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SEMI n-IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity. A proper ideal I is said to be an n-ideal of R if for $a, b \in R$, $ab \in I$ and $a \notin \sqrt{0}$ imply $b \in I$. We give a new generalization of the concept of n-ideals by defining a proper ideal I of R to be a semi n-ideal if whenever $a \in R$ is such that $a^2 \in I$, then $a \in \sqrt{0}$ or $a \in I$. We give some examples of semi n-ideal and investigate semi n-ideals under various contexts of constructions such as direct products, homomorphic images and localizations. We present various characterizations of this new class of ideals. Moreover, we prove that every proper ideal of a zero dimensional general ZPI-ring R is a semi n-ideal if and only if R is a UN-ring or $R \cong F_1 \times F_2 \times \ldots \times F_k$, where F_i is a field for $i = 1, \ldots, k$. Finally, for a ring homomorphism $f\colon R \to S$ and an ideal J of S, we study some forms of a semi n-ideal of the amalgamation $R \bowtie^f J$ of R with S along J with respect to f.

Keywords: semi n-ideal; semiprime ideal; n-ideal

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1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity. We recall that a proper ideal I of a ring R is called *semiprime* if whenever $a \in R$ is such that $a^2 \in I$, then $a \in I$. It is well-known that I is semiprime in R if and only if I is a radical ideal, that is $I = \sqrt{I} = \{x \in R\colon\ x^m \in I \text{ for some } m \in \mathbb{Z}\}$. More generally, Badawi in [1] defined I to be weakly semiprime if $a \in R$ and $0 \neq a^2 \in I$ imply $a \in I$. In 2017, Tekir, Koc and Oral in [10] introduced the concept of n-ideals of commutative rings. A proper ideal I of a ring R is called an *n-ideal* if whenever $a, b \in R$ are such that $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$. Recently, Khashan and Bani-Ata in [7] generalized n-ideals by defining and studying the class of J-ideals. A proper ideal I of R is called a *J-ideal* if $ab \in I$ and $a \notin J(R)$ imply $b \in I$ for $a, b \in R$, where $J(R)$ denotes the Jacobson radical of R. Later, some other generalizations of n-ideals and J-ideals have been introduced, see for example [3], [8] and [9].

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In this paper, we define a proper ideal I of a ring R to be a semi n-ideal if whenever $a \in R$ is such that $a^2 \in I$, then either $a \in \sqrt{0}$ or $a \in I$. The class of semi n-ideals is a generalization of nil, semiprime and n-ideals. We start Section 2 by giving some examples (see Example 2.1) to show that this generalization is proper. Next, we determine several characterizations of semi n-ideals, see Theorem 2.1. Among many other results, in Theorem 2.3, we prove that every proper ideal of a zero dimensional general ZPI-ring R is a semi n-ideal if and only if R is a UN-ring (every nonunit element a of R is a product of a unit and a nilpotent) or $R \cong F_1 \times F_2 \times \ldots \times F_k$, where F_i is a field for $i = 1, \ldots, k$. Moreover, we characterize all ideals of the ring $\mathbb{Z}_m$ that are not semi n-ideals. In Section 2, we investigate semi n-ideals under various contexts of constructions such as homomorphic images and localizations, see Propositions 3.1 and 3.4. Moreover, for a direct product of rings $R = R_1 \times R_2 \times \ldots \times R_k$, we determine all semi n-ideals of R, see Theorems 3.2 and 3.3.

Recall that the idealization of an R-module M, denoted by $R(+)M$, is the commutative ring $R \times M$ with the coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. It is well known that if $I(+)N$ is an ideal of $R(+)M$, then $\sqrt{I(+)N} = \sqrt{I}(+)M$ and, in particular, $\sqrt{0_{R(+)M}} = \sqrt{0}(+)M$. In Proposition 3.5, we clarify the relation between semi n-ideals of the idealization ring $R(+)M$ and those of R. Let R and S be two rings, J be an ideal of S and $f\colon R \to S$ be a ring homomorphism. We finally study some forms of a semi n-ideal of the amalgamation ring $R \bowtie^f J$ of R with S along J with respect to f.

2. Properties of semi n-ideals

This section deals with many properties of semi n-ideals. We justify the relations among the concepts of semiprime ideals, n-ideals, nil ideals and our new class of ideals. Moreover, several characterizations and examples are presented. In particular, we characterize zero dimensional general ZPI-rings for which every proper ideal is semi n-ideal.

Definition 2.1. Let R be a ring. A proper ideal I of R is called a *semi n-ideal* if whenever $a \in R$ with $a^2 \in I$, then $a \in \sqrt{0}$ or $a \in I$.

The following properties of semi n-ideals can be easily observed.

Proposition 2.1. *For a ring R, the following statements hold.*

1. *Every n-ideal is a semi n-ideal.*
2. *Every (weakly) semiprime ideal I is a semi n-ideal. The converse also holds if $\sqrt{0} \subseteq I$.*

(3) *For every proper ideal I of R, $\sqrt{I}$ is a (semiprime) semi n-ideal. In particular, $\sqrt{0}$ is a semi n-ideal of R.*
(4) *Any nil ideal I of R (that is $I \subseteq \sqrt{0}$) is a semi n-ideal.*
(5) *An ideal I of a reduced ring R is a semi n-ideal if and only if I is semiprime.*

However, the converses of (1) and (2) in Proposition 2.1 are not true in general.

Example 2.1.
(1) Any nonzero prime ideal of a reduced ring (in particular, an integral domain) is a semi n-ideal that is not an n-ideal.
(2) The ideal $\langle\overline{16}\rangle$ of $\mathbb{Z}_{32}$ is a semi n-ideal that is not (weakly) semiprime.

Next, we give some equivalent conditions that characterize semi n-ideals.

Theorem 2.1. *Let I be a proper ideal of a ring R. The following statements are equivalent.*
(1) *I is a semi n-ideal of R.*
(2) *Whenever $a \in R$ with $0 \neq a^2 \in I$, then $a \in \sqrt{0}$ or $a \in I$.*
(3) *Whenever $a \in R$ with $a^m \in I$ for some positive integer m, then $a \in \sqrt{0}$ or $a \in I$.*
(4) *Whenever J is an ideal of R with $J^m \subseteq I$ for some positive integer m, then $J \subseteq \sqrt{0}$ or $J \subseteq I$.*

Proof. (1) $\Leftrightarrow$ (2). Suppose that (2) holds. Let $a^2 \in I$ for some $a \in R \setminus I$. If $a^2 \neq 0$, then we are done. If $a^2 = 0$, then $a \in \sqrt{0}$. The converse part is obvious.

(1) $\Rightarrow$ (3). Let $a^m \in I$ and suppose that $a \notin \sqrt{0}$. To prove the assertion we use the mathematical induction method. If $m \leqslant 2$, then the claim is clear as I is a semi n-ideal. Assume that the claim of (3) holds for all $2 < k < m$. We show that it is also true for m. Suppose m is even, say, $m = 2t$ for some positive integer t. Since I is a semi n-ideal of R, then $a^m = (a^t)^2 \in I$ and $a^t \notin \sqrt{0}$ imply $a^t \in I$. By the induction hypothesis, we conclude that $a \in I$. Now, suppose m is odd. Then $m + 1 = 2s$ for some $s < m$. Similarly, since $(a^s)^2 \in I$ and $a^s \notin \sqrt{0}$, we get $a^s \in I$ and again by the induction hypothesis, we conclude $a \in I$, so we are done.

(3) $\Rightarrow$ (4). Suppose that $J^m \subseteq I$ and $J \nsubseteq \sqrt{0}$ for some ideal J of R. Then clearly $J^m \nsubseteq \sqrt{0}$, so we can choose a nonnilpotent element a in J^m. Let $b \in J$. If $b \notin \sqrt{0}$, then $b^m \in I$ implies that $b \in I$ by (3). Suppose $b \in \sqrt{0}$. Since $(a+b)^m \in I$ and clearly $a + b \notin \sqrt{0}$, we conclude that $a + b \in I$ and so $b \in I$. It follows that $J \subseteq I$ as needed.

(4) $\Rightarrow$ (1). Let $a^2 \in I$ for $a \in R$. The result follows directly by putting $J = \langle a\rangle$ and $m = 2$ in (4). □

Unlike n-ideals, if I is a semi n-ideal of a ring R, then I need not be contained in $\sqrt{0}$, see Example 2.1 (1).

Proposition 2.2. *Let I be a proper ideal of a ring R. Then I is a semi n-ideal if and only if $\sqrt{I}=\sqrt{0}$ or $\sqrt{I}=I$.*

P r o o f. Suppose that I is a semi n-ideal and $a\in\sqrt{I}$. Then $a^n\in I$ for some $n\geqslant 1$. By Theorem 2.1, we conclude that $a\in\sqrt{0}$ or $a\in I$. Thus, $\sqrt{I}\subseteq\sqrt{0}\cup I$. Since the converse inclusion always holds, the claim is clear.

Conversely, if $\sqrt{I}=I$, then I is semiprime, so it is a semi n-ideal by Proposition 2.1. Suppose that $\sqrt{I}=\sqrt{0}$. Let $a\in R$ such that $a^2\in I$ but $a\notin I$. Since clearly $a\in\sqrt{I}=\sqrt{0}$, I is a semi n-ideal. □

Next, we prove the following lemma which we need for the characterization of UN-rings.

Lemma 2.1. *Let I and J be ideals of R with $I,J\nsubseteq\sqrt{0}$. Then*
(1) *If I and J are semi n-ideals with $I^2=J^2$, then $I=J$.*
(2) *If I^2 is a semi n-ideal, then $I^2=I$.*

P r o o f. (1) Since $I^2\subseteq J$ and $I\nsubseteq\sqrt{0}$, then by Theorem 2.1, we have $I\subseteq J$. Similarly, since $J^2\subseteq I$ and $J\nsubseteq\sqrt{0}$, we have $J\subseteq I$. Thus, we have the equality.

(2) Since $I^2\subseteq I^2$, $I\nsubseteq\sqrt{0}$ and I^2 is a semi n-ideal, we have $I\subseteq I^2$ and so $I^2=I$. □

Following [2], a ring R is called a *UN-ring* if every nonunit element a of R is a product of a unit and a nilpotent element. We now characterize UN-rings in terms of semi n-ideals.

Theorem 2.2. *The following statements are equivalent for a ring R.*
(1) *$\sqrt{0}$ is the unique prime ideal of R.*
(2) *R is a UN-ring.*
(3) *Every proper ideal of R is an n-ideal.*
(4) *R is quasi-local and every proper ideal of R is a semi n-ideal.*

P r o o f. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Follows by [10], Proposition 2.25.

(3) $\Rightarrow$ (4). Let M be a maximal ideal of R and $x\in M$. Since $x\cdot 1\in M$ and M is an n-ideal, then we must have $x\in\sqrt{0}$ and so $M\subseteq\sqrt{0}\subseteq J(R)\subseteq M$. It follows that $M=J(R)$ and R is quasi-local. The other part of (4) follows directly by Proposition 2.1 (1).

(4) $\Rightarrow$ (1). Suppose M is the unique maximal ideal of R and P is a prime ideal of R. Assume that $P\nsubseteq\sqrt{0}$. Since P^2 is a semi n-ideal, by Lemma 2.1 (2), we conclude $P^2=P$. By the Krull intersection theorem, we have $P=\bigcap\limits_{n=1}^{\infty}P^n\subseteq\bigcap\limits_{n=1}^{\infty}M^n=0$ which is a contradiction. Thus $P\subseteq\sqrt{0}$, and so $P=\sqrt{0}$. □

We note that the condition "R is quasi-local" in (4) of Theorem 2.2 cannot be omitted. For example, in the ring $\mathbb{Z}_6$ every proper ideal is a semi n-ideal but $\mathbb{Z}_6$ has no n-ideals. Also, it is known from [7], Proposition 2.3 that if a ring R is quasi-local, then every proper ideal of R is a J-ideal. In the following example, we see that we may find a non semi n-ideal in a quasi-local ring.

Example 2.2. Consider the quasi-local ring $R = \mathbb{Z}_{\langle 2\rangle} = \{a/b\colon\ a, b \in \mathbb{Z},\ 2 \nmid b\}$ and let $I = \langle 4\rangle_{\langle 2\rangle} = \{a/b\colon\ a \in \langle 4\rangle,\ 2 \nmid b\}$. Then I is not a semi n-ideal of R. For example, $(\frac{2}{3})^2 \in I$ but $\frac{2}{3} \notin \sqrt{0} = \{0\}$ and $\frac{2}{3} \notin I$.

A ring R is said to be a *ZPI-ring* (Zerlegung Primideale) if each nonzero ideal I of R is uniquely expressible as a product of prime ideals of R. The ring R is said to be a *general ZPI-ring* if each ideal of R can be expressed as a finite product of prime ideals of R. Dedekind domains and special primary rings are particular examples for general ZPI-rings. A general ZPI-ring R is Noetherian and each primary ideal of R is a prime power. For more details, the reader can refer to [6].

Theorem 2.3. *Let R be a zero dimensional general ZPI-ring. The following statements are equivalent.*

(1) *Every proper ideal of R is a semi n-ideal.*
(2) *Either $0 = P^r$ for some prime ideal P of R and $r \in \mathbb{N}$ or $0 = P_1P_2\dots P_k$, where $P_1, P_2, \dots, P_k$ are distinct prime ideals of R.*
(3) *Either $0 = M^r$ for some maximal ideal M of R and $r \in \mathbb{N}$ or $0 = M_1M_2\dots M_k$, where $M_1, M_2, \dots, M_k$ are distinct maximal ideals of R.*
(4) *R is a UN-ring or $R \cong F_1 \times F_2 \times \dots \times F_k$, where F_i is a field for $i = 1, \dots, k$.*

Proof. (1) $\Rightarrow$ (2). Suppose every proper ideal of R is a semi n-ideal. If $0 = P^r$ for some prime ideal P of R and $r \in \mathbb{N}$, then we are done. Suppose $0 = P_1^{r_1}P_2^{r_2}\dots P_k^{r_k}$, where $P_1, P_2, \dots, P_k$ are distinct prime ideals and $k \geqslant 2$. Assume that $r_i \geqslant 2$ for some i. Since $P_i^{r_i} \subseteq P_i^{r_i}$ but $P_i \nsubseteq \sqrt{0}$ and $P_i \nsubseteq P_i^{r_i}$, then $P_i^{r_i}$ is a non semi n-ideal of R by Theorem 2.1. By this contradiction, we conclude that 0 is a product of distinct prime ideals as required.

(2) $\Rightarrow$ (3). It follows immediately by the assumption that R is zero dimensional.

(3) $\Rightarrow$ (4). Suppose that $0 = M^r$ for some maximal ideal M of R and $r \in \mathbb{N}$. If there is another maximal ideal M_1 of R, then $0 = M^r \subseteq M_1$ implies $M \subseteq M_1$, so we conclude the equality. Since $M \subseteq \sqrt{0}$, then by the maximality of M, we have $M = \sqrt{0}$. It follows that $(R, \sqrt{0})$ is a quasi-local ring, and so R is a UN-ring. Now suppose that $0 = M_1M_2\dots M_k$, where $M_1, M_2, \dots, M_k$ are distinct maximal ideals of R. Apply the Chinese Remainder Theorem to get $R \cong R/0 = R/M_1M_2\dots M_k \cong R/M_1 \times R/M_2 \times \dots \times R/M_k$. Now, we conclude the claim by putting $F_i = R/M_i$ for all $i = 1, \dots, k$.

(4) $\Rightarrow$ (1). If R is a UN-ring, then every proper ideal of R is a semi n-ideal by Theorem 2.2. Suppose that $R \cong F_1 \times F_2 \times \ldots \times F_k$, where F_i's are fields. Let I be a proper ideal of R. Then $I = I_1 \times I_2 \times \ldots \times I_k$, where $I_i = 0$ or $I_i = F_i$ for all $i = 1, \ldots, k$. Since I is clearly radical, it is a semi n-ideal of R, which completes the proof. □

By [10], Theorem 2.12, a ring R has n-ideals if and only if $\sqrt{0}$ is a prime ideal. Thus, R has n-ideals if and only if 0 is a power of a prime ideal of R. Therefore, we have the following corollary of Theorem 2.3.

Corollary 2.1. *Every proper ideal of a zero dimensional general ZPI-ring R is a semi n-ideal that is not an n-ideal if and only if $R \cong F_1 \times F_2 \times \ldots \times F_k$, where F_i is a field for $i = 1, \ldots, k$.*

As a particular case, in the following corollary, we determine all $m \in \mathbb{N}$ such that every proper ideal of the ring $\mathbb{Z}_m$ is a semi n-ideal.

Corollary 2.2. *Every proper ideal of the ring $\mathbb{Z}_m$ is a semi n-ideal if and only if either m is a power of a prime or m is a product of distinct primes.*

In the following, we precisely characterize all ideals of the ring $\mathbb{Z}_m$ that are not semi n-ideals.

Theorem 2.4. *Consider the ring $\mathbb{Z}_m$, where $m = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$, $k \geqslant 2$ and $r_i \geqslant 2$ for at least one $i \in \{1, 2, \ldots, k\}$. Then $I = \langle p_1^{s_1} p_2^{s_2} \ldots p_k^{s_k} \rangle$ is not a semi n-ideal of $\mathbb{Z}_m$ if and only if $s_i \geqslant 2$ and $s_j = 0$ for at least one i and one j.*

Proof. Suppose $I = \langle p_1^{s_1} p_2^{s_2} \ldots p_k^{s_k} \rangle$ is not a semi n-ideal of $\mathbb{Z}_m$. If $s_j \neq 0$ for all $j \in \{1, 2, \ldots, k\}$, then $I \subseteq \langle p_1 p_2 \ldots p_k \rangle = \sqrt{0}$ is nil, so it is a semi n-ideal by Proposition 2.1 (5) which is a contradiction. If $s_i \leqslant 1$ for all i, then clearly I is a radical ideal, so it is a semi n-ideal which is also a contradiction. Conversely, assume with no loss of generality that $s_k = 0$ and $s_1 \geqslant 2$. Then $(p_1^{s_1-1} p_2^{s_2} \ldots p_{k-1}^{s_{k-1}})^2 \in I$ but $p_1^{s_1-1} p_2^{s_2} \ldots p_{k-1}^{s_{k-1}} \notin \sqrt{0}$ and $p_1^{s_1-1} p_2^{s_2} \ldots p_{k-1}^{s_{k-1}} \notin I$. Therefore, I is not a semi n-ideal of $\mathbb{Z}_m$. □

In view of Corollary 2.1, we have the following result for the ring $\mathbb{Z}_m$.

Corollary 2.3. *Every proper ideal of the ring $\mathbb{Z}_m$ is a semi n-ideal that is not an n-ideal if and only if m is a product of distinct primes.*

3. Quotients, localizations, products and amalgamations

In this section, we study the behavior of semi n-ideals under homomorphisms, quotient rings, localizations, direct product of rings, idealization rings and amalgamation rings. We also observe that the intersection and the sum of two semi n-ideals are also semi n-ideals but their product and the Cartesian product need not be so.

Proposition 3.1. *Let $f\colon R_1 \to R_2$ be a ring epimorphism. Then the following statements hold.*

(1) *If I_1 is a semi n-ideal of R_1 with $\operatorname{Ker}(f) \subseteq I_1$, then $f(I_1)$ is a semi n-ideal of R_2.*

(2) *If I_2 is a semi n-ideal of R_2 and $\operatorname{Ker}(f) \subseteq \sqrt{0_{R_1}}$, then $f^{-1}(I_2)$ is a semi n-ideal of R_1.*

Proof. (1) Let $a \in R_2$ such that $a^2 \in f(I_1)$ and $a \notin f(I_1)$. Then there exists $x \in R_1 \setminus I_1$ such that $a = f(x)$. Since $f(x^2) = a^2 \in f(I_1)$ and $\operatorname{Ker}(f) \subseteq I_1$, we have $x^2 \in I_1$. As I_1 is a semi n-ideal of R_1, we get $x \in \sqrt{0_{R_1}}$ and so clearly $a = f(x) \in \sqrt{0_{R_2}}$ as required.

(2) Suppose I_2 is a semi n-ideal of R_2. Let $x \in R_1$ such that $x^2 \in f^{-1}(I_2)$ and $x \notin f^{-1}(I_2)$. Then $f(x^2) = f(x)^2 \in I_2$ and $f(x) \notin I_2$ imply $f(x) \in \sqrt{0_{R_2}}$. Hence, $f(x)^m = f(x^m) = 0_{R_2}$ for some positive integer m. Since $\operatorname{Ker}(f) \subseteq \sqrt{0_{R_1}}$, we conclude that $x \in \sqrt{0_{R_1}}$. □

In view of Proposition 3.1, we have the following result for quotient rings.

Corollary 3.1. *Let I and J be ideals of a ring R with $J \subseteq I$.*

(1) *If I is a semi n-ideal of R, then I/J is a semi n-ideal of R/J.*

(2) *If I/J is a semi n-ideal of R/J and J is a semi n-ideal of R, then I is a semi n-ideal of R.*

Proof. (1) Consider the natural epimorphism $\pi\colon R \to R/J$ with $\operatorname{Ker}(\pi) = J$ and apply Proposition 3.1.

(2) Let $a \in R$ such that $a^2 \in I$ and $a \notin \sqrt{0}$. Then $(a+J)^2 = a^2 + J \in I/J$. If $a + J \in \sqrt{0_{I/J}}$, then $a^k + J = (a+J)^k = J$ for some integer k and so $a^k \in J$. Since J is a semi n-ideal of R, we get $a \in J \subseteq I$. If $a + J \notin \sqrt{0_{I/J}}$, then $a + J \in I/J$ as I/J is a semi n-ideal of R/J and so again $a \in I$. □

In particular, (2) in Corollary 3.1 holds if $J \subseteq \sqrt{0}$.

Proposition 3.2. *If $\{I_\alpha\colon \alpha \in \Lambda\}$ is a family of semi n-ideals of a ring R, then so is $\bigcap\limits_{\alpha\in\Lambda} I_\alpha$.*

P r o o f. Let $a^2 \in \bigcap\limits_{\alpha\in\Lambda} I_\alpha$ and $a \notin \bigcap\limits_{\alpha\in\Lambda} I_\alpha$. Then $a \notin I_\gamma$ for some $\gamma \in \Lambda$. Since I_γ is a semi n-ideal, we have $a \in \sqrt{0}$ and so $\bigcap\limits_{\alpha\in\Lambda} I_\alpha$ is a semi n-ideal. □

Proposition 3.3. *Let I and J be two semi n-ideals in a ring R. If $I + J$ is proper in R, then $I + J$ is a semi n-ideal of R.*

P r o o f. By (1) of Corollary 3.1, $I/I \cap J$ is a semi n-ideal of $R/I \cap J$. Thus, $(I + J)/J \cong I/I \cap J$ is also a semi n-ideal of R/J. Therefore, by (2) of Corollary 3.1, we conclude that $I + J$ is a semi n-ideal of R. □

However, if I and J are two semi n-ideals in a ring R, then IJ need not be a semi n-ideal. For example, while $\langle 2\rangle$ is a semi n-ideal of $\mathbb{Z}$, $\langle 2\rangle^2 = \langle 4\rangle$ is not so.

Let I be a proper ideal of R. In the following proposition, the notations $Z(R)$ and $Z_I(R)$ denote the set of zero divisors of R and the set $\{r \in R\colon\ rs \in I$ for some $s \in R \setminus I\}$, respectively.

Proposition 3.4. *Let S be a multiplicatively closed subset of a ring R. Then the following holds.*

(1) *If I is a semi n-ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a semi n-ideal of $S^{-1}R$.*

(2) *If $S^{-1}I$ is a semi n-ideal of $S^{-1}R$ and $S \cap Z(R) = S \cap Z_I(R) = \emptyset$, then I is a semi n-ideal of R.*

P r o o f. (1) Suppose for $a/s \in S^{-1}R$ that $(a/s)^2 \in S^{-1}I$ and $(a/s) \notin S^{-1}I$. Then there exists $u \in S$ such that $ua^2 \in I$ and so $(ua)^2 \in I$. Since clearly $ua \notin I$ and I is a semi n-ideal, we have $ua \in \sqrt{0}$. Thus, $(ua)^m = 0$ for some positive integer m. It follows that $(a/s)^m = (ua/(us))^m = 0_{S^{-1}R}$ and so $a/s \in \sqrt{0_{S^{-1}R}}$.

(2) Suppose $a^2 \in I$ for $a \in R \setminus I$. Since $S^{-1}I$ is a semi n-ideal of $S^{-1}R$ and $(a/1)^2 \in S^{-1}I$, we have either $a/1 \in S^{-1}I$ or $a/1 \in \sqrt{0_{S^{-1}R}}$. If $a/1 \in S^{-1}I$, then there exists $u \in S$ such that $ua \in I$. Since $S \cap Z_I(R) = \emptyset$, we conclude that $a \in I$. If $a/1 \in \sqrt{0_{S^{-1}R}}$, then there is a positive integer k such that $(ua)^k = u^k a^k = 0$. Since $S \cap Z(R) = \emptyset$, we conclude that $a^k = 0$ and so $a \in \sqrt{0}$. Therefore, I is a semi n-ideal of R. □

Lemma 3.1. *Let R be a ring and S be a nonempty subset of R where $S \cap Z_{\sqrt{0}}(R) = \emptyset$. If I is a semi n-ideal of R with $S \nsubseteq I$, then $(I : S)$ is a semi n-ideal of R.*

P r o o f. Let $a \in R$ such that $a^2 \in (I : S)$ but $a \notin \sqrt{0}$. Then $(as)^2 \in I$ for all $s \in S$. As I is a semi n-ideal of R, we have either $as \in \sqrt{0}$ or $as \in I$ for all $s \in S$. If $as \in \sqrt{0}$, then $S \cap Z_{\sqrt{0}}(R) \neq \emptyset$, a contradiction. Thus, $as \in I$ for all $s \in S$ and so $a \in (I : S)$ as required. □

Theorem 3.1. *If an ideal I of a ring R is a maximal semi n-ideal satisfying $Z_{\sqrt{0}}(R) \subseteq I$, then I is semiprime in R. Additionally, if $I \subseteq \sqrt{0}$, then $I = \sqrt{0}$ is a prime ideal.*

Proof. Let $a \in R$ such that $a^2 \in I$. Suppose $a \notin I$. Then by assumption, $\{a\} \cap Z_{\sqrt{0}}(R) = \emptyset$ and so $(I : a)$ is a semi n-ideal of R by Lemma 3.1. By the maximality of I, we get $a \in (I : a) = I$, a contradiction. Therefore, $a \in I$ and I is semiprime in R. Now, suppose that $I \subseteq \sqrt{0}$. Let $ab \in \sqrt{0}$ and $a \notin \sqrt{0}$. Since $Z_{\sqrt{0}}(R) \subseteq I$, we have $b \in I \subseteq \sqrt{0}$ and so $\sqrt{0}$ is prime. Since $I \subseteq \sqrt{0}$ and $\sqrt{0}$ is clearly a semi n-ideal satisfying $Z_{\sqrt{0}}(R) \subseteq \sqrt{0}$, the maximality of I implies $I = \sqrt{0}$. □

Let $R = R_1 \times R_2$, where R_1 and R_2 are two rings. It is known from [10], Proposition 2.25 that there are no n-ideals in R. By characterizing semi n-ideals of R, the next theorem allows us to build some examples for semi n-ideals which are not n-ideals.

Theorem 3.2. *Let R_1 and R_2 be two rings and $R = R_1 \times R_2$. Then a proper ideal $I = I_1 \times I_2$ is a semi n-ideal of R if and only if one of the following statements holds.*

(1) *I is a semiprime ideal of R.*

(2) *I_1 is a semi n-ideal of R_1 and $I_2 = \sqrt{0_{R_2}}$.*

(3) *I_2 is a semi n-ideal of R_2 and $I_1 = \sqrt{0_{R_1}}$.*

Proof. ($\Rightarrow$) Suppose that $I = I_1 \times I_2$ is a semi n-ideal of R that is not semiprime. Then $\sqrt{I} \neq I$ and there exists an element $(x, y) \in \sqrt{I} \setminus I$ which means $(x, y)^m \in I$ for some positive integer m and $(x, y) \notin I$. First, we show that $I_1 = \sqrt{0_{R_1}}$ or $I_2 = \sqrt{0_{R_2}}$. Assume not. If $I_1 \neq \sqrt{0_{R_1}}$ and $I_2 \neq \sqrt{0_{R_2}}$, then there exist some elements $a \in I_1 \setminus \sqrt{0_{R_1}}$ and $b \in I_2 \setminus \sqrt{0_{R_2}}$. Hence, $(x + a)^m \in I_1$ and $(y + b)^m \in I_2$ which implies that $(x + a\ ,\ y + b)^m \in I$. Since $(x, y) \notin I$, without loss of generality we may assume that $x \notin I_1$. Then, clearly $x + a \notin I_1$ and $(x + a\ ,\ y + b) \notin I$. Since I is a semi n-ideal, we conclude that $(x + a,\ y + b) \in \sqrt{0_{R_1\times R_2}} = \sqrt{0_{R_1}} \times \sqrt{0_{R_2}}$. It follows that $x + a \in \sqrt{0_{R_1}}$ and $y + b \in \sqrt{0_{R_2}}$ which imply $x \notin \sqrt{0_{R_1}}$ and $y \notin \sqrt{0_{R_2}}$ as $a \notin \sqrt{0_{R_1}}$ and $b \notin \sqrt{0_{R_2}}$. Therefore, $(x, y) \notin \sqrt{0_{R_1}} \times \sqrt{0_{R_2}}$, a contradiction. Thus, $I_1 = \sqrt{0_{R_1}}$ or $I_2 = \sqrt{0_{R_2}}$.

Suppose without loss of generality that $I_1 \neq \sqrt{0_{R_1}}$ and $I_2 = \sqrt{0_{R_2}}$. Let $a^2 \in I_1$ and $a \notin I_1$. Since $(a, 0)^2 \in I$ and $(a, 0) \notin I$, we have $(a, 0) \in \sqrt{0_{R_1}} \times \sqrt{0_{R_2}}$. Hence, $a \in \sqrt{0_{R_1}}$ and I_1 is a semi n-ideal of R_1. Similarly, if $I_1 = \sqrt{0_{R_1}}$ and $I_2 \neq \sqrt{0_{R_2}}$, we conclude that I_2 is a semi n-ideal of R_2.

(⇐) If I is semiprime in R, then it is a semi n-ideal by Proposition 2.1. Suppose that $I = I_1 \times \sqrt{0_{R_2}}$, where I_1 is a semi n-ideal of R_1. Let $(a,b)^2 \in I$ and $(a,b) \notin \sqrt{0_{R_1}} \times \sqrt{0_{R_2}}$. Since $b^2 \in \sqrt{0_{R_2}}$, we get $b \in \sqrt{0_{R_2}}$ and so $a \notin \sqrt{0_{R_1}}$. Since $a^2 \in I_1$, $a \notin \sqrt{0_{R_1}}$ and I_1 is a semi n-ideal of R_1, we conclude that $a \in I_1$. Thus, $(a,b) \in I$ and I is a semi n-ideal of R. □

Generalizing Theorem 3.2, we have the following result.

Theorem 3.3. *Let $R_1, R_2, \ldots, R_m$ be rings and $R = R_1 \times \ldots \times R_m$, where $m \geqslant 2$. Then a proper ideal I of R is a semi n-ideal if and only if one of the following statements is satisfied.*

(1) *I is a semiprime ideal of R.*

(2) *$I = I_1 \times \ldots \times I_m$, where I_k is a semi n-ideal of R_k for some $k \in \{1, \ldots, m\}$ and $I_j = \sqrt{0_{R_j}}$ for all $j \in \{1, \ldots, m\} \setminus \{k\}$.*

Proof. Suppose I is a semi n-ideal of R that is not semiprime. We prove (2) by using mathematical induction on m. By Theorem 3.2, the result holds for $m = 2$. Now let $3 \leqslant m < \infty$ and assume that the result holds for $R' = R_1 \times \ldots \times R_{m-1}$. We show that the result also holds for $R = R' \times R_m$. We have the following two cases by Theorem 3.2:

Case I: $I = J \times \sqrt{0_{R_m}}$ where J is a semi n-ideal of R'. If J is a semiprime ideal of R', then clearly I is so which is a contradiction. Hence, by our induction hypothesis, $J = I_1 \times \ldots \times I_{m-1}$, where I_k is a semi n-ideal of R_k for some $k \in \{1, \ldots, m-1\}$ and $I_j = \sqrt{0_{R_j}}$ for all $j \in \{1, \ldots, m-1\} \setminus \{k\}$. Therefore, $I = J \times \sqrt{0_{R_m}}$ is in the desired form.

Case II: $J = \sqrt{0_{R'}} \times I_m$, where I_m is a semi n-ideal of R. In this case, we have $I = \sqrt{0_{R_1}} \times \sqrt{0_{R_2}} \times \ldots \times \sqrt{0_{R_{m-1}}} \times I_m$, so we are done.

The converse part is similar to the proof of Theorem 3.2. □

Next, for an R-module M, we justify the relation between semi n-ideals of R and those of the idealization ring $R(+)M$.

Proposition 3.5. *Let M be an R-module and I be a proper ideal of R. Then I is a semi n-ideal of R if and only if $I(+)M$ is a semi n-ideal of the idealization ring $R(+)M$.*

Proof. Suppose that $(a,m)^2 \in I(+)M$ and $(a,m) \notin \sqrt{0_{R(+)M}} = \sqrt{0}(+)M$. Then $a^2 \in I$ and $a \notin \sqrt{0}$. Since I is a semi n-ideal, we conclude that $a \in I$. Thus, $(a,m) \in I(+)M$. Conversely, suppose that $I(+)M$ is a semi n-ideal of $R(+)M$ and let $a^2 \in I$ but $a \notin I$. Then $(a,0)^2 \in I(+)M$ and $(a,0) \notin I(+)M$ imply that $(a,0) \in \sqrt{0_{R(+)M}}$. Thus, $a \in \sqrt{0_R}$ and we are done. □

Remark 3.1. Let N be a proper submodule of an R-module M and I be an ideal of R with $IM \subseteq N$. In general, if I is a semi n-ideal of R, then $I(+)N$ need not be a semi n-ideal of $R(+)M$. For example, consider the idealization ring $\mathbb{Z}(+)\mathbb{Z}_4$. Then while $\langle 2\rangle$ is a semi n-ideal of $\mathbb{Z}$, $\langle 2\rangle(+)\langle\bar{2}\rangle$ is not a semi n-ideal of $\mathbb{Z}(+)\mathbb{Z}_4$. Indeed, $(2,\bar{1})^2 = (4,\bar{0}) \in \langle 2\rangle(+)\langle\bar{2}\rangle$ but $(2,\bar{1}) \notin \sqrt{0_{\mathbb{Z}(+)\mathbb{Z}_4}} = 0(+)\mathbb{Z}_4$ and $(2,\bar{1}) \notin \langle 2\rangle(+)\langle\bar{2}\rangle$.

Let R and S be two rings, J be an ideal of S and $f\colon R \to S$ be a ring homomorphism. The amalgamation of R and S along J with respect to f is the subring $R \bowtie^f J = \{(a, f(a)+j)\colon a \in R, j \in J\}$ of $R \times S$. If f is the identity homomorphism on R, then we get the amalgamated duplication of R along an ideal J, $R \bowtie J = \{(a, a+j)\colon a \in R,\ j \in J\}$. For more related definitions and several properties of this kind of rings, one can see [5]. If I is an ideal of R and K is an ideal of $f(R)+J$, then $I \bowtie^f J = \{(i, f(i)+j)\colon i \in I,\ j \in J\}$ and $\overline{K}^f = \{(a, f(a)+j)\colon a \in R,\ j \in J,\ f(a)+j \in K\}$ are ideals of $R \bowtie^f J$, see [4]. Next, we determine conditions under which $I \bowtie^f J$ and $\overline{K}^f$ are semi n-ideals of $R \bowtie^f J$.

Theorem 3.4. *Let R, S, J and f be as above. If $I \bowtie^f J$ is a semi n-ideal of $R \bowtie^f J$, then I is a semi n-ideal of R. Moreover, the converse is true if $J \subseteq \sqrt{0_S}$.*

Proof. Suppose $I \bowtie^f J$ is a semi n-ideal of $R \bowtie^f J$ and let $a^2 \in I$ for $a \in R$ and $a \notin \sqrt{0}$. Then $(a, f(a)) \in R \bowtie^f J$ with $(a, f(a))^2 = (a^2, f(a^2)) \in I \bowtie^f J$ and clearly $(a, f(a)) \notin \sqrt{0_{R\bowtie^f J}}$. As $I \bowtie^f J$ is a semi n-ideal, we have $(a, f(a)) \in I \bowtie^f J$ and so $a \in I$. Now, suppose $J \subseteq \sqrt{0_S}$ and I is a semi n-ideal of R. Let $(a, f(a)+j) \in R \bowtie^f J$ such that $(a, f(a)+j)^2 = (a^2, f(a^2)+2jf(a)+j^2) \in I \bowtie^f J$ and $(a, f(a)+j) \notin \sqrt{0_{R\bowtie^f J}}$. Then $a^2 \in I$ and $a \notin \sqrt{0}$ since otherwise $f(a)+j$ is nilpotent in S as $J \subseteq \sqrt{0_S}$ and so $(a, f(a)+j) \in \sqrt{0_{R\bowtie^f J}}$, a contradiction. Since I is a semi n-ideal of R, then $a \in I$ and so $(a, f(a)+j) \in I \bowtie^f J$. □

Theorem 3.5. *Let $f\colon R \to S$ be a ring epimorphism and J, K be ideals of S.*

(1) *If $\overline{K}^f$ is a semi n-ideal of $R \bowtie^f J$, then K is a semi n-ideal of S.*

(2) *If $J \subseteq \sqrt{0_S}$, $\mathrm{Ker}(f) \subseteq \sqrt{0_R}$ and K is a semi n-ideal of S, then $\overline{K}^f$ is a semi n-ideal of $R \bowtie^f J$.*

Proof. (1) Suppose $\overline{K}^f$ is a semi n-ideal of $R \bowtie^f J$. Let $b \in S$ such that $b^2 \in K$ and $b \notin \sqrt{0_S}$. Choose $a \in R$ such that $f(a) = b$. Then $(a, f(a)) \in R \bowtie^f J$ with $(a, f(a))^2 \in \overline{K}^f$ and clearly $(a, f(a)) \notin \sqrt{0_{R\bowtie^f J}}$. By assumption, we have $(a, f(a)) \in \overline{K}^f$ and so $b = f(a) \in K$.

(2) Suppose $J \subseteq \sqrt{0_S}$, $\mathrm{Ker}(f) \subseteq \sqrt{0_R}$ and K is a semi n-ideal of S. Let $(a, f(a)+j) \in R \bowtie^f J$ be such that $(a, f(a)+j)^2 = (a^2, (f(a)+j)^2) \in \overline{K}^f$ and $(a, f(a)+j) \notin \sqrt{0_{R\bowtie^f J}}$. If $f(a)+j \in \sqrt{0_S}$, then $f(a) \in \sqrt{0_S}$ as $J \subseteq \sqrt{0_S}$. Thus, $f(a^m) = (f(a))^m = 0_S$ for some $m \in \mathbb{Z}$ and then $a \in \sqrt{0}$ since $\mathrm{Ker}(f) \subseteq \sqrt{0_R}$. It follows that $(a, f(a)+j) \in \sqrt{0_{R\bowtie^f J}}$, a contradiction. Hence, $(f(a)+j)^2 \in K$ and $f(a)+j \notin \sqrt{0_S}$ imply that $f(a)+j \in K$. Therefore, $(a, f(a)+j) \in \overline{K}^f$ and $\overline{K}^f$ is a semi n-ideal of $R \bowtie^f J$. □

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