Mechanisms without transfers for fully biased agents[*]

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Abstract

A principal must decide between two options. Which one she prefers depends on the private information of two agents. One agent always prefers the first option; the other always prefers the second. Transfers are infeasible. One application of this setting is the efficient division of a fixed budget between two competing departments. We first characterize all implementable mechanisms under arbitrary correlation. Second, we study when there exists a mechanism that yields the principal a higher payoff than she could receive by choosing the ex-ante optimal decision without consulting the agents. In the budget example, such a profitable mechanism exists if and only if the information of one department is also relevant for the expected returns of the other department. We generalize this insight to derive necessary and sufficient conditions for the existence of a profitable mechanism in the n-agent allocation problem with independent types.

1 Introduction

A principal has to decide between two options. Which one she prefers depends on the private information of two agents. One agent always prefers the first option; the other always prefers the second. Transfers are infeasible. The principal designs and commits to a mechanism: a mapping from reported information profiles to a – potentially randomized – decision. One prominent example of a setting without transfers is the allocation of a fixed amount of money:

Example 1 (Budget allocation). *Upper management has endowed a division manager with a fixed budget. She can divide these funds between her two departments L, R. Her objective is to maximize expected returns. Department heads $i = \ell, r$ hold private information θ_i about the future marginal returns y_L, y_R and want to maximize their department's budget. Formally, $(\theta_\ell, \theta_r, y_L, y_R)$ follows some joint distribution and conditional on the private information the manager's marginal return from allocating 1\$ to L is $v(\theta_\ell, \theta_r) = E[y_L - y_R | \theta_\ell, \theta_r]$.*

To the best of our knowledge this is the first paper that characterizes all implementable mechanisms without transfers under arbitrary correlation. We find a connection between our mechanism design setting and a zero-sum game. Incentive compatibility of a mechanism given a type distribution corresponds to this distribution being a correlated equilibrium in the game induced by the mechanism.

Crémer and McLean's (1985; 1988) results for the corresponding setting with transfers suggest that the principal should be able to exploit correlation to induce truthful reporting. We define a preorder on type distributions and find that correlation has the opposite effect in our setting: it restricts the set of implementable mechanisms. Under their full-rank condition the set of implementable mechanisms collapses completely and the principal can never do better than choosing her ex-ante preferred option. We give necessary and sufficient conditions for the existence of a "profitable" mechanism that allows the principal to do better. When she is ex-ante indifferent the existence of a profitable mechanism is equivalent to a non-additive payoff structure. When she is not ex-ante indifferent a key insight is that choosing a mechanism corresponds to introducing endogenous correlation. Existence of a profitable mechanism depends on the value of a related optimal transport problem in which the principal chooses this endogenous correlation structure. Incentive constraints translate into an equal marginals condition and an orthogonality constraint between the exogenously given type distribution and the endogenously chosen one.

One application of our results is the problem of allocating a single nondisposable good between two agents. In section 6, we extend our setting and study the problem of allocating a (potentially disposable) good among n agents under independence. When the good has to be allocated, we find that a profitable mechanism exists if and only if a generalized version

of the additivity condition is violated. Under free disposal, a profitable mechanism exists if and only if there is an agent such that the principal's value from allocating to that agent depends on the types of other agents.

More broadly, our results convey that there is large class of settings without transfers where the principal can profit from designing a mechanism that elicits the agent's information despite their opposed interests. This scope for communication does not rely on any correlation of the agents' information but instead on types interdependence in the principal's preferences.

2 Model

There is a principal, two agents $i = \ell, r$ and a decision: L or R. Agent ℓ always prefers L; agent r always prefers R. Agents enjoy utility 1 if their favored decision is taken and 0 otherwise.[1] Each agent has a private *type* $\theta_i \in \Theta_i$ ($|\Theta_i| < \infty$) and the type profile $\theta = (\theta_\ell, \theta_r)$ is drawn from a commonly known distribution $\pi(\theta_\ell, \theta_r)$ with positive[2] marginals π_ℓ, π_r. Let Π be the set of joint type distributions with positive marginals and let $\Pi(\pi_\ell, \pi_r)$ be the set of joint type distributions with marginals π_i. The independent type distribution with marginals π_i is denoted by $\pi_\ell \pi_r$.

The principal designs and commits to a mechanism. By the revelation principle she can restrict attention to direct, incentive-compatible[3] mechanisms $x \colon \Theta = \Theta_\ell \times \Theta_r \to [0, 1]$, where $x(\theta_\ell, \theta_r)$ denotes the probability that L is chosen if agent ℓ reports θ_ℓ and agent r reports θ_r. From now on we refer to direct mechanisms simply as mechanisms.

If the realized type profile is θ then the principal receives a payoff of $v_L(\theta)$ from L and of $v_R(\theta)$ from R, her (without loss) ex-ante preferred option. We normalize $v_R = 0$ so that $E_\pi[v_L(\boldsymbol{\theta})] \leq 0$.[4] From now on we denote $v_L = v$.

The principal's problem then reads:

$$\max_{0 \leq x(\theta) \leq 1} \quad E_\pi[v(\boldsymbol{\theta}) x(\boldsymbol{\theta})]$$

$$\text{s.t.} \quad E_\pi[x(\theta_\ell, \boldsymbol{\theta}_r)|\theta_\ell] \geq E_\pi[x(\theta_\ell', \boldsymbol{\theta}_r)|\theta_\ell] \qquad \forall \theta_\ell, \theta_\ell' \qquad (IC_\ell)$$

$$E_\pi[x(\boldsymbol{\theta}_\ell, \theta_r)|\theta_r] \leq E_\pi[x(\boldsymbol{\theta}_\ell, \theta_r')|\theta_\ell] \qquad \forall \theta_r, \theta_r' \qquad (IC_r)$$

Given $\pi \in \Pi$, let the set of IC mechanisms be $\mathcal{X}(\pi)$. A mechanism is said to be *profitable*

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[1] All of our results would apply unchanged if agents receive utility $\bar{u}_i(\theta_i)$ from their preferred decision and utility $\underline{u}_i(\theta_i)$ from their less preferred decision, where $\bar{u}_i > \underline{u}_i$.

[2] This is without loss. Note that we do not assume full support.

[3] More precisely: Bayesian IC. In this setting, the only ex-post IC mechanisms are constant mechanisms.

[4] Bold symbols denote random variables.

if it is IC and yields the principal a strictly greater payoff than choosing her ex-ante preferred option R without consulting the agents. Given our normalization of the principal's payoff, an IC mechanism x is profitable if and only if $E_\pi[v(\boldsymbol{\theta})x(\boldsymbol{\theta})] > 0$.

3   Implementation

In this section we characterize the set of IC mechanisms given a type-distribution. The proof is based on the observation that incentive-compatibility can be phrased in terms of the correlated equilibria of an auxiliary two-player zero-sum game.

Let $\pi \in \Pi$ and let x be an IC mechanism. The IC conditions read

$$\sum_{\theta_r} \pi(\theta_r|\theta_\ell)x(\theta_\ell,\theta_r) \geq \sum_{\theta_r} \pi(\theta_r|\theta_\ell)x(\theta_\ell',\theta_r) \quad \forall \theta_\ell, \theta_\ell' \qquad (IC_\ell')$$

$$\sum_{\theta_\ell} \pi(\theta_\ell|\theta_r)x(\theta_\ell,\theta_r) \leq \sum_{\theta_\ell} \pi(\theta_\ell|\theta_r)x(\theta_\ell,\theta_r') \quad \forall \theta_r, \theta_r'. \qquad (IC_r')$$

Consider now the auxiliary two-player zero-sum game G in which the Maximizer chooses θ_ℓ, the Minimizer chooses θ_r and the objective (i.e. the Maximizer's payoff if θ_ℓ and θ_r is chosen) is $x(\theta_\ell,\theta_r)$. In this game we can interpret π as a *correlated strategy* under which the Maximizer's payoff is $\sum_{\theta_\ell}\sum_{\theta_r}\pi(\theta_\ell,\theta_r)x(\theta_\ell,\theta_r)$. With this interpretation the IC conditions become *obedience conditions* and π becomes a *correlated equilibrium* of G:

Lemma 1. *A mechanism x is IC under some type distribution $\pi \in \Pi$ if and only if π is a correlated equilibrium of the auxiliary two-player zero-sum game in which the Maximizer chooses $\theta_\ell \in \Theta_\ell$, the Minimizer chooses $\theta_r \in \Theta_r$ and the Maximizer's payoff is $x(\theta_\ell,\theta_r)$.*

Note that under the mechanism design interpretation π is an exogenous part of the environment while x is endogenous. In the auxiliary game the roles are exactly flipped: x is an exogenous while π is endogenous.

Proposition 1. *Let $\pi \in \Pi$ and let x be some mechanism. The following are equivalent.*

(i) x is IC under π.

(ii) For each type θ_i of each agent i, $E_\pi[x(\theta_i', \boldsymbol{\theta}_{-i})|\theta_i] = E_\pi[x(\boldsymbol{\theta})]$, for any report θ_i'.

Moreover, if x is IC then $E_\pi[x(\boldsymbol{\theta})] = \bar{x}$, where $\bar{x} = \max\limits_{\sigma_\ell \in \Delta\Theta_\ell} \min\limits_{\sigma_r \in \Delta\Theta_r} \sum\limits_{\theta_\ell}\sum\limits_{\theta_r} \sigma_\ell(\theta_\ell)\sigma_r(\theta_r)x(\theta_\ell,\theta_r)$ is the maximin value of the auxiliary game.

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Proposition 1 says that a mechanism is IC if and only if each type of each agent is indifferent between every possible report and each type's expectations of x are given by the distribution-independent constant $\bar{x}$.

Proof. Any mechanism that satisfies (ii) is clearly IC. To show the converse let $\pi \in \Pi$ and let x be an IC mechanism. By Lemma 1, π is a correlated equilibrium of the auxiliary game G in which the Maximizer chooses θ_ℓ, the Minimizer chooses θ_r and the Maximizer's payoff from such an action profile is $x(\theta_\ell, \theta_r)$.

Suppose now that the Minimizer obeys while the Maximizer ignores his recommendation under π and instead plays the mixed strategy π_ℓ. As this cannot be profitable to him we get

$$\sum_{\theta_\ell} \sum_{\theta_r} \pi(\theta_\ell, \theta_r) x(\theta_\ell, \theta_r) \geq \sum_{\theta_\ell} \sum_{\theta_r} \pi(\theta_\ell, \theta_r) \sum_{\theta'_\ell} \pi_\ell(\theta'_\ell) x(\theta'_\ell, \theta_r).$$

But the last term is simply $\sum_{\theta_\ell} \sum_{\theta_r} \pi_\ell(\theta_\ell) \pi_r(\theta_r) x(\theta_\ell, \theta_r)$ and so we obtain

$$\sum_{\theta_\ell} \sum_{\theta_r} \pi(\theta_\ell, \theta_r) x(\theta_\ell, \theta_r) \geq \sum_{\theta_\ell} \sum_{\theta_r} \pi_\ell(\theta_\ell) \pi_r(\theta_r) x(\theta_\ell, \theta_r). \tag{1}$$

The symmetric argument for the Minimizer implies that the opposite inequality to (1) must also hold. We conclude that (1) must hold with equality. Finally, since π_ℓ has full support if there were some pair $\theta_\ell, \theta'_\ell$ for which the inequality in the obedience constraint (IC'_ℓ) were strict then (1) could not hold with equality. Hence (IC'_ℓ) and (IC'_r) must always bind. Thus, if player i is recommended some action θ_i then he is indifferent between all actions and his interim expectation of x is $\bar{x}_i(\theta_i) = E_\pi[x(\boldsymbol{\theta})|\theta_i]$. We will now show that the interim expectations $\bar{x}_i(\theta_i)$ are actually all the same.

Let $\sigma = (\sigma_\ell, \sigma_r)$ be a Nash equilibrium of G and let $\bar{x} = \sum_{\theta_\ell} \sigma_\ell(\theta_\ell) \sum_{\theta_r} \sigma_r(\theta_r) x(\theta_\ell, \theta_r)$ be the Maximizer's expected payoff under σ. Then for any θ_r it holds that

$$\bar{x} \geq \sum_{\theta_\ell} \pi(\theta_\ell|\theta_r) \sum_{\tilde{\theta}_r} \sigma_r(\tilde{\theta}_r) x(\theta_\ell, \tilde{\theta}_r) = \sum_{\tilde{\theta}_r} \sigma_r(\tilde{\theta}_r) \sum_{\theta_\ell} \pi(\theta_\ell|\theta_r) x(\theta_\ell, \tilde{\theta}_r)$$

$$\geq \sum_{\theta_\ell} \pi(\theta_\ell|\theta_r) x(\theta_\ell, \theta_r) = \bar{x}_r(\theta_r),$$

where the first inequality holds since the mixed strategy $\pi(\cdot|\theta_r)$ is not a profitable deviation from σ_ℓ; the second inequality follows from type θ_r's IC constraint for $\tilde{\theta}_r$ and the last equality is by definition. Combining this inequality with the corresponding inequality for the other player we thus have that

$$\bar{x}_r(\theta_r) \leq \bar{x} \leq \bar{x}_\ell(\theta_\ell) \quad \forall \theta_\ell, \theta_r.$$

Since the terms on the left and the right hand side of the above inequalities are equal in expectation and all θ_ℓ and θ_r occur with positive probability both inequalities above must always bind. That is to say, for each agent i, $E_\pi[x(\theta_i', \boldsymbol{\theta}_{-i})|\theta_i] = \bar{x}$ for any θ_i and θ_i'. Finally, note that $\bar{x} = \max_{\sigma_\ell \in \Delta\Theta_\ell} \min_{\sigma_r \in \Delta\Theta_r} \sum_{\theta_\ell} \sum_{\theta_r} \sigma_\ell(\theta_\ell)\sigma_r(\theta_r)x(\theta_\ell, \theta_r)$ holds since (σ_ℓ, σ_r) is a Nash equilibrium of the zero sum game G. $\qquad\square$

4　Comparative statics for implementation

In this section, we study how the set of implementable mechanisms depends on the type distribution. We define a preorder on distributions and derive a monotone comparative statics result for the correspondence $\pi \mapsto \mathcal{X}(\pi)$. We conclude that correlation has a restrictive effect.

Definition 1. *Let* $\tau^0, \tau^1, \ldots, \tau^k \in \Delta\Theta_{-i}$ *be any beliefs over types of agent* $-i$. *Then* $\{\tau^1, \ldots, \tau^k\}$ *is said to span* τ^0 *if there exist coefficients* $\alpha_j \in \mathbb{R}$ *such that*

$$\tau^0(\theta_{-i}) = \sum_{j=1}^k \tau^j(\theta_{-i})\alpha_j \quad \forall \theta_{-i}.$$

Given joint type distributions $\pi, \tilde{\pi} \in \Pi$, π *is said to span* $\tilde{\pi}$ *if for all* θ_i, $\{\pi(\cdot|\tilde{\theta}_i) \colon \tilde{\theta}_i \in \Theta_i\}$ *spans* $\tilde{\pi}(\cdot|\theta_i)$, $i = \ell, r$.

Hence π spans $\tilde{\pi}$ if each interim belief an agent can hold under $\tilde{\pi}$ is a linear combination of some interim beliefs that he can hold under π. Spanning is reflexive and transitive but not anti-symmetric and therefore defines a preorder.

Example 2. *Let* $\pi \in \Pi$ *with marginals* π_i. *Then* π *spans the independent type distribution* $\tilde{\pi} = \pi_\ell \pi_r$ *because*
$$\pi_i(\theta_i) = \sum_{\theta_{-i}} \pi(\theta_i|\theta_{-i})\pi_{-i}(\theta_{-i}) \quad \forall \theta_i \forall i.$$

Example 3. *A joint distribution* $\pi \in \Pi$ *spans every other joint distribution* $\tilde{\pi} \in \Pi$ *if and only if the matrix* $(\pi(\theta_\ell, \theta_r)) \in \mathbb{R}^{\Theta_\ell \times \Theta_r}$ *has full column-rank and full row-rank. This is exactly the condition introduced by Crémer and McLean (see Assumption 4 in their 1985 paper and Theorem 1 in their 1988 paper).*

Our first application of the spanning relation shows that the set of IC mechanisms cannot shrink when passing from π to some other type distribution $\tilde{\pi}$ that is spanned by π.

Proposition 2. *Let $\pi, \tilde{\pi} \in \Pi$ be type distributions. If π spans $\tilde{\pi}$ then*

$$\mathcal{X}(\pi) \subset \mathcal{X}(\tilde{\pi}).$$

Proof. By Proposition 1 a mechanism x is IC under π if and only if

$$\sum_{\theta_{-i}} \pi(\theta_{-i}|\theta_i)(x(\theta_i', \theta_{-i}) - \bar{x}) = 0 \quad \forall \theta_i, \theta_i', i = \ell, r.$$

Now let x be IC under π and consider some $\tilde{\pi} \in \Pi$ spanned by π. By definition, there exist coefficients $\alpha_i(\theta_i, \tilde{\theta}_i)$ such that

$$\tilde{\pi}(\theta_{-i}|\theta_i) = \sum_{\tilde{\theta}_i} \pi(\theta_{-i}|\tilde{\theta}_i)\alpha_i(\theta_i, \tilde{\theta}_i) \quad \forall \theta_i, \theta_{-i}, i = \ell, r.$$

But then x must also be IC under $\tilde{\pi}$ because for all θ_i, θ_i':

$$\sum_{\theta_{-i}} \tilde{\pi}(\theta_{-i}|\theta_i)(x(\theta_i', \theta_{-i}) - \bar{x}) = \sum_{\tilde{\theta}_i} \alpha_i(\theta_i, \tilde{\theta}_i) \sum_{\theta_{-i}} \pi(\theta_{-i}|\tilde{\theta}_i)(x(\theta_i', \theta_{-i}) - \bar{x}) = 0.$$

$\square$

The proof (appendix) of the next result is another application of the spanning relation.

Proposition 3. *Let $\pi \in \Pi$ with marginals π_i and let x be some mechanism. Then:*

1. If x is IC under π then x is also IC under the independent type distribution $\tilde{\pi} = \pi_\ell \pi_r$.

2. If the matrix $(\pi(\theta_\ell, \theta_r)) \in \mathbb{R}^{\Theta_\ell \times \Theta_r}$ has full rank then only constant mechanisms are IC.

The maximal elements of the spanning preorder are exactly the full-rank distributions and its minimal elements are exactly the independent distributions.

Crémer and McLean (1985) show in a setting with transfers that full rank correlation makes it possible to implement any allocation rule while paying zero information rents.[5]

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[5]The full-rank condition is often seen as generic. In many applications, though, it is not satisfied even when types are correlated. For example, assume that there exists a finite underlying state of the world $\omega \in \{1, \ldots k\}$ such that θ_ℓ and θ_r are independent given ω. That is, $\pi(\theta_\ell, \theta_r|\omega) = \pi_\ell(\theta_\ell|\omega)\pi_r(\theta_r|\omega)$ $\quad \forall \theta_\ell, \theta_r, \omega$. Then

$$\pi(\theta_\ell, \theta_r) = \sum_{\omega=1}^{k} \pi_\ell(\theta_\ell|\omega)\pi_r(\theta_r|\omega) \Pr(\omega).$$

and so each column $\pi(\cdot, \theta_r)$ of the matrix $(\pi(\theta_\ell, \theta_r))_{\theta_\ell, \theta_r}$ is a linear combination of the k vectors $\pi_\ell(\cdot|\omega)$, $\omega = 1, \ldots, k$ (with coefficients $\alpha_{\theta_r}(\omega) = \pi_r(\theta_r|\omega) \Pr(\omega)$). Hence $\text{rank}(\pi) \leq k$.

We show that under the same full-rank condition only mechanisms that ignore the agents' reports are IC. Absent full rank correlation, any mechanism that is IC under correlation must also be IC when types are independent. This shows that the spirit of Crémer and McLean's results is inverted in our setting. The next example illustrates this difference.

Example 4. *Assume $\Theta_\ell = \Theta_r = \{-1, 1\}$, $\pi_\ell = \pi_r = \frac{1}{2}$ and $v(\theta) = \theta_\ell \theta_r$. Both options yield the principal an ex-ante expected payoff of 0 while the first best mechanism x^* would choose L iff $\theta_\ell = \theta_r$ and yield $E[v(\boldsymbol{\theta}) x^*(\boldsymbol{\theta})] = \frac{1}{2} > 0$. If types are independent then x^* is actually IC because from each agent's perspective, any report will lead to the same probability of L. Now assume instead that types are correlated and that π is given by*

	-1	1
-1	$0.25 - \varepsilon$	$0.25 + \varepsilon$
1	$0.25 + \varepsilon$	$0.25 - \varepsilon$

where $0 < \varepsilon \leq \frac{1}{4}$ is arbitrary. Then x^* is not IC anymore: For example, type 1 of agent ℓ would infer from his type that the other agent's type is probably -1 and would therefore claim to be type -1 instead of being truthful. Since the distribution matrix has full rank, Proposition 3 shows that the only remaining IC mechanisms are constant.

This example also illustrates how more correlated distributions make implementation harder because agents become more informed about each other's types.

5   Profitable mechanisms

In this section we investigate when the principal can design a profitable mechanism. We attack this question from two different angles. Our first characterization is in terms of the principal's objective and applies when the principal is ex-ante indifferent between the two options. The second characterization is in terms of a related optimal transport problem. It also yields an explicit characterization of incentive-compatible mechanisms under independence.

5.1   The role of the objective

Definition 2. *The principal's objective is said to be additive if there exist functions $v_i \colon \Theta_i \to \mathbb{R}$ such that*

$$v(\theta_\ell, \theta_r) = v_\ell(\theta_\ell) + v_r(\theta_r) \quad \forall \theta_\ell, \theta_r.$$

Given $\pi \in \Pi$ the objective is said to be π-additive if there exist coefficients $\lambda_\ell(\theta_\ell, \tilde{\theta}_\ell)$, $\lambda_r(\theta_r, \tilde{\theta}_r) \in \mathbb{R}$ such that

$$v(\theta_\ell, \theta_r)\pi(\theta_\ell, \theta_r) = \sum_{\tilde{\theta}_\ell} \lambda_\ell(\theta_\ell, \tilde{\theta}_\ell)\pi(\theta_r|\tilde{\theta}_\ell) + \sum_{\tilde{\theta}_r} \lambda_r(\theta_r, \tilde{\theta}_r)\pi(\theta_\ell|\tilde{\theta}_r) \quad \forall \theta_\ell, \theta_r. \tag{2}$$

Additivity is a special case of π-additivity (take $\lambda_i(\theta_i, \tilde{\theta}_i) = v_i(\theta_i)\pi_i(\tilde{\theta}_i)\mathbf{1}_{(\tilde{\theta}_i = \theta_i)}$) and it is easily seen that the two concepts coincide when $\pi = \pi_\ell \pi_r$. To interpret π-additivity let the type distribution by $\pi \in \Pi$ and consider some mechanism x. When v is π-additive we then get from (2) that

$$E_\pi[v(\boldsymbol{\theta})x(\boldsymbol{\theta})] = \sum_{\theta_\ell, \tilde{\theta}_\ell} \lambda_\ell(\theta_\ell, \tilde{\theta}_\ell)E_\pi[x(\theta_\ell, \boldsymbol{\theta}_r)|\tilde{\theta}_\ell] + \sum_{\theta_r, \tilde{\theta}_r} \lambda_r(\theta_\ell, \tilde{\theta}_r)E_\pi[x(\boldsymbol{\theta}_\ell, \theta_r)|\tilde{\theta}_r]$$

so that $E_\pi[v(\boldsymbol{\theta})x(\boldsymbol{\theta})]$ is a linear combination of the potential expected payoffs $E_\pi[x(\theta_i, \boldsymbol{\theta}_{-i})|\tilde{\theta}_i]$ of types $\tilde{\theta}_i$ from any (mis-)report θ_i. If x is IC then $E_\pi[v(\boldsymbol{\theta})x(\boldsymbol{\theta})]$ is the principal's expected payoff from x and the "misreporting expectations" must all coincide with the maximin value $\bar{x}$. Hence replacing x by the constant mechanism $\tilde{x} \equiv \bar{x}$ does not change the principal's payoff and x cannot be profitable. A necessary condition for the existence of a profitable mechanism is thus that the principal's objective is not π-additive. If the principal is ex-ante indifferent between L and R then this condition is also sufficient

Proposition 4. *Let types be distributed according to $\pi \in \Pi$. A profitable mechanism can exist only if the principal's objective is not π-additive. If $E_\pi[v(\boldsymbol{\theta})] = 0$ then a profitable mechanism exists if and only if the principal's objective is not π-additive. In particular, if types are independent then a profitable mechanism exists if and only if the principal's objective is not additive.*

The proof (appendix) works by projecting $v\pi$ on the linear subspace U of functions that can be expressed in the form of the right hand side of (2). Given ex-ante indifference the principal's expected payoff in an IC mechanism depends only on the part of $v\pi$ that is orthogonal to U. We construct a mechanism that yields a strictly positive payoff whenever this projection residual is nonzero.

Example 1 (continued). *Consider again the budget allocation problem. Recall that*

$$v(\theta_\ell, \theta_r) = E[y_L - y_R|\theta_\ell, \theta_r] = E[y_L|\theta_\ell, \theta_r] - E[y_R|\theta_\ell, \theta_r].$$

Let $h_i(\theta_\ell, \theta_r) = E[y_i|\theta_i, \theta_{-i}]$. If $h_i(\theta_\ell, \theta_r)$ depends only on θ_i then Proposition 4 implies that there does not exist a profitable mechanism. Hence a necessary condition for the existence of

a profitable mechanism is that at least one department head has information that is relevant to the future return of the other department. Now assume that types are iid and that $h_\ell = h_r = h$. Then $E[y_\ell] = E[h(\theta_\ell, \theta_r)] = E[h(\theta_r, \theta_\ell)] = E[y_r]$ so that the principal is ex-ante indifferent. Then a profitable mechanism exists if, and only if $h(\theta_\ell, \theta_r) - h(\theta_r, \theta_\ell)$ is not additive.

5.2   The role of correlation

Correlation between agent-types affects the principal through two distinct channels. Firstly, correlation affects the set of mechanisms in which agents find it optimal to be truthful (see Section 4). Secondly, fixing a mechanism and assuming that agents are truthful, correlation can increase or decrease the principal's expected payoff by concentrating more mass on specific type profiles. In this section we show that the principal's problem can be viewed as a problem of choosing an "optimal correlation structure".

We start by reinterpreting incentive-compatibility. The proof is in the appendix.

Lemma 2. *Let the type distribution be $\pi \in \Pi$. A mechanism x is IC if and only if*

1. *Agents are ex-ante indifferent between reports:* $E_\pi[x(\theta_i', \boldsymbol{\theta}_{-i})] = E_\pi[x(\theta_i'', \boldsymbol{\theta}_{-i})] \quad \forall \theta_i', \theta_i'' \, \forall i$

2. *Their type realizations are uninformative:* $E_\pi[x(\theta_i', \boldsymbol{\theta}_{-i})|\theta_i] = E_\pi[x(\theta_i', \boldsymbol{\theta}_{-i})] \quad \forall \theta_i, \theta_i' \, \forall i.$

Ex-ante indifference is equivalent to IC under the independent type distribution $\pi_\ell \pi_r$. Uninformativeness implies that agents cannot gain any payoff-relevant information from their type about their opponent's type. Note that this is automatically satisfied if types are independent. Correlation therefore restricts the set of IC mechanisms by making the agents more informed which adds additional incentive-constraints. From this perspective, IC under correlation lies mid-way between IC under independence and IC under full information.

Lemma 2 allows us to derive a necessary and sufficient criterion for the existence of a profitable mechanism. We need the following definition.

Definition 3. *Two joint type distributions $\pi, \tilde{\pi} \in \Pi(\pi_\ell, \pi_r)$ with the same marginals $\pi_i > 0$ are said to be orthogonal if*

$$\mathrm{Cov}(\pi(\theta_i|\boldsymbol{\theta}_{-i}), \tilde{\pi}(\theta_i'|\boldsymbol{\theta}_{-i})) = 0 \quad \forall \theta_i, \theta_i' \, \forall i.$$

Hence π and $\tilde{\pi}$ are orthogonal if the random variables $\pi(\theta_i|\boldsymbol{\theta}_{-i})$ and $\tilde{\pi}(\theta_i'|\boldsymbol{\theta}_{-i})$ are uncorrelated for all θ_i, θ_i', $i = \ell, r$. Note that

$$\mathrm{Cov}(\pi(\theta_i|\boldsymbol{\theta}_{-i}), \tilde{\pi}(\theta_i'|\boldsymbol{\theta}_{-i})) = \sum_{\theta_{-i}} [\pi(\theta_i|\boldsymbol{\theta}_{-i}) - \pi_i(\theta_i)][\tilde{\pi}(\theta_i'|\boldsymbol{\theta}_{-i}) - \pi_i(\theta_i')]\pi_{-i}(\boldsymbol{\theta}_{-i})$$

and $\pi(\theta_i|\theta_{-i}) - \pi_i(\theta_i)$ is the update of type θ_{-i} about the probability of type θ_i under π. Clearly, if one of π or $\tilde{\pi}$ is the independent type distribution $\pi_l \pi_r$ then orthogonality is automatically satisfied. Otherwise the condition says that for all θ_i, θ_i', the vector $\pi(\theta_i|\cdot) - \pi_i(\theta_i) \in \mathbb{R}^{\Theta_{-i}}$ of possible belief updates of agent $-i$ about the probability of type θ_i under π must be orthogonal to the vector of updates $\tilde{\pi}(\theta_i'|\cdot) - \pi_i(\theta_i) \in \mathbb{R}^{\Theta_{-i}}$ about the probability of θ_i' under $\tilde{\pi}$ under the inner product $\langle a, b \rangle = \sum_{\theta_{-i}} a(\theta_{-i})b(\theta_{-i})$ on $\mathbb{R}^{\Theta_{-i}}$.

The next result shows that the problem of finding a profitable mechanism is intricately related to the choice of an "optimal correlation strucuture": A profitable mechanism exists if and only if it is possible to find some alternative correlation structure that is orthogonal to the exogenously given one and such that — under the alternative correlation structure (and with a suitably transformed objective) — L becomes the principal's ex-ante strictly preferred option. This can be phrased as a constrained optimal transport problem.[6]

Proposition 5. *Let the type distribution be $\pi \in \Pi$ and denote its marginals by π_i. Let $\hat{v} = v\pi/\pi_\ell \pi_r$. A profitable mechanism exists if and only if*

$$\left(\max_{\tilde{\pi} \in \Pi(\pi_\ell, \pi_r)} E_{\tilde{\pi}}[\hat{v}(\boldsymbol{\theta})] \ \ s.t. \ \tilde{\pi} \ is \ orthogonal \ to \ \pi \right) > 0. \tag{3}$$

In particular, if types are independent then a profitable mechanism exists if, and only if

$$\max_{\tilde{\pi} \in \Pi(\pi_\ell, \pi_r)} E_{\tilde{\pi}}[v(\boldsymbol{\theta})] > 0.$$

To explain how the constrained optimal transport problem in Proposition 5 is related to the principal's problem let x be some mechanism. Together with π, x induces a density $g(\theta) = \pi(\theta)x(\theta) = \frac{\pi(\theta)}{\pi_\ell(\theta_\ell \pi_r(\theta_r)} f(\theta)$ of a measure on Θ whose "correlation structure" depends on an exogenous part $\frac{\pi}{\pi_\ell \pi_r}$ and an endogenous part f. Instead of in terms of mechanisms, the principal's problem can also be phrased in terms of f. Requiring ex-ante indifference for the agents then translates into requiring that f should be a nonnegative multiple of some probability distribution $\tilde{\pi} \in \Pi$ with the same marginals as π: $f = q\tilde{\pi}$ ($q \in [0,1]$). Uninformativeness translates into $\tilde{\pi}$ being orthogonal to π. Under this reparametrization the principal's objective becomes $qE_{\tilde{\pi}}[\hat{v}(\theta)]$ and the (upper) feasibility constraint on the mechanism becomes a correlation constraint: $q\frac{\tilde{\pi}}{\pi_\ell \pi_r} \leq 1$. If there is a profitable $\tilde{\pi}$ then the principal therefore faces a tradeoff between up-scaling her objective (by increasing q) and the ability to concentrate more mass on type profiles with a positive objective value (by decreasing q). The mere existence of a profitable $\tilde{\pi}$ does not depend on the correlation constraint, however, and after dropping this constraint and dividing everything by q we

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[6]For an in-depth treatment of optimal transport see Villani (2009).

arrive at the formulation in Proposition 5.

The proof of the above proposition (in the appendix) yields another characterization of incentive compatible mechanisms when types are independent.

Corollary 1. *If types are independent then a mechanism x is IC if and only if there exist nonnegative coefficients $\{\gamma_j\}_{j=1}^k$ ($k \geq 0$) and extreme points[7] π^j of $\Pi(\pi_\ell, \pi_r)$ such that*

$$x = \sum_{j=1}^k \frac{\pi^j}{\pi_\ell \pi_r} \gamma_j.$$

Consider an example where both agents have the same number of types (without loss $\Theta_\ell = \Theta_r$) and where marginals are uniform. Together with the Birkhoff-von Neumann Theorem the characterization then implies that a mechanism is IC if and only if it can be decomposed into mechanisms where, up to relabeling of the types, the principal chooses L if and only if both agents make the same report. This illustrates how incentive-compatibility is fundamentally based on the inability (and unwillingness) of the agents to coordinate.

Example 5. *Assume $\Theta_\ell = \Theta_r = \{1, \ldots, m\}$. A mechanism x is said to be a match-your opponent mechanism if there exists a matching[8] $m : \Theta_\ell \to \Theta_r$ such that*

$$x(\theta_\ell, \theta_r) = \begin{cases} 1, & \text{if } \theta_r = m(\theta_\ell) \\ 0, & \text{otherwise} \end{cases}$$

Assume that types are independent with $\pi_i = \frac{1}{N}$. Using the Birkhoff-von Neumann Theorem and Corollary 1, a mechanism x is IC if and only if there exist match-your-opponent mechanisms x^j and nonnegative coefficients γ_j such that

$$x = \sum_{j=1}^k x^j \gamma_j.$$

Thus, a profitable mechanism exists if and only if there exists a profitable match-your-opponent mechanism. If the principal's objective is supermodular it follows[9] that a profitable mechanism exists if and only if

$$\sum_{t=1}^m v(t, t) > 0.$$

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[7]Recall that an element of a convex set is an extreme point of the set if is is not the midpoint of a line-segment connecting two distinct points in the set. For a characterization of the extreme points of $\Pi(\pi_\ell, \pi_r)$ see Brualdi (2006), Theorem 8.1.2.

[8]A matching is a bijective function.

[9]See Becker (1973); Vince (1990); Hardy et al. (1952).

Indeed, as long as types are independent and agents are symmetric (i.e. $\pi_\ell(t) = \pi_r(t)$, $t = 1, \ldots, m$), it can be shown that a profitable mechanism exists if and only if $\sum_{t=1}^{m} \pi_\ell(t)v(t,t) > 0$.[10]

6   Allocation with more than two agents and disposal

One application of our results is the problem of allocating a single nondisposable good between two agents. In this section, we extend our setting and study the problem of allocating a (potentially disposable) good among n agents $i = 1, \ldots, n$.

Agents are again expected utility maximizers and enjoy utility 1 from receiving the good and 0 otherwise.[11] Every agent has a private type $\theta_i \in \Theta_i$. The set of type profiles $\Theta = \Pi_i \Theta_i$ is finite. Throughout this section we assume that types are independent; the joint type distribution is denoted by $\pi(\theta_1, \ldots, \theta_n) = \pi_1(\theta_1) \ldots \pi_n(\theta_n)$[12].

The principal's value from allocating the good to agent i can depend on the types $\theta = (\theta_1, \ldots, \theta_n)$ of all agents and is denoted by $v_i(\theta) \in \mathbb{R}$. A (direct) mechanism specifies for each agent i and every profile θ the probability of allocating the good to this agent when the report profile is θ.

We distinguish between the case where the principal can commit to dispose the good from the case where she is forced to allocate to one of the agents.[13] We normalize the principal's utility from disposing the good to 0. If the principal must allocate the good the feasibility constraint reads: $\sum_{i=1}^{n} x_i(\theta) = 1$; under free disposal it reads: $\sum_{i=1}^{n} x_i(\theta) \leq 1$. In either case, the principal's problem is to find a feasible, incentive compatible mechanism that maximizes $E[\sum_{i=1}^{n} v_i(\boldsymbol{\theta})x_i(\boldsymbol{\theta})]$. As before, we will be interested in whether the principal can do better than choosing her ex-ante preferred option.

We first characterize the set of incentive compatible mechanisms. Whether or not disposal is possible, a mechanism is incentive compatible if and only if each agent's interim probability of obtaining the good does not depend on his report:

Lemma 3. *Assume there are n agents with independent types and let x be a mechanism (with or without disposal). Then x is incentive compatible if and only if*

$$E[x_i(\theta_i, \boldsymbol{\theta}_{-i})] = E[x_i(\boldsymbol{\theta})] \quad \forall i \, \forall \theta_i.$$

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[10]See Hoffman et al. (1963).

[11]All results apply unchanged if agents receive utility $\bar{u}_i(\theta_i) > 0$ from getting the good and 0 otherwise.

[12]As before, we assume without loss of generality that $\pi_i > 0$, $i = 1, \ldots, n$.

[13]An alternative interpretation of disposal is that the principal allocates the good to herself.

Let $\bar{v} = \max_i E[v_i(\boldsymbol{\theta})]$ be the principal's expected payoff from allocating to her ex-ante preferred agent.

An incentive compatible mechanism is profitable if it yields the principal strictly more than choosing her ex-ante preferred option (ignoring type reports). Formally, when there is free disposal, an incentive compatible mechanism is profitable if $\sum_i E[v_i(\boldsymbol{\theta})x_i(\boldsymbol{\theta})] > \max\{0, \bar{v}\}$. Without disposal it is profitable if $\sum_i E[v_i(\boldsymbol{\theta})x_i(\boldsymbol{\theta})] > \bar{v}$.

The following proposition generalizes the scope of Proposition 4 to the n agent case under independence.

Proposition 6. *Assume there are n agents with independent types and that the principal has to allocate to some agent. If the principal is unbiased then a profitable mechanism exists if and only if there do not exist functions $u_1(\theta_1), \ldots, u_n(\theta_n)$ such that*

$$v_i(\theta) - v_j(\theta) = u_i(\theta_i) - u_j(\theta_j) \quad \forall i, j \, \forall \theta. \tag{4}$$

In the proof (in the appendix) we show that a violation of condition (4) remains a necessary condition for the existence of a profitable mechanism when the principal is not unbiased. The above proposition also allows us to state a necessary and sufficient condition for the existence of a profitable mechanism when the principal is allowed to discard the good:

Proposition 7. *Assume there are n agents with independent types and that the principal does not have to allocate the good to the agents. If the principal is unbiased and $\bar{v} = 0$ then a profitable mechanism exists if and only there is an agent j such that $v_j(\theta_j, \theta_{-j})$ is not constant in θ_{-j}.*

7 Related Literature

Our main setting can be interpreted as an allocation problem without disposal. It therefore relates to Myerson (1981) who characterizes the set of IC-mechanisms with transfers under independence. Crémer and McLean (1985, 1988) show that with transfers, full rank correlation makes any allocation rule implementable.

Börgers and Postl (2009) study a setting without transfers and two agents with opposed interests. Their setting has a third option that acts as a compromise and types are iid. They consider utilitarian welfare and study second-best rules using numerical tools. Since utilitarian welfare is additive, our results underline the importance of the compromise option for their results. Kim (2017) considers a related setting with at least three ex-ante symmetric alternatives and several agents with iid private values whose interests are not necessarily opposed.

Feng and Wu (2019) ask in a setting without transfer with a perfect conflict of interests not between the agents but between the agents and the principal if the later can do better than choosing her ex-ante preferred option. Goldlücke and Tröger (2020) study "threshold mechanisms" with binary message spaces to assign an unpleasant task without transfers in a setting with symmetric agents with iid types.

The proof of Proposition 1 connects implementation of mechanisms with the properties of correlated equilibrium Aumann (1974, 1987) in zero sum games Rosenthal (1974).

Our comparative statics result for the set of IC mechanisms with respect to the spanning preorder relates to Blackwell's (1953; 1951) comparison of experiments (see also Börgers et al. (2013)) and Bergemann and Morris's (2016) comparison of information structures in games. A difference is that they compare signals which are informative about a payoff-relevant state while in our setting the signals themselves are payoff-relevant.

Since the set of DIC mechanisms in our setting coincides with the set of constant mechanisms, the existence of non-constant IC mechanisms is related to BIC-DIC equivalence Gershkov et al. (2013); Manelli and Vincent (2010). In our setting, IC under correlation can be viewed as a mid-way point between IC under independence and DIC.

A   Appendix

A.1   Proof of Lemma 2

Proof. Let $\pi \in \Pi$ and let x be IC. By Proposition 1, agents must be ex-ante indifferent between reports and their type realizations must be uninformative. Conversely, suppose that x satisfies the assumptions of Lemma 2. Ex-ante indifference combined with the law of iterated expectations implies that $E_\pi[x(\theta_i', \boldsymbol{\theta}_{-i})] = E_\pi[x(\boldsymbol{\theta})] \, \forall i, \theta_i'$. Hence for any i and θ_i, θ_i':

$$E_\pi[x(\theta_i', \boldsymbol{\theta}_{-i})|\theta_i] = E_\pi[x(\theta_i', \boldsymbol{\theta}_{-i})]$$
$$= E_\pi[x(\boldsymbol{\theta})],$$

where the first equality follows from uninformativeness. □

A.2   Proof of Proposition 3

Proof. The first assertion is an immediate consequence of Proposition 2 and Example 2. To see why the second assertion holds note that if $\tau^0, \tau^1, \ldots, \tau^k \in \Delta\Theta_{-i}$ and $y(\theta_{-i}) : \Theta_{-i} \to \mathbb{R}$

is any function such that

$$\sum_{\theta_{-i}} \tau^j(\theta_{-i}) y(\theta_{-i}) = 0 \quad \forall j = 1, \dots, k.$$

then also $\sum_{\theta_{-i}} \tau^0(\theta_{-i}) y(\theta_{-i}) = 0$ if τ^0 is spanned by $\{\tau^1, \dots, \tau^k\}$.

Now assume without loss that $|\Theta_\ell| \geq |\Theta_r|$. By the full rank-condition the vectors $(\pi(\cdot|\theta_\ell))_{\theta_\ell}$ contain a basis of $\mathbb{R}^{\Theta_r}$. In particular, for any θ_r, they span the belief $\mathbb{1}_{\theta_r}$ which puts mass 1 on θ_r. But then ℓ's IC constraints must be satisfied under that belief (consider the function $y_{\theta'_\ell}(\theta_r) = x(\theta'_\ell, \theta_r) - \bar{x}$). But that means that

$$x(\theta'_\ell, \theta_r) = \bar{x} \quad \forall \theta'_\ell.$$

Since θ_r was arbitrary it follows that x must be constant.

Next we will show that $\pi \in \Pi$ is maximal iff it has full rank and minimal iff it is an independent type distribution. We need to show that (i) π has full rank iff it spans every $\tilde{\pi}$ that spans it and (ii) π is an independent type distribution iff for every $\tilde{\pi}$ spanned by π it is also the case that $\tilde{\pi}$ spans π. Throughout, assume without loss of generality that $|\Theta_\ell| \geq |\Theta_r|$.

First note that for $\pi, \tilde{\pi} \in \Pi$, π spans $\tilde{\pi}$ if and only if the row space and the column space of $\tilde{\pi}$ are contained in the row space and the column space, respectively, of π.

Assume that $\pi \in \Pi$ has full rank and that $\tilde{\pi} \in \Pi$ spans π. Denote the column and row spaces of π and $\tilde{\pi}$ by V, W and $\tilde{V}, \tilde{W}$, respectively. Since $\tilde{\pi}$ spans π it holds that $V \subset \tilde{V}$ and $W \subset \tilde{W}$. Since π has full rank, V and W both have dimension $|\Theta_r|$. $\tilde{\pi}$ is an $|\Theta_\ell| \times |\Theta_r|$ matrix and so its column and row spaces cannot have a dimension larger than $|\Theta_r|$. Hence we must have $V = \tilde{V}$ and $W = \tilde{W}$. But that implies that π also spans $\tilde{\pi}$. Thus π is maximal. Conversely, assume that π is maximal. Let $\tilde{\pi} \in \Pi$ be some full rank distribution that spans π. Since π is maximal, π must then also span $\tilde{\pi}$. Hence the row space (and also the column space) of π must have dimension $|\Theta_r|$. This means that π has full rank.

Now let π be an independent distribution. Let $\tilde{\pi}$ be spanned by π. Then, for all θ_i, $\tilde{\pi}(\cdot|\theta_i)$ is a linear combination of the vectors $\pi(\cdot|\tilde{\theta}_i)$ ($\tilde{\theta}_i \in \Theta_i$). But since types are independent under π, the latter vectors all coincide with $\pi_i(\cdot)$. Hence $\tilde{\pi}(\cdot|\theta_i) = \pi_i(\cdot)$ for all θ_i, $i = \ell, r$. Thus $\tilde{\pi} = \pi$; in particular $\tilde{\pi}$ spans π. Hence π is a minimal element. Now let $\tilde{\pi}$ be a minimal element. Let $\tilde{\pi}$ be the independent distribution with the same marginals as π. Then π spans $\tilde{\pi}$ and since π is minimal, $\tilde{\pi}$ must also span π. But $\tilde{\pi}$ has rank one and so π must also have rank one. But that means that π must be an independent type distribution (thus $\pi = \tilde{\pi}$). $\quad\square$

A.3 Proof of Proposition 4

Proof. First consider a general $\pi \in \Pi$. Define $w(\theta) = v(\theta)\pi(\theta)$. If $E_\pi[v(\boldsymbol{\theta})] = 0$ and agents are truthful then the principal's payoff from some mechanism x is

$$\sum_\theta v(\theta)\pi(\theta)x(\theta) = E_\pi[v(\boldsymbol{\theta})]E_\pi[x(\boldsymbol{\theta})] + \sum_\theta v(\theta)\pi(\theta)(x(\theta) - E_\pi[x(\boldsymbol{\theta})])$$

$$= \sum_\theta w(\theta)(x(\theta) - E_\pi[x(\boldsymbol{\theta})]).$$

By Proposition 1, x is IC if and only if

$$\sum_{\theta_{-i}} \pi(\theta_{-i}|\theta_i)(x(\theta_i', \theta_{-i}) - E_\pi[x(\boldsymbol{\theta})]) = 0 \quad \forall \theta_i, \theta_i' \forall i. \tag{3}$$

First assume that there exist coefficients $\lambda_\ell(\theta_\ell, \tilde{\theta}_\ell)$, $\lambda_r(\theta_r, \tilde{\theta}_r) \in \mathbb{R}$ such that

$$w(\theta) = \sum_{\tilde{\theta}_\ell} \lambda_\ell(\theta_\ell, \tilde{\theta}_\ell)\pi(\theta_r|\tilde{\theta}_\ell) + \sum_{\tilde{\theta}_r} \lambda_r(\theta_r, \tilde{\theta}_r)\pi(\theta_\ell|\tilde{\theta}_r) \quad \forall \theta_\ell, \theta_r. \tag{4}$$

Then by (3), any IC mechanism satisfies $\sum_\theta w(\theta)(x(\theta) - E_\pi[x(\boldsymbol{\theta})]) = 0$ and so there is no profitable mechanism.

Now assume instead that there exist no coefficients λ_ℓ and λ_r such that w satisfies (4). Let U be the set of all $u \in \mathbb{R}^{\Theta_\ell \times \Theta_r}$ for which there exist coefficients $\lambda_\ell(\theta_\ell, \tilde{\theta}_\ell)$, $\lambda_r(\theta_r, \tilde{\theta}_r) \in \mathbb{R}$ such that

$$u(\theta) = \sum_{\tilde{\theta}_\ell} \lambda_\ell(\theta_\ell, \tilde{\theta}_\ell)\pi(\theta_r|\tilde{\theta}_\ell) + \sum_{\tilde{\theta}_r} \lambda_r(\theta_r, \tilde{\theta}_r)\pi(\theta_\ell|\tilde{\theta}_r) \quad \forall \theta_\ell, \theta_r.$$

Note that U is a linear subspace of $\mathbb{R}^{\Theta_\ell \times \Theta_r}$ (Indeed, $0 \in U$ and U is closed under addition and multiplication by scalars). Let $\hat{u}$ be the orthogonal projection of w onto U and let $\hat{w}$ be the orthogonal projection of w onto the orthogonal complement of U so that $w = \hat{w} + \hat{u}$. By assumption $w \notin U$, and so $\hat{w} \neq 0$.

As before, given any IC mechanism x it holds that $\sum_\theta \hat{u}(\theta)(x(\theta) - E_\pi[x(\boldsymbol{\theta})]) = 0$ and so the principal's payoff from any IC mechanism x is

$$\sum_\theta w(\theta)(x(\theta) - E_\pi[x(\boldsymbol{\theta})]) = \sum_\theta \hat{w}(\theta)(x(\theta) - E_\pi[x(\boldsymbol{\theta})]).$$

We will now construct a profitable mechanism, i.e. an IC mechanism for which the latter expression is positive. Define

$$\hat{x}(\theta) = \varepsilon(\hat{w}(\theta) - \min \hat{w})$$

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where $\varepsilon > 0$ is sufficiently small such that $\hat{x} \leq 1$. First note that $\hat{x}$ is IC. Indeed, for any $\theta'_\ell, \theta''_\ell$,

$$\sum_{\theta_r} \pi(\theta_r|\theta''_\ell)\hat{x}(\theta'_\ell, \theta_r) = \varepsilon \sum_{\theta_r} \pi(\theta_r|\theta''_\ell)\hat{w}(\theta'_\ell, \theta_r) - \varepsilon \min \hat{w}$$

$$= \varepsilon \sum_{\theta_\ell} \sum_{\theta_r} \underbrace{\left(\sum_{\tilde{\theta}_\ell} \mathbb{1}_{\theta_\ell = \theta'_\ell, \tilde{\theta}_\ell = \theta''_\ell} \pi(\theta_r|\tilde{\theta}_\ell) \right)}_{\in U} \hat{w}(\theta_\ell, \theta_r) - \varepsilon \min \hat{w}$$

$$= -\varepsilon \min \hat{w},$$

where the last equality follows because $\hat{w}$ lies in the orthogonal complement of U. Hence $\hat{x}$ is IC for agent ℓ. The proof that $\hat{x}$ is IC for agent r is symmetric.

We now show that $\hat{x}$ is actually a profitable mechanism. First note that the above incentive-compatibility calculation implies that $E_\pi[\hat{x}(\boldsymbol{\theta})|\theta_\ell] = -\varepsilon \min \hat{w}$ and in particular $E_\pi[x(\boldsymbol{\theta})] = -\varepsilon \min \hat{w}$. Thus the principal's payoff from $\hat{x}$ is

$$\sum_\theta \hat{w}(\theta)(x(\theta) - E_\pi[x(\boldsymbol{\theta})]) = \varepsilon \sum_\theta \hat{w}(\theta)\hat{w}(\theta)$$

$$> 0.$$

Hence $\hat{x}$ is a profitable mechanism.

Finally, assume that types are independent. Note that

$$v(\theta_\ell, \theta_r)\pi_\ell(\theta_\ell)\pi_r(\theta_r) = \sum_{\tilde{\theta}_\ell} \lambda_\ell(\theta_\ell, \tilde{\theta}_\ell)\pi(\theta_r) + \sum_{\tilde{\theta}_r} \lambda_r(\theta_r, \tilde{\theta}_r)\pi(\theta_\ell) \quad \forall \theta_\ell, \theta_r$$

if and only if

$$v(\theta_\ell, \theta_r) = \underbrace{\sum_{\tilde{\theta}_\ell} \tilde{\lambda}_\ell(\theta_\ell, \tilde{\theta}_\ell)}_{v_\ell(\theta_\ell)} + \underbrace{\sum_{\tilde{\theta}_r} \tilde{\lambda}_r(\theta_r, \tilde{\theta}_r)}_{v_r(\theta_r)},$$

where $\tilde{\lambda}_i(\theta_i, \tilde{\theta}_i) = \frac{\lambda_i(\theta_i, \tilde{\theta}_i)}{\pi_i(\theta_i)}$ and so the earlier condition reduces to additivity. $\square$

A.4 Proof of Proposition 5

Proof. By Lemma 2, the principal's problem can be written as

$$\max \quad \sum_\theta \hat{v}(\theta)\pi_\ell(\theta_\ell)\pi_r(\theta_r)x(\theta)$$

$$\text{s.t.} \quad \sum_{\theta_{-i}} \pi_{-i}(\theta_{-i})x(\theta_i', \theta_{-i}) = \sum_\theta \pi_\ell(\theta_\ell)\pi_r(\theta_r)x(\theta_\ell, \theta_r) \qquad \forall \theta_i' \, \forall i \qquad (I_i)$$

$$\sum_{\theta_{-i}} \pi(\theta_{-i}|\theta_i)x(\theta_i', \theta_{-i}) = \sum_{\theta_{-i}} \pi_{-i}(\theta_{-i})x(\theta_i', \theta_{-i}) \qquad \forall \theta_i, \theta_i' \, \forall i \qquad (U_i)$$

$$0 \le x(\theta) \le 1 \qquad \forall \theta. \qquad (F)$$

Here, (I_i) are the ex-ante indifference constraints (or, equivalently, the IC constraints under the independent type distribution $\pi_\ell \pi_r$) and (U_i) are the uninformativeness constraints. Now define

$$f(\theta_\ell, \theta_r) = \pi_\ell(\theta_\ell)\pi_r(\theta_r)x(\theta_\ell, \theta_r).$$

Using this substitution the principal's objective becomes $\sum_\theta v(\theta)f(\theta)$, the ex-ante indifference constraints become

$$\sum_{\theta_{-i}} f(\theta_i', \theta_r) = \pi_i(\theta_i') \sum_\theta f(\theta) \qquad \forall \theta_i' \, \forall i \qquad (6)$$

and the uninformativeness constraints can be written as

$$\sum_{\theta_{-i}} (\pi(\theta_i|\theta_{-i}) - \pi_i(\theta_i))f(\theta_i', \theta_{-i}) = 0 \qquad \forall \theta_i, \theta_i' \, \forall i \qquad (7)$$

while the feasibility constraints become

$$0 \le f(\theta_\ell, \theta_r) \le \pi_\ell(\theta_\ell)\pi_r(\theta_r) \qquad \forall \theta_\ell, \theta_r.$$

Equation (6) says that the marginals of f are proportional to π_i. Since f is nonzero, it is thus a nonnegative multiple of some joint probability distribution $\tilde{\pi}$ with marginals π_i. Hence the

principal's problem can be written as

$$\max_{q \in [0,1]} \max_{\tilde{\pi} \in \Pi(\pi_l, \pi_r)} \quad q \sum_{\theta} \hat{v}(\theta) \tilde{\pi}(\theta)$$

$$\text{s.t.} \qquad \sum_{\theta_{-i}} (\pi(\theta_i | \theta_{-i}) - \pi_i(\theta_i)) \tilde{\pi}(\theta_i', \theta_{-i}) = 0 \qquad \forall \theta_i, \theta_i' \, \forall i$$

$$q \tilde{\pi}(\theta_\ell, \theta_r) \leq \pi_l(\theta_\ell) \pi_r(\theta_r) \qquad \forall \theta_\ell, \theta_r$$

where $\Pi(\pi_l, \pi_r)$ is the set of joint type distributions with marginals π_i. A profitable mechanism therefore exists if and only if the latter problem has a positive optimal value.

Since any $\tilde{\pi} \in \Pi(\pi_\ell, \pi_r)$ can be made to satisfy the constraint $q \tilde{\pi}(\theta_\ell, \theta_r) \leq \pi_l(\theta_\ell) \pi_r(\theta_r)$ after appropriate scaling, the problem's optimal value is positive if and only if the value of the relaxed problem in which that constraint is left out is positive. Finally, note that

$$\sum_{\theta_{-i}} (\pi(\theta_i | \theta_{-i}) - \pi_i(\theta_i)) \tilde{\pi}(\theta_i', \theta_{-i}) = \sum_{\theta_{-i}} (\pi(\theta_i | \theta_{-i}) - \pi_i(\theta_i)) \tilde{\pi}(\theta_i' | \theta_{-i}) \pi_{-i}(\theta_{-i})$$

$$= \sum_{\theta_{-i}} (\pi(\theta_i | \theta_{-i}) - \pi_i(\theta_i))(\tilde{\pi}(\theta_i' | \theta_{-i}) - \pi_i(\theta_i')) \pi_{-i}(\theta_{-i})$$

because $\sum_{\theta_{-i}} (\pi(\theta_i | \theta_{-i}) - \pi_i(\theta_i)) \pi_i(\theta_i') \pi_{-i}(\theta_{-i}) = 0$. Therefore a profitable $\tilde{\pi}$ exists if and only if the optimal value of the following problem is positive:

$$\max_{\tilde{\pi} \in \Pi(\pi_l, \pi_r)} \quad \sum_{\theta} \hat{v}(\theta) \tilde{\pi}(\theta)$$

$$\text{s.t.} \qquad \sum_{\theta_{-i}} (\pi(\theta_i | \theta_{-i}) - \pi_i(\theta_i))(\tilde{\pi}(\theta_i' | \theta_{-i}) - \pi_i(\theta_i')) \pi_{-i}(\theta_{-i}) = 0 \qquad \forall \theta_i, \theta_i' \, \forall i.$$

This concludes the proof of Proposition 5. $\qquad \square$

A.5 Proof of Corollary 1

Proof. The proof of Proposition 5 shows that under independence, a mechanism x is IC if and only if there exists some $q \in [0, 1]$ and $\tilde{\pi} \in \Pi(\pi_l, \pi_r)$ such that $\pi_l \pi_r x = q \tilde{\pi}$. The set $\Pi(\pi_l, \pi_r)$ is a polytope (known as the transportation polytope), hence by the Weyl-Minkowski Theorem it is the convex hull of its finitely many extreme points. This implies the claim. $\quad \square$

A.6 Proof of Lemma

Let $x = (x_1, \ldots, x_n)$ be an incentive compatible mechanism. Let i be an agent and let $\theta_i, \theta_i' \in \Theta_i$. In order for type θ_i to be truthful, it must hold that $E[x_i(\theta_i, \boldsymbol{\theta}_{-i})] \geq E[x_i(\theta_i', \boldsymbol{\theta}_{-i})]$.

In order for type θ_i' to be truthful, $E[x_i(\theta_i, \boldsymbol{\theta}_{-i})] \leq E[x_i(\theta_i', \boldsymbol{\theta}_{-i})]$ must hold. Hence in any incentive compatible mechanism $E[x_i(\theta_i, \boldsymbol{\theta}_{-i})]$ is constant in θ_i, for all i. Since any mechanism satisfying the latter is also incentive compatible, the condition is equivalent to incentive compatibility. Finally, if $E[x_i(\theta_i, \boldsymbol{\theta}_{-i})]$ is constant in θ_i then it must equal $E[x_i(\boldsymbol{\theta})]$.

A.7 Proof of Proposition 6

Proof. First assume that (4) holds. It follows that there exist functions $u_i(\theta_i)$ such that $v_i(\theta) - v_n(\theta) = u_i(\theta_i) - u_n(\theta_n)$ for all i and θ. Recall that in an IC mechanism x, $\bar{x}_i := E[x_i(\theta_i, \boldsymbol{\theta}_{-i})]$ does not depend on θ_i. The principal's payoff from an incentive compatible mechanism x is therefore

$$
\begin{aligned}
\sum_i E[v_i(\boldsymbol{\theta}) x_i(\boldsymbol{\theta})] &= \sum_{i<n} E[(v_i(\boldsymbol{\theta}) - v_n(\boldsymbol{\theta})) x_i(\boldsymbol{\theta})] + E[v_n(\boldsymbol{\theta}) \sum_i x_i(\boldsymbol{\theta})] \\
&= \sum_{i<n} E[(u_i(\boldsymbol{\theta}_i) - u_n(\boldsymbol{\theta}_n)) x_i(\boldsymbol{\theta})] + E[v_n(\boldsymbol{\theta})] \\
&= \sum_{i<n} E[u_i(\boldsymbol{\theta}_i) E[x_i(\boldsymbol{\theta})|\boldsymbol{\theta}_i]] - E[u_n(\boldsymbol{\theta}_n) E[\sum_{i<n} x_i(\boldsymbol{\theta})|\boldsymbol{\theta}_n]] + E[v_n(\boldsymbol{\theta})] \\
&= \sum_{i<n} E[u_i(\boldsymbol{\theta}_i)] \bar{x}_i - E[u_n(\boldsymbol{\theta}_n)(1 - \bar{x}_n)] + E[v_n(\boldsymbol{\theta})] \\
&= \sum_i E[u_i(\boldsymbol{\theta}_i)] \bar{x}_i + E[v_n(\boldsymbol{\theta}) - u_n(\boldsymbol{\theta}_n)] \\
&= \sum_i E[v_i(\boldsymbol{\theta}) + u_n(\boldsymbol{\theta}_n) - v_n(\boldsymbol{\theta})] \bar{x}_i + E[v_n(\boldsymbol{\theta}) - u_n(\boldsymbol{\theta}_n)] \\
&= \sum_i E[v_i(\boldsymbol{\theta})] \bar{x}_i.
\end{aligned}
$$

Hence, if (4) holds then the principal's expected payoff from an incentive compatible mechanism x is the same as her expected payoff from the constant mechanism y given by $y_i(\theta) \equiv \bar{x}_i$. In particular, the principal cannot do better than allocating to her ex-ante preferred agent and so no profitable mechanism exists. Note that we have not used the unbiasedness assumption and so the following is true even if the principal is not unbiased: A profitable mechanism can only exist if (4) is violated.

Now let the principal be unbiased. Assume that (4) is violated. Then there do no exist functions $u_i(\theta_i)$ $(i = 1, \ldots, n)$ such that $v_i(\theta) - v_n(\theta) = u_i(\theta_i) - u_n(\theta_j)$ $(i < n)$. (If such functions did exist then it would follow that for any i, j: $v_i(\theta) - v_j(\theta) = v_i(\theta) - v_n(\theta) - (v_j(\theta) - v_n(\theta)) = u_i(\theta_i) - u_n(\theta_n) - (u_j(\theta_j) - u_n(\theta_n)) = u_i(\theta_i) - u_j(\theta_j)$ and so (4) would hold). We will now construct a profitable mechanism.

Let Ω be the vector space of functions from $\{1, \ldots, n-1\} \times \Theta$ to $\mathbb{R}$ and let U_i be the set of

functions from Θ_i to $\mathbb{R}$. Moreover, let $W \subset \Omega$ be the set of functions from $\{1, \ldots, n-1\} \times \Theta$ to $\mathbb{R}$ for which there exist functions $u_i(\theta_i)$ with $w_i(\theta) = \pi(\theta)(u_i(\theta_i) - u_n(\theta_n)) \; \forall i < n \, \forall \theta$. Now consider the following minimization problem

$$\min_{u_1 \in U_1, \ldots, u_n \in U_n} \sum_{i<n} \sum_{\theta} [\pi(\theta)(v_i(\theta) - v_n(\theta)) - \pi(\theta)(u_i(\theta) - u_n(\theta))]^2$$

$$= \min_{w \in W} \sum_{i<n} \sum_{\theta} [\tilde{v}_i(\theta) - w_i(\theta)]^2,$$

where $\tilde{v}_i(\theta) = \pi(\theta)(v_i(\theta) - v_n(\theta))$. Note that W is a linear subspace of Ω and hence the solution $\hat{w}$ to the above minimization problem is the orthogonal projection of $\tilde{v} \in \Omega$ onto W (all spaces are finite-dimensional and so existence is not an issue). Let $\hat{\varepsilon} = \tilde{v} - \hat{w}$ be the projection residual. Note that the optimal value of the minimization problem is zero if and only if (4) holds. By assumption, (4) is violated and hence in particular $\hat{\varepsilon}$ must be nonzero. Moreover, since $\hat{\varepsilon}$ is orthogonal to W, for any $h \in W$ it must hold that

$$\sum_{i<n} \sum_{\theta} h_i(\theta) \hat{\varepsilon}(\theta) = 0.$$

We will now use $\hat{\varepsilon}$ to construct a profitable mechanism. Let $\underline{\hat{\varepsilon}} = \min_{i<n,\theta} \hat{\varepsilon}_i(\theta)$ and let

$$\hat{z}_i(\theta) = \hat{\varepsilon}_i(\theta) - \underline{\hat{\varepsilon}} \quad \forall i < n \, \forall \theta.$$

By construction, $\hat{z} \in \Omega$ is nonnegative. Define

$$\hat{x}_i(\theta) = \alpha \hat{z}_i(\theta),$$

where $\alpha > 0$ is chosen sufficiently small such that $\sum_{i<n} \hat{x}_i(\theta) \leq 1$ for all θ. Also, define $\hat{x}_n(\theta) = 1 - \sum_{i<n} \hat{x}_i(\theta)$. Then $\hat{x}$ is a feasible mechanism.

In the remainder of the proof we show that $\hat{x}$ is a profitable mechanism. We first verify that $\hat{x}$ is IC. Let $j < n$ be an agent. Then for any report θ'_j it holds that

$$\sum_{\theta_{-j}} \pi_{-j}(\theta_{-j}) \hat{x}_j(\theta'_j, \theta_{-j}) = \sum_{i<n} \sum_{\theta} \pi(\theta) \frac{1}{\pi_j(\theta_j)} 1(\theta_j = \theta'_j) 1(i = j) \hat{\varepsilon}_i(\theta) - \alpha \underline{\hat{\varepsilon}}$$

$$= -\alpha \underline{\hat{\varepsilon}},$$

because the function $\pi(\theta) \frac{1}{\pi_j(\theta_j)} 1(\theta_j = \theta'_j) 1(i = j)$ lies in W and the function $\hat{\varepsilon}_i(\theta)$ is orthogonal to W. Since $-\alpha \underline{\hat{\varepsilon}}$ does not depend on θ'_j, $\hat{x}$ is IC for agent j. It remains to check IC for

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agent n. Let θ'_n be a report. Then

$$\sum_{\theta_{-n}} \pi_{-n}(\theta_{-n})\hat{x}_n(\theta'_n, \theta_{-n}) = \sum_{\theta_{-n}} \pi_{-n}(\theta_{-n})(1 - \sum_{i<n} \hat{x}_i(\theta'_n, \theta_{-n}))$$

$$= 1 + (n-1)\alpha\hat{\underline{\varepsilon}} - \alpha \sum_{i<n}\sum_{\theta} \pi(\theta)\frac{1}{\pi_n(\theta_n)}1(\theta_n = \theta'_n)\hat{\varepsilon}_i(\theta)$$

$$= 1 + (n-1)\alpha\hat{\underline{\varepsilon}},$$

because the function $\pi(\theta)\frac{1}{\pi_n(\theta_n)}1(\theta_n = \theta'_n)$ lies in W and the function $\hat{\varepsilon}_i(\theta)$ is orthogonal to W. Hence $\hat{x}$ is an IC mechanism. It only remains to show that the principal's expected payoff from $\hat{x}$ is greater than $\hat{v}$.

The principal's expected payoff from $\hat{x}$ is

$$\sum_i \sum_\theta \pi(\theta)v_i(\theta)x_i(\theta) = \sum_{i<n}\sum_\theta \pi(\theta)(v_i(\theta) - v_n(\theta))\hat{x}_i(\theta) + \sum_\theta \pi(\theta)v_n(\theta)\sum_i \hat{x}_i(\theta)$$

$$= \sum_{i<n}\sum_\theta \tilde{v}_i(\theta)\hat{x}_i(\theta) + \bar{v}$$

$$= \alpha \sum_{i<n}\sum_\theta (\hat{w}_i(\theta) + \hat{\varepsilon}_i(\theta))\hat{\varepsilon}_i(\theta) - \alpha \sum_i \sum_\theta \tilde{v}_i(\theta)\hat{\underline{\varepsilon}} + \bar{v}.$$

By assumption, $\sum_\theta \pi(\theta)v_i(\theta)$ is the same for all i and hence for any $i < n$: $\sum_\theta \tilde{v}_i(\theta) = 0$. This means that the second term in the last line above is zero. Because in addition $\hat{w} \in W$ and $\hat{\varepsilon}$ is orthogonal to W, the principal's expected payoff now simplifies to

$$\alpha \sum_{i<n}\sum_\theta \hat{\varepsilon}_i(\theta)^2 + \bar{v}.$$

By assumption $\tilde{v}$ does not lie in W and so the projection residual $\hat{\varepsilon}$ is nonzero. It follows that the first term above is positive and therefore that the principal's expected payoff from $\hat{x}$ is greater than $\bar{v}$. That is to say, $\hat{x}$ is a profitable mechanism. $\qquad\square$

A.8 Proof of Proposition 7

Proof. The result follows from Proposition 6 by interpreting the disposal option as an additional agent. Formally, let there be an agent 0 with a singleton type space $\Theta_0 = \{\theta^0\}$ and $v_0 \equiv 0$. A mechanism without disposal in this setting corresponds to a mechanism with disposal in the original setting. By Proposition 6, a profitable mechanism without disposal exists in the setting with the additional agent if and only if there do not exist functions $u_i(\theta_i)$ $(i = 0, \ldots, n)$ such that $v_i(\theta) - v_0(\theta) = u_i(\theta_i) - u_0(\theta_0)$ $\forall i > 0 \, \forall \theta$. Since Θ_0 is a singleton

and $v_0 \equiv 0$ the condition simplifies the following: A profitable mechanism exists if and only there do not exist functions $u_1(\theta_1), \ldots, u_n(\theta_n)$ and a constant c such that $v_i(\theta) = u_i(\theta_i) - c$ $\forall i > 0 \, \forall \theta$. But the latter simply means that there does not exist an agent $i > 0$ such that $v_i(\theta_i, \theta_{-i})$ is not constant in θ_{-i}. $\qquad \square$

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