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# Iterative methods for solving Riccati differential equations

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# Iterative methods for solving Riccati differential equations

**O. P. Ogundile[1] and S. O. Edeki[1], D. G. Olaniregun[2]**

[1]Department of Mathematics, Covenant University Ota, Nigeria

[2]Department of Mathematics, University of Ibadan, Ibadan, Nigeria

Corresponding Author Email: soedeki@yahoo.com

**Abstract** This work considers the approximate solution of the Riccati differential equations (RDEs). For ease of computation, the iterative methods applied are the Daftardar-Gejji and Jafari Method (DJM) and the Picard Iteration Method (PIM). The results obtained via the DJM are compared with those from PIM. The comparison shows that both methods are in agreement with the corresponding exact form. The Picard approach transforms the differential equation into an interconnected form; though, Lipschitz's criterion of consistency but satisfied.

**Keywords:** Iterative method; differential equations; quadratic riccati

## 1. Introduction

The differential equation of the form:

$$\begin{cases} x'(t) = a_1 + a_2 x(t) + a_3 x^2(t), \quad a_3 \neq 0, \\ x(0) = \eta, \end{cases} \tag{1.1}$$

is called the Riccati differential equation.

In equation (1.1), $a_1, a_2,$ and $a_3$ are real constants. The differential equation was derived by the Italian Scholar Francesco Riccati in the 17th Century [1]. This particular class of equations are encountered in many applied sciences and engineering, and they play a significant role in nonlinear control theory, stochastic processes, etc. [2]. The solutions of (1.1) can be obtained by using different effective numerical techniques. Recently, several authors have investigated this equation in order to obtain the approximate solutions. In [3], the authors used ADM to solve the RDEs and obtained the approximate solutions. He's VIM, HPM, iterated He's HPM was used by Abbsabandy to solve the RDEs and compared his result to that of ADM [4-6]. Tsai and Chen in [7] used LADM and Pade's approximations technique to study this particular type of equation. Their study shows acceptable accuracy results. The Legendre Wavelet approach was used by Mohammadi and Hosseini to solve the RDEs and compared their results with other existing methods in the literature [8]. Several other authors have used other numerical methods to handle this particular equation. See for references [9-12]. A book written by Murphy contains several methods that can be applied to solve the Riccati differential equation [13 and14]. In this work, the Riccati differential equation solution will be considered using two iterative methods, namely: Daftardar-Gejji and Jafari (DJ), and Picard Iterative (PI) methods.

For solvability purposes, efficient solution methods are needed for obtaining the solutions of differential equations of real-life situations. Historically, the DJM was introduced by two researchers Daftarda-Gejji and Jafari, in 2006 for obtaining the solutions of linear and nonlinear functional equations [15]. Different scholars have since used this method for solving differential equations of the various forms [16-22]. The main merits of the DJM are not limited to ease of application and reduction of computational work. In [24], PIM was used with Legendre Wavelets methods to solve nonlinear initial value problems (IVPs). Saeed *et al.* applied PIM to linearize the system of differential equations [24]. Ogundile and Edeki [25], in their recent work, presented PIM to obtain approximate-analytical solutions of SDE. Other relevant literature on the Picard iteration method is linked to references [26-30].

## 2. The Methods of solution

This section presents the fundamental concept of DJM [15-22] and PIM [23-30] as fast accuracy and highly efficient methods.

### *2.1. Daftar-Gejji-Jafari method (DJM)*

Supposed the following general functional equation is considered:

$$x = b + L(x) + N[x], \tag{2.1}$$

where $b$ is a given known function, and $N[\cdot]$ and $L[\cdot]$ are the nonlinear and linear operators, respectively. Suppose we define $\bar{N}[x]$ as:

$$\bar{N}[x] = L[x] + N[x], \tag{2.2}$$

then (2.1) becomes:

$$x = b + \bar{N}[x] \tag{2.3}$$

such that the solution, $x$ of (2.2) takes an infinite series pattern of the form:

$$\begin{cases} x = \sum_{i=0}^{\infty} x_i, \\ \bar{N}[x] = \bar{N}\left[\sum_{i=0}^{\infty} x_i\right]. \end{cases} \tag{2.4}$$

Consiquently, the nonlinear operator $\bar{N}$ is decomposed as

$$\bar{N}\left(\sum_{i=0}^{\infty} x_i\right) = \bar{N}[x_0] + \sum_{i=1}^{\infty}\left[\bar{N}\left(\sum_{i=0}^{m} x_i\right) - \bar{N}\left(\sum_{i=0}^{m-1} x_i\right)\right], m = 1,\ 2... \tag{2.5}$$

Therefore, putting (2.4) and (2.5) into (2.3), we obtain

$$\sum_{i=0}^{\infty} x_i = b + \bar{N}[x_0] + \sum_{i=1}^{\infty}\left[\bar{N}\left(\sum_{i=0}^{m} x_i\right) - \bar{N}\left(\sum_{i=0}^{m-1} x_i\right)\right]\ m = 1,\ 2, \ldots\ , \tag{2.6}$$

Hence, the recurrence relation is:

$$\begin{cases} x_0 = b \\ x_1 = \bar{N}(x_0) \\ z_{m+1} = \bar{N}\left[\sum_{i=0}^{m} x_i\right] - \bar{N}\left[\sum_{i=0}^{m-1} x_i\right],\ m = 1,\ 2, \ldots \end{cases} \tag{2.7}$$

such that:

$$x = b + \sum_{i=1}^{\infty} x_i = \sum_{i=0}^{\infty} x_i. \tag{2.8}$$

*2.2 Picard Iterative Method (PIM)*

The Picard Iterative method as an integral method is used for differential equations with emphasis on the existence and uniqueness of solutions of the linear and nonlinear differential equations; therefore, an equation to be handled by this method must satisfy the Lipchitz continuity condition.

*2.3 The Lipschitz Continuity Condition*

The function $r(a,b)$ is said to satisfy the Lipschitz condition with respect to $b$ in a region $D^*$ in the XY-plane, if $\exists$ a positive constant $K$ .
Such that:

$$\left|r(a,b_1)-r(a-b_2)\right| \le K\left|b_1-b_2\right| \tag{2.9}$$

whenever $(a,b_1)$ and $(a,b_1)$ are in $D^*$ , then $K$ is called the Lipschitz condition.
The Picard Iterative method connected to the differential equation the form

$$\begin{cases} x' = g(t,x), \\ x(t_0) = x_0, \end{cases} \tag{2.10}$$

is given below [29-31]:

$$\psi_{m+1}(t) = x(t_0) + \int_{t_0}^{t} f(s,\psi_m(s))ds, \tag{2.11}$$

where $x(t) = \psi_{m+1}$ , $x(s) = \psi_m(s)$ .

## 3. Numerical Examples

**Example 3.1**
Consider the Riccati differential equation [32]

$$x'(t) = 1 + x^2(t), \tag{3.1}$$

with I.C. $x(0) = 0.$
The exact solution of (3.0) is presented as:

$$x^*(0) = \tan(t). \tag{3.2}$$

The numerical solutions of (3.0) via the two methods of solution are presented in table and graph below.

**Table 1:** DJM, PIM versus Exact solutions

| $t$ | $DJM^6$ | $PIM^6$ | EXACT |
|---|---|---|---|
| 0.0 | 0.000000000000000 | 0.000000000000000 | 0.000000000000000 |
| 0.1 | 0.100334672085451 | 0.100334672085451 | 0.100334672085451 |
| 0.2 | 0.202710035508671 | 0.202710035508470 | 0.202710035508673 |
| 0.3 | 0.309336249609116 | 0.309336249567961 | 0.309336249609623 |
| 0.4 | 0.422793218696547 | 0.422790712339784 | 0.422793218738162 |
| 0.5 | 0.546302488509300 | 0.546302451536160 | 0.546302489843790 |
| 0.6 | 0.684136784374701 | 0.684136340501477 | 0.684136808341692 |

| | | | |
|---|---|---|---|
| 0.7 | 0.842288087786640 | 0.842284292051472 | 0.842288380463079 |
| 0.8 | 1.029635819385023 | 1.029610217610683 | 1.029638557050364 |
| 0.9 | 1.260136940795833 | 1.259991097538113 | 1.260158217550339 |
| 1.0 | 1.557261577356836 | 1.556523840478488 | 1.557407724654902 |

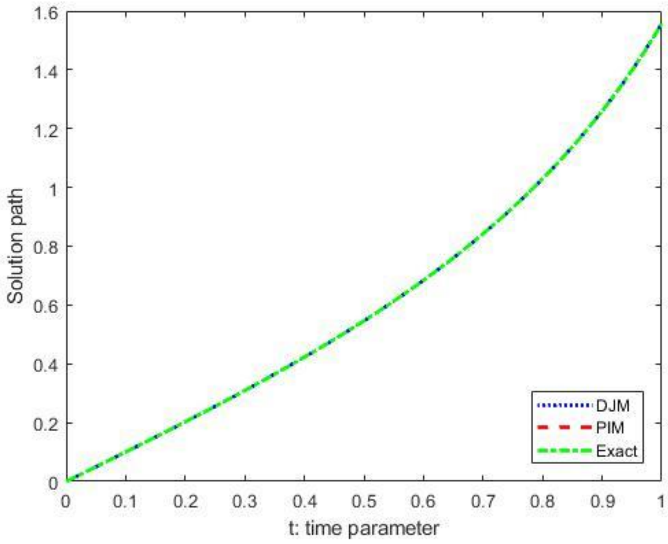


**Table 2**: Approximate solutions (DJM and PIM) at 6th term versus exact solutions

## 4. Results Discussion and Conclusion

In conclusion, the iterative methods (DJM and the PIM) have been used to solve the Riccati differential equations successfully. These two methods converge to the exact solution at the sixth term respectively and produced good approximation as shown in Figure and Table 2 respectively; though, Picard converts the differential equation to its equivalent in the integral form provided the Lipschitz continuity condition is satisfied. By extension, these methods can be applied to other linear and nonlinear models of higher-order and degree. The techniques are accurate in comparison with other discussed methods.

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