COMPOSITE RATIONAL FUNCTIONS AND ARITHMETIC PROGRESSIONS

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Abstract. In this paper we deal with composite rational functions having zeros and poles forming consecutive elements of an arithmetic progression. We also correct a result published in [12] related to composite rational functions having a fixed number of zeros and poles.

1. Introduction

We consider a problem related to decompositions of polynomials and rational functions. In this subject a classical result obtained by Ritt [13] says that if there is a polynomial $f \in \mathbb{C}[X]$ satisfying certain tameness properties and

$$f = g_1 \circ g_2 \circ \cdots \circ g_r = h_1 \circ h_2 \circ \cdots \circ h_s,$$

then $r = s$ and $\{\deg g_1, \dots, \deg g_r\} = \{\deg h_1, \dots, \deg h_r\}$. Ritt's fundamental result has been investigated, extended and applied in various wide-ranging contexts (see e.g. [4, 6, 7, 9, 10, 11, 14, 15]). The above mentioned result is not valid for rational functions. Gutierrez and Sevilla [9] provided the following example

$$f = \frac{x^3(x+6)^3(x^2-6x+36)^3}{(x-3)^3(x^2+3x+9)^3},$$

$$f = g_1 \circ g_2 \circ g_3 = x^3 \circ \frac{x(x-12)}{x-3} \circ \frac{x(x+6)}{x-3},$$

$$f = h_1 \circ h_2 = \frac{x^3(x+24)}{x-3} \circ \frac{x(x^2-6x+36)}{x^2+3x+9}.$$

To determine decompositions of a given rational function there were developed algorithms (see e.g. [1, 2, 3]). In [2], Ayad and Fleischmann

2000 *Mathematics Subject Classification.* Primary 11R58; Secondary 14H05, 12Y05.

Key words and phrases. composite rational functions, lacunary polynomials, arithmetic progressions.

Research supported in part by the OTKA grants NK104208 and K100339.

implemented a MAGMA [5] code to find decompositions, they provided the following example

$$f = \frac{x^4 - 8x}{x^3 + 1}$$

and they obtained that $f(x) = g(h(x))$, where

$$g = \frac{x^2 + 4x}{x + 1} \quad \text{and} \quad h = \frac{x^2 - 2x}{x + 1}.$$

Fuchs and Pethő [8] proved the following theorem.

Theorem A. *Let k be an algebraically closed field of characteristic zero. Let n be a positive integer. Then there exists a positive integer J and, for every $i \in \{1, \ldots, J\}$, an affine algebraic variety V_i defined over $\mathbb{Q}$ and with $V_i \subset \mathbb{A}^{n+t_i}$ for some $2 \le t_i \le n$, such that:*

(i) If $f, g, h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \ge 2, g$ not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in PGL_2(k)$, and f has at most n zeros and poles altogether, then there exists for some $i \in \{1, \ldots, J\}$ a point $P = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{t_i}) \in V_i(k)$, a vector $(k_1, \ldots, k_{t_i}) \in \mathbb{Z}^{t_i}$ with $k_1 + k_2 + \ldots + k_{t_i} = 0$ depending only [1] on V_i , a partition of $\{1, \ldots, n\}$ in $t_i + 1$ disjoint sets $S_\infty, S_{\beta_1}, \ldots, S_{\beta_{t_i}}$ with $S_\infty = \emptyset$ if $k_1 + k_2 + \ldots + k_{t_i} = 0$, and a vector $(l_1, \ldots, l_n) \in \{0, 1, \ldots, n-1\}^n$, also both depending only on V_i, such that

$$f(x) = \prod_{j=1}^{t_i} (\omega_j / \omega_\infty)^{k_j}, \quad g(x) = \prod_{j=1}^{t_i} (x - \beta_j)^{k_j}$$

and

$$h(x) = \begin{cases} \beta_j + \frac{\omega_j}{\omega_\infty} \quad (j = 1, \ldots, t_i), & \text{if } k_1 + k_2 + \ldots + k_{t_i} \neq 0 \\ \frac{\beta_{j_1}\omega_{j_2} - \beta_{j_2}\omega_{j_1}}{\omega_{j_2} - \omega_{j_1}} \quad (1 \le j_1 < j_2 \le t_i), & \text{otherwise,} \end{cases}$$

where

$$\omega_j = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m}, \quad j = 1, \ldots, t_i$$

and

$$\omega_\infty = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m}.$$

Moreover, we have deg $h \le (n-1)/\max\{t_i - 2, 1\} \le n - 1$.

(ii) Conversely for given data $P \in V_i(k), (k_1, \ldots, k_{t_i}), S_\infty, S_{\beta_1}, \ldots, S_{\beta_{t_i}}$, $(l_1, \ldots, l_n)$ as described in (i) one defines by the same equations rational functions f, g, h with f having at most n zeros and poles altogether for which $f(x) = g(h(x))$ holds.

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[1] in [8] it is written as "or not depending", this typo is corrected here.

(iii) The integer J and equations defining the varieties V_i are effectively computable only in terms of n.

Pethő and Tengely [12] provided some computational experiments that they obtained by using a MAGMA [5] implementation of the algorithm of Fuchs and Pethő [8].

If the zeros and poles of a composite rational function form an arithmetic progression, then we have the following result.

Theorem 1. *Let f, g, h be rational functions as in Theorem A. Assume that the zeros and poles of f form an arithmetic progression, that is*

$$\alpha_i = \alpha_0 + T_i d$$

for some $\alpha_0, d \in k$ and $T_i \in \{0, 1, \ldots, n-1\}$. If $k_1 + k_2 + \ldots + k_t \neq 0$, then either the difference d satisfies an equation of the form

$$d^N = M$$

for some $N \in \mathbb{Z}, M \in \mathbb{Q}$ or $(l_1, \ldots, l_n) \in \{0, 1, \ldots, n-1\}^n$ satisfies a system of linear equations

$$\sum_{r \in S_{\beta_i}} l_r = \sum_{s \in S_{\beta_j}} l_s, \quad i, j \in \{1, \ldots, t\}, i \neq j.$$

If $k_1 + k_2 + \ldots + k_t = 0$ and $1 \leq j_1 < j_2 < j_3 \leq t$, then

$$d^{\sum_{m_1 \in S_{\beta_{j_1}}} l_{m_1}}, d^{\sum_{m_2 \in S_{\beta_{j_2}}} l_{m_2}}, d^{\sum_{m_3 \in S_{\beta_{j_3}}} l_{m_3}}$$

satisfy a system of linear equations and $\beta_{j_1}, \beta_{j_2}, \beta_{j_3}$ satisfy a system of linear equations.

We will apply the above theorem to determine composite rational functions having 4 zeros and poles. We prove the following statement.

Proposition 1. *Let k be an algebraically closed field of characteristic zero. If $f, g, h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2$, g not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in PGL_2(k)$, and f has 4 zeros and poles altogether forming an arithmetic progression, then f is equivalent to the following rational function*

$$(x - \alpha_0)^{k_1} (x - \alpha_0 - d)^{k_2} (x - \alpha_0 - 2d)^{k_2} (x - \alpha_0 - 3d)^{k_1},$$

for some $\alpha_0, d \in k$ and $k_1, k_2 \in \mathbb{Z}, k_1 + k_2 \neq 0$.

In this paper we correct results obtained in [12], where the computations related to the case $k_1 + k_2 + \ldots + k_t \neq 0, S_\infty = \emptyset$ are missing. The following theorem is the corrected version of Theorem 1 from [12], where part (c) was missing. We define equivalence of rational functions. Two rational functions $f_1(x) = \prod_{u=1}^{n} (x - \alpha_u^{(1)})^{f_u^{(1)}}$ and

$f_2(x) = \prod_{u=1}^{n}(x-\alpha_u^{(2)})^{f_u^{(2)}}$ are equivalent if there exist $a_{u,v} \in \mathbb{Q}, u \in \{1,2,\ldots,n\}, v \in \{1,2,\ldots,n+1\}$ such that

$$\alpha_u^{(1)} = a_{u,1}\alpha_1^{(2)} + a_{u,2}\alpha_2^{(2)} + \ldots + a_{u,n}\alpha_n^{(2)} + a_{u,n+1},$$

for all $u \in \{1,2,\ldots,n\}$.

Theorem 2. *Let k be an algebraically closed field of characteristic zero. If $f,g,h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2, g$ not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in PGL_2(k)$, and f has 3 zeros and poles altogether, then f is equivalent to one of the following rational functions*

(a) $\frac{(x-\alpha_1)^{k_1}(x+1/4-\alpha_1)^{2k_2}}{(x-1/4-\alpha_1)^{2k_1+2k_2}}$ *for some $\alpha_1 \in k$ and $k_1,k_2 \in \mathbb{Z}, k_1+k_2 \neq 0$,*

(b) $\frac{(x-\alpha_1)^{2k_1}(x+\alpha_1-2\alpha_2)^{2k_2}}{(x-\alpha_2)^{2k_1+2k_2}}$ *for some $\alpha_1,\alpha_2 \in k$ and $k_1,k_2 \in \mathbb{Z}, k_1+k_2 \neq 0$,*

(c) $\left(x-\frac{\alpha_1+\alpha_2}{2}\right)^{2k_1}(x-\alpha_1)^{k_2}(x-\alpha_2)^{k_2}$ *for some $\alpha_1,\alpha_2 \in k$ and $k_1,k_2 \in \mathbb{Z}, k_1+k_2 \neq 0$.*

Remark. The MAGMA procedure `CompRatFunc.m` can be downloaded from `http://shrek.unideb.hu/~tengely/CompRatFunc.m`. All systems in cases of $n \in \{3,4,5\}$ can be downloaded from `http://shrek.unideb.hu/~tengely/CFunc345.tar.gz`.

Remark. It is interesting to note that in the above formulas the zeros and poles form an arithmetic progression

$$\begin{aligned}
&\text{(a):}\quad \alpha_1-\frac{1}{4},\alpha_1,\alpha_1+\frac{1}{4}\quad \text{difference: } \frac{1}{4},\\
&\text{(b):}\quad \alpha_1,\alpha_2,-\alpha_1+2\alpha_2\quad \text{difference: } \alpha_2-\alpha_1,\\
&\text{(c):}\quad \alpha_1,\frac{\alpha_1+\alpha_2}{2},\alpha_2\quad \text{difference: } \frac{\alpha_2-\alpha_1}{2}.
\end{aligned}$$

2. Auxiliary results

We repeat some parts of the proof of Theorem A from [8] that will be used here later on. Without loss of generality we may assume that f and g are monic. Let

$$f(x) = \prod_{i=1}^{n}(x-\alpha_i)^{f_i}$$

with pairwise distinct $\alpha_i \in k$ and $f_i \in \mathbb{Z}$ for $i = 1,\ldots,n$. Similarly, let

$$g(x) = \prod_{j=1}^{t}(x-\beta_j)^{k_j}$$

with pairwise distinct $\beta_j \in k$ and $k_j \in \mathbb{Z}$ for $j = 1, \ldots, t$ and $t \in \mathbb{N}$. Hence we have

$$\prod_{i=1}^{n}(x-\alpha_i)^{f_i} = f(x) = g(h(x)) = \prod_{j=1}^{t}(h(x)-\beta_j)^{k_j}.$$

We shall write $h(x) = p(x)/q(x)$ with $p, q \in k[x], p, q$ coprime. Fuchs and Pethő [8] showed that if $k_1 + k_2 + \ldots + k_t \neq 0$, then there is a subset S_∞ of the set $\{1, \ldots, n\}$ for which

$$q(x) = \prod_{m \in S_\infty} (x-\alpha_m)^{l_m}$$

and there is a partition of the set $\{1, \ldots, n\} \setminus S_\infty$ in t disjoint non empty subsets $S_{\beta_1}, \ldots, S_{\beta_t}$ such that

$$h(x) = \beta_j + \frac{1}{q(x)} \prod_{m \in S_{\beta_j}} (x-\alpha_m)^{l_m}, \tag{1}$$

where $l_m \in \mathbb{N}$ satisfies $l_m k_j = f_m$ for $m \in S_{\beta_j}$, and this holds true for every $j = 1, \ldots, t$. We get at least two different representations of h, since we assumed that g is not of the special shape $(\lambda(x))^m$. Therefore we get at least one equation of the form

$$\beta_i + \frac{1}{q(x)} \prod_{r \in S_{\beta_i}} (x-\alpha_r)^{l_r} = \beta_j + \frac{1}{q(x)} \prod_{s \in S_{\beta_j}} (x-\alpha_s)^{l_s}. \tag{2}$$

If $k_1 + k_2 + \ldots + k_t = 0$, then we have

$$(p(x) - \beta_j q(x))^{k_j} = \prod_{m \in S_{\beta_j}} (x-\alpha_m)^{f_m}.$$

Now we have that $t \geq 3$, otherwise g is in the special form we excluded. Siegel's identity provides the equations in this case. That is if $1 \leq j_1 < j_2 < j_3 \leq t$, then we have

$$v_{j_1,j_2,j_3} + v_{j_3,j_1,j_2} + v_{j_2,j_3,j_1} = 0, \tag{3}$$

where

$$v_{j_1,j_2,j_3} = (\beta_{j_1} - \beta_{j_2}) \prod_{m \in S_{\beta_{j_3}}} (x-\alpha_m)^{l_m}.$$

3. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. If $k_1 + k_2 + \ldots + k_t \neq 0$ and there exist $r_1 \in S_{\beta_i}, s_1 \in S_{\beta_j}$ for some $i \neq j$ such that $l_{r_1} \neq 0$ and $l_{s_1} \neq 0$, then it follows from (2) that

$$\beta_i - \beta_j = \frac{\prod_{s\in S_{\beta_j}}(\alpha_{r_1}-\alpha_s)^{l_s}}{\prod_{m\in S_\infty}(\alpha_{r_1}-\alpha_m)^{l_m}}, \tag{4}$$

$$\beta_i - \beta_j = -\frac{\prod_{r\in S_{\beta_i}}(\alpha_{s_1}-\alpha_r)^{l_r}}{\prod_{m\in S_\infty}(\alpha_{s_1}-\alpha_m)^{l_m}} \tag{5}$$

for any appropriate $\alpha_{r_1} \in S_{\beta_i}$ and $\alpha_{s_1} \in S_{\beta_j}$. Hence we obtain that

$$C_1(i,j,r_1,s_1) = d^{\sum_{r\in S_{\beta_i}} l_r - \sum_{s\in S_{\beta_j}} l_s},$$

where $C_1(i,j,r_1,s_1) \in \mathbb{Q}$. If there exist S_{β_i} and S_{β_j} for which $\sum_{r\in S_{\beta_i}} l_r - \sum_{s\in S_{\beta_j}} l_s \neq 0$, then the possible values of d satisfy equations of the form $x^N = M$. Otherwise we get that

$$\sum_{r\in S_{\beta_i}} l_r = \sum_{s\in S_{\beta_j}} l_s, \quad i,j \in \{1,\ldots,t\}, i \neq j.$$

Let us consider the special case when $l_r = 0$ for all $r \in S_{\beta_i}$. If $l_s = 0$ for all $s \in S_{\beta_j}$, then we get that

$$h(x) = \beta_i + \frac{1}{q(x)} = \beta_j + \frac{1}{q(x)}.$$

Hence $\beta_i = \beta_j$ for some $i \neq j$, a contradiction. Thus we may assume that there exists $s_1 \in S_{\beta_j}$ for which $l_{s_1} \neq 0$. In a similar way as in the above case it follows that

$$\beta_i - \beta_j = \frac{\prod_{s\in S_{\beta_j}}(\alpha_{r_1}-\alpha_s)^{l_s}}{\prod_{m\in S_\infty}(\alpha_{r_1}-\alpha_m)^{l_m}} - \frac{1}{\prod_{m\in S_\infty}(\alpha_{r_1}-\alpha_m)^{l_m}}, \tag{6}$$

$$\beta_i - \beta_j = -\frac{1}{\prod_{m\in S_\infty}(\alpha_{s_1}-\alpha_m)^{l_m}}. \tag{7}$$

Therefore

$$d^{\sum_{s\in S_{\beta_j}} l_s} = C_2(i,j,r_1,s_1),$$

where $C_2(i,j,r_1,s_1) \in \mathbb{Q}$. Since $s_1 > 0$ we have that $\sum_{s\in S_{\beta_j}} l_s \neq 0$, that is d satisfies an appropriate polynomial equation.

If $k_1+k_2+\ldots+k_t = 0$, then there are at least 3 partitions and for any appropriate $r_1 \in S_{\beta_{j_1}}, r_2 \in S_{\beta_{j_2}}, r_3 \in S_{\beta_{j_3}}$ (that is $l_{r_i} \neq 0, i = 1,2,3$)

equation (3) implies that

$$\begin{aligned}(\beta_{j_3}-\beta_{j_1})\prod_{m_2\in S_{\beta_{j_2}}}(\alpha_{r_3}-\alpha_{m_2})^{l_{m_2}}+(\beta_{j_2}-\beta_{j_3})\prod_{m_1\in S_{\beta_{j_1}}}(\alpha_{r_3}-\alpha_{m_1})^{l_{m_1}}&=0\\(\beta_{j_1}-\beta_{j_2})\prod_{m_3\in S_{\beta_{j_3}}}(\alpha_{r_2}-\alpha_{m_3})^{l_{m_3}}+(\beta_{j_2}-\beta_{j_3})\prod_{m_1\in S_{\beta_{j_1}}}(\alpha_{r_2}-\alpha_{m_1})^{l_{m_1}}&=0\\(\beta_{j_1}-\beta_{j_2})\prod_{m_3\in S_{\beta_{j_3}}}(\alpha_{r_1}-\alpha_{m_3})^{l_{m_3}}+(\beta_{j_3}-\beta_{j_1})\prod_{m_2\in S_{\beta_{j_2}}}(\alpha_{r_1}-\alpha_{m_2})^{l_{m_2}}&=0,\end{aligned}$$

that is a system of linear equations in d_1, d_2, d_3, where $d_i = d^{\sum_{m_i\in S_{\beta_{j_i}}} l_{m_i}}, i \in \{1,2,3\}$ and the statement follows. In a very similar way we obtain a system of equations if $l_r = 0$ for all $r \in S_{\beta_{j_3}}$, the last two equations are as before, while on the left-hand side of the first one there is an additional term $\beta_{j_1} - \beta_{j_2}$. $\square$

Proof of Theorem 2. In [12] all cases are given with $k_1+k_2+\ldots+k_t = 0$ and also with $k_1+k_2+\ldots+k_t \neq 0, S_\infty \neq \emptyset$. Therefore it remains to deal with those cases with $k_1+k_2+\ldots+k_t \neq 0, S_\infty = \emptyset$. First let $t = 2$. There are 18 systems of equations. Among these systems there are two types. The first one has only a single equation, e.g. when $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3\}, (l_1, l_2, l_3) = (1,0,1)$, this equation is as follows

$$\alpha_1 - \alpha_3 - \beta_1 + \beta_2 = 0.$$

Hence

$$h(x) = \beta_1 + (x-\alpha_1) = \beta_2 + (x-\alpha_3)$$

is a linear function. A system from the second type is given by $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3\}, (l_1, l_2, l_3) = (1,1,2)$ and equations as follows

$$\begin{aligned}\alpha_1+\alpha_2-2\alpha_3&=0\\(\alpha_2-\alpha_3)^2-\beta_1+\beta_2&=0.\end{aligned}$$

That is we obtain that

$$\begin{aligned}h(x)&=\beta_2+\left(x-\frac{\alpha_1+\alpha_2}{2}\right)^2,\\g(x)&=\left(x-\beta_2-\left(\frac{\alpha_2-\alpha_1}{2}\right)^2\right)^{k_1}(x-\beta_2)^{k_2},\\f(x)&=\left(x-\frac{\alpha_1+\alpha_2}{2}\right)^{2k_1}(x-\alpha_1)^{k_2}(x-\alpha_2)^{k_2}.\end{aligned}$$

It is a decomposition of type (c) in the theorem. Let $t = 3$. There are 6 systems of equations, all of the same type, e.g. $S_{\beta_1} = \{1\}, S_{\beta_2} =$

$\{2\}, S_{\beta_3} = \{3\}, (l_1, l_2, l_3) = (1, 1, 1)$ and

$$\begin{aligned} \alpha_1 - \alpha_3 - \beta_1 + \beta_3 &= 0 \\ \alpha_2 - \alpha_3 - \beta_2 + \beta_3 &= 0. \end{aligned}$$

Hence the degree of h is 1, that yields a trivial decomposition. $\square$

4. Proof of Proposition 1

Proof of Proposition 1. In this section we apply Theorem 1 to determine composite rational functions having zeros and poles as consecutive elements of certain arithmetic progressions. We need to handle the following cases

$$\begin{aligned} (I) &: \quad n = 4 \text{ and } t \in \{2, 3, 4\}, k_1 + k_2 + \ldots + k_t \neq 0, S_\infty = \emptyset, \\ (II) &: \quad n = 4 \text{ and } t \in \{2, 3\}, k_1 + k_2 + \ldots + k_t \neq 0, S_\infty \neq \emptyset, \\ (III) &: \quad n = 4 \text{ and } t \in \{3, 4\}, k_1 + k_2 + \ldots + k_t = 0, S_\infty = \emptyset. \end{aligned}$$

In the proof we use the notation of Theorem 1, that is we write

$$\alpha_i = \alpha_0 + T_i d,$$

where $\alpha_0, d \in k$ and $\{T_1, T_2, T_3, T_4\} = \{0, 1, 2, 3\}$.

$\underline{(I) : t = 2, \{|S_{\beta_1}|, |S_{\beta_2}|\} = \{1, 3\}}.$ We may assume that $S_{\beta_1} = \{1\}, S_{\beta_2} = \{2, 3, 4\}$. We obtain that

$$\begin{aligned} h(x) &= \beta_1 + (x - \alpha_1)^{l_1}, \\ h(x) &= \beta_2 + (x - \alpha_2)^{l_2} (x - \alpha_3)^{l_3} (x - \alpha_4)^{l_4}. \end{aligned}$$

Substituting $x = \alpha_2, \alpha_3, \alpha_4$ yields (assuming $l_2 l_3 l_4 \neq 0$)

$$(\alpha_2 - \alpha_1)^{l_1} = (\alpha_3 - \alpha_1)^{l_1} = (\alpha_4 - \alpha_1)^{l_1}.$$

Since the zeros and poles form an arithmetic progression one gets that either $d = 0$ or $l_1 = 0$. In the former case the zeros and poles are not distinct, a contradiction. In the latter case the degree of h is less than 2, a contradiction as well. If two out of l_2, l_3, l_4 are equal to zero, then it follows that $l_1 = 1$, hence the degree of h is 1, a contradiction. If exactly one out of l_2, l_3, l_4 is zero, then $l_1 = 2$ and the corresponding f has only 3 zeros and poles. As an example we consider the case $l_4 = 0$. We obtain that

$$\alpha_1 = \frac{\alpha_2 + \alpha_3}{2} \quad \text{and} \quad \beta_2 = \beta_1 + \left(\frac{\alpha_2 - \alpha_3}{2} \right)^2.$$

It follows that $f(x) = \left(x - \frac{\alpha_2 + \alpha_3}{2}\right)^2 f_1(x)$, where $\deg f_1 = 2$.

$\underline{(I): t=2, \{|S_{\beta_1}|, |S_{\beta_2}|\} = \{2\}}$. Here we may assume that $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3,4\}$. We get that

$$\begin{aligned} h(x) &= \beta_1 + (x-\alpha_1)^{l_1}(x-\alpha_2)^{l_2}, \\ h(x) &= \beta_2 + (x-\alpha_3)^{l_3}(x-\alpha_4)^{l_4}. \end{aligned}$$

It follows that (assuming that $0 \notin \{l_1, l_2, l_3, l_4\}$)

$$(\alpha_1-\alpha_3)^{l_3}(\alpha_1-\alpha_4)^{l_4} = (\alpha_2-\alpha_3)^{l_3}(\alpha_2-\alpha_4)^{l_4}$$

and

$$(\alpha_3-\alpha_1)^{l_1}(\alpha_3-\alpha_2)^{l_2} = (\alpha_4-\alpha_1)^{l_1}(\alpha_4-\alpha_2)^{l_2}.$$

Using the fact that the zeros and poles form an arithmetic progression it turns out that one has to deal with 80 cases.

- There are 8 cases with $(l_1, l_2, l_3, l_4) = (1,1,1,1)$. We obtain equivalent solutions, so we only consider one of these. Let $\alpha_1 = \alpha_0, \alpha_2 = \alpha_0 + 3d$. It follows that $\beta_2 = \beta_1 - 2d^2$. That is we have

$$\begin{aligned} g(x) &= (x-\beta_1)(x-\beta_1+2d^2), \\ h(x) &= \beta_1 + (x-\alpha_0)(x-\alpha_0-3d), \\ f(x) &= (x-\alpha_0)(x-\alpha_0-d)(x-\alpha_0-2d)(x-\alpha_0-3d). \end{aligned}$$

- There are 16 equivalent cases with $(l_1, l_2, l_3, l_4) \in \{(1,1,2,2),(2,2,1,1)\}$. One obtains that $d^2 = \pm\frac{1}{2}$ and $\beta_2 = \beta_1 \pm 1$. One example from this family is given by

$$\begin{aligned} g(x) &= (x-\beta_1)(x-\beta_1-1), \\ h(x) &= \beta_1 + (x-\alpha_0-\sqrt{2}/2)^2(x-\alpha_0-\sqrt{2})^2, \\ f(x) &= (x-\alpha_0)\left(x-\alpha_0-\frac{\sqrt{2}}{2}\right)^2(x-\alpha_0-\sqrt{2})^2\left(x-\alpha_0-\frac{3\sqrt{2}}{2}\right)f_2(x), \end{aligned}$$

  where $f_2(x)$ is a quadratic polynomial such that f has more than 4 zeros and poles. We remark that if we use the equations related to β_2 we have

$$\begin{aligned} g(x) &= (x-\beta_2)(x-\beta_2+1), \\ h(x) &= \beta_2 + (x-\alpha_0)(x-\alpha_0-3\sqrt{2}), \\ f(x) &= (x-\alpha_0)\left(x-\alpha_0-\frac{\sqrt{2}}{2}\right)(x-\alpha_0-\sqrt{2})\left(x-\alpha_0-\frac{3\sqrt{2}}{2}\right), \end{aligned}$$

  that is we obtain a "solution" covered by the family given by the case $(l_1, l_2, l_3, l_4) = (1,1,1,1)$.

- There are 8 equivalent cases with $(l_1, l_2, l_3, l_4) = (2, 2, 2, 2)$. All of these cases can be eliminated in the same way. From the equation

$$(\alpha_1 - \alpha_3)^{l_3}(\alpha_1 - \alpha_4)^{l_4} = -(\alpha_3 - \alpha_1)^{l_1}(\alpha_3 - \alpha_2)^{l_2} \tag{8}$$

it follows that

$$d^{l_1+l_2-l_3-l_4} = \frac{(T_1 - T_3)^{l_3}(T_1 - T_4)^{l_4}}{-(T_3 - T_1)^{l_1}(T_3 - T_2)^{l_2}},$$

where $\{T_1, T_2, T_3, T_4\} = \{0, 1, 2, 3\}$. The left-hand side is $d^0 = 1$ and the right-hand side is -1, a contradiction.
- There are 16 equivalent cases with $(l_1, l_2, l_3, l_4) \in \{(1, 1, 3, 3), (3, 3, 1, 1)\}$. As an example we handle the one with $(l_1, l_2, l_3, l_4) = (3, 3, 1, 1)$ and

$$\begin{aligned}
\alpha_1 &= \alpha_0, \\
\alpha_2 &= \alpha_0 + 3d, \\
\alpha_3 &= \alpha_0 + 2d, \\
\alpha_4 &= \alpha_0 + d.
\end{aligned}$$

Equation (8) implies that either $d = 0$ or $d^4 = \frac{1}{4}$. If $d^2 = \frac{1}{2}$, then we get

$$\begin{aligned}
g(x) &= (x - \beta_1)(x - \beta_1 + 1), \\
h(x) &= \beta_1 + (x - \alpha_0)^3(x - \alpha_0 - 3\sqrt{2}/2)^3, \\
f(x) &= (x - \alpha_0)^3 \left(x - \alpha_0 - \frac{\sqrt{2}}{2}\right)(x - \alpha_0 - \sqrt{2})\left(x - \alpha_0 - \frac{3\sqrt{2}}{2}\right)^3 f_3(x),
\end{aligned}$$

where $f_3(x)$ is a quartic polynomial resulting an f having more than 4 zeros and poles. If $d^2 = -\frac{1}{2}$, then we get

$$\begin{aligned}
g(x) &= (x - \beta_1)(x - \beta_1 - 1), \\
h(x) &= \beta_1 + (x - \alpha_0)^3(x - \alpha_0 - 3\sqrt{-2}/2)^3, \\
f(x) &= (x - \alpha_0)^3 \left(x - \alpha_0 - \frac{\sqrt{-2}}{2}\right)(x - \alpha_0 - \sqrt{-2})\left(x - \alpha_0 - \frac{3\sqrt{-2}}{2}\right)^3 f_4(x),
\end{aligned}$$

where f_4 is a quartic polynomial and we get a contradiction in the same way as before.

- There are 16 equivalent cases with $(l_1, l_2, l_3, l_4) \in \{(2, 2, 3, 3), (3, 3, 2, 2)\}$. We handle the case with $(l_1, l_2, l_3, l_4) = (3, 3, 2, 2)$ and

$$\begin{aligned}
\alpha_1 &= \alpha_0 + 3d, \\
\alpha_2 &= \alpha_0, \\
\alpha_3 &= \alpha_0 + 2d, \\
\alpha_4 &= \alpha_0 + d.
\end{aligned}$$

It follows from equation (8) that $d = 0$ or $d^2 = \frac{1}{2}$. Also we have that $\beta_2 = \beta_1 - 1$. In a similar way as in the above cases we obtain a composite function f having 4 zeros and poles forming an arithmetic progression, but an additional quartic factor appears, a contradiction.

- There are 8 equivalent cases with $(l_1, l_2, l_3, l_4) = (3, 3, 3, 3)$. Here we consider the case with

$$\begin{aligned}
\alpha_1 &= \alpha_0, \\
\alpha_2 &= \alpha_0 + 3d, \\
\alpha_3 &= \alpha_0 + d, \\
\alpha_4 &= \alpha_0 + 2d.
\end{aligned}$$

It follows that $\beta_2 = \beta_1 - 8d^6$. As in the previous cases $g(h(x))$ has 4 zeros and poles coming from an arithmetic progression, but there is an additional quartic factor yielding a contradiction.

If $0 \in \{l_1, l_2, l_3, l_4\}$, then we have three possibilities. Either $\{l_1, l_2\} = \{l_3, l_4\} = \{0, 1\}$ or $\{l_1, l_2\} = \{1\}, \{l_3, l_4\} = \{0, 2\}$ or $\{l_1, l_2\} = \{0, 2\}, \{l_3, l_4\} = \{1\}$. In the first case the degree of h is 1, a contradiction. The last two cases can be handled in the same way, therefore we only deal with the case $\{l_1, l_2\} = \{1\}, \{l_3, l_4\} = \{0, 2\}$. Without loss of generality we may assume that $l_3 = 2, l_4 = 0$. It follows that $\alpha_1 = 2\alpha_3 - \alpha_2$ and $\beta_2 = \beta_1 - (\alpha_2 - \alpha_3)^2$. Thus

$$\begin{aligned}
h(x) &= \beta_1 + (x - 2\alpha_3 + \alpha_2)(x - \alpha_2), \\
g(x) &= (x - \beta_1)(x - \beta_1 + (\alpha_2 - \alpha_3)^2), \\
f(x) &= (x - \alpha_2)(x - \alpha_3)^2(x - 2\alpha_3 + \alpha_2).
\end{aligned}$$

We conclude that $f(x)$ has only 3 zeros and poles, a contradiction.

<u>$(I): t=3, |S_{\beta_1}|=|S_{\beta_2}|=1, |S_{\beta_3}|=2$.</u> Here we may assume that $S_{\beta_1}=\{1\}, S_{\beta_2}=\{2\}, S_{\beta_3}=\{3,4\}$, that is one has

$$\begin{aligned} h(x) &= \beta_1+(x-\alpha_1)^{l_1}, \\ h(x) &= \beta_2+(x-\alpha_2)^{l_2}, \\ h(x) &= \beta_3+(x-\alpha_3)^{l_3}(x-\alpha_4)^{l_4}, \end{aligned}$$

where $l_1, l_2 \in \{2,3\}$. Let us consider the case $l_3 \neq 0, l_4 \neq 0$. Substitute α_3, α_4 into the above system of equations to get

$$\begin{aligned} \beta_3 &= \beta_1+(\alpha_3-\alpha_1)^{l_1}, \\ \beta_3 &= \beta_2+(\alpha_3-\alpha_2)^{l_2}, \\ \beta_3 &= \beta_1+(\alpha_4-\alpha_1)^{l_1}, \\ \beta_3 &= \beta_2+(\alpha_4-\alpha_2)^{l_2}. \end{aligned}$$

These equations imply that $\alpha_i=\alpha_j$ for some $i \neq j$, a contradiction. Now assume that $l_4=0$, hence $l_3=2$ or 3. We can reduce the system as follows

$$\begin{aligned} (\alpha_1-\alpha_2)^{l_2}+(\alpha_2-\alpha_1)^{l_1} &= 0, \\ (\alpha_1-\alpha_3)^{l_3}+(\alpha_3-\alpha_1)^{l_1} &= 0, \\ (\alpha_2-\alpha_3)^{l_3}+(\alpha_3-\alpha_2)^{l_2} &= 0, \end{aligned}$$

where $l_1, l_2, l_3 \in \{2,3\}$. We get a contradiction in all these cases.
<u>$(I): t=4, S_{\beta_1}=\{1\}, S_{\beta_2}=\{2\}, S_{\beta_3}=\{3\}, S_{\beta_4}=\{4\}$.</u> We obtain the system of equations

$$\begin{aligned} h(x) &= \beta_1+(x-\alpha_1)^{l_1}, \\ h(x) &= \beta_2+(x-\alpha_2)^{l_2}, \\ h(x) &= \beta_3+(x-\alpha_3)^{l_3}, \\ h(x) &= \beta_4+(x-\alpha_4)^{l_4}, \end{aligned}$$

where $l_i \geq 2$ (since $\deg h \geq 2$.) Here we prove that this type of composite rational function cannot exist. One has that for any different i, j

$$(\alpha_i-\alpha_j)^{l_j-l_i}=(-1)^{l_i+1}.$$

If $l_i=l_j=2$, then we have a contradiction. Assume that $l_i=2$. There exist $l_j=l_k=3$. Hence $\alpha_i=\alpha_j-1$ and $\alpha_i=\alpha_k-1$, a contradiction. Let us deal with the case $(l_1,l_2,l_3,l_4)=(3,3,3,3)$. Substituting $\alpha_1+\alpha_2$ into the system of equations yields $\beta_1 = \beta_2 + \alpha_1^3 - \alpha_2^3$. We also have that $\beta_1 = \beta_2 + (\alpha_1 - \alpha_2)^3$. By combining these equations we get that

$$-3\alpha_1\alpha_2(\alpha_1 - \alpha_2) = 0.$$

In a similar way we obtain

$$-3\alpha_3\alpha_4(\alpha_3 - \alpha_4) = 0.$$

It follows that for some different i, j one has $\alpha_i = \alpha_j$, a contradiction.

$\underline{(II) : t = 2, |S_\infty| = 2, |S_{\beta_1}| = |S_{\beta_2}| = 1.}$ We may assume that $S_\infty = \{1, 2\}, S_{\beta_1} = \{3\}, S_{\beta_2} = \{4\}$. The system of equations in this case is as follows

$$\begin{aligned} h(x) &= \beta_1 + \frac{(x - \alpha_3)^{l_3}}{(x - \alpha_1)^{l_1}(x - \alpha_2)^{l_2}}, \\ h(x) &= \beta_2 + \frac{(x - \alpha_4)^{l_4}}{(x - \alpha_1)^{l_1}(x - \alpha_2)^{l_2}}. \end{aligned}$$

If $l_3 = l_4 = 0$, then it follows that $\beta_1 = \beta_2$, a contradiction. Let us deal with the case $l_3 = 0, l_4 \neq 0$ (in a similar way one can handle the case $l_3 \neq 0, l_4 = 0$). There are only three systems to consider. If $(l_1, l_2, l_3, l_4) = (0, 1, 0, 1)$ or $(1, 0, 0, 1)$, then $\beta_1 - 1 = \beta_2$ and the composite function f has only 2 zeros and poles, a contradiction. If $(l_1, l_2, l_3, l_4) = (1, 1, 0, 2)$, then $\beta_1 - 1 = \beta_2$ and $\alpha_4 = \alpha_2 \pm 1, \alpha_1 = \alpha_2 \pm 2$. In all these cases we obtain a composite function f having only 3 zeros and poles, a contradiction. Let us consider the cases with $l_3 \neq 0, l_4 \neq 0$. There are 18 systems to deal with. It turns out that d satisfies the equation

$$d^{l_4 - l_3} = -\frac{(T_4 - T_3)^{l_3}(T_3 - T_1)^{l_1}(T_3 - T_2)^{l_2}}{(T_4 - T_1)^{l_1}(T_4 - T_2)^{l_2}(T_3 - T_4)^{l_4}},$$

where $\alpha_i = \alpha_0 + T_i d$ for some $T_i \in \{0, 1, 2, 3\}$. If $(l_1, l_2, l_3, l_4) = (1, 0, 2, 2)$, then

$$(T_1, T_2, T_3, T_4) \in \{(1, 3, 0, 2), (1, 3, 2, 0), (2, 0, 1, 3), (2, 0, 3, 1)\}.$$

In all these cases we obtain a composite function f having only 3 zeros and poles, a contradiction. As an example we compute f when $(T_1, T_2, T_3, T_4) = (1, 3, 0, 2)$. We get that $\beta_2 = \beta_1 + 4d$

and

$$\begin{aligned} h(x) &= \beta_1 + \frac{(x-\alpha_0)^2}{(x-\alpha_0-d)}, \\ g(x) &= (x-\beta_1)(x-\beta_1-4d), \\ f(x) &= \frac{(x-\alpha_0-2d)^2(x-\alpha_0)^2}{(x-\alpha_0-d)^2}. \end{aligned}$$

We exclude the tuple $(l_1, l_2, l_3, l_4) = (0, 1, 2, 2)$ following the same lines. If $(l_1, l_2, l_3, l_4) = (1, 1, 1, 2)$, then we also have that $d = \frac{1}{T_1+T_2-2T_4}$ and $d = \frac{T_2-T_3}{(T_2-T_4)^2}$, it is easy to check that such tuple (T_1, T_2, T_3, T_4) does not exist. In a very similar way if $(l_1, l_2, l_3, l_4) = (1, 1, 2, 1)$ we obtain that

$$d = \frac{1}{T_1 + T_2 - 2T_3} = \frac{T_2 - T_4}{(T_2 - T_3)^2}$$

and such tuple (T_1, T_2, T_3, T_4) does not exist. If $(l_1, l_2, l_3, l_4) = (2, 1, 2, 3)$, then

$$\begin{aligned} \frac{(T_3-T_4)^3}{(T_3-T_1)^2(T_3-T_2)} &= 1, \\ -\frac{(T_4-T_3)^2}{(T_4-T_1)^2(T_4-T_2)} &= \frac{4}{27(T_3-T_4)}. \end{aligned}$$

There is no solution in $T_i \in \{0, 1, 2, 3\}, T_i \neq T_j, i \neq j$. We obtain a very similar system of equations in case of $(l_1, l_2, l_3, l_4) = (1, 2, 3, 2), (1, 2, 2, 3), (2, 1, 3, 2)$. If $(l_1, l_2, l_3, l_4) = (1, 1, 3, 3)$, then we get

$$\begin{aligned} T_1 + T_2 &= T_3 + T_4, \\ (T_4-T_1)(T_4-T_2) &= (T_3-T_1)(T_3-T_2), \\ 27(T_2-T_4)^4(T_4-T_1)^2 &= 9(T_4-T_3)^3(T_2-T_4)^2(T_4-T_1) - (T_4-T_3)^6. \end{aligned}$$

The above system has no solution in (T_1, T_2, T_3, T_4). If $(l_1, l_2, l_3, l_4) = (1, 2, 3, 1)$, then

$$\begin{aligned} T_1 - 4T_3 + 3T_4 &= 0, \\ 2T_2 + T_3 - 3T_4 &= 0, \\ (T_4-T_3)^3 &= (T_4-T_1)(T_4-T_2)^2. \end{aligned}$$

The system has no solution. The same argument works in case of $(l_1, l_2, l_3, l_4) = (1, 2, 1, 3), (2, 1, 1, 3), (2, 1, 3, 1)$. If $(l_1, l_2, l_3, l_4) =$

$(0,2,2,1)$, then we have

$$\begin{aligned} \alpha_2 &= \alpha_4 + \frac{1}{4}, \\ \alpha_3 &= \alpha_4 - \frac{1}{4}, \end{aligned}$$

hence

$$\begin{aligned} h(x) &= \beta_1 + \frac{(x-\alpha_4+\frac{1}{4})^2}{(x-\alpha_4-\frac{1}{4})^2}, \\ g(x) &= (x-\beta_1)(x-\beta_1-1), \\ f(x) &= \frac{(x-\alpha_4)(x-\alpha_4+\frac{1}{4})^2}{(x-\alpha_4-\frac{1}{4})^4}. \end{aligned}$$

That is f has only 3 zeros and poles, a contradiction. We handle in the same way the tuples $(l_1,l_2,l_3,l_4) = (2,0,2,1), (2,0,1,2), (0,2,1,2)$. If $(l_1,l_2,l_3,l_4) = (0,0,1,1)$, then $\deg h(x) = 1$, a contradiction.

$\underline{(II): t = 3, |S_\infty| = |S_{\beta_1}| = |S_{\beta_2}| = |S_{\beta_3}| = 1.}$ We may assume that $S_\infty = \{1\}, S_{\beta_1} = \{2\}, S_{\beta_2} = \{3\}, S_{\beta_3} = \{4\}$. In this case $h(x)$ can be written as follows

$$\begin{aligned} h(x) &= \beta_1 + \frac{(x-\alpha_2)^{l_2}}{(x-\alpha_1)^{l_1}}, \\ h(x) &= \beta_2 + \frac{(x-\alpha_3)^{l_3}}{(x-\alpha_1)^{l_1}}, \\ h(x) &= \beta_3 + \frac{(x-\alpha_4)^{l_4}}{(x-\alpha_1)^{l_1}}. \end{aligned}$$

The only possible exponent tuple (l_1,l_2,l_3,l_4) is $(0,1,1,1)$. Thus $\deg h(x) = 1$, a contradiction.

$\underline{(III): t = 3, |S_{\beta_1}| = 2, |S_{\beta_2}| = |S_{\beta_3}| = 1.}$ We may assume that $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3\}, S_{\beta_3} = \{4\}$. The only exponent tuple for which $\deg h(x) > 1$ is given by (l_1,l_2,l_3,l_4) is $(1,1,2,2)$. We obtain the following system of equations if $d \neq 0$:

$$\begin{aligned} (\beta_3-\beta_1)(T_4-T_3)^2 + (\beta_2-\beta_3)(T_4-T_1)(T_4-T_2) &= 0 \\ (\beta_1-\beta_2)(T_3-T_4)^2 + (\beta_2-\beta_3)(T_3-T_1)(T_3-T_2) &= 0 \\ (\beta_1-\beta_2)(T_1-T_4)^2 + (\beta_3-\beta_1)(T_1-T_3)^2 &= 0 \\ (\beta_1-\beta_2)(T_2-T_4)^2 + (\beta_3-\beta_1)(T_2-T_3)^2 &= 0, \end{aligned}$$

where $\{T_1,T_2,T_3,T_4\} = \{0,1,2,3\}$. Solving the above system of equations for all possible tuples (T_1,T_2,T_3,T_4) one gets that $\beta_i = \beta_j$ for some $i \neq j$, a contradiction.

$\underline{(III): t=3, |S_{\beta_1}| = |S_{\beta_2}| = |S_{\beta_3}| = |S_{\beta_4}| = 1}$. We may assume that $S_{\beta_1} = \{1\}, S_{\beta_2} = \{2\}, S_{\beta_3} = \{3\}, S_{\beta_4} = \{4\}$. The only possible exponent tuple is $(l_1, l_2, l_3, l_4) = (1,1,1,1)$. Thus the corresponding $h(x)$ has degree 1, a contradiction. As an example we consider the case

$$\begin{aligned} \alpha_1 &= \alpha_0 + d, \\ \alpha_2 &= \alpha_0, \\ \alpha_3 &= \alpha_0 + 3d, \\ \alpha_4 &= \alpha_0 + 2d. \end{aligned}$$

We use equation (3) here with $(j_1, j_2, j_3) = (1,2,3)$ and $(j_1, j_2, j_3) = (1,2,4)$. If $d \neq 0$, then we have

$$\begin{aligned} \beta_3 &= 3\beta_1 - 2\beta_2, \\ \beta_4 &= 2\beta_1 - \beta_2. \end{aligned}$$

Let $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ such that $k_1 + k_2 + k_3 + k_4 = 0$. Theorem A implies that

$$\begin{aligned} g(x) &= (x-\beta_1)^{k_1}(x-\beta_2)^{k_2}(x-3\beta_1+2\beta_2)^{k_3}(x-2\beta_1+\beta_2)^{k_4}, \\ h(x) &= \frac{1}{d}(\beta_1(x-\alpha_0) - \beta_2(x-\alpha_0-d)), \\ f(x) &= (x-\alpha_0-d)^{k_1}(x-\alpha_0)^{k_2}(x-\alpha_0-3d)^{k_3}(x-\alpha_0-2d)^{k_4}. \end{aligned}$$

□

5. Cases with $n = 4$

In this section we provide some details of the computation corresponding to cases with $n = 4, t \in \{2,3,4\}, k_1 + k_2 + \ldots + k_t \neq 0, S_\infty = \emptyset$. These are the cases which are not mentioned in Section 5 in [12].

$\underline{\text{The case } n = 4, t = 2 \text{ and } S_\infty = \emptyset}$. There are 134 systems to deal with. We treat only a few representative examples.

If $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3,4\}$ and $(l_1, l_2, l_3, l_4) = (2,1,2,1)$, then we have

$$\begin{aligned} \alpha_1 + 1/2\alpha_2 - \alpha_3 - 1/2\alpha_4 &= 0 \\ \alpha_2 - 4/3\alpha_3 + 1/3\alpha_4 &= 0 \\ \alpha_2\alpha_3^2 - 2\alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_4^2 - \alpha_3^2\alpha_4 + 2\alpha_3\alpha_4^2 - \alpha_4^3 - 9\beta_1 + 9\beta_2 &= 0 \\ \alpha_2 - 4/3\alpha_3 + 1/3\alpha_4 &= 0 \\ \alpha_3^3 - 3\alpha_3^2\alpha_4 + 3\alpha_3\alpha_4^2 - \alpha_4^3 - 27/4\beta_1 + 27/4\beta_2 &= 0. \end{aligned}$$

The corresponding rational functions are as follows

$$\begin{aligned} f(x) &= (x-\alpha_1)^{2k_1}(x-\alpha_2)^{k_1}(x-\frac{1}{3}\alpha_1-\frac{2}{3}\alpha_2)^{2k_2}(x-\frac{4}{3}\alpha_1+\frac{1}{3}\alpha_2)^{k_2}, \\ g(x) &= (x-\beta_1)^{k_1}(x-\beta_1-\frac{4}{27}(\alpha_1-\alpha_2)^3)^{k_2} \\ h(x) &= \beta_1+(x-\alpha_1)^2(x-\alpha_2), \end{aligned}$$

where $k_1+k_2\neq 0$. We note that the zeros and poles of f do not form an arithmetic progression for all values of the parameters as the choice $\alpha_1=0, \alpha_2=3$ shows.

If $S_{\beta_1}=\{1,2\}, S_{\beta_2}=\{3,4\}$ and $(l_1,l_2,l_3,l_4)=(1,1,0,2)$, then we get the system of equations

$$\begin{aligned} \alpha_1+\alpha_2-2\alpha_4 &= 0 \\ (\alpha_2-\alpha_4)^2-\beta_1+\beta_2 &= 0. \end{aligned}$$

It yields a decomposable rational function f having only 3 zeros and poles altogether.

If $S_{\beta_1}=\{1,2\}, S_{\beta_2}=\{3,4\}$ and $(l_1,l_2,l_3,l_4)=(1,1,1,1)$, then we obtain

$$\begin{aligned} \alpha_1+\alpha_2-\alpha_3-\alpha_4 &= 0 \\ \alpha_2^2-\alpha_2\alpha_3-\alpha_2\alpha_4+\alpha_3\alpha_4-\beta_1+\beta_2 &= 0. \end{aligned}$$

It yields the following solution

$$\begin{aligned} f(x) &= (x+\alpha_2-\alpha_3-\alpha_4)^{k_1}(x-\alpha_2)^{k_1}(x-\alpha_3)^{k_2}(x-\alpha_4)^{k_2}, \\ g(x) &= (x-\beta_1)^{k_1}(x-\beta_1+\alpha_2^2-\alpha_2\alpha_3-\alpha_2\alpha_4+\alpha_3\alpha_4)^{k_2} \\ h(x) &= \beta_1+(x-\alpha_3-\alpha_4+\alpha_2)(x-\alpha_2), \end{aligned}$$

where $k_1+k_2\neq 0$.

If $S_{\beta_1}=\{1,2,3\}, S_{\beta_2}=\{4\}$ and $(l_1,l_2,l_3,l_4)=(1,1,1,3)$, then we have

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3-3\alpha_4 &= 0 \\ \alpha_2^2+\alpha_2\alpha_3-3\alpha_2\alpha_4+\alpha_3^2-3\alpha_3\alpha_4+3\alpha_4^2 &= 0 \\ \alpha_3^3-3\alpha_3^2\alpha_4+3\alpha_3\alpha_4^2-\alpha_4^3-\beta_1+\beta_2 &= 0. \end{aligned}$$

We obtain the following rational functions

$$\begin{aligned} f(x) &= (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_1}(x-\alpha_3)^{k_1}(x-\alpha_4)^{3k_2}, \\ g(x) &= (x-\beta_2-(\alpha_3-\alpha_4)^3)^{k_1}(x-\beta_2)^{k_2} \\ h(x) &= \beta_2+(x-\alpha_4)^3, \end{aligned}$$

where $k_1 + k_2 \neq 0$ and

$$\begin{aligned}\alpha_1 &= \frac{1}{2}\alpha_4\Big(-i\sqrt{3}+3\Big) - \frac{1}{2}\alpha_3\Big(-i\sqrt{3}+1\Big)\\ \alpha_2 &= \frac{1}{2}\alpha_4\Big(i\sqrt{3}+3\Big) + \frac{1}{2}\alpha_3\Big(-i\sqrt{3}-1\Big).\end{aligned}$$

The case $n = 4, t = 3$ and $S_\infty = \emptyset$. There are 48 systems to handle in this case. We consider one of these. Let $S_{\beta_1} = \{1\}, S_{\beta_2} = \{2, 3\}, S_{\beta_3} = \{4\}$ and $(l_1, l_2, l_3, l_4) = (1, 1, 0, 1)$. We obtain the system of equations

$$\begin{aligned}\alpha_1 - \alpha_4 - \beta_1 + \beta_3 &= 0\\ \alpha_2 - \alpha_4 - \beta_2 + \beta_3 &= 0.\end{aligned}$$

It follows that h is a linear function, which only provides trivial decomposition. In the remaining cases we have the same conclusion.

The case $n = 4, t = 4$ and $S_\infty = \emptyset$. Here we get 24 systems to consider. In all cases we have that

$$\{S_{\beta_1}, S_{\beta_2}, S_{\beta_3}, S_{\beta_4}\} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$$

and $(l_1, l_2, l_3, l_4) = (1, 1, 1, 1)$. Therefore h is linear, a contradiction.

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