Standard Formalization

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Abstract

A *standard formalization* of a scientific theory is a system of axioms for that theory in a first-order language (possibly many-sorted; possibly with the membership primitive $\in$). Patrick Suppes (Suppes (1992)) expressed skepticism about whether there is a "simple or elegant method" for presenting mathematicized scientific theories in such a standard formalization, because they "assume a great deal of mathematics as part of their substructure".

The major difficulties amount to these. First, as the theories of interest are *mathematicized*, one must specify the underlying *applied mathematics base theory*, which the physical axioms live on top of. Second, such theories are typically *geometric*, concerning quantities or trajectories in space/time: so, one must specify the underlying *physical geometry*. Third, the differential equations involved generally refer to *coordinate representations* of these physical quantities with respect to some implicit coordinate chart, not to the original quantities.

These issues may be resolved. Once this is done, constructing standard formalizations is not so difficult—at least for the theories where the mathematics has been worked out rigorously. Here we give what may be claimed to be a simple and elegant means of doing that. This is for mathematicized scientific theories comprising differential equations for $\mathbb{R}$-valued quantities Q (that is, scalar fields), defined on n ("spatial" or "temporal") dimensions, taken to be isomorphic to the usual Euclidean space $\mathbb{R}^n$. For illustration, I give standard (in a sense, "text-book") formalizations: for the simple harmonic oscillator equation in one-dimension and for the Laplace equation in two dimensions.

Contents

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1 Introduction

> A major point I want to make is that a simple standard formalization of most theories in the empirical sciences is not possible. The source of the difficulty is easy to describe. Almost all systematic scientific theories of any interest or power assume a great deal of mathematics as part of their substructure. There is no simple or elegant way to include this mathematical substructure in a standard formalization that assumes only the apparatus of elementary logic. (Suppes (1992): 207)

1.1 Differential Equations in Physics

In mathematical physics, one is often dealing with laws expressed as differential equations. Familiar examples are:

$$\text{Harmonic oscillator} \qquad \frac{d^2F}{dx^2} = -\omega^2 F \tag{1}$$

$$\text{Heat equation} \qquad \frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \tag{2}$$

$$\text{Wave equation} \qquad \left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) U = 0 \tag{3}$$

$$\text{Laplace's equation} \qquad \nabla^2 U = 0 \tag{4}$$

$$\text{Gauss's Law} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}. \tag{5}$$

$$\text{Schrödinger equation (harmonic oscillator)} \qquad \left(-\frac{\hbar^2}{2m}\partial_{xx} + \frac{1}{2}m\omega^2 x^2\right)\psi = E\psi \tag{6}$$

$$\text{Schrödinger equation (Hydrogen)} \qquad \left(-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r}\right)\psi = E\psi \tag{7}$$

In mathematics and the mathematical sciences, these differential equations are fairly well-understood.[1] This is especially so when the quantities involved are considered to be functions defined on $\mathbb{R}$ or $\mathbb{R}^n$, or perhaps on a differentiable manifold M.[2] Suppes, though, is worrying about how to formalize these laws in the context of mathematical *physics*. Indeed, Suppes is expressing a form of scepticism about such. Indeed, this scepticism is fairly widespread.

My aim here is to show that this scepticism is misplaced. The requirement is to provide standard formalization, or a machinery to generate standard formalizations, for a large class of theories in theoretical physics—and not just "toy theories". This what I shall do.

One might contrast the differential equations above with very simple, perhaps even trivial, examples:

$$\forall x(P(x) \to Q(x)) \tag{8}$$

These are, mathematically speaking, often completely trivial.[3] If we conflate the predicate symbols and corresponding set constants, (8) just says:

$$P \subseteq Q \tag{9}$$

However, it's by no means clear how one might do something similar for, say, (1):

$$\partial_{xx}F(x) + \omega^2 F(x) = 0 \tag{10}$$

We ignore, at least for a moment, that these differential equations are meant to "express" *physical* laws, and focus first on the issue of *formalization*: i.e., their encoding into a formalized language. First, note that the equation (1) has an implicit universal quantifier hiding there. Making it explicit, the equation becomes:[4]

$$(\forall x : \mathbb{R}) \left(\partial_{xx}F(x) + \omega^2 F(x) = 0\right) \tag{11}$$

Second, we see this equation governs a function

$$F : \mathbb{R} \to \mathbb{R}, \tag{12}$$

[1]There is a vast literature on differential equations in mathematical physics. Classics are Sommerfeld (1964), Courant & Hilbert (1953/1962) and Jeffreys & Jeffreys (1966), and Arnold (1989). There are many standard undergraduate level texts, which I would recommend: e.g, Arfken & Weber (2005) for physicists and Olver (2014) for mathematicians. There exist very good advanced undergraduate/graduate level texts which develop the framework of differential geometry: e.g., Schutz (1980); and the quite advanced Frankel (2011). Additionally, there are "differential equation solvers" for software packages (e.g., Matlab, Mathematica and `R`). I use one of these below in §7.2 to compare an analytic solution of Laplace's equation to a numerically integrated one. The interested reader who knows `R` programming might consult the book Soetaert et al. (2012).

[2]Nice explanations of differential geometry may be found in, say, Schutz (1980), Wald (1984), Frankel (2011), or Robin & Salamon (2012).

[3]Epistemologically, perhaps not so trivial, I need to add: as, in their own ways, Hume, Popper, Hempel and Goodman have all pointed out.

[4]Here I use type/sort notation: $(\forall x : \mathbb{R})$. It's equivalent to $(\forall x \in \mathbb{R})$.

assumed to be at least twice differentiable. I.e., F is an element of the function space $\mathrm{C}^2(\mathbb{R})$:

$$F \in \mathrm{C}^2(\mathbb{R}) \tag{13}$$

This means that whatever formalization we should like to provide, it will certainly require the resources to define and name the function space $\mathrm{C}^2(\mathbb{R})$ (and many others too, obviously).[5]

1.2 Formalizability in $L_\in$

It is clear that there is an $L_\in$-formula $\Phi_{\mathrm{C}^2(\mathbb{R})}(F)$ expressing (13), and there is an $L_\in$-formula $\Phi_{\mathsf{sho}}(F, \omega)$ expressing the Simple Harmonic Oscillator equation (11):[6]

$$\Phi_{\mathrm{C}^2(\mathbb{R})}(F) \qquad F \in \mathrm{C}^2(\mathbb{R}) \tag{14}$$

$$\Phi_{\mathsf{sho}}(F, \omega) \quad (\forall x : \mathbb{R})((\partial_{xx} + \omega^2)F = 0) \tag{15}$$

Below, after we have settled several other important matters, we incorporate these into the standard formalization.

2 Mixed Functions and Impure Structures

2.1 The Idea of a Number at Every Point

We return to equation (11),

$$(\forall x : \mathbb{R})\left(\partial_{xx}F(x) + \omega^2 F(x) = 0\right) \tag{16}$$

Here we see what I take to be the first *conceptual problem*. The equation (11) is assumed to hold for some function, in $\mathrm{C}^2(\mathbb{R})$:

[5]On the usual reduction of reals to sets, each real lives inside $V_{\omega+1}$ (which is, more or less, $\mathcal{P}(\mathbb{N})$), where the V_α's are "the von Neumann levels" in the hierarchy of pure (well-founded) sets. V_0 is defined to be $\varnothing$; and, for any ordinal α, $V_{\alpha+1}$ is defined to be $P(V_\alpha)$ (i.e., the power set of V_α); and, if λ is a limit ordinal (i.e., λ is not equal to $\alpha + 1$, for any ordinal α), then V_λ is defined to be the union of all the V_α's with $\alpha < \lambda$. This is the definition of the von Neumann hierarchy. It is analogous to defining the factorial function by, $F(0) = 1$ and $F(n + 1) = (n + 1)F(n)$, except that it uses ordinals instead of numbers and it has a special "infinity clause". So, since ω is the smallest infinite ordinal, V_ω is the smallest infinite level: it is the set of all well-founded finite pure sets. From the definition, $V_{\omega+1}$ is $P(V_\omega)$. There are several ways to encode $\mathbb{N}$ into V_ω (including a bijective way, called the Ackermann encoding; computer scientists call it the BIT predicate). And there are ways to encode each real number as a subset $X \subseteq \mathbb{N}$, and conversely too, and indeed bijectively. So, one can think of $V_{\omega+1}$ as (encoding) the set of reals. Each *function* $F : \mathbb{R} \to \mathbb{R}$ lives inside $V_{\omega+2}$; and thus a *class* of such functions lives inside $V_{\omega+3}$. Somewhat loosely speaking, the arena of almost all theoretical physics—the function spaces, Lie groups & algebras, topological spaces, manifolds, distribution spaces, bundles, etc.—is $V_{\omega+n}$, for some smallish n. This material is explained in a set theory textbook, such as Jech (2002), Potter (2004), or Enderton (1977).

[6]A sketch of the construction of the formula $\Phi_{\mathrm{C}^2(\mathbb{R})}(F)$ is given below, in §5, Example 3.

$$F : \mathbb{R} \to \mathbb{R}. \tag{17}$$

But there is an important sense in which the law that the *physicist* is interested in governs a closely related *mixed, physical function*, $\mathbf{F}$, which I'll express somewhat schematically as follows:

$$\mathbf{F} : \mathsf{point} \to \mathbb{R} \tag{18}$$

This is called a *mixed* function because its domain is the set of non-mathematical *space (or time, or spacetime) points*.

If I have, say, a flat sheet of metal and using sensors, I measure the temperature $T(p)$ at 100 points, $p_1, \ldots, p_{100}$, on this sheet, then I have a database approximating the mixed function T. If the function doesn't vary too rapidly over distance, a "smoothing" of this database may give a very accurate approximation of this function.[7] This is a physical function; its domain is physical. Typically, the laws of physics are about mixed functions—scalar fields, vector fields, tensor fields, wavefunctions, and so on.

If the physical function were not mixed, the relevant law could have no physical, or even empirical, content! For one cannot derive a *physical prediction* from some known property of, let's say, e^{-x^2} or a Bessel function.[8] One needs physical assumptions, relating some *physical function* to that Bessel function. The *domain* of the Bessel function is $\mathbb{R}$. The *domain* of the physical function is physical space. If the Bessel function has a certain property P, then this is a *necessity.* If a physical function $\mathbf{F}$—e.g., the radiation energy density at points in spacetime—has a certain property P, this is *contingent.* When Penzias and Wilson in 1964 discovered an all-pervading energy density in space (corresponding to a black body radiation distribution, at temperature around 3K), this was an empirical discovery.

To simplify the discussion a bit, instead of writing the domain as above, I prefer to lift the corresponding "sort" point to a *set constant*, "$\mathbb{P}$" (the set of points), using comprehension:

$$\forall a(\mathsf{point}(a) \leftrightarrow a \in \mathbb{P}) \tag{20}$$

This reformulates (18), giving:

$$\mathbf{F} : \mathbb{P} \to \mathbb{R} \tag{21}$$

[7] In data science, to "smooth" a dataset (usually a time series), one interpolates a smooth curve between the datapoints, usually not lying on those points, but rather averages. For example, the usual visualizations for the global temperature time series are smoothed using a moving average. A "low pass filter" is also a special kind of smoothing function (which eliminates high frequencies). Likewise, image processing filters.

[8] A real-valued Bessel function (on the reals) is a function $F : \mathbb{R} \to \mathbb{R}$ satisfying:

$$x^2 \frac{d^2F}{dx^2} + x\frac{dF}{dx} + \left(x^2 - \alpha^2\right) F = 0 \tag{19}$$

for some constant α. Such functions turn up in many applications involving waves.

The domain set $\mathbb{P}$, as defined by (20), is an "*impure set*": its elements are *physical points*. Thus, the physical function $\mathbf{F}$ assigns a real number $\mathbf{F}(p)$ to each space or temporal point $p \in \mathbb{P}$. As I see it, this raises three interesting puzzles. The first, though, is purely technical:

(T) How are the pure mathematical function $F : \mathbb{R} \to \mathbb{R}$ and the physical field $\mathbf{F} : \mathbb{P} \to \mathbb{R}$ "connected"?

The answer to this is fairly simple: *coordinate representation* with respect to a *chart* $\varphi : \mathbb{P} \to \mathbb{R}^n$. I shall explain it in §4.6.

The second and third puzzles I have in mind are of a more metaphysical nature. They were raised by Richard Feynman:[9]

> From a mathematical view, there is an electric field vector and a magnetic field vector at every point in space; that is, there are six numbers associated with every point. ... But I honestly do not understand the idea of a number at every point. (Feynman (1970): §20-3: "The Scientific Imagination").

Feynman went to say that there is no real way of avoiding the assumption here, even if we can't "understand" it. For it lies at the basis of the extra-ordinary predictive success of science. So Feynman is arguing, much in the spirit of Quine and Putnam, that what Hartry Field calls "Heavy Duty Platonism" is unavoidable.[10] A few years later, Hilary Putnam made a very similar point to Feynman's:

> ...one wants to say that the Law of Gravitation makes an objective statement about bodies—not just about sense data or meter readings. What is this statement? It is just that bodies behave in such a way that the quotient of two numbers associated with the bodies is equal to a third number associated with the bodies. But how can such a statement have any objective content at all if numbers and associations (i.e., functions) are alike mere fictions? It is like trying to maintain that God does not exist and angels do not exist while maintaining at the same time that it is an objective fact that God has put an angel in charge of each star and the angels in charge of each binary star were always created at the same time! If talk of numbers and associations between masses, etc., and numbers is theology (in the pejorative sense), then the Law of Universal Gravitation is likewise theology. (Putnam (1975): 74–75)

Putnam's conclusion is:

> ...mathematics and physics are integrated in such a way that it is not possible to be a realist with respect to physical theory and a nominalist with respect to mathematical theory. (Putnam (1975): 74)

[9]Feynman's puzzle here corresponds to what Hartry Field calls "Heavy Duty Platonism". See Field (1989): 186–200. The rejection of Heavy Duty Platonism plays a central role also in Field's main anti-nominalist arguments in Field (1980). See also Knowles (2015) for a recent discussion of this. Knowles defines it as "the view that physical magnitudes, such as mass and temperature, are cases of physical objects being related to numbers".

[10]I allude here to the Quine-Putnam Indispensability Argument (see Colyvan (2019)). Quine's views on this are scattered across several decades (e.g., Quine (1948), Quine (1986)). Putnam's canonical formulations are given in Putnam (1971) and Putnam (1975).

Feynman's second puzzle is the following:

> I find it quite amazing that it is possible to predict what will happen by mathematics, which is simply following rules which really have nothing to do with the original thing. (Feynman (1965): 171.)

I hope that the discussion below in §6, §7 and §8, especially the discussion of application conditionals, does indeed help explain the logical machinery of how this occurs. In mathematics, one can certainly obtain conclusions about some initial object, where the conclusion seems amazing: for example, if we are given an affine incidence plane $(A, \mathcal{L})$, satisfying just a few axioms, one may show that each line $\ell \in \mathcal{L}$ is isomorphic to the field $\mathbb{F}_n$, where n is the number of points on the line (with respect to any two distinct points, $O, I \in \ell$ and certain addition and multiplication operations one can define on the line ℓ).[11] This indeed carries over to the infinite case too, and with suitable axioms in place, each line is isomorphic to the standard real field $\mathbb{R}$.

The situation is entirely analogous in mathematical physics. It is indeed amazing, but it is quite analogous.

2.2 Impure Structure

A related, secondary point is that, from the perspective of physics, the domain set $\mathbb{P}$ of points is not merely treated as an *unstructured* impure set. It is an impure set carrying distinguished physical structure. For example, there are *physical geometric relationships* between the points in $\mathbb{P}$, such as the physical 3-place betweenness relation $B \subseteq \mathbb{P}^3$ and physical 4-place equidistance relation $\equiv \subseteq \mathbb{P}^4$. These are then "impure relations". Taken together, Nature gives us a *structured system* P_{geom}, defined on the point set $\mathbb{P}$:[12]

$$P_{geom} \quad := \quad (\mathbb{P}, B, \equiv) \tag{22}$$

One may call such a structured system an "*impure structure*". It is a mathematical structure, to be sure. But it is an *impure structure*.

2.3 Summary

The ontology of applied mathematics and mathematicized science contains these objects:

Object of applied mathematics	Example	
Mixed function	$Q : \mathbb{P} \to \mathbb{R}$	Physical field
Mixed function	$\varphi : \mathbb{P} \to \mathbb{R}^n$	Coordinate chart
Impure relation	$B \subseteq \mathbb{P}^3$	Geometric relation
Impure structure	$(\mathbb{P}, B, \equiv)$	Physical system

[11] For details, see, for example, Bennett (1995), Theorem 1, p. 72.

[12] I must stress that the claim that the points of space do indeed carry such relations is an approximation. Yet it holds to an extremely good approximation.

More generally, the mixed functions appearing in theoretical physics include: measurement scales, coordinate charts and field functions. As we just noted, the structured system P_{geom} has an impure carrier set $\mathbb{P}$ and two distinguished impure relations, B and $\equiv$. When we use mathematicized scientific theories, the mathematical objects that in some sense "mediate" between "purely non-mathematical" content (if this even makes sense) and "purely mathematical" content are precisely these mixed and impure mathematical objects.

3 Applied Mathematics Base Theory

3.1 Urelements/atoms

Let us assume we fully recognize the import of all this. Physical quantities and fields are mixed mathematical functions Q on a subdomain $\mathbb{P}$ of non-mathematical urelements or atoms (here, geometric points) to the reals $\mathbb{R}$ (or sometimes to $\mathbb{C}$, or to some vector space $\mathbb{V}$). Similar considerations tell us that the *physical structures or systems* involved are *impure*: they contain "non-mathematical atoms" (here, geometric points): $P_{geom} = (\mathbb{P}, B, \equiv)$.

It follows that, if we wish to carry out what Suppes is somewhat sceptical about—to formalize such theories—one needs a specification of the underlying *applied mathematics base theory*, which the physical theory lives "on top of", as it were. Such a *formalized mathematicized physical theory* T will then take the generic form

$$T = \mathsf{Appl.Math} + \Theta \tag{23}$$

where $\mathsf{Appl.Math}$ is our desired *applied mathematics base theory*, and where Θ is the *set of physical axioms for the theory in question.* We shall need the base theory to prove that certain (physical) conclusions follow from certain (physical) assumptions. Otherwise, the phenomenon that Feynman noted, and called "amazing", becomes impossible.

3.2 Specifying the Applied Mathematics Base Theory

So: what then is $\mathsf{Appl.Math}$?

Loosely speaking, there are four main approaches that one might wish to consider:

Sets	Set existence axioms.
Sorts	Many-sorted system.
Types	Type theory, including arithmetic.
Orders	Higher-order logic, including arithmetic.

There are also fairly detailed known interpretability connections between these foundational systems. In general, any of these can be interpreted into a sufficiently rich set-existence approach, indeed an entirely first-order approach. The set-existence approach is given in Field (1980), Ketland (1998), Leng (2010) and Ketland (2021). There is a close relationship between type theories and higher-order logics, and so it is not clear how to classify, say, the approach sketched in Carnap (1928), Carnap (1939), Carnap (1956) and Carnap (1966) (it could be understood either way). Roughly speaking, Carnap's base theory is a version of Russell & Whitehead's *Principia Mathematica*, *PM* (Russell & Whitehead (1912)). And higher-order systems can be considered many-sorted first-order systems. For example, it is routine to treat second-order arithmetic PA_2 as a *first-order two-sorted theory*: indeed, this is how all mathematical logicians study PA_2 and its subsystems (see, e.g., Hájek & Pudlák (2017) or Simpson (2009)).[13] The many-sorted approach is developed in Burgess (1984), Burgess & Rosen (1997), Andréka et al. (2012) (their base theories are algebraic in character, containing a system of axioms for real numbers), and incorporated into Ketland (2021) too.

One may compare these choices to the choice of the *machine code language* or *assembly language*, into which high-level programming language is compiled. On a standard computer, one generally programs in something at a moderately high level—such as Python, or R, or Javascript, or C++ and so on. The compiler or interpreter then automatically translates this down into underlying C code and FORTRAN code, and ultimately into assembly and machine code. This translation is automated and built-in, and is the result of many decades of work by software designers and developers. In this way, a high-level instruction, like

$$\texttt{plot}(\texttt{db}, \text{type} = \text{"l"}) \tag{24}$$

gets translated into barely intelligible machine code and carried out by the hardware, producing a nice visual screen plot of your database `db`. The (high-level) programmer, or computer user, does not need to worry about how the *automated translation* works: that has all been worked out already and is built-in. Somewhat different compilers may generate quite different machine code renditions for the same high-level instruction: but this non-uniqueness is harmless.

Similarly, the working mathematician or physicist, does not need to worry about the machine code translation of, for example, "f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$" when deriving various results and theorems. This notion does indeed have a machine code—i.e., $L_\in$—translation,

$$\Phi_{\mathsf{cont.fun.reals}}(f) \tag{25}$$

And then theorems such as the Intermediate Value Theorem,

$$(\Phi_{\mathsf{cont.fun.reals}}(f) \wedge a < b \wedge f(a) < 0 \wedge f(b) > 0) \to (\exists c \in \mathbb{R})(f(c) = 0) \tag{26}$$

[13]The reason for this is that when PA_2 is treated as a two-sorted first-order theory, one may use The Completeness Theorem—Henkin's Completeness Theorem (Henkin (1950))—to transfer between model-theoretic and proof-theoretic results.

are provable using the axioms of set theory. The working mathematician or physicist does not need to advert to such facts. However, when we are examining the meta-theory of mathematical or scientific theories, we must point out that this is how it works.

So, a formalized scientific theory sits on top of its underlying base mathematical theory. The relevant "high-level" predicates (used by the scientist) are, or could in principle be, translated into machine code through a long and complicated series of (usually implicit) definitions. This means that the base theory has been extended with all of these explicit definitions. So that, for any standard mathematical theorem ϕ, one can write,

$$\mathsf{Appl.Math} \vdash \phi \tag{27}$$

The situation is somewhat analogous to formalized truth theories, where the base theory is Peano arithmetic, PA, whose basic "machine code" symbols are $0, S, +, \times$.[14] The axioms are, say, those for compositional truth:

$$\begin{array}{lll} \mathsf{CT1} & (\forall t, u : \mathtt{ClTerm})\,[\mathsf{T}(\mathtt{equal}(t, u)) \leftrightarrow \mathtt{val}(t) = \mathtt{val}(t))] & (28) \\ \mathsf{CT2} & (\forall x : \mathtt{Sent})\,[\mathsf{T}(\mathtt{neg}(x)) \leftrightarrow \neg \mathsf{T}(x)] & (29) \\ \mathsf{CT3} & (\forall x, y : \mathtt{Sent})\,[\mathsf{T}(\mathtt{and}(x, y)) \leftrightarrow \mathsf{T}(x) \wedge \mathsf{T}(y)] & (30) \\ \mathsf{CT4} & (\forall x : \mathtt{Form}_1)(\forall v : \mathtt{Var})\,[\mathsf{T}(\mathtt{forall}(v, x)) \leftrightarrow \forall n(\mathsf{T}(\mathtt{sub}(\mathtt{num}(n), v, x)))] & (31) \end{array}$$

where $\mathtt{ClTerm}(t)$, $\mathtt{equal}(t, u)$, $\mathtt{val}(t)$, ... are *defined function symbols and predicates*, ultimately expressible in terms of $0, S, +, \times$. Without such definitions, we should be unable to study formalized truth theories.[15]

Admittedly, there are squabbles amongst workers in these fields as regards which machine-code level approach is "truly foundational". I do not wish to participate in these frog-mouse battles. For it's clear that *detailed interpretability mappings and inter-translations* exist. These largely settle the matter of "fundamentality", modulo further understanding of the concepts involved. It is akin to arguing about whether a blue electric guitar is better than a red electric guitar. So long as you can tune it up, plug it in and pluck the strings with the amplified notes coming out, it does its job.

There are remaining differences, which come down to *implementation features*. How complex are the definitions? How complex are the rules? How natural, or awkward, are the axioms? What is the basic language and its syntax? These may vary significantly and, indeed, surprisingly.[16]

[14] See, e.g., Halbach (2014) or Cieśliński (2017).

[15] I have seen it remarked that a formalized theory, in particular for scientific theories, must always be *stated directly in machine code*. This is incorrect. The requirement is only that it uses predicates and function symbols which are *definable*. Axioms "higher-up", so to speak, are stated using these *defined predicates*. $\mathsf{CT1}$-$\mathsf{CT4}$ above are examples. But there are many others, well-known in mathematical logic.

[16] One example is the following: Georg Kreisel conjectured that if there is some fixed k such that, for all $n \in \mathbb{N}$, PA proves $\phi(S^n(0))$ in at most k steps, then PA proves $\forall x \phi(x)$. Does Kreisel's Conjecture (KC) hold for Peano arithmetic? It turns out to be implementation dependent. Parikh (1973) showed that

But all that said, I shall focus on a foundational system which incorporates set existence axioms, including atoms (urelements), in a many-sorted logic.[17] In Ketland (2021), such an applied mathematics base theory is specified out in some detail, adopting a combination of the first two listed approaches: a many-sorted set theory with atoms, with a variable application signature. The slimmed down approach given in Ketland (2021) has four "basic sorts", $\{\mathsf{global}, \mathsf{atom}, \mathsf{bool}, \mathsf{set}\}$ and defined the four-sorted (first-order) set theory with atoms on top of that. Here, it turns out to be convenient to include a few more basic sorts:

$$S_0 = \{\mathsf{global}, \mathsf{atom}, \mathsf{bool}, \mathsf{set}, \mathsf{point}, \mathsf{nat}, \mathsf{real}\}$$

It will simplify things if we rewrite these base sorts (types) as follows:

$$S_0 = \{\mathbb{G}, \mathbb{A}, \mathbb{B}, \mathbb{S}, \mathbb{P}, \mathbb{N}, \mathbb{R}\}$$

Associated with each sort is special set of variables:

Base Sort	Variables	Range over
$\mathbb{G}$	$x, y, x_1, x_2, \ldots$	everything (at first-order level)
$\mathbb{A}$	$a, a_1, a_2, \ldots$	atoms
$\mathbb{B}$	$\mathbf{b}, \mathbf{b}_1, \mathbf{b}_2, \ldots$	booleans $(\top, \bot)$
$\mathbb{S}$	$X, Y, Z, X_1, X_2, \ldots$	sets
$\mathbb{P}$	$p, p_1, p_2, \ldots$	points
$\mathbb{N}$	$n, m, n_1, n_2, \ldots$	natural numbers
$\mathbb{R}$	$r, r_1, r_2, \ldots$	reals

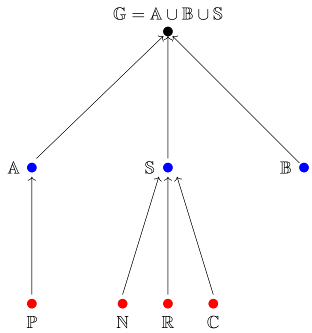
I allow equations between any sort variables. In this way (following Barwise (1975)), I may *explicitly define* each sort, using a global variable and a variable for the relevant sort in S_0:

$$\begin{array}{lcl@{\qquad}lcl}
\mathsf{atom}(x) & := & \exists a(a = x) & \mathsf{bool}(x) & := & \exists \mathbf{b}(\mathbf{b} = x) \\
\mathsf{set}(x) & := & \exists X(X = x) & \mathsf{point}(x) & := & \exists p(p = x) \\
\mathsf{nat}(x) & := & \exists n(n = x) & \mathsf{real}(x) & := & \exists r(r = x).
\end{array}$$

In addition, we impose the following subsort structure (the arrows represent subsort):

KC holds for an implementation called PA^*, where the function symbols $+, \bullet$ are replaced by appropriate predicates, say, $A(x, y, z)$ and $M(x, y, z)$, with suitable axioms stating existence and uniqueness. The proof of this is quite difficult. But, cleverly choosing $\phi(x)$, it is easy to give a formulation PA^ϕ of PA, in the same language $L(0, S, +, \bullet)$ and with the same theorems as standard PA, such that KC does not hold. See Cavagnetto (2009), Proposition 1.2 (we choose $\phi(x)$ to be any formula witnessing ω-incompleteness).

[17] The theories of Russell and Zermelo included atoms (Zermelo (1908), Russell (1903), Russell (1908)) as did Church's type theory (Church (1940)). Recent examples are: Mendelson (2010) (§4.6.5); Jech (2002) (p. 250 ff.); Potter (2004). A detailed exposition of many-sorted logic appears in Manzano (1996) and recent applications to questions concerning theory formalization in Halvorson (2019).



And the following "exclusion structure"

$$\mathbb{A} \cap \mathbb{B} = \varnothing = \mathbb{A} \cap \mathbb{S} = \mathbb{B} \cap \mathbb{S}$$

An application signature σ over the seven sorts has the form,

$$\sigma = \{\mathsf{c}_i\}_{i \in I_1} \cup \{\mathsf{P}_i\}_{i \in I_2} \cup \{\mathsf{F}_i\}_{i \in I_3} \tag{32}$$

where c_i are primitive constants, P_i are primitive predicates, and F_i are primitive function symbols, whose sorts are declared with explicit "type declarations" of the form:

$$\mathsf{c} \quad : \quad s \tag{33}$$
$$\mathsf{P} \quad : \quad s_1 \Rightarrow \cdots \Rightarrow s_n \Rightarrow \mathtt{bool} \tag{34}$$
$$\mathsf{F} \quad : \quad s_1 \Rightarrow \cdots \Rightarrow s_n \Rightarrow s_{n+1} \tag{35}$$

where the s_i are the basic sorts/types. If the type of predicate P is $s_1 \Rightarrow \cdots \Rightarrow s_n \Rightarrow \mathtt{bool}$, we say it is an n-place predicate. If the type of function symbol F is $s_1 \Rightarrow \cdots \Rightarrow s_n \Rightarrow s_{n+1}$, we say it is an n-place function symbol. If all the sorts, the s_i's, in a type declaration (except `bool`) are `atom`, we say that the symbol is "on atoms" or is "purely atomic".

I treat identity and membership as *global binary predicates*:[18]

$$= \quad : \quad \mathsf{global} \Rightarrow \mathsf{global} \Rightarrow \mathtt{bool} \tag{36}$$
$$\in \quad : \quad \mathsf{global} \Rightarrow \mathsf{global} \Rightarrow \mathtt{bool} \tag{37}$$

The *extended signature*, written $\sigma_\in$, is obtained by adding the binary membership predicate $\in$. $L(\sigma)$ is the first-order language over σ, and $L(\sigma_\in)$ is the first-order language over $\sigma_\in$.

3.3 Axioms for ZCA_σ

Definition 1 (Set theory with atoms over a given signature, ZCA_σ)**.** Given a signature σ, the axioms of ZCA_σ in $L(\sigma_\in)$ are:[19]

[18] So $v_i = v_j$ and $v_i \in v_j$ are well-formed primitive formulas irrespective of the sorts of the variables v_i, v_j.

[19] See, e.g., Jech (2002) or Enderton (1977) for a detailed exposition of these axioms.

$$\text{Atoms set.} \quad \exists X_1 \forall x_1 (x_1 \in X_1 \leftrightarrow \mathtt{atom}(x_1)). \tag{38}$$

$$\text{Extensionality.} \quad \forall x_1 (x_1 \in X_1 \leftrightarrow x_1 \in X_2) \to X_1 = X_2. \tag{39}$$

$$\text{Separation.} \quad \exists X_1 \forall x_1 \big(x_1 \in X_1 \leftrightarrow (x_1 \in X_2 \wedge \phi(x_1, \dots))\big). \tag{40}$$

$$\text{Pairing.} \quad \exists X_1 \forall x_1 \big(x_1 \in X \leftrightarrow (x_1 = x_2 \vee x_1 = x_3)\big). \tag{41}$$

$$\text{Union.} \quad \exists X_1 \forall x_1 \big(x_1 \in X_1 \leftrightarrow \exists X_2 (x_1 \in X_2 \wedge X_2 \in X_3)\big). \tag{42}$$

$$\text{Power set.} \quad \exists X_1 \forall x_1 \big(x_1 \in X_1 \leftrightarrow \exists X_2 (x_1 = X_2 \wedge X_2 \subseteq X_3)\big). \tag{43}$$

$$\text{Infinity.} \quad \exists X_1 \big(\varnothing \in X_1 \wedge \forall X_2 (X_2 \in X_1 \to X_2 \cup \{X_2\} \in X_1)\big). \tag{44}$$

$$\text{Choice.} \quad \varnothing \notin X_1 \to \exists X_2 (\forall X_3 \in X_1) \exists x_1 (X_3 \cap X_2 = \{x_1\}) \tag{45}$$

In addition, there are various explicit definitions of required $L(\sigma_\in)$-formulas (see below), and a number of "sort axioms" implementing the above sort structure, and the following two axioms:

$$(a) \quad \mathsf{nat}(x) \leftrightarrow \forall X (\mathsf{inductive}(X) \to x \in X) \tag{46}$$

$$(b) \quad \mathsf{real}(x) \leftrightarrow \mathbb{R}_\in(x) \tag{47}$$

(where the $L(\sigma_\in)$-formula $\mathsf{inductive}(X)$ is defined below, and the $L(\sigma_\in)$-formula $\mathbb{R}_\in(x)$ can be reconstructed from any decent set theory textbook: e.g., by stating that x is a Dedekind section of rationals).

This corresponds, more or less, to what is called ZCA_σ in Ketland (2021), except that there I included Foundation, which in this context, is unnecessary and Empty Set, but that is redundant, as it follows from Separation. We assume below that all the usual notions from core mathematics have been explicitly defined using $L(\sigma_\in)$-formulas. For example,

New Formula	Explicit Definition
$\mathsf{ord.pair}(X_1, x_1, x_2)$	$X_1 = \{\{x_1\}, \{x_1, x_2\}\}$.
$\mathsf{prod}(X_1, X_2, X_3)$	$\forall x_1 (x_1 \in X_1 \leftrightarrow (\mathsf{ord.pair}(x_1, x_2, x_3) \wedge x_2 \in X_2 \wedge x_3 \in X_3))$.
$X_1 = X_2 \times X_3$	$\mathsf{prod}(X_1, X_2, X_3)$.
$\mathsf{bin.rel}(X_1, X_2)$	$X_1 \subseteq X_2 \times X_2$.
$\mathsf{inductive}(X_1)$	$\varnothing \in X_1 \wedge (\forall X_2 \in X_1)(X_2 \cup \{X_2\} \in X_1)$.
etc.	

The additional sort-matching axioms (a) and (b) ensure that the (elements of the) primitive sorts $\mathtt{nat}$ and real, match their explicit definitions in set-theoretical language. These ensure that, in a (seven-sorted) model M for $L(\sigma_\in)$, the interpretation $\mathtt{nat}^M$ of the sort $\mathtt{nat}$ will be exactly what the model "thinks" the elements of $\mathbb{N}$ are (i.e., finite ordinals), and the interpretation real^M of the sort real will be exactly what the model "thinks" the elements of $\mathbb{R}$ are (i.e., Dedekind sections of rationals). Henceforth, I assume these axioms, along with the sort axioms, are contained in ZCA_σ. So, the system ZCA_σ contains the various sort axioms, along with the usual set-existence axioms above, along with some implicit collection of explicit definitions of the usual notions of mathematics, as sketched above, and sort-matching axioms (a) and (b).[20]

ZCA_σ is a powerful mathematical theory which interprets a large amount of core mathematics. For example, it interprets higher-order logic (HOL) for all finite orders; it interprets nth-order arithmetic, PA_n, for all n. Because ZCA_σ is a set theory with atoms (over application signature σ), it will allow us easily to apply mathematics to *impure mathematical objects*—impure sets, relations and structures—and to *mixed objects*—mixed relations and functions (e.g., coordinate charts, measurement scales and various field quantities).

I include the *Axiom of Choice* (one may omit it if one wishes, to obtain ZA_σ). The degree to which scientifically applicable mathematics and theoretical physics rests on Choice is, at least at present, difficult to establish.[21] Certainly, functional analysis requires versions of Choice and functional analysis is central to parts of theoretical physics. An example of the use of Choice to prove a result important to theoretical physics is the Initial Value Problem in General Relativity, with the first standard result being Choquet-Bruhat & Geroch (1969), who showed that a specification of initial data on a spacelike hypersurface leads to a unique full solution—a Cauchy maximal development—satisfying Einstein's field equations.[22] A recent contribution to the question concerning "how much" Choice is required is Sbierski (2016), where a slightly weaker form of Choice is used in establishing the Cauchy maximal development results for GR.

4 Physical Geometry

4.1 Euclidean Geometry

The differential equations listed above are fairly well-understood when the quantities involved are considered to be functions defined on $\mathbb{R}$ or $\mathbb{R}^n$. So, the equations, on which such physical theories are based, are, typically, living on a *geometric background* which "modelled" by $\mathbb{R}^n$.[23] But this geometric background (in the physical case) is *physical geometry*—not simply $\mathbb{R}^n$, say. This means that, in order to provide a proper formalization, we shall need to somehow incorporate physical geometry, along with required *coordinate*

[20] So, what is called "ZCA_σ" below is, strictly speaking, a definitional extension of the given axioms.

[21] An interesting discussion of the mathematical needs of scientifically applicable mathematics is Feferman (1992), in which Feferman discusses a predicative system W (descended from that set out by Hermann Weyl in Weyl (1918)) which is proof-theoretically conservative over PA. However, this all seems besides the *central point*, which derives from Quine and Putnam: physical quantities, fields, and so on, are themselves mixed functions and impure structures. It seems irrelevant then whether the set theory one uses, say, is conservative over PA or not. Even if it is, these physical, mixed quantities exist (and their values, etc.) and that contradicts nominalism. The realism of Quine and Putnam in question *concerns these physical, mixed quantities.*

[22] See Hawking & Ellis (1973), Ch. 7, especially pp. 249–251; see Wald (1984), Theorem 10.2.2, p. 264.

[23] Note that, using numerical integration, one may also model geometry on a *discrete grid*. An example of this is briefly explained in §7.2, where we look at the analytic solution for Laplace equation in 2 dimensions (with Dirichlet boundary conditions on a unit square) and compare that with the numerical solution obtained using a software partial differential equation solver.

charts (which map points to elements of $\mathbb{R}^n$). So, how do we specify, axiomatically, the physical geometry?

There are a great many approaches to the formalization of Euclidean geometry (and variants). For our purposes, the most important are the axiomatizations of Euclidean geometry, both first-order and second-order, associated with David Hilbert (Hilbert (1899)), Alfred Tarski (Tarski (1959), Tarski & Givant (1999)), Karol Borsuk & Wanda Szmielew (Borsuk & Szmielew (1960)) and related work. I shall not write out the Tarski axioms for n-dimensional Euclidean geometry, as there are slight differences accounting for dimensionality, and there is a slight peculiarity in the one-dimensional case. Instead, I refer the reader to the detailed discussion in Tarski & Givant (1999). And for a comprehensive survey of axiomatizations of "ordered geometry" based, usually, on "betweenness" notions, I refer to the reader to the survey Pambuccian (2011).

For our purposes, we are interested in (physical) geometry formulated with primitives:

$\mathsf{B}(x, y, z)$	the point y lies inclusively between points x and z.
$xy \equiv zu$	the segment xy has the same length as the segment zu.

and we are interested in the *second-order axiomatizations*: this means that the theory includes *set variables and quantifiers*, ranging over sets of points. Below, I shall call this (finite) axiom system $\mathsf{EG}^{(n)}$ (excluding the comprehension axioms).

Definition 2 (Signature for Euclidean Geometry)**.** The signature

$$\sigma = \{\mathsf{B}, \equiv\}$$

with type declarations:

$$\begin{array}{lll} \mathsf{B}: & \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{B} & \text{(geometric betweenness primitive)} \\ \equiv: & \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{B} & \text{(geometric congruence/equidistance primitive)} \end{array}$$

is the Tarskian *betweenness-congruence* (or *betweenness-equidistance*) signature for Euclidean geometry. (Below, Definition 8, we call signatures containing B and $\equiv$ "bet-cong signatures".)

Given $\sigma = \{\mathsf{B}, \equiv\}$, let $L_2(\sigma)$ be the second-order language over σ.[24]

Definition 3 (Euclidean Geometry)**.** $\mathsf{EG}^{(n)}$ (second-order Euclidean geometry in n dimensions) is the theory in $L_2(\sigma)$, whose axioms are Tarski's axioms (for dimensionality n), along with the second-order Dedekind Continuity Axiom:

$$\begin{array}{ll} \mathsf{Cont} & \exists p_3 \forall p_1 \forall p_2 \, (p_1 \in X_1 \wedge p_2 \in X_2 \to \mathsf{B}(p_3, p_1, p_2)) \\ & \to \exists p_4 \forall p_1 \forall p_2 \, (p_1 \in X_1 \wedge p_2 \in X_2 \to \mathsf{B}(p_1, p_4, p_2)) \end{array}$$

[24] To obtain $L_2(\sigma)$, we add second-order variables $X_i^{(j)}$ (i.e., the ith variable for j-place relations), and atomic formulas $X_i^{(j)}(p_1, \dots, p_j)$ (and equations $X_i^{(j)} = X_k^{(j)}$), and corresponding second-order quantifiers. In a clear sense, one can interpret $L_2(\sigma)$ into the full set-theoretic extended language $L(\sigma_\in)$. For the set-theoretic translation $(X_i^{(j)}(p_1, \dots, p_j))^\circ$ of an atomic formula $X_i^{(j)}(p_1, \dots, p_j)$ is given as: $(p_1, \dots, p_j) \in X_i^{(j)}$, using pairing.

Definition 4. Given $\sigma = \{\mathsf{B}, \equiv\}$, we define (within ZCA_σ, using Separation, Extensionality and Pairing) the following sets and relations:[25]

$$\begin{aligned}
\mathbb{A} &:= \{x \mid \exists a(x = a)\} && (48)\\
\mathbb{P} &:= \{p \in \mathbb{A} \mid \exists x(x = p)\} && (49)\\
\mathsf{B} &:= \{(a_1, a_2, a_3) \in \mathbb{P}^3 \mid \mathsf{B}(a_1, a_2, a_3)\} && (50)\\
\equiv &:= \{(a_1, a_2, a_3, a_4) \in \mathbb{P}^4 \mid a_1 a_2 \equiv a_3 a_4)\} && (51)\\
P_{\mathsf{geom}} &:= (\mathbb{P}, \mathsf{B}, \equiv) && (52)
\end{aligned}$$

These are examples of the enormously useful Comprehension feature of applied mathematics. We may then, whenever need be, use the equivalences:

$$\begin{aligned}
x \in \mathbb{A} &\leftrightarrow \mathsf{atom}(x) && (53)\\
x \in \mathbb{P} &\leftrightarrow \mathsf{point}(x) && (54)\\
(x_1, x_2, x_3) \in \mathsf{B} &\leftrightarrow \mathsf{B}(x_1, x_2, x_3) && (55)\\
(x_1, x_2, x_3, x_4) \in \equiv &\leftrightarrow x_1 x_2 \equiv x_3 x_4 && (56)
\end{aligned}$$

4.2 Models

Definition 5 (Euclidean spaces)**.** Let $\sigma = \{\mathsf{B}, \equiv\}$ be the betweenness-congruence signature for Euclidean geometry. Let $M = (D, B, C)$ be an $L_2(\sigma)$-structure, with $D = \mathrm{dom}(M)$, $B = \mathsf{B}^M$ and $C = \equiv^M$. Let $\mathsf{EG}^{(n)}$ in in $L_2(\sigma)$ be Tarski's system of *second-order Euclidean geometry*, for n dimensions. Then a *Euclidean space* (of n dimensions) is a full model

$$(D, B, C) \models_2 \mathsf{EG}^{(n)}. \qquad (57)$$

(We write "$\models_2$" to emphasize we mean a full model.)

Next, I define the "standard coordinate models" for Euclidean geometry.

Definition 6 ($\mathbb{E}^{(n)}(\mathbb{F})$)**.** Let $\mathbb{F} = (F, 0, 1, +, \bullet, \leq)$ be a Euclidean ordered field (an ordered field in which each positive element is a square). The "Cartesian spaces" over $\mathbb{F}$ are defined as follows. We use the notation "$\mathbf{x} = (x_1, \dots, x_n)$" to mean that $\mathbf{x}$ is a tuple of length n,

[25] I conflate the function symbol B and the set constant B naming its extension; likewise $\equiv$. But this conflation is harmless, since context always disambiguates.

where each $x_i \in F$. We define:

$$\langle \mathbf{x}, \mathbf{y} \rangle_{F^n} := \sum_i x_i \bullet y_i \tag{58}$$
$$\|\mathbf{x}\|_{F^n} := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_{F^n}} \tag{59}$$
$$d_{F^n}(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_{F^n} \tag{60}$$
$$B_{F^n}\mathbf{xyz} := (\exists \lambda \in [0,1])(\mathbf{y} - \mathbf{x} = \lambda(\mathbf{z} - \mathbf{x})) \tag{61}$$
$$\mathbf{xy} \equiv_{F^n} \mathbf{zu} := d(\mathbf{x}, \mathbf{y}) = d(\mathbf{z}, \mathbf{u}) \tag{62}$$
$$\mathbb{E}^n(\mathbb{F}) := (F^n, B_{F^n}, \equiv_{F^n}) \tag{63}$$
$$\mathbb{E}^n := \mathbb{E}^n(\mathbb{R}) = (\mathbb{R}^n, B_{\mathbb{R}^n}, \equiv_{\mathbb{R}^n}) \tag{64}$$

Though I have not given the axioms, it is not difficult to show that:

Lemma 1. $\mathbb{E}^n \models_2 \mathsf{EG}^{(n)}$.

What is more interesting is that a powerful *converse* holds:

Theorem 1 (Representation Theorem: Tarski)**.** Let $M = (D, B, C)$ be a $L(\sigma)$-structure. Then $M \models_2 \mathsf{EG}^{(n)}$ if and only if there is an isomorphism $\varphi : M \to \mathbb{E}^n$.

So, $\mathbb{E}^n$ is the standard coordinate model (over $\mathbb{R}$) for Euclidean geometry. All (full) models of $\mathsf{EG}^{(n)}$ are isomorphic to $\mathbb{E}^n$. And these isomorphisms are in fact (standard) coordinate charts on $\mathbb{E}^n$. They are *standard* because they respect isomorphism conditions for betweenness and congruence on $\mathbb{R}^n$. That is, given a model $M \models_2 \mathsf{EG}^{(n)}$, we have, for any $p_1, \dots, p_4 \in D$,

$$(p_1, p_2, p_3) \in B \quad \text{iff} \quad (\varphi(p_1), \varphi(p_2), \varphi(p_3)) \in B_{\mathbb{R}^n} \tag{65}$$
$$(p_1, p_2, p_3, p_4) \in C \quad \text{iff} \quad (\varphi(p_1), \varphi(p_2), \varphi(p_3), \varphi(p_4)) \in \equiv_{\mathbb{R}^n} \tag{66}$$

4.3 Geometry Axiom

The Representation Theorem 1 for Euclidean geometry can be formalized and proved *inside* ZCA_σ.[26] Nowadays, such a result is called an *Internal* Representation Theorem.[27] For geometry, we have the corresponding:[28]

Theorem 2. $\mathsf{ZCA}_\sigma \vdash \left[(\exists \varphi : \mathbb{P} \to \mathbb{R}^n) \ (P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^n) \right] \leftrightarrow \mathsf{EG}^{(n)}$.

Proof. The notion of "satisfaction" can be formalized inside ZCA_σ, yielding a disquotation sentence:

$$\text{(a)} \quad \mathsf{ZCA}_\sigma \vdash \ \mathsf{Sat}_2(P_{\mathsf{geom}}, \ulcorner \mathsf{EG}^{(n)} \urcorner) \leftrightarrow \mathsf{EG}^{(n)}$$

[26]This is analogous to the fact that one can prove a Representation Theorem for second-order Peano arithmetic PA_2 inside set theory: any (full) model $M \models \mathsf{PA}_2$ is isomorphic to $(\omega, \varnothing, (.)^+, +_\omega, \bullet_\omega)$.

[27]See also Väänänen & Wang (2015); Button & Walsh (2018) (pp. 223–50); Ketland (2021).

[28]For us, it is important that $\mathsf{EG}^{(n)}$ be expressed as a *single axiom*, and one can do this because the Continuity Axiom is a single second-order sentence quantifying over sets of points.

The Tarski Representation Theorem 1 can be formalized inside ZCA_σ (using satisfaction):

$$\text{(b)}\quad \mathsf{ZCA}_\sigma \vdash \quad (\forall M : \mathsf{Str}_\sigma)\left(\mathsf{Sat}_2(M, \ulcorner \mathsf{EG}^{(n)} \urcorner) \leftrightarrow (\exists \varphi : \mathrm{dom}(M) \to \mathbb{R}^n)(M \overset{\varphi}{\cong} \mathbb{E}^n)\right)$$

where $\mathsf{Str}_\sigma(x)$ is a defined predicate, "x is a structure of signature σ". Now ZCA_σ can certainly prove that P_{geom} is a structure satisfying $\mathsf{Str}_\sigma(x)$. I.e., it proves $\mathsf{Str}_\sigma(P_{\mathsf{geom}})$. And it proves that its domain is $\mathbb{P}$. So, instantiating (b) with the term "P_{geom}":

$$\text{(c)}\quad \mathsf{ZCA}_\sigma \vdash \quad \mathsf{Sat}_2(P_{\mathsf{geom}}, \ulcorner \mathsf{EG}^{(n)} \urcorner) \leftrightarrow (\exists \varphi : \mathbb{P} \to \mathbb{R}^n)(P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^n)$$

Immediately, applying disquotation (a), we obtain:

$$\text{(d)}\quad \mathsf{ZCA}_\sigma \vdash \quad \left[(\exists \varphi : \mathbb{P} \to \mathbb{R}^n)(P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^n)\right] \leftrightarrow \mathsf{EG}^{(n)}$$

as claimed.

□

This says, provably inside ZCA_σ, that the (full) models of $\mathsf{EG}^{(n)}$ are isomorphic to the corresponding Cartesian space $\mathbb{E}^n$, with the isomorphism given by some relevant coordinate chart. And thus, within $\mathsf{ZCA}_\sigma + \mathsf{EG}^{(n)}$, we can prove the isomorphism claim. This corresponds, more or less, to the following comment by John Burgess:

> Within ZFU one can define the pure sets as those none of whose elements, elements of elements, and so on, is an urelement or non-set. ... Within G_0+ZFU it can be proved that the system of point surrogates (and sets thereof, and sets of sets thereof) is *isomorphic* to the system of points (and sets thereof, and sets of sets thereof). (Burgess (1988): 461)

4.4 Reformulation

Definition 7. We extend $\sigma = \{B, \equiv\}$ to $\sigma^+ = \{B, \equiv, \varphi\}$, introducing a new function symbol φ, with type/sort:[29]

$$\varphi : \quad \mathbb{P} \Rightarrow \mathbb{R}^n$$

We define the n-dimensional geometry axiom $\mathsf{Geom}^{(n)}$, in $L(\sigma^+_\in)$ as follows:[30]

$$\mathsf{Geom}^{(n)} : \quad P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^n \tag{67}$$

I will occasionally call this the "Representation Hypothesis" (for geometry).

[29] Strictly speaking, this is a bit sloppy as I have not defined the product sort $\mathbb{R}^n$. In our setting, with the given basic sorts, the correct sort of φ is $\mathbb{P} \Rightarrow \mathbb{S}$; but we add a new axiom (called a *codomain axiom* in Ketland (2021)): $(\forall p \in \mathbb{P})(\varphi(p) \in \mathbb{R}^n)$.

[30] One may then regard the symbol "φ" as a skolem constant for the existential quantifier in "$(\exists \varphi : \mathbb{P} \to \mathbb{R}^n)$ $(P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^n)$".

4.5 Bet-Cong Application Signatures

Definition 8. A *bet-cong* signature is an application signature of the following form,

$$\sigma = \{\overbrace{\mathsf{B}, \equiv, \varphi}^{\text{geometry}}, \overbrace{Q_1, \dots, Q_k}^{\text{quantity primitives}}, \overbrace{\omega_1, \dots, \omega_m}^{\text{parameters}}\} \tag{68}$$

with type/sort declarations:[31]

$$\begin{array}{rcl}
\mathsf{B}: & \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{B} & \text{(geometric betweenness primitive)} \\
\equiv: & \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{P} \Rightarrow \mathbb{B} & \text{(geometric congruence primitive)} \\
\varphi: & \mathbb{P} \Rightarrow \mathbb{R}^n & \text{(a fixed (global) coordinate chart)} \\
Q_i: & \mathbb{P} \Rightarrow \mathbb{V}_i & \text{(for } i = 1 \text{ to } k; \mathbb{V}_i \text{ is usually } \mathbb{R} \text{ or } \mathbb{C}) \\
\omega_i: & \mathbb{R} & \text{(for } i = 1 \text{ to } m)
\end{array}$$

4.6 Coordinate Representation

The physical laws involved, concerning physical quantities (and fields), are usually expressed as differential equations referring to *coordinate representations* of physical quantities (and fields) with respect to some implicit coordinate chart, not the original mixed quantities/fields which are defined on the physical points of space or time or spacetime. How is the original field related to its coordinate representation?

In the simplest case (where there is a global coordinate chart), the relationship is given by an *explicit definition*,

$$F \quad := \quad \mathbf{F} \circ \varphi^{-1}, \tag{69}$$

where

$$\varphi : \mathbb{P} \to \mathbb{R} \tag{70}$$

is a *coordinate chart* (1-dimensional, in this case). The functions F and $\mathbf{F}$ have different domains, clearly; but the same codomain ($\mathbb{R}$):

$$\begin{tikzcd}
 & \mathbb{R} \\
\mathbb{P} \arrow[ur, "\mathbf{F}"] \arrow[r, "\varphi"'] & \mathbb{R} \arrow[u, "F"']
\end{tikzcd}$$

[31] Strictly speaking, again, the sort declaration given for Q_i is a bit sloppy as I have not defined the relevant sort $\mathbb{V}$ (i.e., elements of some vector space). In our setting, with the given basic sorts, the correct sort of Q_i is $\mathbb{P} \Rightarrow \mathbb{S}$; but, for each Q_i, we add a new codomain axiom: $(\forall p \in \mathbb{P})(Q_i(p) \in \mathbb{V})$, where $X \in \mathbb{V}$ is the $L_\in$-formula defining $\mathbb{V}$ (e.g., $\mathbb{R}$ or $\mathbb{C}$, ...).

Definition 9. Given a bet-cong signature σ, for each quantity primitive function symbol Q_i, with type declaration,

$$Q_i : \quad \mathbb{P} \Rightarrow \mathbb{V}_i \tag{71}$$

we introduce a new function symbol Q_i^{φ} with type declaration

$$Q_i^{\varphi} : \quad \mathbb{R}^n \Rightarrow \mathbb{V}_i \tag{72}$$

which is explicitly defined by:

$$\mathsf{Dfn}(Q_i^{\varphi}) : \quad (\forall \mathbf{x} : \mathbb{R}^n) \big(Q_i^{\varphi}(\mathbf{x}) = (Q_i \circ \varphi^{-1})(\mathbf{x})\big) \tag{73}$$

Q_i^{φ} is called the *coordinate representation* of Q_i wrt φ.

5 State Space Axioms and Dynamical Axioms

The remaining aspects concern the far more specific features of the detailed theories in question. In moderately simple cases, the remainder of constructing a standard formalization amount to specifying a "state space axiom" and specifying the "dynamical axioms", which will be differential equation(s) governing the quantities involved.

Definition 10 (State Space Axiom)**.** A state space axiom takes the following form:

$$\mathsf{State} \quad : \quad Q^{\varphi} \in \mathsf{StateSpace} \tag{74}$$

where $\mathsf{StateSpace}$ is defined separately.

Example 1. An example of a state space is $\mathrm{C}^k(\mathbb{R}^n)$ $(k \in \mathbb{N})$, or $\mathrm{C}^{\infty}(\mathbb{R}^n)$. △

Example 2. Another standard example of a state space is $L^2(\mathbb{R}^n)$. △

It is clear that such state spaces are definable using rather complicated $L_{\in}$-formulas.

Example 3. *Defining* $\mathrm{C}^2(\mathbb{R})$
Consider a state space axiom saying

$$Q^{\varphi} \in \mathrm{C}^2(\mathbb{R}). \tag{75}$$

We have defined Q^{φ} explicitly, given Q and φ (by functional composition). So, we wish to find an $L_{\in}$ formula expressing:

$$X \in \mathrm{C}^2(\mathbb{R}) \tag{76}$$

To do this, first, I assume we already defined $\mathbb{R}$. Then I need to explain what it is for X to be a *function* with the right domain and codomain: i.e., $\mathbb{R}$; and then I need to explain what it is for such an X to be *continuous* and *(twice) differentiable.*

(Function) We first require that X be a *function* with domain $\mathbb{R}$ and codomain $\mathbb{R}$. That is, X is a set of pairs (r_1, r_2), where $r_1, r_2 \in \mathbb{R}$, and this set X is such that if $(r_1, r_2) \in X$ and $(r_1, r_3) \in X$, then $r_2 = r_3$, and also is such that, for all $r \in \mathbb{R}$, there exists $r_1 \in \mathbb{R}$, such that $(r, r_1) \in \mathbb{R}$. X is thus a total function $\mathbb{R} \to \mathbb{R}$.
So, whenever $r \in \mathbb{R}$, we can write $X(r)$ as the value of X at r: i.e., the unique $r' \in \mathbb{R}$ such that $(r, r') \in X$.

Next we need to express that X is continuous (at all points $r \in \mathbb{R}$).

(Continuous) Suppose we begin by considering a point $r \in \mathbb{R}$. Consider any open interval (a, b) containing the value $X(r)$. We work out the *preimage* of (a, b) wrt X. If this preimage is always open in $\mathbb{R}$, then X is continuous at r, and if this holds for all r, then X is continuous.

(Differentiable) Next one, in effect, defines a second set, X' (of ordered pairs), which is also a function, and which is the derivative of this function X. So, fixing a point r, and small positive real $\epsilon \in \mathbb{R}$, we work out $X(r + \epsilon) - X(r)$ divided by ϵ. In order for X to belong to $\mathrm{C}^1(\mathbb{R})$, we need, for any r, the limit, as ϵ goes to zero, to exist. If this limit does exist, it defines our new function X' (i.e., $X'(r) := \lim_{\epsilon \to 0} \frac{X(r+\epsilon)-X(r)}{\epsilon}$). We next repeat this for X', and if this limit exists (for all r) we obtain a new function X'' (i.e., $X''(r) := \lim_{\epsilon \to 0} \frac{X'(r+\epsilon)-X'(r)}{\epsilon}$).

I don't wish to write out this $L_\in$-formula, which we can call $\Phi_{\mathrm{C}^2(\mathbb{R})}(X)$, expressing that X is a twice differentiable function $\mathbb{R} \to \mathbb{R}$, in detail, as it would take probably a few hours to check the gruesome details (and I'd probably make a mistake) and, what is more, would result in an *extremely long formula.* This would be akin to rewriting `plot(db)` down into FORTRAN, or even machine code. But it is perfectly clear that such a formula *does indeed exist.* As noted above, we are perfectly entitled to use *defined* predicates in formalizations, just as mathematical logicians, truth theorists and so on, routinely do.

Finally, note that, since any such function X is a function $\mathbb{R} \to \mathbb{R}$, it follows that $X \subseteq \mathbb{R} \times \mathbb{R}$. So any such function is an element of $\mathcal{P}(\mathbb{R} \times \mathbb{R})$. Then we define the function space itself by Separation:

$$\mathrm{C}^2(\mathbb{R}) := \{X \in \mathcal{P}(\mathbb{R} \times \mathbb{R}) \mid \Phi_{\mathrm{C}^2(\mathbb{R})}(X)\} \tag{77}$$

$\triangle$

The final part of the jigsaw is to specify the formalization of the relevant differential equation or dynamical law. I call such a specification a "dynamical axiom":

Definition 11 (Dynamical Axiom)**.** Assuming a state space $\mathcal{F}$ is fixed, a dynamical axiom takes the following form:

$$\mathsf{Dyn} \quad : \quad \mathcal{D}(Q^\varphi, \dots) = 0 \tag{78}$$

where $\mathcal{D} : \mathcal{F} \to \mathcal{F}$ is a differential operator on the state space.

Notice, by the way, how the dynamical law, Dyn, has an explicit φ dependence: in other words, Dyn can be thought of as a formula

$$\mathsf{Dyn}(\varphi, \ldots), \tag{79}$$

containing φ as a variable. Though I shall not go into details, one can explain quite precisely what is meant by "covariance under a certain class of coordinate transformations". It means that one can prove the biconditional:

$$\mathsf{Dyn}(\varphi, \ldots) \leftrightarrow \mathsf{Dyn}(\varphi', \ldots), \tag{80}$$

for all coordinate charts φ, φ' related by the relevant group transformations (in our case, the Euclidean group relating Cartesian coordinate charts).

Example 4. ∂_x is a differential operator on $\mathrm{C}^k(\mathbb{R})$ (with $k \geq 1$). △

Example 5. $\partial_{xx} + \partial_{yy}$ is the Laplace differential operator on $\mathrm{C}^k(\mathbb{R}^2)$ (with $k \geq 2$). △

I have simplified the above discussion a little bit, in order to avoid getting drawn into rather subtle questions about singularities, Green's functions, distributions and so on, and instead sticking to cases where the quantities and functions we are interested in are sufficiently smooth. How to deal with such important cases is, of course, well-understood mathematically. They are excluded here entirely for reasons of expositional simplicity.

6 Constructing Standard Formalizations: Two Examples

6.1 Assembly

Many conceptually awkward, and philosophically interesting, aspects of constructing standard formalizations of theories that physicists give as differential equations are located in what we've discussed above:

(1) The applied mathematics base theory.
(2) The geometric representation hypothesis.
(3) Coordinate representation (definition).

These have been discussed in some depth, and, I believe, settled in Sections 3, 4 and 4.6 above. I next restate some of Suppes's concerns regarding formalization:

> It is a natural thing to talk about theories as linguistic entities, that is, to speak explicitly of the precisely defined set of sentences of the theory and the like, when the theories are given what is called standard formalization. Theories are ordinarily said to have a standard formalization when they are formulated within first-order logic. Roughly speaking, when a theory assumes more than first-order logic, it neither natural nor simple to formalize it in this fashion. For example, if in axiomatizing geometry we want to define lines as certain *sets* of points, we must work within a framework that already includes the ideas of set theory. (Suppes (2002), §1.2, p. 4)

And indeed that is precisely what we have done above in Sections 3 and 4. Suppes' passage continues:

> To be sure, it is theoretically possible to axiomatize simultaneously geometry and the relevant portions of set theory, but this is awkward and unduly laborious. Theories of a more complicated structure, like quantum mechanics, classical thermodynamics, or a modern quantitative version of learning theory, need to use not only general ideas of set theory but also many results concerning the real numbers. Formalization of such theories in first-order logic is utterly impractical. (Suppes (2002), §1.2, p. 4)

Regarding the first point and second points, we have done precisely this in Section 4. One of our main results, Theorem 2, is:

$$\mathsf{ZCA}_\sigma \vdash \left[(\exists \varphi : \mathbb{P} \to \mathbb{R}^n)\ (P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^n)\right] \leftrightarrow \mathsf{EG}^{(n)} \tag{81}$$

In my view, this is not "awkward or unduly laborious". And we shall give several formalizations below using real numbers. It is simply not true that such formalization is "utterly impractical". It is true that it is, in some sense, laborious and requires quite a bit of interdisciplinary knowledge: concerning axiomatic set theory, axiomatic geometry, model theory, certain parts of physics and the like.

In addition to (1)–(3) above, we noted:

(4) The state (space) axiom(s).
(5) The dynamical axiom(s).

Once this is all—at least to some moderate degree of clarity—properly identified and specified, we can next assemble these pieces, and begin to construct examples of standard formalizations, of the kind Suppes referred to, using ZCA_σ as underlying applied mathematical base theory. And, indeed, this is all first-order.

Our applied mathematics base theory ZCA_σ, in the extended signature $\sigma_\in$, sits in the background. We fix a bet-cong signature σ (over the system given above of base sorts/types),

$$\sigma = \{\overbrace{\mathsf{B}, \equiv, \varphi}^{\text{geometry}},\ \overbrace{Q_1, \dots, Q_k}^{\text{quantity primitives}}\ , \overbrace{\omega_1, \dots, \omega_m}^{\text{parameters}}\} \tag{82}$$

We then specify the underlying geometric axiom, the state space axiom(s) and the dynamical axiom(s). Schematically, the standard formalization, in $L(\sigma_\in)$, will then look like this:

STANDARD FORMALIZATION

Physical axioms	Θ	$:=$	$\mathsf{Geom}^{(n)} \wedge \mathsf{Dfn}(Q_i^\varphi) \wedge \mathsf{State} \wedge \mathsf{Dyn}$	(83)
Mathematicized physical theory	T	$:=$	$\mathsf{ZCA}_\sigma + \Theta$	(84)

Here the state space axiom(s) State and the dynamical axiom(s) Dyn are fixed on a "case by case" basis.

A large class of mathematicized theories from theoretical physics can be given a standard formalization exactly as above.

6.2 Example 1: The Simple Harmonic Oscillator

Definition 12. The signature for the Simple Harmonic Oscillator theory is an bet-cong signature:

$$\sigma_{\mathsf{sho}} = \{\mathsf{B}, \equiv, \varphi, Q, \omega\} \tag{85}$$

with additional type declarations:

$$\begin{aligned} Q :&\quad \mathbb{P} \Rightarrow \mathbb{R} \\ \omega :&\quad \mathbb{R} \end{aligned}$$

The extended signature is $\sigma_{\mathsf{sho},\in} = \sigma_{\mathsf{sho}} \cup \{\in\}$.

We have a (function symbol for a) single mixed function Q (the quantity that is "oscillating") and a (constant for a) single real-valued parameter ω (which fixes its frequency, relative to the scale implicit in φ). Recall that in §1.2, we agreed that there is an $L_\in$-formula expressing the Simple Harmonic Oscillator equation:

$$\Phi_{\mathsf{sho}}(F, \omega) \quad (\forall x : \mathbb{R})((\partial_{xx} + \omega^2)F = 0) \tag{86}$$

But since we know that this is expressible, one may simply write $(\forall x : \mathbb{R})((\partial_{xx} \wedge z^2)F = 0)$ in good conscience, bearing its expressibility in mind. Recall also, from §1.2 and §5 (Example 3), that the State Space Axiom

$$(Q^\varphi \in \mathrm{C}^2(\mathbb{R})) \tag{87}$$

is also expressible as a $L_\in$-formula, and we may also write $(Q^\varphi \in \mathrm{C}^2(\mathbb{R}))$ in good conscience, bearing its expressibility in mind.

We may now assemble all this—along with the geometry axiom, the mathematics base theory and the definition of the coordinate representation—to obtain the standard formalization:

Definition 13 (Simple Harmonic Oscillator Standard Formalization)**.** To simplify notation, let $\sigma = \sigma_{\mathsf{sho}}$ and let $\sigma_\in = \sigma_{\mathsf{sho},\in}$. We can now define the mathematicized physical theory T_{sho} (with ZCA_σ as underlying applied mathematics base theory) in $L(\sigma_\in)$ as follows. The physical axioms in in $L(\sigma_\in)$ are:

$$\Theta_{\mathsf{sho}} := \overbrace{\left(P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^1\right)}^{\mathsf{Geom}^{(1)}} \wedge \overbrace{\left(Q^\varphi = Q \circ \varphi^{-1}\right)}^{\mathsf{Dfn}(Q^\varphi)} \wedge \overbrace{\left(Q^\varphi \in \mathrm{C}^2(\mathbb{R})\right)}^{\mathsf{State}_{\mathsf{sho}}} \wedge \overbrace{(\forall x : \mathbb{R})(\partial_{xx} \wedge \omega^2)Q^\varphi = 0)}^{\mathsf{Dyn}_{\mathsf{sho}}} \tag{88}$$

And the overall mathematicized theory is:

$$T_{\mathsf{sho}} := \mathsf{ZCA}_\sigma + \Theta_{\mathsf{sho}} \tag{89}$$

In my view, (89) is a "simple and elegant formalization".[32]

The formalized theory T_{sho} in $L(\sigma_{\in})$ isolates the physical content of the theory of the simple harmonic oscillator. It is "standard" because it is first-order (but, of course, it contains a built-in set theory). Admittedly, all the required definitions—especially to encode core mathematics inside ZCA_{σ}—have not been explicitly set out. But such work *is* standard, well-known and presented in any set theory textbook, like Jech (2002), etc. For example, Potter (2004), Chapters 5–12 provides a detailed exposition of the required definitions: pairs, relations, functions, natural numbers, lines, real numbers, structures and so on.

Given the methods indicated above, it is relatively straightforward to transfer them to a multitude of other examples. It is true that I have ignored time, because properly dealing with time introduces certain complexities related to the underlying *space-time* geometry. Yet, it is indeed possible to present the basic geometric axioms required for Galilean spacetime and for Minkowski spacetime, just as was done for Euclidean geometry above. And one indeed can provide "Representation Hypothesis" axiomatizations, just as above, for the spacetime case too. For example, for Galilean spacetime (and thus classical mechanics and field theory), the single axiom needed is:[33]

$$(\mathbb{P}, B, \sim, \equiv^{\sim}) \cong (\mathbb{R}^4, B_{\mathbb{R}^4}, \sim_{\mathbb{R}^4}, \equiv^{\sim}_{\mathbb{R}^4}) \tag{90}$$

I prefer to simply bypass all these complexities for the time being.[34]

That said, the range of cases—differential equations on $\mathbb{R}^n$ assumed to have its Euclidean structure, given by a coordinate chart on space—to which one can apply the above methods remains extremely large; and, when the above extension is added, I'm not sure there are any mathematicized theories in theoretical physics to which these methods don't apply.

6.3 Example 2: Laplace's Equation in Two Dimensions

I rerun the construction next, for Laplace's equation in two dimensions:

$$\text{Laplace's equation (2D)} \quad (\forall x, y : \mathbb{R})\,(\partial_{xx}U + \partial_{yy}U = 0) \tag{91}$$

Recall again, from §1.2, that we agreed that there is an $L_{\in}$-formula $\Phi_{\mathsf{lap},2}(U)$ expressing Laplace's Equation:

$$\Phi_{\mathsf{lap},2}(U) \quad (\forall x, y : \mathbb{R})((\partial_{xx} + \partial_{yy})U = 0) \tag{92}$$

And likewise, there is an $L_{\in}$-formula expressing $U^{\varphi} \in \mathrm{C}^2(\mathbb{R}^2)$.

[32] Henceforth I shall suppress reference to explicit definitions such as $\mathsf{Dfn}(Q^{\varphi})$: one might consider them absorbed into a definitional extension of ZCA_{σ}.

[33] It seems fairly clear that much the same can be done for General Relativity, where the geometric axiom will be: $(\mathbb{P}, \texttt{chart}, \texttt{metric})$ is a relativistic spacetime.

[34] As I noted above, I have also deliberately ignored singularities and distributions. There are ways to handle these, but they introduce orthogonal and, in a sense, "higher-level" complexities not relevant to the current level of analysis: i.e., geometric structure and point fields on the geometry satisfying some differential equation.

Definition 14. The application signature for the Laplace equation (in two dimensions) is a bet-cong signature:

$$\sigma_{\mathsf{lap},2} = \{\mathsf{B}, \equiv, \varphi, U\} \tag{93}$$

with additional type declaration, for the physical quantity involved:

$$U : \quad \mathbb{P} \Rightarrow \mathbb{R} \tag{94}$$

As before, we now assemble all this—the mathematics base theory, the geometry axiom, the definition of the coordinate representation, the state axiom and the dynamical axiom—to obtain the standard formalization:

Definition 15. The physical axioms for the standard formalization of Laplace's equation is the following:

$$\Theta_{\mathsf{lap},2} \quad := \quad \overbrace{(P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^2)}^{\mathsf{Geom}^{(2)}} \quad \wedge \quad \overbrace{U^{\varphi} \in \mathrm{C}^2(\mathbb{R}^2)}^{\mathsf{State}_{\mathsf{lap}}} \quad \wedge \quad \overbrace{(\forall x, y : \mathbb{R})\,((\partial_{xx} + \partial_{yy})U^{\varphi} = 0)}^{\mathsf{Dyn}_{\mathsf{lap}}} \tag{95}$$

And the overall mathematicized theory is then:

$$T_{\mathsf{lap},2} := \mathsf{ZCA}_{\sigma_{\mathsf{lap},2}} + \Theta_{\mathsf{lap},2} \tag{96}$$

Again, in my view, (96) *is* a simple and elegant formalization.

7 Solution Statements & Boundary Conditions

7.1 Solution Statements

The formalized mathematicized theories given above have the form $\mathsf{ZCA}_{\sigma} + \Theta$, where Θ is some set of $L(\sigma_{\in})$-sentences expressing the physical content of the theory. Yet, as everyone who has worked through the standard parts of a theoretical physics degree knows, once we have gotten the differential equation(s) set down, then the hard work has only just begun: we must try to find *solutions*. It is all very well, and not especially difficult, to *define* the model class corresponding to a differential equation. But it is far harder to see what lives inside this model class. The undergraduate physics student spends several years learning countless tricks and methods to solve differential equations. My aim here is solely to discuss the *meta-theory* of this acquired knowledge.

Statements expressing those solutions, or expressing properties of those solutions, may be thought of as "solution formulas" in $L(\sigma_{\in})$ too. What is important from the meta-theoretic point of view is that there are always *mathematical derivations* of such formulas from the assumptions of the theory: i.e., from Θ (usually with boundary conditions and/or initial value conditions).

Consider a fairly simple result in applied mathematics:[35]

[35]This is taught to first-year undergraduates, as how to solve the differential equation $(\partial_{xx} + k^2)F = 0$, though not quite in the way we describe it here!

Theorem 3. Let σ_{sho} be the signature for the formalized theory of the simple harmonic oscillator and let Θ_{sho} be the physical axioms. Then:

$$\mathsf{ZCA}_{\sigma_{\mathsf{sho}}} \quad \vdash \quad \Theta_{\mathsf{sho}} \to (\exists x_1, x_2 : \mathbb{R})(\forall x_3 : \mathbb{R})(Q^{\varphi}(x_3) = x_1 \sin(\omega x_3) + x_2 \cos(\omega x_3)).$$

I've expressed this in a very peculiar way, which would no doubt make no sense to a physicist. But even so, *this* is what the physicist establishes (or just routinely assumes). The more conventional way to put this is to say that a solution of the SHO has the form:

$$Q(x) = A \sin(\omega x) + B \cos(\omega x) \tag{97}$$

An immediate corollary is:

Corollary 1. $T_{\mathsf{sho}} \vdash (\exists r_1, r_2 : \mathbb{R})(\forall r : \mathbb{R})(Q^{\varphi}(r) = r_1 \sin(\omega r) + r_2 \cos(\omega r))$.

I give this example mainly to illustrate the metalogic here. But this result itself is very simple, and not particularly interesting. A perhaps more interesting example is that one can rerun our standard formalization technique for the time-independent Schrödinger equation: essentially an eigenvalue problem:

$$\mathcal{L}\Psi = E\Psi \tag{98}$$

For example, we may do this for the quantum simple harmonic oscillator (in one dimension), or for the hydrogen atom, with the Coulomb potential. In both cases, the bet-cong signature includes a number of parameters as primitives and the basic quantity primitive Ψ is a function from points to $\mathbb{C}$, and the state axiom says that $\Psi^{\varphi} \in L^2(\mathbb{R})$ (for the quantum harmonic oscillator) or $\Psi^{\varphi} \in L^2(\mathbb{R}^3)$ (for the electron orbitals).[36]

I call the corresponding physical axioms Θ_{qho} and Θ_{hyd} and the overall standard formalization (i.e., mathematicized theory) T_{qho} and T_{hyd}.

Theorem 4. The following metatheorems hold:

$$\mathsf{ZCA}_{\sigma_{\mathsf{qho}}} \quad \vdash \quad \Theta_{\mathsf{qho}} \to \left((\exists n : \mathbb{N})(\Psi(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega^2 x}{\hbar}} H_n(\sqrt{\frac{m\omega}{\hbar}} x)\right) \tag{99}$$

$$\mathsf{ZCA}_{\sigma_{\mathsf{qho}}} \quad \vdash \quad \Theta_{\mathsf{qho}} \to ((\exists n : \mathbb{N})(E = \hbar\omega(n + 1/2))) \tag{100}$$

$$\mathsf{ZCA}_{\sigma_{\mathsf{hyd}}} \quad \vdash \quad \Theta_{\mathsf{hyd}} \to \left(E < 0 \to (\exists n : \mathbb{N}^+)(E = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2})\right). \tag{101}$$

where the functions $H_n(x)$ are the Hermite polynomials, defined as follows (Arfken & Weber (2005): 430):[37]

$$H_n(x) := (-1)^n e^{x^2} (\partial_x)^n e^{-x^2} \tag{102}$$

[36] E.g., for the quantum harmonic oscillator, these additional parameters are m, $\hbar$, ω, E.

[37] The first four are: $H_0(x) = 1$; $H_1(x) = 2x$; $H_2(x) = 4x^2 - 2$; $H_3(x) = 8x^3 - 12x$; ...

We have the corollaries:

Corollary 2. $T_{\mathsf{qho}} \vdash (\exists n : \mathbb{N})\,(E = \hbar\omega(n + 1/2))$.

Corollary 3. $T_{\mathsf{hyd}} \vdash E < 0 \to (\exists n : \mathbb{N}^+)\left(E = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}\right)$.

Thus, statements expressing the discrete energy spectrum in each case are *derivable theorems* of the mathematicized theory, T_{qho} or T_{hyd}.

7.2 Boundary Conditions

I consider the solution of Laplace's equation in two dimensions, with Dirichlet boundary conditions. Recall that we defined the physical axioms $\Theta_{\mathsf{lap},2}$ for the Laplace equation in 2-dimensions as follows:

$$\Theta_{\mathsf{lap},2} := \mathsf{Geom}^{(2)} \wedge (U^{\varphi} \in \mathrm{C}^2(\mathbb{R}^2)) \wedge (\forall x, y : \mathbb{R})((\partial_{xx} + \partial_{yy})U^{\varphi} = 0) \tag{103}$$

Let's see how we use the standard applied mathematics tricks to solve the Laplace equation, subject to Dirichlet boundary conditions.[38] Suppose that the (coordinate representation) U^{φ} vanishes along the lines $y = 0$ (i.e., the x-axis), $y = 1$ and $x = 0$ (the y-axis):

$$U^{\varphi}(x, 0) = 0 \tag{104}$$
$$U^{\varphi}(x, 1) = 0 \tag{105}$$
$$U^{\varphi}(0, y) = 0 \tag{106}$$

Assume U^{φ} coincides with some function $f(y)$ along the line $x = 1$, say:

$$U^{\varphi}(1, y) \quad = \quad f(y) \tag{107}$$

We aim to solve

$$\partial_{xx}U^{\varphi} + \partial_{yy}U^{\varphi} = 0 \tag{108}$$

subject to the above Dirichlet boundary conditions on $[0, 1] \times [0, 1]$. We assume separation of variables:

$$U^{\varphi}(x, y) = X(x)Y(y) \tag{109}$$

This yields two simple ODEs, one for X and one for Y.

$$\frac{1}{X}\frac{d^2X}{dx^2} \quad = \quad k^2 \tag{110}$$
$$\frac{1}{Y}\frac{d^2Y}{dy^2} \quad = \quad -k^2 \tag{111}$$

[38] These methods are described in any textbook on differential equations. See, for example, Arfken & Weber (2005), Ch. 9, §9.4 (p. 554), for explanation of separation of variables.

By imposing three of the boundary conditions and using linearity, a general solution takes the form of a superposition:

$$U^{\varphi}(x,y)=\sum_{n=1}^{\infty}A_n\sinh(n\pi x)\sin(n\pi y) \tag{112}$$

The remaining boundary condition is: $U(1,y)=f(y)$. Hence

$$U^{\varphi}(1,y)=\sum_{n=1}^{\infty}A_n\sinh(n\pi)\sin(n\pi y)=f(y) \tag{113}$$

We should like to solve this for the coefficients A_n. To do this, we use the orthogonality of the functions $\sin(n\pi y)$ over the interval $[0,1]$ (see Arfken & Weber (2005), p. 884). In our case, this is:

$$\int_0^1 \sin(n\pi y)\sin(m\pi y)dy=\begin{cases}\frac{1}{2}\delta_{n,m} & \text{for } m\neq 0\\ 0 & \text{for } m=0\end{cases} \tag{114}$$

This implies:

$$A_n=\frac{2}{\sinh(n\pi)}\int_0^1 f(y)\sin(n\pi y)dy \tag{115}$$

Let us now set the specific boundary function (on the line $x=0$):

$$f(y)=\sin(\pi y) \tag{116}$$

we see the only coefficient which is non-zero is A_1. So:

$$U^{\varphi}(x,y)=\frac{1}{\sinh(\pi)}\sinh(\pi x)\sin(\pi y) \tag{117}$$

It is clear that this mathematical reasoning could, in principle, be formalized.[39] Let us introduce a statement expressing the Dirichlet boundary conditions for U^{φ}:

$$\mathsf{BC}:=\forall x(U^{\varphi}(x,0)=0\wedge U^{\varphi}(x,1)=0)\wedge\forall y(U^{\varphi}(0,y)=0\wedge U^{\varphi}(1,y)=\sin(\pi y)) \tag{118}$$

By the reasoning above, we have derived the conclusion:

$$\big(\Theta_{\mathsf{lap},2}\wedge\mathsf{BC}\big)\to(\forall x,y:\mathbb{R})\left(U^{\varphi}(x,y)=\frac{1}{\sinh(\pi)}\sinh(\pi x)\sin(\pi y)\right) \tag{119}$$

And since we have derived it, in semi-formal background mathematics:

[39] Though it is clear that this is so, it is certainly not *easily* so—I do not volunteer to specify every detail and lemma. One could, in principle, formalize all this in a theorem-proving assistant like Isabelle, Lean and so on. But it would be something like a year-long MSc research project for a mathematics or computer science student.

Theorem 5. Formalizing the reasoning above inside the applied mathematics base theory, we have:

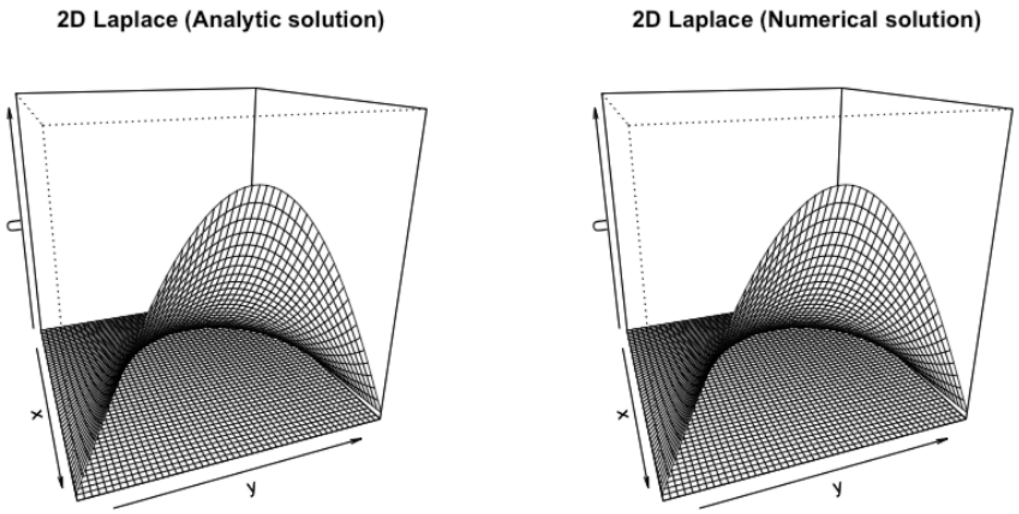
$$\mathsf{ZCA}_\sigma \vdash \left(\Theta_{\mathsf{lap},2} \wedge \mathsf{BC}\right) \rightarrow (\forall x, y : \mathbb{R}) \left(U^\varphi(x,y) = \frac{1}{\sinh(\pi)} \sinh(\pi x) \sin(\pi y) \right) \tag{120}$$

The conditional (119) is a *theorem* of your favourite foundational system for applied mathematics.[40]

7.3 Numerical Integration

Above, we showed how to derive the analytic solution (117) for Laplace's equation on $\mathbb{R}^2$ with Dirichlet boundary conditions defined on $[0,1] \times [0,1]$. We have noted that (119) is *derivable* from the axioms $\Theta_{\mathsf{lap},2} \wedge \mathsf{BC}$, describing the field satisfying the equation and its boundary conditions, within the applied mathematics base theory.

But note also that we can simulate this on a *finite discrete grid*: a numerical integration of the partial differential equation, satisfying the boundary conditions (say on a grid of dimensions 50×50). If we do this, we obtain a very good approximation to the analytic solution. This is easiest to see visually:[41]



[40]In this case, the Axiom of Choice isn't needed. As I have tried to stress above, such metatheoretic conclusions will be largely invariant with respect the choice of foundational system, so long as it is not too weak. For example, I am reasonably certain this could be formalized inside the HOL/Isabelle theorem-proving assistant, or within Mizar, Coq or Lean.

[41]The numerical solution was worked out using the package `ReacTran` (for the programming language `R`), based on modifying code provided in Soetaert et al. (2012) (Ch. 9, "Solving Partial Differential Equations in `R`", p. 167).

There is a sense in which discretized versions of our formalized axioms—$\Theta_{\mathsf{lap},2}$ and BC—are implemented within the computational system, which then computes a discretized version of the solution. And note well, that the results are *extremely good approximations to each other*. Moreover, more careful analysis can indeed examine the convergence of such approximations: the numerical solution more closely approximates the analytic one, as the grid becomes finer. This is probably why such continuum methods will almost certainly work perfectly fine and give correct answers to high approximation, even if space (or spacetime) is somehow granular at a tiny level.

8 Application Conditionals

The (quantified) conditionals referred to in Theorem 3, Theorem 4 and Theorem 5 are:

$$\Theta_{\mathsf{sho}} \to (\exists r_1, r_2 : \mathbb{R})(\forall r : \mathbb{R})(Q^{\varphi}(r) = r_1 \sin(\omega r) + r_2 \cos(\omega r)) \tag{121}$$

$$\Theta_{\mathsf{qho}} \to \Big((\exists n : \mathbb{N})(\Psi(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega^2 x}{\hbar}} H_n(\sqrt{\frac{m\omega}{\hbar}} x)\Big) \tag{122}$$

$$\Theta_{\mathsf{qho}} \to ((\exists n : \mathbb{N})(E = \hbar\omega(n + 1/2)) \tag{123}$$

$$\Theta_{\mathsf{hyd}} \to \Big(E < 0 \to (\exists n : \mathbb{N}^+)(E = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} \frac{1}{n^2})\Big) \tag{124}$$

$$\big(\Theta_{\mathsf{lap},2} \wedge \mathsf{BC}\big) \to (\forall x, y : \mathbb{R}) \Big(U^{\varphi}(x, y) = \frac{1}{\sinh(\pi)} \sinh(\pi x) \sin(\pi y)\Big) \tag{125}$$

I take no credit, of course, for these! They are well-known, but often quite difficult to derive, results in mathematical physics. My point, however, is meta-theoretic: these are *theorems* of the applied mathematics base theory ZCA_σ.

Definition 16. I refer to (possibly, universally quantified) conditionals which are provable in applied mathematics as *application conditionals*.

I admit that this definition, at first sight, seems a bit silly, as it includes countless redundant cases: e.g., $\mathsf{ZCA}_\sigma \vdash \forall x(x = x) \to \forall x(x = x)$; and $\mathsf{ZCA}_\sigma \vdash \phi \to \theta$, if $\mathsf{ZCA}_\sigma \vdash \theta$; and $\mathsf{ZCA}_\sigma \vdash \phi \to \theta$, if $\mathsf{ZCA}_\sigma \vdash \neg\phi$.

Nonetheless, theorems of the form $\phi \to \theta$, where neither θ nor $\neg\phi$ is provable in ZCA_σ, show up *everywhere in applied mathematics*. In fact, it is not too much of an exaggeration to say that a significant bulk of applied mathematics consists in the discovery and proof of interesting application conditionals. For example, the logical form of an application conditional such as (125) is $\big[\Theta_{\mathsf{lap},2} \wedge \mathsf{BC}\big] \to \mathsf{Solution}$. So, meta-theorems stating the provability of application conditionals frequently have the form:

$$\mathsf{ZCA}_\sigma \ \vdash \ \text{physical axioms} \ \wedge \ \text{boundary conditions} \ \to \text{solution statement} \tag{126}$$

9 Summary

To recap, in §1, we introduced the problem of providing *standard formalizations* of scientific theories, and especially the case of theories formulated as differential equations—Laplace's equation, the wave equation and so on. We recalled Patrick Suppes' scepticism (Suppes (1992)) about whether there could be "simple or elegant method" for presenting such theories in standard formalization, as they "assume a great deal of mathematics as part of their substructure". How might we resolve this challenge?

In §2, we noted that, strictly speaking, the equation is given for the *coordinate representation* Q^φ of the physical quantity Q involved, relative to a coordinate chart φ; we noted, with Feynman, that quantities/fields such as Q are mixed functions on a domain of space or time or spacetime points; and likewise, that the structures referred to in physics are, typically, impure structures. And we noted there are two further obstacles in constructing simple or elegant formalizations:

(I) The specification of the underlying applied mathematics base theory.
(II) The specification of the underlying geometric structure of physical points (or instants, or events, ...) on which the physical quantity is defined.

In §3, we resolved the first of these obstacles by invoking ZCA_σ as applied mathematics base theory: Definition 1. In §4, we resolved the second by specifying "quasi-synthetic" axioms $\mathsf{EG}^{(n)}$ for the underlying physical geometry for n-dimensions, and showed how these can be condensed to a *single geometric axiom* of the form

$$\mathsf{Geom}^{(n)}: \quad P_{\mathsf{geom}} \overset{\varphi}{\cong} \mathbb{E}^n, \tag{127}$$

where φ is a chart and $\mathbb{E}^n$ is the standard Cartesian coordinate structure for n-dimensional Euclidean geometry, using the "internal" version (Theorem 2) of Tarski's Representation Theorem (Theorem 1).

In §5, we noted that one needs to state at least two further kinds of physical axiom. One must specify a *state space*, $\mathcal{F}$, to which the coordinate representation belongs. Second, one needs to state the "dynamical" axiom, a formula expressing that the coordinate representation satisfies the relevant dynamical equation. Usually, this has the form $\mathcal{D}f = 0$, where $\mathcal{D}$ is a differential operator on the state space $\mathcal{F}$.

In §6, we assembled the above pieces, and gave two examples of standard formalizations. Definition 13, for the simple harmonic oscillator theory; and Definition 15, for Laplace's equation in two dimensions:

$$\begin{array}{lccccc}
\Theta_{\mathsf{sho}}: & \overbrace{\mathsf{Geom}^{(1)}}^{\text{geometric axiom}} & \wedge & \overbrace{(Q^\varphi \in \mathrm{C}^2(\mathbb{R}))}^{\text{state space axiom}} & \wedge & \overbrace{(\forall x:\mathbb{R})((\partial_{xx}+\omega^2)Q^\varphi = 0)}^{\text{dynamical axiom}} \\
\Theta_{\mathsf{lap},2}: & \mathsf{Geom}^{(2)} & \wedge & (U^\varphi \in \mathrm{C}^2(\mathbb{R}^2)) & \wedge & (\forall x, y:\mathbb{R})((\partial_{xx}+\partial_{yy})U^\varphi = 0)
\end{array}$$

The corresponding *standard formalizations* are then simply:

$$\begin{aligned}
T_{\mathsf{sho}} &:= \mathsf{ZCA}_{\sigma_{\mathsf{sho}}} + \Theta_{\mathsf{sho}} \\
T_{\mathsf{lap},2} &:= \mathsf{ZCA}_{\sigma_{\mathsf{lap},2}} + \Theta_{\mathsf{lap},2}
\end{aligned}$$

In §7 and §8, we looked, albeit briefly, at solution statements and application conditionals. Examples are (121)–(125):

$$\mathsf{ZCA}_{\sigma_{\mathsf{sho}}} \vdash \Theta_{\mathsf{sho}} \to (\exists r_1, r_2 : \mathbb{R})(\forall r : \mathbb{R})(Q^{\varphi}(r) = r_1 \sin(\omega r) + r_2 \cos(\omega r)).$$

$$\mathsf{ZCA}_{\sigma_{\mathsf{qho}}} \vdash \Theta_{\mathsf{qho}} \to (\exists n : \mathbb{N})(\Psi(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega^2 x}{\hbar}} H_n(\sqrt{\frac{m\omega}{\hbar}} x))$$

$$\mathsf{ZCA}_{\sigma_{\mathsf{qho}}} \vdash \Theta_{\mathsf{qho}} \to (\exists n : \mathbb{N})(E = \hbar\omega(n + 1/2))$$

$$\mathsf{ZCA}_{\sigma_{\mathsf{hyd}}} \vdash \Theta_{\mathsf{hyd}} \to (E < 0 \to (\exists n : \mathbb{N}^+)(E = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} \frac{1}{n^2})).$$

$$\mathsf{ZCA}_{\sigma_{\mathsf{lap,2}}} \vdash \left[\Theta_{\mathsf{lap,2}} \wedge \mathsf{BC}\right] \to (\forall x, y : \mathbb{R})(U^{\varphi}(x, y) = \frac{1}{\sinh(\pi)} \sinh(\pi x) \sin(\pi y))$$

I stress that $\Theta_{\mathsf{sho}}, \Theta_{\mathsf{qho}}, \Theta_{\mathsf{hyd}}$ and $\Theta_{\mathsf{lap,2}} \wedge \mathsf{BC}$ are *physical axioms* (assuming the language is interpreted). The above application conditionals show how *physical axioms* lead to *physical conclusions/predictions*, using *mathematical* reasoning.[42]

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