

ON A FUNCTIONAL EQUATION RELATED TO TWO-VARIABLE WEIGHTED QUASI-ARITHMETIC MEANS

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ABSTRACT. In this paper, we are going to describe the solutions of the functional equation

$$\varphi\left(\frac{x+y}{2}\right)(f(x) + f(y)) = \varphi(x)f(x) + \varphi(y)f(y)$$

concerning the unknown functions φ and f defined on an open interval. In our main result only the continuity of the function φ and a regularity property of the set of zeroes of f are assumed. As application, we determine the solutions of the functional equation

$$G(g(u) - g(v)) = H(h(u) + h(v)) + F(u) + F(v)$$

under monotonicity and differentiability conditions on the unknown functions F, G, H, g, h .

1. INTRODUCTION

The theory of means have provided a rich source and background for the introduction and investigation of functional equations in several variables. The characterization of quasi-arithmetic means was solved independently by de Finetti [15], Kolmogorov [27], and Nagumo [43] for the case when the number of variables is non-fixed. For the two-variable case, Aczél [1], [2], [3], [4], proved a characterization theorem involving the notion of bisymmetry. This result was extended to the n -variable case by Maksa–Münnich–Mokken [41], [42]. A recent characterization theorem of generalized quasi-arithmetic means have been obtained by Matkowski–Páles [40]. Characterization theorems for quasideviation means and for Bajraktarević means were obtained by the author [44], [45], [46].

The equality problem and the so-called invariance equation in various classes of means have been investigated in the papers Aczél–Kuczma [5], Baják–Páles [9], Berrone [10], Berrone–Lombardi [11], Daróczy–Maksa–Páles [12], Daróczy–Páles [13], [14], Jarczyk [17], [18], [19], [20], [21], [22], Jarczyk–Matkowski [23], [24], Kahlig–Matkowski [26], Leonetti–Matkowski–Tringali [28], Losonczy [30], [31], [32], Losonczy–Páles [33], Makó–Páles [34], Matkowski [35], [36], [37], [38], Matkowski–Nowicka–Witkowski [39], Páles [49].

In order to solve the equality problem of two-variable functionally weighted quasi-arithmetic means (called Bajraktarević means [8]) and quasi-arithmetic means, Z. Daróczy, Gy. Maksa and Zs. Páles [12] investigated and solved the functional equation

$$(1) \quad \varphi\left(\frac{x+y}{2}\right)(f(x) + f(y)) = \varphi(x)f(x) + \varphi(y)f(y), \quad (x, y \in I),$$

concerning the unknown functions $\varphi : I \rightarrow \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$. (Here and throughout this paper, let $I \subseteq \mathbb{R}$ stand for a nonempty open interval.) They determined the solutions under the natural conditions needed for the definition of weighted quasi-arithmetic means, namely, they assumed that φ is strictly monotone and continuous, furthermore, that f is positive on its domain. The main idea of their approach was to prove that such solutions of (1) are infinitely many times differentiable. Thus, using differentiability, and

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the results of Losonczi [29] on the equality of two variable Bajraktarević means, they determined the above equation.

Although the equation (1) appeared first related to means, its solutions can be interesting in general, without any means in the background. Motivated by this, we are going to solve (1) assuming only the continuity of φ and a regularity property of the zeroes of f . After eliminating the non-regular solutions, our approach is parallel to what was followed in [12]: first, we are going to improve the regularity properties of the unknown functions φ and f , then, we prove that f and $g := \varphi f$ are solutions of a second-order homogeneous linear differential equation with constant coefficients. Solving this differential equation, the solutions of the functional equation (1) is finally obtained. Our main result stated in Theorem 11 is proper generalization of [12, Theorem 2] eventhough, for a first glance, the two formulations look very different from each other. In the construction of the second-order homogeneous linear differential equation with constant coefficients we also avoided the direct application of the result of Losonczi [29], instead we followed a completely independent argument.

As an application of our results, we will solve the functional equation

$$(2) \quad G(g(u) - g(v)) = H(h(u) + h(v)) + F(u) + F(v)$$

under monotonicity and differentiability conditions on the unknown functions F, G, H, g, h . It remains an open problem to reach the same conclusion as in Theorem 15 without assuming differentiability of the unknown functions. A possible way is to follow the regularity improving methods developed in the papers Aczél–Maksa–Páles [6], [7], Gilányi–Páles [16], Járαι–Maksa–Páles [25], Páles [47], [48].

2. AUXILIARY RESULTS

In the sequel, denote by \mathcal{Z}_f the set of zeros of the function f , that is, let

$$\mathcal{Z}_f := \{x \in I \mid f(x) = 0\} = f^{-1}(\{0\}).$$

For a subset $S \subseteq I$, we shall also use the notation $S^c := I \setminus S$. Similarly, \overline{S} will denote the closure of the set S relative to I . In the next theorem we give a sufficient condition for the solutions of equation (1).

Theorem 1. *Let $\varphi, f : I \rightarrow \mathbb{R}$ be functions such that φ is constant on the set $\frac{1}{2}(\mathcal{Z}_f^c + I)$. Then the pair (φ, f) solves equation (1).*

Proof. Let $x, y \in I$ be arbitrary. Then, obviously, we can distinguish the following four cases.

- (i) If $x, y \in \mathcal{Z}_f$, then $f(x) = f(y) = 0$ and hence (1) holds trivially.
- (ii) If $x \in \mathcal{Z}_f$ and $y \in \mathcal{Z}_f^c$, then $y, \frac{y+x}{2} \in \frac{1}{2}(\mathcal{Z}_f^c + I)$ thus $\varphi(y) = \varphi(\frac{x+y}{2})$ and $f(x) = 0$. These properties imply (1) directly.
- (iii) The case $y \in \mathcal{Z}_f$ and $x \in \mathcal{Z}_f^c$ is analogous to that of (ii).
- (iv) Finally, if $x, y \in \mathcal{Z}_f^c$, then $x, y, \frac{y+x}{2} \in \frac{1}{2}(\mathcal{Z}_f^c + I)$ whence $\varphi(x) = \varphi(y) = \varphi(\frac{x+y}{2})$ follows implying the validity of (1).

This completes the proof of the statement. □

In order to have a computational formula for the set $\frac{1}{2}(\mathcal{Z}_f^c + I)$, we need the following lemma.

Lemma 2. *For any subset $A \subseteq I$, the sum $A + I$ is an open interval, furthermore, if $A \neq \emptyset$, then*

$$(3) \quad A + I =] \inf A + \inf I, \sup A + \sup I[.$$

Proof. If $A \subseteq I$ is empty, then $A + I$ is also empty. Now, assume that A is nonempty and let $x, y \in A + I$ be arbitrary with $x < y$. Then there exist $u, v \in A$ such that $x \in u + I$ and $y \in v + I$. Then, obviously, $u + v \in (u + I) \cap (v + I)$, furthermore, the sets $u + I$ and $v + I$ are intervals. Therefore, if either $x < u + v$ or $u + v < y$, we have either

$$[x, u + v] \subseteq u + I \quad \text{or} \quad [u + v, y] \subseteq v + I,$$

respectively. Using these inclusions, if $u + v \leq x$, we obtain that

$$[x, y] \subseteq [u + v, y] \subseteq v + I \subseteq A + I.$$

Similarly, if $y \leq u + v$, then

$$[x, y] \subseteq [x, u + v] \subseteq u + I \subseteq A + I.$$

Finally, if $x < u + v < y$, we get

$$[x, y] = [x, u + v] \cup [u + v, y] \subseteq (u + I) \cup (v + I) \subseteq A + I.$$

Hence, the inclusion $[x, y] \subseteq A + I$ holds in each of the above cases, proving that $A + I$ is an interval. The openness of $A + I$ is a consequence of the openness of I . This directly implies (3). \square

Lemma 3. *Let $A, B \subseteq I$ be subsets such that $A \subseteq B \subseteq \overline{\text{conv}}(A)$ holds. Then $A + I = B + I$.*

Proof. If A is empty, then B is also empty and there is nothing to prove. Therefore, we may restrict ourselves to the case when A is not empty. Then, in view of formula (3) of Lemma 2, it is sufficient to show that $\inf A = \inf B$ and $\sup A = \sup B$ hold.

By the condition $A \subseteq B$, it follows that $\inf B \leq \inf A$. On the other hand, it is easy to see that $\inf \overline{\text{conv}}(A) = \inf A$, therefore the inclusion $B \subseteq \overline{\text{conv}}(A)$ implies that $\inf B \geq \inf \overline{\text{conv}}(A) = \inf A$, which completes the proof of $\inf A = \inf B$. The proof of $\sup A = \sup B$ is analogous. \square

In the next lemma, we establish a consequence of functional equation (1).

Lemma 4. *If a pair of functions (φ, f) solves (1), then, for all $(x, y) \in \mathcal{Z}_f \times \mathcal{Z}_f^c$, we have*

$$(4) \quad \varphi(y) = \varphi\left(\frac{x + y}{2}\right).$$

In addition, if φ is continuous, then (4) also holds for all $(x, y) \in \overline{\mathcal{Z}}_f \times \mathcal{Z}_f^c$.

Proof. If $(x, y) \in \mathcal{Z}_f \times \mathcal{Z}_f^c$, then $f(x) = 0$ and $f(y) \neq 0$, hence (1) reduces to (4). In the case the continuity of φ , the validity of (4) for $(x, y) \in \overline{\mathcal{Z}}_f \times \mathcal{Z}_f^c$ follows by a standard limiting argument. \square

In the following proposition, we give a necessary condition for the solution pairs of equation (1).

Proposition 5. *If the pair of functions (φ, f) solves (1), φ is continuous on I and \mathcal{Z}_f is nonempty, then φ is constant on the set $\frac{1}{2}(\overline{\mathcal{Z}}_f^c + I)$.*

Proof. The statement is trivial if $\overline{\mathcal{Z}}_f^c = \emptyset$, thus we may assume that $\overline{\mathcal{Z}}_f^c$ is nonempty.

The set $\overline{\mathcal{Z}}_f^c$ is open, thus it can be written as the union of its components, that is, there exists a nonempty set Γ and, for all $\gamma \in \Gamma$, there exist extended real numbers $a_\gamma < b_\gamma$ such that

$$\overline{\mathcal{Z}}_f^c = \bigcup_{\gamma \in \Gamma}]a_\gamma, b_\gamma[.$$

In the first step we prove that φ is constant on any component of $\overline{\mathcal{Z}}_f^c$. Let $J_\gamma :=]a_\gamma, b_\gamma[$ be an arbitrary component of $\overline{\mathcal{Z}}_f^c$. Because of that $\overline{\mathcal{Z}}_f$ is nonempty, one of the endpoints of J_γ must belong to I . Without loss of generality, we may assume that $a_\gamma \in I$. Then, the maximality of J_γ implies that $a_\gamma \in \overline{\mathcal{Z}}_f$. Now, let $u \in J_\gamma$ be arbitrarily fixed and define the sequence $(u_n) \subseteq J_\gamma$ as $u_n := 2^{1-n}u + (1 - 2^{1-n})a_\gamma$ whenever $n \in \mathbb{N}$. It is easy to see, that $u_1 = u$, $u_n \rightarrow a_\gamma$ as $n \rightarrow \infty$, and that the recursive formula $u_{n+1} = \frac{a_\gamma + u_n}{2}$ holds. Now, we are going to show that

$$(5) \quad \varphi(u) = \varphi(u_n), \quad (n \in \mathbb{N}).$$

This is trivial for $n = 1$. Now, assume that (5) holds for some $n = k \in \mathbb{N}$. To prove (5) for $n = k + 1$, apply (4) for $y := u_k \in \overline{\mathcal{Z}}_f^c$ and $x := a_\gamma \in \overline{\mathcal{Z}}_f$. Then it follows that $\varphi(u_k) = \varphi\left(\frac{a_\gamma + u_k}{2}\right) = \varphi(u_{k+1})$. By the inductive hypothesis, this means that $\varphi(u) = \varphi(u_{k+1})$, which is the required identity.

Upon taking the limit $n \rightarrow \infty$ in equation (5) and using the continuity of φ at a_γ , it follows that $\varphi(u) = \varphi(a_\gamma)$. Since the point $u \in J_\gamma$ was arbitrary, it follows that φ is constant on the component J_γ .

Now, we show that φ is constant on the set $\frac{1}{2}(\overline{\mathcal{Z}}_f^c + \overline{\mathcal{Z}}_f^c)$. It is easy to check that

$$\frac{1}{2}(\overline{\mathcal{Z}}_f^c + \overline{\mathcal{Z}}_f^c) = \bigcup_{\gamma, \kappa \in \Gamma} \left] \frac{a_\gamma + a_\kappa}{2}, \frac{b_\gamma + b_\kappa}{2} \right[.$$

If $\overline{\mathcal{Z}}_f^c$ has only one component, then we are done. Therefore we may assume that it has at least two components, say $J_\gamma =]a_\gamma, b_\gamma[$ and $J_\kappa =]a_\kappa, b_\kappa[$ and we can also assume that $b_\gamma < a_\kappa$. In this case, because of the maximality of the components J_γ and J_κ , we have that $b_\gamma, a_\kappa \in \overline{\mathcal{Z}}_f$. Using what we have already proved, there exist $\alpha_\gamma, \alpha_\kappa \in \mathbb{R}$, such that $\varphi|_{J_\gamma} = \alpha_\gamma$ and $\varphi|_{J_\kappa} = \alpha_\kappa$, moreover, according to the continuity of φ , it follows that $\varphi(b_\gamma) = \alpha_\gamma$ and $\varphi(a_\kappa) = \alpha_\kappa$. Now, applying the equation (4) for $x := a_\kappa \in \overline{\mathcal{Z}}_f$ and $y \in J_\gamma \subseteq \overline{\mathcal{Z}}_f^c$, we get that $\varphi(\frac{a_\kappa+y}{2}) = \varphi(y) = \alpha_\gamma$ for all $y \in J_\gamma$. In other words, $f(u) = \alpha_\gamma$ for $u \in \frac{1}{2}(J_\gamma + a_\kappa) =]\frac{a_\gamma+a_\kappa}{2}, \frac{b_\gamma+a_\kappa}{2}[$. Similarly, for $u \in \frac{1}{2}(b_\gamma + J_\kappa) =]\frac{b_\gamma+a_\kappa}{2}, \frac{b_\gamma+b_\kappa}{2}[$, we get that $f(u) = \alpha_\kappa$. By the continuity of φ at the point $\frac{b_\gamma+a_\kappa}{2}$, it follows that $\alpha_\gamma = \alpha_\kappa$ and that φ equals the constant $\alpha_\gamma = \alpha_\kappa$ on the interval $]\frac{a_\gamma+a_\kappa}{2}, \frac{b_\gamma+b_\kappa}{2}[$. Because $\gamma, \kappa \in \Gamma$ was arbitrary, it follows that α_γ is the same constant for all $\gamma \in \Gamma$, that is, there exists $\alpha \in \mathbb{R}$ such that $\varphi(x) = \alpha$ if $x \in \frac{1}{2}(\overline{\mathcal{Z}}_f^c + \overline{\mathcal{Z}}_f)$.

To complete the proof, we show that φ equals α on $\frac{1}{2}(\overline{\mathcal{Z}}_f^c + I)$. Let

$$v \in \frac{1}{2}(\overline{\mathcal{Z}}_f^c + I) = (\frac{1}{2}(\overline{\mathcal{Z}}_f^c + \overline{\mathcal{Z}}_f^c)) \cup (\frac{1}{2}(\overline{\mathcal{Z}}_f^c + \overline{\mathcal{Z}}_f))$$

be arbitrary. We may assume that $v \in \frac{1}{2}(\overline{\mathcal{Z}}_f^c + \overline{\mathcal{Z}}_f)$. Then there exists $y \in \overline{\mathcal{Z}}_f^c \subseteq (\frac{1}{2}(\overline{\mathcal{Z}}_f^c + \overline{\mathcal{Z}}_f^c)) \cap \mathcal{Z}_f^c$ and $x \in \overline{\mathcal{Z}}_f$, such that $v = \frac{1}{2}(x + y)$. By the second assertion of Lemma 4, we have $\varphi(v) = \varphi(y) = \alpha$, which finishes the proof. \square

3. IMPROVING OF THE REGULARITY

Proposition 6. *If the pair of functions (φ, f) solves (1), φ is continuous on I and φ is not constant on the set $\frac{1}{2}(\overline{\mathcal{Z}}_f^c + I)$, then f is nowhere zero and continuous, φ and f are infinitely many times differentiable, the function φ is strictly monotone on I , and there exists a nonzero real constant λ such that*

$$(6) \quad \varphi' f^2 = \lambda$$

holds on I .

Proof. In view of Proposition 5, if φ is not constant on $\frac{1}{2}(\overline{\mathcal{Z}}_f^c + I)$ then \mathcal{Z}_f is empty or, equivalently, f is nowhere zero on I .

We claim that there is no nonempty subinterval of I where φ would be constant. Indirectly, assume that this is not the case, that is, there exists a maximal proper subinterval $J :=]a, b[$ of I such that, for some $\alpha \in \mathbb{R}$ we have $\varphi(x) = \alpha$ whenever $x \in J$. The interval J cannot be equal to I , hence one of its endpoints, say $a \in I$, must belong to I . The continuity of φ implies that $\varphi(a) = \alpha$. Let $u \in I$ and $y \in J$ be arbitrarily fixed such that $u < a$ and $\frac{1}{2}(u + y) \in J$. Then applying (1) for $x \in [u, a]$ and y , then using that $\varphi(\frac{x+y}{2}) = \varphi(y) = \alpha$ and $f(x) \neq 0$, we immediately obtain that $\varphi(x) = \alpha$. This contradicts the maximality of J . Therefore φ cannot be constant on any nonempty subinterval of I .

Now, we are able to show that f is continuous on I . In fact, we are going to show that any point of I has a neighborhood where f coincides with a proper continuous function. To do this, let $x_0 \in I$ be arbitrarily fixed. Then there exists $r > 0$ such that $[x_0 - 2r, x_0 + 2r] \subseteq I$. Based on the previous part of the proof, φ cannot be constant on the interval $[x_0 - r, x_0 + r]$. Consequently, there exists $u_0 \in [x_0 - r, x_0 + r]$ such that $\varphi(x_0) \neq \varphi(u_0)$. Let $y_0 := 2u_0 - x_0$. Then $y_0 \in [x_0 - 2r, x_0 + 2r]$, the function $x \mapsto \varphi(\frac{x+y_0}{2}) - \varphi(x)$ is continuous on I , and, by the choice of u_0 , it is different from zero at the point $x := x_0$. Moreover, there exists $\delta > 0$ such that this function is different from zero also on the entire interval $]x_0 - \delta, x_0 + \delta[\subseteq I$. Using this, equation (1) directly implies that

$$(7) \quad f(x) = f(y_0) \frac{\varphi(y_0) - \varphi(\frac{x+y_0}{2})}{\varphi(\frac{x+y_0}{2}) - \varphi(x)}, \quad (x \in]x_0 - \delta, x_0 + \delta[).$$

Thus f equals to a continuous function on the neighborhood $]x_0 - \delta, x_0 + \delta[$ of x_0 , therefore, particularly, f is continuous at x_0 . Because x_0 was arbitrarily chosen, it follows that f is continuous on I . This implies that f is either positive or negative on I .

Thereafter we show that φ and f are continuously differentiable on I . The argument followed here is parallel to that of in the paper [12]. Let $I_0 \subseteq I$ be any open subinterval and $r > 0$ such that the endpoints

of the intervals I_0 , $I_0 - r$ and $I_0 + r$ belong to I . Let further $u \in I_0$ and $h \in]-r, r[$ be arbitrary. Writing $u + h$ and $u - h$ instead of x and y into the equation (1), respectively, we get that

$$\varphi(u)(f(u + h) + f(u - h)) = \varphi(u + h)f(u + h) + \varphi(u - h)f(u - h)$$

holds for all $u \in I_0$ and for all $h \in]-r, r[$. Integrating both sides of the above equation on the interval $]0, r[$, then using that f is either positive or negative on I , a standard calculation yields that

$$(8) \quad \varphi(u) = \left(\int_{u-r}^{u+r} f(t) dt \right)^{-1} \int_{u-r}^{u+r} \varphi(t)f(t) dt, \quad (u \in I_0).$$

By the continuity of φ and f on I_0 , equation (8) implies that $\varphi|_{I_0}$ is continuously differentiable. Hence φ is continuously differentiable on I . Now, by (7), we easily get that f possesses this property on I too.

In the next step we show that φ and f are twice continuously differentiable. After differentiating (1) with respect to x , we get that

$$(9) \quad \frac{1}{2}\varphi'\left(\frac{x+y}{2}\right)(f(x) + f(y)) = (\varphi \cdot f)'(x) - \varphi\left(\frac{x+y}{2}\right)f'(x), \quad (x, y \in I).$$

Keeping the definitions and notations of the previous part, substitute $x := u + h$ and $y := u - h$ into the equation (9). Integrating the equation so obtained on the interval $]0, r[$, we get that

$$(10) \quad \frac{1}{2}\varphi'(u) \int_{u-r}^{u+r} f(t) dt = \int_u^{u+r} (\varphi \cdot f)'(t) dt - \varphi(u) \int_u^{u+r} f'(t) dt, \quad (u \in I_0)$$

holds. In view of (10), it follows that φ' is continuously differentiable on I_0 , and hence also on I , therefore φ is twice continuously differentiable on I . Again, due to (7), we obtain the same conclusion for f .

Finally, we prove that φ and f are infinitely many times differentiable. To do this, differentiate (9) with respect to y , and then write $x = y$ into the equation so obtained. We get that

$$\varphi''(x)f(x) + 2\varphi'(x)f'(x) = 0, \quad (x \in I).$$

Multiplying this equation by $f(x)$, we can deduce that $(\varphi' \cdot f^2)' = 0$. Therefore there exists a constant $\lambda \in \mathbb{R}$ such that $\varphi'(x)f^2(x) = \lambda$ for all $x \in I$, where λ cannot be zero, because φ is non-constant. Consequently, (6) is valid.

Applying equations (6) and (7) repeatedly, it can be seen that φ and f are indeed infinitely many times differentiable. Moreover, in view of (6), we also obtained that φ is strictly monotone on I . \square

4. THE SOLUTIONS OF THE EQUATION (1)

In order to solve (1), we rewrite it first into an equivalent form. For two given functions $f, g : I \rightarrow \mathbb{R}$, define the two-variable function $\mathcal{D}_{f,g}$ by

$$\mathcal{D}_{f,g}(x, y) := \det \begin{pmatrix} f\left(\frac{x+y}{2}\right) & f(x) + f(y) \\ g\left(\frac{x+y}{2}\right) & g(x) + g(y) \end{pmatrix}, \quad (x, y \in I).$$

The following lemma establishes an equivalent form of equation (1) in terms of $\mathcal{D}_{f,g}$.

Lemma 7. *Let $\varphi, f : I \rightarrow \mathbb{R}$ such that f is nowhere zero and define $g : I \rightarrow \mathbb{R}$ by $g(x) := \varphi(x)f(x)$. Then (φ, f) solves (1) if and only if*

$$(11) \quad \mathcal{D}_{f,g}(x, y) = 0, \quad (x, y \in I).$$

Proof. The equivalence of equations (1) and (11) can be seen by a short and simple calculation. \square

In order to solve the latter equation for the unknown functions f and g , we are going to differentiate it twice and four times with respect to the variable x . So we obtain differential equations for f and g . To perform the differentiations, we shall need the following extension of the Leibniz Product Rule.

Lemma 8. *If $B : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ is bilinear and $F, G : I \rightarrow \mathbb{R}^m$ are n times differentiable functions, then $H : I \rightarrow \mathbb{R}$ defined by $H(x) := B(F(x), G(x))$ is also n times differentiable and*

$$(12) \quad H^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} B(F^{(k)}(x), G^{(n-k)}(x)), \quad (x \in I).$$

Proof. In the case when $m = 1$ and $B(u, v) = uv$, the above rule is exactly the Leibniz Product Rule. However, the proof in the more general case can be carried out (using induction with respect to n) exactly in the same way as for the case $m = 1$. \square

Lemma 9. *Let $f, g : I \rightarrow \mathbb{R}$ be n times differentiable functions. Then, for all $x, y \in I$, we have*

$$\partial_1^n \mathcal{D}_{f,g}(x, y) = \sum_{k=0}^{n-1} \frac{1}{2^k} \binom{n}{k} \det \begin{pmatrix} f^{(k)}\left(\frac{x+y}{2}\right) & f^{(n-k)}(x) \\ g^{(k)}\left(\frac{x+y}{2}\right) & g^{(n-k)}(x) \end{pmatrix} + \frac{1}{2^n} \det \begin{pmatrix} f^{(n)}\left(\frac{x+y}{2}\right) & f(x) + f(y) \\ g^{(n)}\left(\frac{x+y}{2}\right) & g(x) + g(y) \end{pmatrix}.$$

Proof. To prove the assertion of the lemma, let $y \in I$ be fixed, define $B : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$(13) \quad B((u_1, u_2), (v_1, v_2)) := \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix},$$

furthermore define $F, G : I \rightarrow \mathbb{R}^2$ by

$$F(x) := \begin{pmatrix} f\left(\frac{x+y}{2}\right) \\ g\left(\frac{x+y}{2}\right) \end{pmatrix} \quad \text{and} \quad G(x) := \begin{pmatrix} f(x) + f(y) \\ g(x) + g(y) \end{pmatrix}.$$

Observe that, with the above notations, $H(x) = \mathcal{D}_{f,g}(x, y)$ holds and now the equality (12) reduces to the identity to be proved. \square

Finally, given a pair (f, g) of sufficiently smooth functions on I , define their *generalized Wronskian* $\mathcal{W}_{f,g}^{k,\ell} : I \rightarrow \mathbb{R}$ for $k, \ell \geq 0$ by

$$(14) \quad \mathcal{W}_{f,g}^{k,\ell}(x) := \det \begin{pmatrix} f^{(k)}(x) & f^{(\ell)}(x) \\ g^{(k)}(x) & g^{(\ell)}(x) \end{pmatrix}.$$

Obviously, due to the basic properties of the determinant, $\mathcal{W}_{f,g}^{k,\ell} = -\mathcal{W}_{f,g}^{\ell,k}$, therefore, the function $\mathcal{W}_{f,g}^{k,\ell}$ is identically zero on I if $k = \ell$.

Theorem 10. *Let $f, g : I \rightarrow \mathbb{R}$ be 4 times differentiable functions such that $\mathcal{W}_{f,g}^{0,1}$ is not identically zero on I . Then (f, g) solves (11) if and only if there exist constants $a, b, c, d, \gamma \in \mathbb{R}$ with $ad \neq bc$ such that*

$$(15) \quad \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where, for all $x \in I$,

- (1) $\psi_1(x) = \sin(\sqrt{-\gamma}x)$ and $\psi_2(x) = \cos(\sqrt{-\gamma}x)$ if $\gamma < 0$,
- (2) $\psi_1(x) = x$ and $\psi_2(x) = 1$ if $\gamma = 0$, and
- (3) $\psi_1(x) = \sinh(\sqrt{\gamma}x)$ and $\psi_2(x) = \cosh(\sqrt{\gamma}x)$ if $\gamma > 0$.

Proof. Assume that the pair (f, g) solves (11). First we are going to show that there exist constants $\alpha, \beta \in \mathbb{R}$ such that

$$(16) \quad \mathcal{W}_{f,g}^{0,1}(x) = \alpha \quad \text{and} \quad \mathcal{W}_{f,g}^{1,2}(x) = \beta, \quad (x \in I).$$

Let $y \in I$ be fixed. Differentiating (11) with respect to the first variable twice and four times, then, applying Lemma 9 for $n = 2$ and $n = 4$, we get that

$$\det \begin{pmatrix} f\left(\frac{x+y}{2}\right) & f''(x) \\ g\left(\frac{x+y}{2}\right) & g''(x) \end{pmatrix} + \det \begin{pmatrix} f'\left(\frac{x+y}{2}\right) & f'(x) \\ g'\left(\frac{x+y}{2}\right) & g'(x) \end{pmatrix} + \frac{1}{4} \det \begin{pmatrix} f''\left(\frac{x+y}{2}\right) & f(x) + f(y) \\ g''\left(\frac{x+y}{2}\right) & g(x) + g(y) \end{pmatrix} = 0$$

and

$$\begin{aligned} \det \begin{pmatrix} f\left(\frac{x+y}{2}\right) & f^{(4)}(x) \\ g\left(\frac{x+y}{2}\right) & g^{(4)}(x) \end{pmatrix} + 2 \det \begin{pmatrix} f'\left(\frac{x+y}{2}\right) & f'''(x) \\ g'\left(\frac{x+y}{2}\right) & g'''(x) \end{pmatrix} + \frac{3}{2} \det \begin{pmatrix} f''\left(\frac{x+y}{2}\right) & f''(x) \\ g''\left(\frac{x+y}{2}\right) & g''(x) \end{pmatrix} \\ + \frac{1}{2} \det \begin{pmatrix} f''' \left(\frac{x+y}{2}\right) & f'(x) \\ g''' \left(\frac{x+y}{2}\right) & g'(x) \end{pmatrix} + \frac{1}{16} \det \begin{pmatrix} f^{(4)}\left(\frac{x+y}{2}\right) & f(x) + f(y) \\ g^{(4)}\left(\frac{x+y}{2}\right) & g(x) + g(y) \end{pmatrix} = 0 \end{aligned}$$

holds for all $x \in I$. Substituting $x = y \in I$ into the previous equations, they reduce to

$$(17) \quad \mathcal{W}_{f,g}^{0,2}(x) = 0 \quad \text{and} \quad 7\mathcal{W}_{f,g}^{0,4} + 12\mathcal{W}_{f,g}^{1,3}(x) = 0,$$

respectively.

For $k, \ell \geq 0$, define the functions $F_k, G_\ell : I \rightarrow \mathbb{R}^2$ as

$$F_k(x) := \begin{pmatrix} f^{(k)}(x) \\ g^{(k)}(x) \end{pmatrix} \quad \text{and} \quad G_\ell(x) := \begin{pmatrix} f^{(\ell)}(x) \\ g^{(\ell)}(x) \end{pmatrix},$$

and, in the rest of the proof, let $B : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as in (13).

Applying Lemma 8 for $n = 2$ and for the functions $B, F := F_0$ and $G := G_2$, we get that the identity $(\mathcal{W}_{f,g}^{0,2})'' = 2\mathcal{W}_{f,g}^{1,3} + \mathcal{W}_{f,g}^{0,4}$ holds on I , thus the system of equations (17) is equivalent to the following one:

$$(18) \quad \mathcal{W}_{f,g}^{0,2}(x) = 0 \quad \text{and} \quad \mathcal{W}_{f,g}^{1,3}(x) = 0.$$

By obvious application of Lemma 8 (for $n = 1, B, F := F_0, G := G_1$ and $n = 1, B, F := F_1, G := G_2$), we obtain that $(\mathcal{W}_{f,g}^{0,1})' = \mathcal{W}_{f,g}^{0,2}$ and $(\mathcal{W}_{f,g}^{1,2})' = \mathcal{W}_{f,g}^{1,3}$ hold on I . In view of (18), this means that there exist constants $\alpha, \beta \in \mathbb{R}$ such that (16) holds for all $x \in I$. By our assumption α is different from zero. Now, consider the equation

$$(19) \quad \det \begin{pmatrix} \psi & \psi' & \psi'' \\ f & f' & f'' \\ g & g' & g'' \end{pmatrix} = 0$$

on I with the unknown function $\psi : I \rightarrow \mathbb{R}$. Expanding the determinant with respect to its first row, in view of (16), we get that (19) is a homogeneous second-order linear differential equation and it is equivalent to

$$(20) \quad \psi'' = \gamma\psi,$$

where γ denotes the constant $-\beta/\alpha$. Then, based on the theory of linear differential equations with constant coefficients, we have that the functions ψ_1 and ψ_2 form a solution basis of (20) in each of the possibilities (1), (2) or (3). On the other hand, the functions f and g trivially solve (19), and hence also (20). Thus, they must be linear combinations of ψ_1 and ψ_2 , that is, (15) holds for some constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$.

To prove the reversed statement, assume now that there exist $a, b, c, d, \gamma \in \mathbb{R}$ with $ad \neq bc$ such that (15) holds. By the product rule for determinants, we have that

$$\mathcal{D}_{f,g}(x, y) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{D}_{\psi_1, \psi_2}(x, y), \quad (x, y \in I),$$

therefore, to check the validity of (11), it is sufficient to prove that

$$(21) \quad \mathcal{D}_{\psi_1, \psi_2}(x, y) = 0, \quad (x, y \in I).$$

This equality is obvious if $\gamma = 0$. To see this in the remaining cases, assume first that $\gamma > 0$. Then, by the well-known identities for hyperbolic functions, we get that

$$\begin{aligned} \psi_1(x) + \psi_1(y) &= \sinh(\sqrt{\gamma}x) + \sinh(\sqrt{\gamma}y) \\ &= 2 \sinh\left(\sqrt{\gamma}\frac{x+y}{2}\right) \cosh\left(\sqrt{\gamma}\frac{x-y}{2}\right) = 2\psi_1\left(\frac{x+y}{2}\right) \cosh\left(\sqrt{\gamma}\frac{x-y}{2}\right) \end{aligned}$$

and

$$\begin{aligned} \psi_2(x) + \psi_2(y) &= \cosh(\sqrt{\gamma}x) + \cosh(\sqrt{\gamma}y) \\ &= 2 \cosh\left(\sqrt{\gamma}\frac{x+y}{2}\right) \cosh\left(\sqrt{\gamma}\frac{x-y}{2}\right) = 2\psi_2\left(\frac{x+y}{2}\right) \cosh\left(\sqrt{\gamma}\frac{x-y}{2}\right) \end{aligned}$$

hold for all $x, y \in I$. Therefore, using these formulae, we can see that the columns of the determinant $\mathcal{D}_{\psi_1, \psi_2}(x, y)$ are linearly dependent, which implies (21).

The proof of (21) in the case $\gamma < 0$ is based on similar identities for trigonometric functions, and therefore it is completely analogous. \square

Theorem 11. *Let $\varphi, f : I \rightarrow \mathbb{R}$ such that φ is continuous and $\mathcal{Z}_f^c \subseteq \overline{\text{conv}}(\overline{\mathcal{Z}_f^c})$. Then (φ, f) solves functional equation (1) if and only if either*

(a) *there exists an interval $J \subseteq I$ such that $f(x) = 0$ for all $x \in I \setminus J$ and φ is constant on $\frac{1}{2}(J + I)$,*
or

(b) *there exist constants $a, b, c, d, \gamma \in \mathbb{R}$ with $ad \neq bc$ such that*

$$\begin{aligned} (1) \quad & f(x) = a \sin(\sqrt{-\gamma}x) + b \cos(\sqrt{-\gamma}x) \neq 0 \text{ and } \varphi(x) = \frac{c \sin(\sqrt{-\gamma}x) + d \cos(\sqrt{-\gamma}x)}{a \sin(\sqrt{-\gamma}x) + b \cos(\sqrt{-\gamma}x)} \text{ if } \gamma < 0, \\ (2) \quad & f(x) = a + bx \neq 0 \text{ and } \varphi(x) = \frac{c + dx}{a + bx} \text{ if } \gamma = 0, \\ (3) \quad & f(x) = a \sinh(\sqrt{\gamma}x) + b \cosh(\sqrt{\gamma}x) \neq 0 \text{ and } \varphi(x) = \frac{c \sinh(\sqrt{\gamma}x) + d \cosh(\sqrt{\gamma}x)}{a \sinh(\sqrt{\gamma}x) + b \cosh(\sqrt{\gamma}x)} \text{ if } \gamma > 0 \end{aligned}$$

for all $x \in I$.

Proof. We can distinguish the following two main cases:

- (i) either $\mathcal{Z}_f \neq \emptyset$ or $\mathcal{Z}_f = \emptyset$ and φ is constant on I , or
- (ii) $\mathcal{Z}_f = \emptyset$ and φ is non-constant on I .

Consider first the case (i) and assume that (φ, f) solves equation (1). If \mathcal{Z}_f^c is empty, then $\overline{\mathcal{Z}_f^c}$ is also empty and we trivially have $\frac{1}{2}(\overline{\mathcal{Z}_f^c} + I) = \frac{1}{2}(\mathcal{Z}_f^c + I)$. If \mathcal{Z}_f^c is nonempty then the condition $\mathcal{Z}_f^c \subseteq \overline{\text{conv}}(\overline{\mathcal{Z}_f^c})$ implies that the set $\overline{\mathcal{Z}_f^c}$ is also nonempty. Applying Lemma 3 for $A := \overline{\mathcal{Z}_f^c}$ and $B := \mathcal{Z}_f^c$, we obtain again that $\frac{1}{2}(\overline{\mathcal{Z}_f^c} + I) = \frac{1}{2}(\mathcal{Z}_f^c + I)$ hold.

Using this, we show that φ is constant on the set $\frac{1}{2}(\mathcal{Z}_f^c + I)$. If \mathcal{Z}_f is nonempty, then, due to Proposition 5, the function φ is constant on $\frac{1}{2}(\overline{\mathcal{Z}_f^c} + I) = \frac{1}{2}(\mathcal{Z}_f^c + I)$. If \mathcal{Z}_f is empty, then $\frac{1}{2}(\mathcal{Z}_f^c + I) = I$, thus, by the second condition of case (i), we also have that φ is constant on $\frac{1}{2}(\mathcal{Z}_f^c + I)$.

Let J denote the convex hull of the set \mathcal{Z}_f^c . Then $I \setminus J \subseteq I \setminus \mathcal{Z}_f^c = \mathcal{Z}_f$, which implies that $f(x) = 0$ whenever $x \in I \setminus J$. Finally, we show that φ is constant on $\frac{1}{2}(J + I)$. If \mathcal{Z}_f^c is empty then J is also empty, hence φ is trivially constant on $\frac{1}{2}(J + I) = \emptyset$. In the other case when \mathcal{Z}_f^c is nonempty, we have that $\{\inf J, \sup J\} = \{\inf \mathcal{Z}_f^c, \sup \mathcal{Z}_f^c\}$, therefore, formula (3) of Lemma 2 implies that $\frac{1}{2}(\mathcal{Z}_f^c + I) = \frac{1}{2}(J + I)$. Thus φ is constant on $\frac{1}{2}(J + I)$, and consequently, (a) holds.

Conversely, assume that the alternative (a) is valid, that is, there exists an interval $J \subseteq I$ such that $I \setminus J \subseteq \mathcal{Z}_f$ and φ is constant on $\frac{1}{2}(J + I)$. Then $\mathcal{Z}_f^c \subseteq J$, hence φ is constant on $\frac{1}{2}(\mathcal{Z}_f^c + I)$, which, in view of Theorem 1, implies that φ and f solve (1).

Now, consider case (ii), namely suppose that \mathcal{Z}_f is empty and φ is non-constant on I . Then, by Proposition 6, the functions φ, f are infinitely many times differentiable such that (6) holds with a

nonzero constant λ . Then, in view of Lemma 7, the functions f and $g := \varphi f$ are infinitely many times differentiable solutions of (11). Furthermore, we have that $\mathcal{W}_{f,g}^{0,1} = \lambda \neq 0$. Applying Theorem 10, we obtain that $\varphi = \frac{g}{f}$ and f must be one of the forms (1), (2) or (3) represented in the alternative (b).

Assuming (b), one can easily see that, in each cases (1), (2) and (3), the functions f and $g := \varphi f$ are solutions of (11). Consequently, φ and f are solutions of (1). \square

Remark 12. The condition $\mathcal{Z}_f^c \subseteq \overline{\text{conv}}(\overline{\mathcal{Z}_f^c})$ is easily fulfilled if either f is nowhere zero (then \mathcal{Z}_f is empty) or f is continuous (then \mathcal{Z}_f is closed). Therefore, Theorem 11 is a proper generalization of the result in [12], where a similar conclusion was reached assuming that φ was strictly monotone and continuous and f was positive. It seems to be an interesting question if the conclusion of Theorem 11 could be obtained without assuming the condition $\mathcal{Z}_f^c \subseteq \overline{\text{conv}}(\overline{\mathcal{Z}_f^c})$.

5. APPLICATION

In this section we are going to solve the functional equation

$$(22) \quad G(g(u) - g(v)) = H(h(u) + h(v)) + F(u) + F(v), \quad (u, v \in J),$$

where $J \subseteq \mathbb{R}$ is a nonempty open interval, furthermore

$$g, h, F : J \rightarrow \mathbb{R}, \quad G : g(J) - g(J) \rightarrow \mathbb{R}, \quad \text{and} \quad H : h(J) + h(J) \rightarrow \mathbb{R}$$

are considered as unknown functions.

Lemma 13. *With the above notations, assume that $h : J \rightarrow \mathbb{R}$ is a continuous strictly monotone function and define*

$$(23) \quad I := 2h(J), \quad \ell(x) := g \circ h^{-1}\left(\frac{x}{2}\right), \quad (x \in I), \quad \text{and} \quad G_0 := G - G(0).$$

If (g, h, F, G, H) is a solution of (22), then $\ell, H : I \rightarrow \mathbb{R}$ and $G_0 : \ell(I) - \ell(I) \rightarrow \mathbb{R}$ solve the functional equation

$$(24) \quad G_0(\ell(x) - \ell(y)) = H\left(\frac{x+y}{2}\right) - \frac{H(x) + H(y)}{2}, \quad (x, y \in I)$$

and

$$(25) \quad F(u) = \frac{1}{2}(G(0) - H(2h(u))), \quad (u \in J).$$

Conversely, if $\ell, H : I \rightarrow \mathbb{R}$ and $G_0 : \ell(I) - \ell(I) \rightarrow \mathbb{R}$ solve the functional equation (24), $G(0) \in \mathbb{R}$ is an arbitrary constant, $J := h^{-1}(\frac{1}{2}I)$, $F : J \rightarrow \mathbb{R}$ is defined by (25), furthermore $g : J \rightarrow \mathbb{R}$ and $G : \ell(I) - \ell(I) \rightarrow \mathbb{R}$ are given by

$$g(u) := \ell(2h(u)) \quad \text{and} \quad G := G(0) + G_0,$$

then (g, h, F, G, H) is a solution of (22).

Proof. Substituting $v = u$ into (22), it immediately follows that F is of the form (25). Therefore, (22) can be rewritten as

$$(26) \quad G(g(u) - g(v)) - G(0) = H(h(u) + h(v)) - \frac{H(2h(u)) + H(2h(v))}{2} \quad (u, v \in J).$$

Since $h : J \rightarrow \mathbb{R}$ is a continuous strictly monotone function, thus its inverse is also a continuous strictly monotone function which is defined on the open interval $h(J)$. With the notation (23), after the substitution $u := h^{-1}(\frac{x}{2})$, $v := h^{-1}(\frac{y}{2})$, equation (26) reduces to (24).

The proof of the reversed implication is a simple computation, therefore, it is omitted. \square

Lemma 14. *Assume that $\ell, H : I \rightarrow \mathbb{R}$ and $G_0 : \ell(I) - \ell(I) \rightarrow \mathbb{R}$ are differentiable solutions of functional equation (24) such that ℓ' does not vanish on I . Then the pair of functions (φ, f) given by*

$$(27) \quad \varphi := H' \quad \text{and} \quad f := \frac{1}{\ell'}$$

solves equation (1).

Proof. Differentiating equation (24) with respect to the variables x and y , we get

$$\begin{aligned} G'_0(\ell(x) - \ell(y)) \cdot \ell'(x) &= \frac{1}{2}H'\left(\frac{x+y}{2}\right) - \frac{H'(x)}{2}, \\ -G'_0(\ell(x) - \ell(y)) \cdot \ell'(y) &= \frac{1}{2}H'\left(\frac{x+y}{2}\right) - \frac{H'(y)}{2} \end{aligned}$$

for all $x, y \in I$. Multiplying the first equation by $\frac{2}{\ell'(x)}$, the second one by $\frac{2}{\ell'(y)}$ and adding up the equations so obtained side by side, we obtain that

$$0 = H'\left(\frac{x+y}{2}\right) \left(\frac{1}{\ell'(x)} + \frac{1}{\ell'(y)}\right) - \frac{H'(x)}{\ell'(x)} - \frac{H'(y)}{\ell'(y)}, \quad (x, y \in I).$$

Therefore, with the notations (27), we can see that (1) is satisfied. \square

Finally we give the complete solution of (24).

Theorem 15. *Let $\ell, H : I \rightarrow \mathbb{R}$ and $G_0 : \ell(I) - \ell(I) \rightarrow \mathbb{R}$ are differentiable functions such that ℓ' does not vanish on I and H' is continuous. Then the triple (G_0, ℓ, H) solves functional equation (24) if and only if there exist constants $A, B, C, D, E, \alpha, \beta$ with $CD\alpha \neq 0$ such that, for all $x \in I$ and $u \in \ell(I) - \ell(I)$ one of the following possibilities holds:*

	$G_0(u) =$	$\ell(x) =$	$H(x) =$
(i)	0	arbitrary	$Ax + B$
(ii)	$C(Du)^2$	$\frac{1}{2D}x + E$	$-Cx^2 + Ax + B$
(iii)	$C(Du)^2$	$\frac{1}{2D}e^{\alpha x} + E$	$-\frac{C}{2}e^{2\alpha x} + Ax + B$
(iv)	$C \ln(\cosh(Du))$	$\frac{1}{2D} \ln \alpha x + \beta + E$	$C \ln \alpha x + \beta + Ax + B$
(v)	$C \ln(\cosh(Du))$	$\frac{1}{2D} \ln \tan(\alpha x + \beta) + E$	$C \ln \sin(2\alpha x + 2\beta) + Ax + B$
(vi)	$C \ln(\cosh(Du))$	$\frac{1}{2D} \ln \tanh(\alpha x + \beta) + E$	$C \ln \sinh(2\alpha x + 2\beta) + Ax + B$
(vii)	$C \ln(\cos(Du))$	$\frac{1}{D} \arctan(\tanh(\alpha x + \beta)) + E$	$C \ln(\cosh(2\alpha x + 2\beta)) + Ax + B$

In addition, in cases (iv), (v), and (vi) we have

$$(28) \quad 0 \notin \alpha I + \beta, \quad \mathbb{Z} \cap \frac{2}{\pi}(\alpha I + \beta) = \emptyset, \quad \text{and} \quad 0 \notin \alpha I + \beta,$$

respectively.

Proof. In the proof we are going to combine the result of Theorem 11 and Lemma 14.

Suppose that (G_0, ℓ, H) solves (24). Under the assumptions of the theorem, $\varphi = H'$ and $f = \frac{1}{\ell'}$ are solutions of the functional equation (1) and f is nowhere zero. Therefore, by Theorem 11, either φ is constant on I or there exist constants a, b, c, d, γ with $ad \neq bc$ such that f and φ are of the forms (1), (2), or (3) in alternative (b) of Theorem 11.

If φ is constant, then H is affine, that is, there exist constants $A, B \in \mathbb{R}$ such that $H(x) = Ax + B$ for $x \in I$. Then the right hand side of (24) is identically zero, therefore, G_0 must equal to zero on $\ell(I) - \ell(I)$ and ℓ can be arbitrary. That is, case (i) holds.

From now on assume that $\varphi = H'$ is non-constant on I . We distinguish six main cases according to the sign of the parameter γ and the parameters a, b . In each cases we are going to solve the differential equations in (27) for the unknown functions H and ℓ , and then we determine G_0 using the functional equation (24).

Case 1. Assume that $\gamma < 0$. By the condition $ad \neq bc$, we have that $a^2 + b^2 > 0$. Denote $\alpha := \frac{\sqrt{-\gamma}}{2} \neq 0$ and define $\beta \in [0, \pi[$ as the unique solution of the system of equations

$$\cos(2\beta) = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin(2\beta) = \frac{b}{\sqrt{a^2 + b^2}}.$$

Then we have that

$$0 \neq \frac{1}{\ell'(x)} = a \sin(\sqrt{-\gamma}x) + b \cos(\sqrt{-\gamma}x) = \sqrt{a^2 + b^2} \sin(2\alpha x + 2\beta), \quad (x \in I).$$

Therefore, $2\alpha x + 2\beta \notin \pi\mathbb{Z}$ for all $x \in I$, which yields that the second condition in (28) is satisfied.

Solving the differential equations in (27) for the unknown functions ℓ and H , with the notations $A := \frac{ac+bd}{a^2+b^2}$, $C := \frac{ad-bc}{\sqrt{-\gamma(a^2+b^2)}} \neq 0$, and $D := \frac{\sqrt{-\gamma(a^2+b^2)}}{2} \neq 0$, we get that there exist constants E and B such that, for all $x \in I$, we have

$$\ell(x) = \frac{1}{2D} \ln |\tan(\alpha x + \beta)| + E \quad \text{and} \quad H(x) = C \ln |\sin(2\alpha x + 2\beta)| + Ax + B.$$

Using these representations and substituting $v := \alpha x + \beta$, $w := \alpha y + \beta$, the functional equation (24) reduces to

$$\begin{aligned} G_0\left(\frac{1}{2D} \ln \frac{\tan v}{\tan w}\right) &= C \ln |\sin(v+w)| - \frac{C \ln |\sin(2v)| + C \ln |\sin(2w)|}{2} \\ &= C \ln \frac{|\sin(v+w)|}{\sqrt{\sin(2v)\sin(2w)}} = C \ln \frac{|\sin v \cos w + \sin w \cos v|}{\sqrt{4 \sin v \cos v \sin w \cos w}} = C \ln \frac{\sqrt{\frac{\tan v}{\tan w}} + \sqrt{\frac{\tan w}{\tan v}}}{2}. \end{aligned}$$

(Note that, due to the condition $\mathbb{Z} \cap \frac{2}{\pi}(\alpha I + \beta) = \emptyset$, in the above computation we have that $\tan v$ and $\tan w$ as well as $\sin(2v)$ and $\sin(2w)$ have the same sign.) Putting $u := \frac{1}{2D} \ln \frac{\tan v}{\tan w}$, it follows that $\sqrt{\frac{\tan v}{\tan w}} = e^{Du}$, hence the above equation yields that

$$G_0(u) = C \ln(\cosh(Du)).$$

That is, we obtain the solutions listed in (v).

Case 2. Assume that $\gamma = 0$ and $b = 0$. Then, in view of the condition $ad \neq bc$, the parameters a and d are different from zero. Solving (27), with the notations $A := \frac{c}{a}$, $C := -\frac{d}{2a} \neq 0$ and $D := \frac{a}{2} \neq 0$, we obtain that there exist constants E and B such that, for all $x \in I$, we have

$$\ell(x) = \frac{1}{2D}x + E \quad \text{and} \quad H(x) = -Cx^2 + Ax + B.$$

Then the equation (24) reduces to the form

$$G_0\left(\frac{x-y}{2D}\right) = -C\left(\frac{x+y}{2}\right)^2 + \frac{Cx^2 + Cy^2}{2} = C\left(\frac{x-y}{2}\right)^2.$$

Now, replacing $\frac{x-y}{2D}$ by u , it follows that $G_0(u) = C(Du)^2$. That is, the solutions obtained are exactly those listed in case (ii).

Case 3. Assume that $\gamma = 0$ and $b \neq 0$. Solving the equations in (27) and using the notations $A := \frac{d}{b}$, $C := \frac{bc-ad}{b^2} \neq 0$, $D := \frac{b}{2} \neq 0$, $\alpha := b \neq 0$ and $\beta := a$, we can see that $0 \notin \alpha I + \beta$ (that is, the first condition in (28) holds) and there exist constants E and B such that, for all $x \in I$,

$$\ell(x) = \frac{1}{2D} \ln |\alpha x + \beta| + E \quad \text{and} \quad H(x) = C \ln |\alpha x + \beta| + Ax + B$$

holds. Consequently, introducing the notations $v := |\alpha x + \beta|$ and $w := |\alpha y + \beta|$, equation (24) reduces to

$$G_0\left(\frac{1}{2D} \ln \frac{v}{w}\right) = C \ln \frac{v+w}{2} - \frac{(C \ln v) + (C \ln w)}{2} = C \ln \frac{v+w}{2\sqrt{vw}} = C \ln \frac{\sqrt{\frac{v}{w}} + \sqrt{\frac{w}{v}}}{2}$$

Substituting $u := \frac{1}{2D} \ln \frac{v}{w}$, it follows that $\sqrt{\frac{v}{w}} = e^{Du}$, hence the above equation yields that

$$G_0(u) = C \ln(\cosh(Du)).$$

That is, we got the solutions listed in (iv).

Case 4. Assume that $\gamma > 0$ and $|a| = |b|$. In this case, due to the condition $ad \neq bc$, both of the parameters a and b are different from zero. In order to get a simpler calculation, using the well-known identities $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$ and $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ for $t \in \mathbb{R}$, we rewrite $\frac{1}{\sqrt{\gamma}}$ and H' to the form

$$\frac{1}{\ell'(x)} = a \sinh(\sqrt{\gamma}x) + b \cosh(\sqrt{\gamma}x) = \frac{a+b}{2}e^{\sqrt{\gamma}x} + \frac{b-a}{2}e^{-\sqrt{\gamma}x} = a \operatorname{sgn}(ab)e^{\operatorname{sgn}(ab)\sqrt{\gamma}x}, \quad (x \in I)$$

and

$$\begin{aligned} H'(x) &= \frac{c \sinh(\sqrt{\gamma}x) + d \sinh(\sqrt{\gamma}x)}{a \sinh(\sqrt{\gamma}x) + b \sinh(\sqrt{\gamma}x)} = \frac{d+c}{2a \operatorname{sgn}(ab)} e^{(1-\operatorname{sgn}(ab))\sqrt{\gamma}x} + \frac{d-c}{2a \operatorname{sgn}(ab)} e^{-(1+\operatorname{sgn}(ab))\sqrt{\gamma}x} \\ &= \frac{\operatorname{sgn}(ab)d-c}{2a} e^{-2\operatorname{sgn}(ab)\sqrt{\gamma}x} + \frac{c+\operatorname{sgn}(ab)d}{2a}, \quad (x \in I). \end{aligned}$$

Thus, solving (27), we obtain that there exist real constants B and E such that, for all $x \in I$, we have

$$\ell(x) = -\frac{1}{a\sqrt{\gamma}} e^{-\operatorname{sgn}(ab)\sqrt{\gamma}x} + E$$

and

$$H(x) = \frac{\operatorname{sgn}(ab)c-d}{4a\sqrt{\gamma}} e^{-2\operatorname{sgn}(ab)\sqrt{\gamma}x} + \frac{c+\operatorname{sgn}(ab)d}{2a}x + B.$$

Now, define A, C, D , and α by

$$A := \frac{c+\operatorname{sgn}(ab)d}{2a}, \quad C := \frac{d-\operatorname{sgn}(ab)c}{2a\sqrt{\gamma}}, \quad D := -\frac{a\sqrt{\gamma}}{2}, \quad \text{and} \quad \alpha := -\operatorname{sgn}(ab)\sqrt{\gamma}.$$

Obviously, $\alpha \neq 0$, furthermore, in view of the condition $ad \neq bc$, the constants C and D are also different from zero. Using these notations, we obtained that the functions ℓ and H are of the form

$$\ell(x) = \frac{1}{2D} e^{\alpha x} + E \quad \text{and} \quad H(x) = -\frac{C}{2} e^{2\alpha x} + Ax + B$$

on the interval I . Therefore, with the substitutions $v := \alpha x$ and $w := \alpha y$, the equation (24) reduces to

$$G_0\left(\frac{1}{2D}(e^v - e^w)\right) = -\frac{C}{2}e^{v+w} + \frac{\frac{C}{2}e^{2v} + \frac{C}{2}e^{2w}}{2} = \frac{C}{4}(e^{2v} - 2e^{v+w} + e^{2w}) = \frac{C}{4}(e^v - e^w)^2.$$

Let $u := \frac{1}{2D}(e^v - e^w)$. Then $e^v - e^w = 2Du$ and

$$G_0(u) = \frac{C}{4}(2Du)^2 = C(Du)^2.$$

That is, we obtain the solutions listed in (iii).

Case 5. Assume that $\gamma > 0$ and $|a| > |b|$. Then it follows that $a^2 - b^2 > 0$ and $a \neq 0$. Denote $\alpha := \frac{\sqrt{\gamma}}{2} \neq 0$ and define $\beta \in \mathbb{R}$ by the equation

$$\sinh(2\beta) = \frac{\operatorname{sgn}(a)b}{\sqrt{a^2 - b^2}}.$$

Then $\cosh(2\beta) = \frac{\operatorname{sgn}(a)a}{\sqrt{a^2 - b^2}}$, therefore the identity

$$0 \neq \frac{1}{\ell'(x)} = a \sinh(\sqrt{\gamma}x) + b \cosh(\sqrt{\gamma}x) = \operatorname{sgn}(a)\sqrt{a^2 - b^2} \sinh(2\alpha x + 2\beta), \quad (x \in I)$$

holds, and we have that $0 \notin \alpha I + \beta$ (that is, the third condition in (28) holds). Solving the differential equations in (27), with the notations $A := \frac{ac-bd}{a^2-b^2}$, $C := \frac{ad-bc}{\sqrt{\gamma}(a^2-b^2)} \neq 0$ and $D := \frac{\sqrt{\gamma}(a^2-b^2)}{2\operatorname{sgn}(a)} \neq 0$, we get that there exist constants B and E such that

$$\ell(x) = \frac{1}{2D} \ln |\tanh(\alpha x + \beta)| + E, \quad \text{and} \quad H(x) = C \ln |\sinh(2\alpha x + 2\beta)| + Ax + B$$

hold for all $x \in I$. Consequently, introducing the notations $v := \alpha x + \beta$ and $w := \alpha y + \beta$, the equation (24) reduces to

$$\begin{aligned} G_0\left(\frac{1}{2D} \ln \frac{\tanh v}{\tanh w}\right) &= C \ln |\sinh(v+w)| - \frac{C \ln |\sinh(2v)| + C \ln |\sinh(2w)|}{2} = C \ln \frac{|\sinh(v+w)|}{\sqrt{|\sinh(2v)\sinh(2w)|}} \\ &= C \ln \frac{|\sinh v \cosh w + \sinh w \cosh v|}{\sqrt{4 \sinh v \cosh v \sinh w \cosh w}} = C \ln \frac{\sqrt{\frac{\tanh v}{\tanh w}} + \sqrt{\frac{\tanh w}{\tanh v}}}{2}. \end{aligned}$$

(Note that, due to the condition $0 \notin \alpha I + \beta$, in the above computation we have that $\tanh v$ and $\tanh w$ as well as $\sinh(2v)$ and $\sinh(2w)$ have the same sign.) Let now $u := \frac{1}{2D} \ln \frac{\tanh v}{\tanh w}$. Then $\sqrt{\frac{\tanh v}{\tanh w}} = e^{Du}$ and

$$G_0(u) = C \ln(\cosh(Du)).$$

That is, we obtain the solutions listed in (vi).

Case 6. Assume finally that $\gamma > 0$ and $|a| < |b|$. Then it follows that $b^2 - a^2 > 0$ and $b \neq 0$. Let $\alpha := \frac{\sqrt{\gamma}}{2} \neq 0$ and define the parameter $\beta \in \mathbb{R}$ by the equation

$$\sinh(2\beta) = \frac{\operatorname{sgn}(b)a}{\sqrt{b^2 - a^2}}.$$

Then we have that $\cosh(2\beta) = \frac{\operatorname{sgn}(b)b}{\sqrt{b^2 - a^2}}$ and therefore

$$\frac{1}{\ell'(x)} = a \sinh(\sqrt{\gamma}x) + b \cosh(\sqrt{\gamma}x) = \operatorname{sgn}(b)\sqrt{b^2 - a^2} \cosh(2\alpha x + 2\beta), \quad (x \in I).$$

In view of the identity above, solving the differential equations in (27), with the notations $A := \frac{bd-ac}{b^2-a^2}$, $C := \frac{cb-ad}{\sqrt{\gamma}(b^2-a^2)}$ and $D := \frac{\sqrt{\gamma}(b^2-a^2)}{2\operatorname{sgn}(b)}$, we get that there exist constants B and E such that

$$\ell(x) = \frac{1}{D} \arctan(\tanh(\alpha x + \beta)) + E \quad \text{and} \quad H(x) = C \ln(\cosh(2\alpha x + 2\beta)) + Ax + B$$

hold for all $x \in I$. By their definitions, we can also see that the parameters C and D are different from zero.

In order to determine G_0 , firstly, we are going to shape the expression in the argument of G_0 . Since, for all $t \in \mathbb{R}$, we have that $-1 < \tanh t < 1$, hence $-\frac{\pi}{4} < \arctan(\tanh t) < \frac{\pi}{4}$. Therefore, for all $x, y \in I$,

$$(29) \quad -\frac{\pi}{2} < \arctan(\tanh(\alpha x + \beta)) - \arctan(\tanh(\alpha y + \beta)) < \frac{\pi}{2}.$$

On the other hand, by the addition theorem of the tangent function, we obtain that

$$\tan(\arctan(\tanh(\alpha x + \beta)) - \arctan(\tanh(\alpha y + \beta))) = \frac{\tanh(\alpha x + \beta) - \tanh(\alpha y + \beta)}{1 + \tanh(\alpha x + \beta)\tanh(\alpha y + \beta)}.$$

By the inequalities of (29), with the substitutions $v := e^{\alpha x + \beta}$ and $w := e^{\alpha y + \beta}$, it follows that

$$\begin{aligned} \arctan(\tanh(\alpha x + \beta)) - \arctan(\tanh(\alpha y + \beta)) &= \arctan \frac{\tanh(\alpha x + \beta) - \tanh(\alpha y + \beta)}{1 + \tanh(\alpha x + \beta)\tanh(\alpha y + \beta)} \\ &= \arctan \frac{\frac{v^2-1}{v^2+1} - \frac{w^2-1}{w^2+1}}{1 + \frac{v^2-1}{v^2+1} \frac{w^2-1}{w^2+1}} = \arctan \frac{v^2 - w^2}{1 + v^2 w^2}. \end{aligned}$$

Hence the equation (24) reduces to

$$\begin{aligned} G_0\left(\frac{1}{D} \arctan \frac{v^2 - w^2}{1 + v^2 w^2}\right) &= C \ln(\cosh(\alpha x + \alpha y + 2\beta)) - \frac{C \ln(\cosh(2\alpha x + 2\beta)) + C \ln(\cosh(2\alpha y + 2\beta))}{2} \\ &= C \ln \frac{\cosh(\alpha x + \beta + \alpha y + \beta)}{\sqrt{\cosh(2\alpha x + 2\beta)\cosh(2\alpha y + 2\beta)}} = C \ln \frac{1 + v^2 w^2}{\sqrt{(v^4 + 1)(w^4 + 1)}} = C \ln \frac{1}{\sqrt{1 + \left(\frac{v^2 - w^2}{1 + v^2 w^2}\right)^2}}. \end{aligned}$$

Substituting $u := \frac{1}{D} \arctan \frac{v^2 - w^2}{1 + v^2 w^2}$, we get that

$$G_0(u) = C \ln \frac{1}{\sqrt{1 + \tan^2(Du)}} = C \ln(\cos(Du)).$$

Therefore, in this case, we get the solutions listed in (vi). \square

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