The marginal firm will earn zero profits. With a Pareto distribution and shape parameter a, the zero-profit productivity cutoff condition applied to the representative firm is as follows

$$f^M_{mrs} c_{mr} - \frac{a + 1 - \sigma^M}{a\sigma^M} \frac{\tilde{p}_{mrs}\tilde{q}_{mrs}}{(1 + t_{mrs})} \geq 0 \quad \perp \quad N_{mrs} \geq 0. \tag{B.13}$$

Firms select into a market up to the point that the cost of setting up operations in that market ($f^M_{mrs} c_{mr}$) equals the expected revenue. Condition (B.13) corresponds to M.12 in table 2 and equation (3.72). For details on the derivation of (B.13), and in particular how we are able to specify it in terms of the representative firms revenues, see the derivation of equation (3.72).

The next condition determines how many firms enter (take a productivity draw). Equilibrium requires that a potential entrant have zero expected profits from potentially multiple markets; then expected profits across multiple markets just equal the annualized sunk costs of establishing a variety. With δ as the rate of firm death, this requires that:

$$\delta f^E_{mr} c_{mr} - \sum_{s \in R} \left(\frac{N_{mrs}}{n_{mr}} \right) \frac{\tilde{p}_{mrs} \tilde{q}_{mrs} (\sigma^M - 1)}{(1 + t_{mrs}) a \sigma^M} \geq 0 \quad \perp \quad n_{mr} \geq 0, \tag{B.14}$$

as derived in equation (3.77) in the text (M.13 in table 2). Again see the derivation of (3.77) to see how we are able to characterize expected profits in terms of the representative firms in each market.

The final variable needed in the Melitz formulation is the productivity of the representative firms in each market.

$$\tilde{\varphi}_{mrs} = b \left(\frac{N_{mrs}}{n_{mr}} \right)^{-1/a} \left(\frac{a + 1 - \sigma^M}{a} \right)^{1/(1-\sigma^M)} \quad \perp \quad \tilde{\varphi}_{mrs} > 0, \tag{B.15}$$

where the parameter b is the lower support on the Pareto distribution and thus $\tilde{\varphi}_{mrs} > 0$. Although $\tilde{\varphi}_{mrs}$ is associated with condition (B.15), the condition is definitional, so we write it as an equality with $\tilde{\varphi}_{mrs} > 0$. Condition (B.15) corresponds to M.14 in table 2 and equation (3.79) in the text.

B.4 Market Clearance

We choose notation that makes explicit that we solve for percentage changes in variables. In the case of the supply of i in region r (from both domestic and imported sources) we write the quantity supplied as $q0_{ir} Q_{ir}$; this is its value of supply in the benchmark times a scalar endogenous variable that has a value of one in the benchmark. Then the endogenous variable, for which we solve, Q_{ir}, is the proportional change in supply. Similarly, we define the production of composite inputs $y0_{ir} Y_{ir}$ and utility $d0_r D_r$, so that the change in the solved variables (times 100) are the percentage changes in composite inputs and utility.

We first establish the market clearance conditions for supply and demand of goods and services available for domestic use. The demand for goods and services is derived by applying Shepard's Lemma to the consumption and production technologies. The market clearance conditions for the goods or service with price P_{ir} are given by:

$$q0_{ir} Q_{ir} - d0_r D_r \frac{\partial e_r}{\partial P_{ir}} - \sum_{j \in I} y0_{jr} Y_{jr} \frac{\partial c_{jr}}{\partial P_{ir}} \geq 0 \quad \perp \quad P_{ir} \geq 0. \tag{B.16}$$

In words the complementary-slack relationship indicates that for positive prices supply (the first term) equals demand (the second and third terms). The market clearance condition (B.16) corresponds to AKM.3 in table 2 and equation (3.81) in the text.

Next, we establish market clearance for the production of composite input i in region r and all of its uses in all markets. Under Armington market clearance is given by:

$$y0_{ir}Y_{ir} - \sum_{s \in R} \tau_{irs} q0_{is} Q_{is} \frac{\partial P_{is}}{\partial[(1+t_{irs})\tau_{irs}c_{ir}]} \geq 0 \quad \perp \quad c_{ir} \geq 0. \tag{B.17}$$

This corresponds to A.9 in table 2 and equation (3.82) in the text. For the monopolistic competition models, we must account for the use of the composite input for fixed costs as well as variable costs. Under Krugman we have

$$y0_{kr}Y_{kr} - n_{kr}\left(f^K_{kr} + \sum_{s \in R} \tau_{krs} q_{krs}\right) \geq 0 \quad \perp \quad c_{kr} \geq 0. \tag{B.18}$$

This corresponds to K.9 in table 2 and equation (3.83) in the text. Under Melitz we have

$$y0_{mr}Y_{mr} - \delta f^E_{mr} n_{mr} - \sum_{s \in R} N_{mrs}\left(f^M_{mrs} + \frac{\tau_{mrs}\tilde{q}_{mrs}}{\tilde{\varphi}_{mrs}}\right) \geq 0 \quad \perp \quad c_{mr} \geq 0. \tag{B.19}$$

This corresponds to M.9 in table 2 and equation (3.85) in the text.

For sector-specific primary factors, the market clearance condition is:

$$\overline{SF}_{fir} - y0_{ir}Y_{ir}\frac{\partial c_{ir}}{\partial \tilde{w}_{fir}} \geq 0 \quad \perp \quad \tilde{w}_{fir} \geq 0, \tag{B.20}$$

which corresponds to AKM.4 in table 2 and equation (3.86) in the text. For primary mobile factors of production, we account for demand across different sectors:

$$\begin{aligned} &\overline{F}_{fr} - \sum_{i \in I} y0_{ir}Y_{ir}\frac{\partial c_{ir}}{\partial w_{fr}} - d0_r D_r \frac{\partial e_r}{\partial w_{fr}} && \geq 0 \quad \perp \quad w_{fi} \geq 0 \quad \text{for} \quad f = \text{L} \\ &\overline{F}_{fr} - \sum_{i \in I} y0_{ir}Y_{ir}\frac{\partial c_{ir}}{\partial w_{fr}} && \geq 0 \quad \perp \quad w_{fi} \geq 0 \quad \text{for} \quad f \neq \text{L}. \end{aligned} \tag{B.21}$$

In the first instance of (B.21) market clearance is for labor, and if we have labor-leisure choice in the model the added term indicates demand for leisure. Equilibrium condition (B.21) corresponds to AKM.5 in table 2 and equation (3.87) in the text.

In all of our model variations except one, intermediates are modeled as in either equation (B.2) or equation (B.3); then the quantity associated with real consumption is fully exhausted in final demand, giving us the market clearance condition for the final demand good as indicated by equation (3.104) in the text (or AKM.6

in table 2):

$$d0_r D_r = \frac{\mathcal{I}_r}{e_r} \geq 0 \quad \perp \quad e_r \geq 0 \tag{B.22}$$

where D_r is an index on utility, $\mathcal{I}_r$ is nominal income, and changes in $\mathcal{I}_r/e_r$ is Hicksian equivalent variation. In the one model where we assume a single composite intermediate input as in equation (B.4), however, some of the consumption good is used in production. In that special case, market clearance for the aggregate consumption good is given by

$$d0_r D_r - \frac{\mathcal{I}_r}{e_r} - \sum_{j \in J} y0_{jr} Y_{jr} \frac{\partial c_{jr}}{\partial e_r} \geq 0 \quad \perp \quad e_r \geq 0 \tag{B.23}$$

and Hicksian equivalent variation remains the change in $\mathcal{I}_r/e_r$.

B.5 Income balance and the numeraire

The model is based on relative prices (the model is homogeneous of degree zero in nominal prices). We define the numeraire as the price of a unit of utility in the United States:

$$e_{\texttt{USA}} \equiv 1. \tag{B.24}$$

All price are relative to this numeraire.

In terms of units of the numeraire, we must have nominal income equal expenditures of the representative agent in region r. Income equals the value of factor endowments plus any tariff revenue plus the value of any capital account surplus. With no labor-leisure choice, the initial total labor supply is the labor endowment. With labor-leisure choice, the endowment of the representative agent is the total time endowment; then income is "full" income, i.e., it includes the imputed value of leisure. In addition to factor income there is tariff revenue across the three types of goods (Armington, Krugman, and Melitz). The model includes a constant balance of trade constraint measured in international transfer units. This enters the agent's income as either a positive or negative nominal transfer depending on if the region has a benchmark trade deficit or surplus. Nominal income is thus

$$\begin{aligned}
\mathcal{I}_r &= \sum_{f \notin \tilde{F}} w_{fr} \overline{F}_{fr} + \sum_{f \in \tilde{F}} \sum_{i \in I} \tilde{w}_{fir} \overline{SF}_{fir} \\
&+ \sum_{i \notin K \cup M} \sum_{s \in R} t_{isr} c_{is} \tau_{isr} q0_{ir} Q_{ir} \frac{\partial P_{ir}}{\partial[(1+t_{isr})\tau_{isr} c_{is}]} \\
&+ \sum_{k \in K} \sum_{s \in R} \frac{t_{ksr} n_{ks} p_{ksr} q_{ksr}}{1+t_{ksr}} \\
&+ \sum_{m \in M} \sum_{s \in R} \frac{t_{msr} N_{msr} \tilde{p}_{msr} \tilde{q}_{msr}}{1+t_{msr}} \\
&+ e_{\text{USA}} \overline{BOP}_r \qquad \perp \quad \mathcal{I}_r > 0
\end{aligned} \tag{B.25}$$

For any nontrivial agent added to the model we can be sure that $\mathcal{I}_r > 0$, so we simply present (B.25) as an equality associated with $\mathcal{I}_r > 0$.

Appendix C. Differentiating Under the Integral

We have not seen a published proof of the property proposed in equation (3.34). Some authors cite Dixit and Stiglitz (1977) for proof, but that paper exclusively employed a finite number of varieties. In an earlier version, Dixit and Stiglitz (1974) employed a continuum of varieties; but (3.34) was asserted without proof in that paper. We first summarize the intuition for this property; the on-line proofs we have found follow. For intuition, suppose we have a sum of a finite number of varieties as in our Krugman model:

$$S = \sum_{\varphi_{mr} \in V_{mrs}} p_s(\varphi_{mr})^{1-\sigma^M}$$

The partial derivative of this sum with respect to a particular term (or variety) $p_s(\varphi'_{mr})$ is

$$\frac{\partial S}{\partial p_s(\varphi'_{mr})} = (1-\sigma^M) p_s(\varphi'_{mr})^{-\sigma^M}$$

which is the right-hand side of equation (3.34). The partial derivative of this sum is unchanged for any finite number of varieties. Heuristically, as the finite number of varieties becomes very large, this partial derivative should approach the partial derivative of the continuous case in (3.34). This, however, is not a proof.

We refer the reader to three proofs available on-line. The first proof applies a theorem in the calculus of variations applied to CES optimization problems with a continuum of commodities. It is due to Bhattacharya (2014), available at: https://economics.stackexchange.com/questions/210/consumer-optimum-in-an-economy-with-a-continuum-of-commodities

The second proof builds on the intuitive argument above. At the link to Bhattacharya (2014) above, Alecos Papadopoulos argues that, based on a theorem by Craven (1970) we may differentiate the integral as if it were a finite sum.

A third proof attributed to Elias (2016) relies on the concept of a functional derivative and is available at: https://economics.stackexchange.com/questions/14068/regarding-a-consumption-aggregator-how-do-i-differentiate-under-the-integral-si?noredirect=1&lq=1.