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Interfaces with Other Disciplines

On the Shapley value of liability games[☆]

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ABSTRACT

In a liability problem, the asset value of an insolvent firm must be distributed among the creditors and the firm itself, when the firm has some freedom in negotiating with the creditors. We model the negotiations using cooperative game theory and analyze the Shapley value to resolve such liability problems. We establish three main monotonicity properties of the Shapley value. First, creditors can only benefit from the increase in their claims or of the asset value. Second, the firm can only benefit from the increase of a claim but can end up with more or with less if the asset value increases, depending on the configuration of small and large liabilities. Third, creditors with larger claims benefit more from the increase of the asset value. Even though liability games are constant-sum games and we show that the Shapley value can be calculated directly from a liability problem, we prove that calculating the Shapley payoff to the firm is NP-hard.

1. Introduction

An insolvent firm (country, state, individual, etc.) with some asset value has liabilities towards a group of creditors. Compared to standard bankruptcy games as studied in the game-theoretical literature (see O'Neill, 1982 for a seminal contribution and Thomson, 2013; Thomson, 2015 for recent surveys) Csóka & Herings (2019) introduced liability problems, by modeling the firm as an explicit player. A liability problem is given by the asset value of the firm to be allocated and the claims of the creditors.

Instead of directly using the values given in a liability problem, Csóka & Herings (2019) defined liability games to indirectly allocate the asset value using a solution concept from cooperative game theory with transferable utility. The worth of a coalition in a liability game is defined as follows. Given a coalition and its complement, the firm first makes payments to the coalition it belongs to, up to the value of the liabilities in the firm's coalition and the asset value of the firm, and then (if possible) pays to the complementary coalition. They remarked that liability games are superadditive: there is no loss of merging disjoint coalitions. Moreover, they proved that the core of a liability game is empty and analyzed one of the two most popular solution concepts, the nucleolus (Schmeidler, 1969).

In this paper, we investigate the Shapley value (Shapley, 1953) of liability games.[1] The numerous applications of the Shapley value include aircraft landing fees (Dubey, 1982; Littlechild & Owen, 1973), minimal cost spanning trees (Bergantinos & Lorenzo-Freire, 2008), a combinatorial structure called augmenting system (Bilbao & Ordóñez, 2009), directed graph games (Khmelnitskaya, Selçuk, & Talman, 2016), risk capital allocation (Balog, Bátyi, Csóka, & Pintér, 2017), and environmental costs in supply chains (Ciardiello, Genovese, & Simpson, 2018), among others.

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[1] We also assume transferable utility (TU), assuming that money has the same utility for all the players and utility functions are linear and separable in it. We believe that the TU assumption is a good approximation in many applications. However, we also note possible generalizations towards nontransferable utility (NTU). For the first formal NTU bankruptcy game see Orshan, Valenciano, & Zarzuelo (2003), for recent advances see, for instance, Dietzenbacher, Borm, & Estévez-Fernández (2020), Dietzenbacher & Peters (2020), Estévez-Fernández, Borm, & Fiestras-Janeiro (2020).

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We show that the Shapley value can also be used as a liability allocation, that is, it allocates the asset value non-negatively among the creditors and the firm in such a way that no creditor gets more than his liability. We establish lower and upper bounds for the Shapley payments. Moreover, we show that (i) creditors can only benefit from the increase in their claims or of the asset value; (ii) the firm can only benefit from the increase of a claim but can end up with more or with less if the asset value increases, depending on the configuration of small and large liabilities; (iii) creditors with larger claims benefit more from the increase of the asset value. In most cases, we even establish sharp upper bounds for the changes in the payments.

It is easy to check that in liability games, for one or two creditors (that is, for two or three players), the Shapley value coincides with the nucleolus. However, for three or more creditors, they give different payoffs in generic examples. Csóka & Herings (2019) showed that at the nucleolus of a liability game, the firm gets a positive payment, which is at most half of the asset value. We show that at the Shapley value, there are cases when the firm can keep almost the whole asset value. Csóka & Herings (2019) also showed that at the nucleolus, creditors with higher liabilities receive higher payments, but they also get higher debt forgiveness (defined as the difference between the liability and the received payments), a result we also have for the Shapley value. They also provided conditions under which the nucleolus coincides with a generalized proportional rule, where the firm gets a positive amount, and the rest is allocated in proportional to the liabilities.

Csóka & Herings (2019) noted that in a liability game, the worth of a coalition plus the worth of the complementary coalition is always equal to the asset value, that is, a liability game is a constant-sum game (Von Neumann & Morgenstern, 1944). Originally, Von Neumann & Morgenstern (1944) analyzed strategic non-cooperative games, where a coalition and the complementary coalition play a constant-sum game. They discussed constant-sum simple games with winning or losing coalitions, where the worth of any coalition can be either zero or one. A prominent application is (weighted) majority voting games, where the worth of the grand coalition is one, and if a coalition is winning, then its complementary coalition is losing. Constant-sum games also play a role in games modeling Bitcoin mining pools (Lewenberg, Bachrach, Sompolinsky, Zohar, & Rosenschein, 2015). For a recent generalization to alpha-constant-sum games, see Wang, van den Brink, Sun, Xu, & Zou (2019). A related new concept is called games of threats (Kohlberg & Neyman, 2018), where the constant sum is zero, but the value of the empty coalition is not always zero. For more details on the value theory of strategic games, see Cai, Candogan, Daskalakis, & Papadimitriou (2016).

Since constant-sum games are exciting on their own, we first study the Shapley value for constant-sum games in general. We propose a basis for the linear vector space of constant-sum games that provides a specialized formula for the Shapley payoff to a player in a constant-sum game. It turns out that some of those general results are very handy for liability games. We obtain a simple computational scheme by which the Shapley value of a liability game is derived directly from the liability problem, that is, from the asset value and the liabilities.

In general, computing the Shapley value based on its definition is practically impossible for large games. Computing the Shapley value in weighted majority games is #P-complete (Deng & Papadimitriou, 1994) and one has to rely on its estimation. Estimation techniques were introduced by Castro, Gómez, & Tejada (2009) and Castro, Gómez, Molina, & Tejada (2017). However, for special classes of games, the Shapley value can be calculated in a polynomial manner (Castro, Gómez, & Tejada, 2008; Granot, Kuipers, & Chopra, 2002; Megiddo, 1978). We show that in liability games, calculating the Shapley value of the insolvent firm is NP-hard. Thus even though the Shapley value can be calculated directly from the liability problem, its application to large liability problems could become computationally laborious.

The paper is organized as follows. In Section 2, we consider general constant-sum games. In Section 3, we introduce liability games, show that the Shapley value can be used as a liability allocation. In Section 4, we prove various properties of the Shapley liability allocation rule. Section 5, we show that calculating the Shapley value of the firm is NP-hard. Section 6 contains concluding remarks and possibilities for further research.

2. The Shapley value of constant-sum games

A *transferable utility cooperative game* (N, v) is a pair where N is a non-empty, finite set of *players* and $v : 2^N \to \mathbb{R}$ is a *coalitional function* satisfying $v(\emptyset) = 0$. The number $v(S)$ is regarded as the worth of the coalition $S \subseteq N$. We identify the game with its coalitional function since the player set N is fixed throughout the paper. The game (N, v) is called *0-normalized* if $v(\{i\}) = 0$ for every $i \in N$; *superadditive* if $S \cap T = \emptyset$ implies $v(S) + v(T) \leq v(S \cup T)$ for every two coalitions $S, T \subseteq N$. The game (N, v) is *constant-sum* if $v(S) + v(N \setminus S) = v(N)$ for every coalition $S \subseteq N$.

Given a game (N, v), a *payoff allocation* $x \in \mathbb{R}^N$ represents the payoffs to the players. The total payoff to coalition $S \subseteq N$ is denoted by $x(S) = \sum_{i \in S} x_i$ if $S \neq \emptyset$ and $x(\emptyset) = 0$. In a game v, we say the payoff allocation x is *efficient*, if $x(N) = v(N)$; *individually rational*, if $x_i = x(\{i\}) \geq v(\{i\})$ for all $i \in N$; *coalitionally rational*, if $x(S) \geq v(S)$ for all $S \subseteq N$. The set of *preimputations*, $I^*(v)$, consists of the efficient payoff vectors, the set of *imputations*, $I(v)$, consists of the individually rational preimputations, and the *core*, $C(v)$, is the set of coalitionally rational (pre)imputations. We call a game *balanced* if its core is non-empty.

We denote the set of all cooperative games on a fixed player set N by $\mathcal{G}^N$. It is well-known that $\mathcal{G}^N$ is a linear vector space of dimension $2^n - 1$ where $n = |N|$. A *value* on $\mathcal{G}^N$ is a map $f : \mathcal{G}^N \to \mathbb{R}^N$, which assigns to every game v on N a vector $f(v)$ with components $f_i(v)$ for all $i \in N$. We say that value f satisfies

- *linearity*: if $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$ holds for all $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathcal{G}^N$.
- *efficiency*: if $\sum_{j \in N} f_j(v) = v(N)$ holds for all $v \in \mathcal{G}^N$.
- *the equal treatment property*: if $j, k \in N$ are *symmetric* players in game $v \in \mathcal{G}^N$, that is if $v(S \cup j) = v(S \cup k)$ $\forall S \subseteq N \setminus \{j, k\}$, then $f_j(v) = f_k(v)$.
- *the null player property*: if $j \in N$ is a *null* player in game $v \in \mathcal{G}^N$, that is if $v(S \cup j) - v(S) = 0$ $\forall S \subseteq N \setminus j$, then $f_j(v) = 0$.

The best known and most frequently used value for general coalitional games was introduced and characterized by a few appealing properties by Lloyd Shapley.

Theorem 1 (Shapley, 1953). *The value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ defined by*

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \gamma_N(S)[v(S \cup i) - v(S)] \qquad (i \in N) \tag{1}$$

where $\gamma_N(S) = \dfrac{s!(n-1-s)!}{n!} = \dfrac{1}{n\binom{n-1}{s}}$ *and* $s = |S|, n = |N|$, *is the only value on $\mathcal{G}^N$ that satisfies linearity, efficiency, the equal treatment property, and the null player property.*

The Shapley value can also be axiomatized using various alternative sets of axioms. A prominent one is due to Young (1985) who replaced linearity and the null player property with *marginality* (which requires that the value of a player i depend only on the player's marginal contributions $v(S \cup i) - v(S)$ in a game v). Pintér (2015) proved that Young's axiomatization also holds for various special classes of games.

For constant-sum games, Khmelnitskaya (2003) characterized the Shapley value with the set of axioms used by Young (1985). In their illuminating paper, Kohlberg & Neyman (2018) (Corollary 2) showed that the Shapley value can also be characterized with Shapley's original set of axioms. For a common generalization of constant-sum coalitional games and games of threats we refer to the paper by Wang et al. (2019) where the aforementioned axiomatizations of the Shapley value are extended to alpha-constant-sum games.

Although in his seminal paper Shapley (1953) has not discussed the axiomatization of his value for special classes of games, he derived a specialized formula for constant-sum games. For the sake of completeness, we also present the short proof. Let $\mathcal{G}^N_{CS}$ denote the set of all constant-sum games on fixed player set N.

Proposition 2 (Shapley, 1953). *The Shapley value of constant-sum game $v \in \mathcal{G}^N_{CS}$ is*

$$\phi_i(v) = -v(N) + 2\sum_{S\subseteq N\setminus i} \gamma_N(S) v(S\cup i) \qquad (i \in N). \tag{2}$$

Proof. Let v be a constant-sum game and $i \in N$ be fixed. For $S \subseteq N\setminus i$, we have $v(S) = v(N) - v(N\setminus S) = v(N) - v((N\setminus i \setminus S)\cup i)$. If we substitute this in the general formula (1), we get $\phi_i(v) = \sum_{S\subseteq N\setminus i} \gamma_N(S)[v(S\cup i) + v((N\setminus i\setminus S)\cup i) - v(N)]$. Since $\gamma_n(s) = \frac{1}{n\binom{n-1}{s}} = \frac{1}{n\binom{n-1}{n-1-s}} = \gamma_n(n-1-s)$ and $N\setminus i\setminus S \subseteq N\setminus i$, each coalition value of type $v(T\cup i)$ for $T\subseteq N\setminus i$ appears twice and is weighted by the same coefficient in the sum. Taking out the constant term $-v(N)$ from the summation, we get (2). □

It is well-known that the weight coefficients $\{\gamma_N(S)\}_{S\subseteq N\setminus i}$ form a probability distribution, we call it the *Shapley distribution*, on the family $2^{N\setminus i}$ of coalitions which do not contain player *i*. Therefore, in general, $\phi_i(v)$ is the expected marginal contribution of player *i* in v to coalitions not containing *i*, when the random formation of such coalitions is described by the Shapley distribution. Notice that in constant-sum games, the Shapley payoff to a player depends only on the values of coalitions the player belongs to, no need to compute his marginal contributions. Since $\gamma_N(S)$ depends only on the cardinalities $n = |N|$ and $s = |S|$ of the two coalitions, we also write $\gamma_n(s)$ when more convenient.

Next, we investigate how the Shapley value of constant-sum games can be computed based on its linearity. Although our arguments would resemble the standard Shapley-type uniqueness proofs (cf. Shapley, 1953, or for constant-sum games, Kohlberg & Neyman (2018) and Wang et al., 2019), our aim is not to give another characterization but to find a basis which facilitates an "easy" decomposition of constant-sum games. Indeed, we work with a "trivial" basis in which the determination of the coefficients (Harsányi dividends) in the linear decomposition require no computation. This computational simplicity comes at the price of not being able to apply the null player property for our basic "trivial" constant-sum games.

It is easily seen that any linear combination of constant-sum games is also a constant-sum game. Thus $\mathcal{G}^N_{CS}$ is a linear subspace of $\mathcal{G}^N$. It is well-known that additive games are the only balanced constant-sum games, so the standard approach of decomposing a game as a linear combination of unanimity games, which are balanced games, cannot be followed for $\mathcal{G}^N_{CS}$. Only the additive unanimity games, that is, the dictator games $u_{\{i\}}$ $(i \in N)$, could be part of a basis for $\mathcal{G}^N_{CS}$, but they are sufficient to span only the n-dimensional linear subspace of $\mathcal{G}^N_{CS}$ consisting of the additive constant-sum games. On the other hand, the average of an unanimity game and its dual game is a constant-sum game (in which the players outside the carrier coalition are null players), and all the aforementioned characterization proofs (Khmelnitskaya, 2003; Kohlberg & Neyman, 2018; Wang et al., 2019) apply these basic constant-sum games.

Foreshadowing the application of the game-theoretic results in this section to a special type of constant-sum games induced by liability problems with an insolvent firm, we arbitrarily choose a player (the insolvent firm) and denote him by $0 \in N$. The set of the $n-1$ other players is denoted by $C = N\setminus\{0\}$. Given this fixed "highlighted" player, the family of all coalitions is decomposed in two parts of equal size: the 2^{n-1} "partner" coalitions containing 0 and the 2^{n-1} "complement" coalitions. Let $\mathcal{P}_0 = \{S\subseteq N : 0 \in S\}$ denote the family of partner coalitions of 0, and $\mathcal{C}_0 = \{S \subseteq N : 0 \notin S\}$ denote the family of coalitions not containing 0. Obviously, $S \in \mathcal{P}_0$ if and only if $N\setminus S \in \mathcal{C}_0$. In particular, $N \in \mathcal{P}_0$ and $\emptyset \in \mathcal{C}_0$, also $\{0\} \in \mathcal{P}_0$ and $C \in \mathcal{C}_0$.

In a constant-sum game $v \in \mathcal{G}^N_{CS}$, we have $v(N\setminus S) = v(N) - v(S)$ for all $S \in \mathcal{P}_0$, thus the values of the partner coalitions $v(S)$ $(S \in \mathcal{P}_0)$ suffice to fully determine v. It follows that the dimension of $\mathcal{G}^N_{CS}$ is at most $2^{n-1} = |\mathcal{P}_0|$. Next, we show that, in fact, equality holds. We present 2^{n-1} linearly independent "elementary" constant-sum games, which form a very "convenient" basis of $\mathcal{G}^N_{CS}$, inasmuch the scalar coefficients in the (unique) linear decompositions are simply the coalitional values.

We define for $0 \in R \subsetneq N$ the constant-sum game $d^R \in \mathcal{G}^N_{CS}$ for all $S \subseteq N$ by

$$d^R(S) = \begin{cases} 1, & \text{if } S = R, \\ -1, & \text{if } S = N\setminus R, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

For $R = N$, the constant-sum game $d^N \in \mathcal{G}^N_{CS}$ is defined for all $S \subseteq N$ as

$$d^N(S) = \begin{cases} 1, & \text{if } S = N \text{ or } 0 \notin S \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

It is easily checked that $d^R(\emptyset) = 0$ and d^R is indeed constant-sum for all $R \in \mathcal{P}_0$. Moreover, $d^N(S) = 1$ but $d^R(S) = 0$ for all $S \neq R \in \mathcal{P}_0$. Notice that for all $R, S \in \mathcal{P}_0$, we have $d^R(S) = 1$ if and only if $R = S$, but $d^R(S) = 0$ otherwise. It follows that the $2^{n-1} = |\mathcal{P}_0|$ games d^R $(R \in \mathcal{P}_0)$ are linearly independent in $\mathcal{G}^N_{CS}$.

We summarize the above discussion in the following proposition.

Proposition 3. *The games $d^R \in \mathcal{G}^N_{CS}$ $(R \in \mathcal{P}_0)$ form a basis of $\mathcal{G}^N_{CS}$, henceforth $\dim(\mathcal{G}^N_{CS}) = 2^{n-1}$. Moreover, $v(S) = \sum_{R\in\mathcal{P}_0} v(R)\cdot d^R(S)$ for all $S \subseteq N$ and $v \in \mathcal{G}^N_{CS}$, consequently, by linearity of the Shapley value, $\phi(v) = \sum_{R\in\mathcal{P}_0} v(R)\cdot\phi(d^R)$.*

The basis game values $d^R(S)$ $(R, S \in \mathcal{P}_0)$ form a unit matrix, thus by formula (2), using $r = |R|$, the Shapley payoffs to our special player 0 in the basis games are

$$\phi_0(d^R) = \begin{cases} 2\gamma_n(r-1), & \text{if } R \neq N, \\ -1 + 2\gamma_n(n-1), & \text{if } R = N. \end{cases} \tag{5}$$

The payoffs to the players in $C = N\setminus\{0\}$ can then be easily obtained from efficiency and the equal treatment property of the Shapley value.

For $R \in \mathcal{P}_0 \setminus \{N\}$, in basis game d^R the players in R are all symmetric, so $\phi_0(d^R) = \phi_i(d^R)$ for all $i \in R$. Similarly, the players in $N\setminus R$ are all symmetric, so $\phi_j(d^R) = \phi_k(d^R)$ for all $j, k \in N\setminus R$. Since $d^R(N) = 0$, efficiency gives $r\phi_0(d^R) + (n-r)\phi_k(d^R) = 0$, where $k \in N\setminus R$. From (5) we easily derive the Shapley payoffs in basis game d^R when $R \neq N$.

$$\phi_i(d^R) = \begin{cases} 2\gamma_n(r-1), & \text{if } i \in R, \\ -2\gamma_n(r), & \text{if } i \in N\setminus R. \end{cases} \tag{6}$$

For $R = N$, in basis game d^N all non-distinguished players in C are symmetric, so $\phi_j(d^N) = \phi_k(d^N)$ for all $j, k \in N\setminus\{0\}$. Since

$d^N(N) = 1$, efficiency gives $\phi_0(d^N) + (n-1)\phi_k(d^N) = 1$, where $k \neq 0$. From (5), we easily get the Shapley payoffs in d^N as

$$\phi_i(d^N) = \begin{cases} -1 + 2\gamma_n(n-1), & \text{if } i = 0, \\ 2\gamma_n(n-1), & \text{if } i \neq 0. \end{cases} \tag{7}$$

Notice that none of the players is a null player in any of the basis games d^R with carrier $R \in \mathcal{P}_0$.

Sharing system (8) schematically summarizes the above formulas. The columns correspond to the partner coalitions of the form $R = \{0\} \cup S$. The first line in the header specifies the number of partners $s = |S|$ of player 0, the second line gives the number of coalitions in that category. The third header line indicates the two subcategories (except for the two boundary cases: for the empty set in the first column, and for the full partner set in the last column) whether an arbitrarily fixed player $i \neq 0$ is a partner of 0 or not. The fourth line gives the number of coalitions in the subcategories. Any given player $i \in C$ can either be a partner of player 0 or not. Thus, except when $S = \emptyset$ or $S = C$, among the $\binom{n-1}{s}$ coalitions $S \subseteq C$ of size $1 \leq s \leq n-2$ there are $\binom{n-2}{s-1}$ coalitions which contain i, the remaining $\binom{n-2}{s}$ coalitions do not contain i. The two (highlighted) rows of the table give the Shapley values of players in the basis constant-sum games with carrier coalitions of the form $R = \{0\} \cup S$, first for our distinguished player 0, second for a generic other player $i \in C$.

size	$s = 0$	$\cdots$	$s = \lvert S\rvert$		$\cdots$	$s = n-1$
number	$\binom{n-1}{0}$	$\cdots$	$\binom{n-1}{s}$		$\cdots$	$\binom{n-1}{n-1}$
type	$i \notin S$	$\cdots$	$i \in S$	$i \notin S$	$\cdots$	$i \in S$
number	$\binom{n-2}{0}$	$\cdots$	$\binom{n-2}{s-1}$	$\binom{n-2}{s}$	$\cdots$	$\binom{n-2}{n-2}$
ϕ_0	$2\gamma_n(0)$	$\cdots$	$2\gamma_n(s)$	$2\gamma_n(s)$	$\cdots$	$-1+2\gamma_n(n-1)$
ϕ_i	$-2\gamma_n(1)$	$\cdots$	$2\gamma_n(s)$	$-2\gamma_n(s+1)$	$\cdots$	$2\gamma_n(n-1)$

(8)

The following features of the Shapley sharing system are easily checked.

Proposition 4. *In the Shapley sharing system (8)*

1. *the ϕ_0 row sum $= 1$, every other ϕ_i ($i \in C$) row sum $= 0$;*
2. *the $s = n-1$ column sum $= 1$, every other $0 \leq s \leq n-2$ column sum $= 0$.*

For illustration, we give the Shapley sharing system for 3-player constant-sum games on $N = \{0\} \cup C$ with $C = \{1, 2\}$:

size	$s = 0$	$s = 1$		$s = 2$
number	$\binom{2}{0} = 1$	$\binom{2}{1} = 2$		$\binom{2}{2} = 1$
partner	$S = \emptyset$	$S = \{1\}$	$S = \{2\}$	$S = C$
carrier	$R = \{0\}$	$R = \{0,1\}$	$R = \{0,2\}$	$R = N$
ϕ_0	2/3	1/3	1/3	−1/3
ϕ_1	−1/3	1/3	−2/3	2/3
ϕ_2	−1/3	−2/3	1/3	2/3

(9)

We replaced the subcategorization in the third and fourth header lines in (8) with the actual set of partners S and the corresponding carrier coalitions R in (9).

The Shapley payoffs are easily computed from sharing system (9) for any 3-player constant-sum game v with distinguished player 0. We simply take the linear combination of the "partner" coalition values weighted with the "shares" of the given player. In formula,

$$\phi_0(v) = \frac{2v_{\overline{0}} + v_{\overline{01}} + v_{\overline{02}} - v_N}{3}, \qquad \phi_i(v) = \frac{-v_{\overline{0}} + v_{\overline{0i}} - 2v_{\overline{0j}} + 2v_N}{3} \quad (i \neq j), \tag{10}$$

where coalitions are described without braces and separating commas but overlined: for example, $\overline{0j}$ means coalition $\{0, j\}$. Its value is shorthanded as $v_{\overline{0j}} = v(\overline{0j})$.

Although in a general constant-sum game distinguishing one arbitrarily picked player served only technical purposes, next, we discuss a special type of constant-sum game where one player is indeed "different" from the other players.

3. Liability games and the Shapley value

We consider a special class of constant-sum games, liability games, introduced by Csóka & Herings (2019).

Let $N = \{0, 1, \ldots, c\}$ denote the set of agents, where agent 0 is a *firm* having a set of *creditors* $C = \{1, \ldots, c\}$ with cardinality $|C| = c \geq 1$. The firm has *asset value* $A \in \mathbb{R}_+$ and *liabilities* $\ell \in \mathbb{R}^C_+$, with $\ell_i \in \mathbb{R}_+$ the liability to creditor $i \in C$. The question is how to allocate the asset value among the creditors and the firm. If the firm is solvent, that is, $\sum_{i \in C} \ell_i \leq A$, then the obvious solution is that every creditor receives its full claim and the firm keeps the rest. Henceforth we only consider the insolvent case, but for ease of presentation, we also allow borderline solvency, that is, $\sum_{i \in C} \ell_i = A$.

Definition 5. A *liability problem* is a pair $(A, \ell) \in \mathbb{R}_+ \times \mathbb{R}^C_+$ such that $\sum_{i \in C} \ell_i \geq A$.

Let $\mathcal{L}^N$ denote the class of liability problems[2] on set of agents $N = \{0\} \cup C$. We seek a liability rule that assigns a unique allocation to each liability problem.

Definition 6 (Csóka & Herings, 2019)**.** A *liability rule* is a function $f : \mathcal{L}^N \to \mathbb{R}^N_+$ such that, for every $(A, \ell) \in \mathcal{L}^N$, the payment vector $f = f(A, \ell) \in \mathbb{R}^N$ is an *allocation*, that is a non-negative vector $f \in \mathbb{R}_+ \times \mathbb{R}^C_+$ satisfying *liabilities boundedness*, that is, $f_i \leq \ell_i$ for all $i \in C$, and *efficiency*, that is, $\sum_{i \in N} f_i = A$.

Note that by non-negativity and efficiency, the payments in allocation $f \in \mathbb{R}^N$ fall between the following bounds:

$$0 \leq f_0 \leq A \quad \text{and} \quad 0 \leq f_i \leq \ell^A_i \quad \text{for all } i \in C,$$

where $\ell^A_i = \min\{A, \ell_i\}$ is the *truncated liability* of creditor $i \in C$. Let $\ell^A \in \mathbb{R}^C_+$ denote the vector of liabilities truncated by the asset value.

Given a subset of creditors $S \subseteq C$, we will use the notation $\ell_S = \ell(S) = \sum_{i \in S} \ell_i$ for the total liabilities of S and $\ell^A(S) = \sum_{i \in S} \ell^A_i$ for the total truncated liabilities of S. On the other hand, we will also use the shorthand $\ell^A_S = \min\{A, \ell(S)\} = \min\{A, \ell^A(S)\}$ for the *truncated total (truncated) liabilities* of creditor group $S \subseteq C$. Clearly, $\ell^A_S \leq \ell^A(S)$.

A liability problem gives rise to a transferable utility cooperative game called liability game (Csóka & Herings, 2019).

Definition 7. Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem. On player set N, the induced *liability game* $v : 2^N \to \mathbb{R}$ is defined by setting, for $S \in 2^N$,

$$v(S) = \begin{cases} \min\{A, \ell(S \setminus \{0\})\} = \ell^A_{S\setminus\{0\}}, & \text{if } 0 \in S, \\ \max\{0, A - \ell(C \setminus S)\}, & \text{if } 0 \notin S. \end{cases}$$

The interpretation of a liability game is as follows. Given a coalition and its complement, the firm first makes payments to the coalition it belongs to, up to the value of the liabilities in the firm's coalition and the asset value of the firm, and then (if possible) pays to the complementary coalition.

Note that $v(\emptyset) = 0$, $0 \leq v(S) \leq A$ for all $S \in 2^N$, and $v(N) = A$. Csóka & Herings (2019) note that liability games are *superadditive*,

[2] Csóka & Herings (2019) consider a slightly restricted class, when all liabilities are at most as large as the asset value, the asset value is strictly positive, there are at least two creditors and the firm is insolvent.

that is, for all $S, T \in 2^N$, $S \cap T = \emptyset$ implies $v(S) + v(T) \le v(S \cup T)$; and *constant-sum*, that is, for all $S \in 2^N$, $v(S) + v(N \setminus S) = v(N)$. Due to their superadditivity and nonnegativity, liability games are *monotonic*, that is, for all $S, T \in 2^N$, $S \subset T$ implies $v(S) \le v(T)$.

We aim to define a liability rule by applying the Shapley value to the induced liability game. This works in practice only if we can compute the Shapley-vector of the liability game directly from the data of the underlying liability problem, that is, from the asset value and the liabilities. The following straightforward observation implies that our indirect approach could only provide a liability rule that ignores excessive parts of the claims. Notice that cutting off the parts of liabilities over the asset value does not make the firm solvent, that is, $\ell(C) \ge A$ implies $\ell^A(C) \ge A$.

Remark 8. Liability problems (A, ℓ) and (A, ℓ^A) induce the same liability game, where ℓ^A denotes the vector of liabilities truncated by the asset value.

It follows that the Shapley rule (or any other liability allocation rule defined via a single-valued solution of the induced game) is different from rules that allocate (some portion of) the asset value among the creditors proportional to their claims (or to their truncated liabilities).

Next, we show that the Shapley value indeed defines a liability rule, that is, the Shapley-vector of the liability game associated with a liability problem is an allocation.

Proposition 9. *Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and let v be the induced liability game on N. Then the Shapley-vector $\phi(v)$ of v satisfies efficiency, non-negativity, and (truncated) liabilities boundedness.*

Proof. The Shapley value assigns an efficient vector to any TU game, so for any liability game (N, v) we have $\sum_{i\in N} \phi_i(v) = v(N) = A$. By monotonicity of liability games all marginal contributions are non-negative, hence the Shapley payoffs are non-negative.

To prove (truncated) liabilities boundedness, let $i \in C$ be a creditor and $S \subseteq N \setminus i$. We have two cases. If $0 \in S$, so $v(S \cup i) - v(S) = \min\{\ell(S \setminus 0) + \ell_i, A\} - \min\{\ell(S \setminus 0), A\}$, then the difference is clearly at most ℓ_i. If $0 \notin S$, so $v(S \cup i) - v(S) = \max\{A - \ell(C \setminus S) + \ell_i, 0\} - \max\{A - \ell(C \setminus S), 0\}$, then again the difference is clearly at most ℓ_i. Thus, we get that all marginal contributions, hence the Shapley payoffs to all creditors are upper bounded by the liabilities. Since non-negativity and efficiency imply $\phi_i \le A$ for all $i \in N$, including the firm, for creditor $i \in C$ we can sharpen the upper bound to $\phi_i \le \ell_i^A$. □

Next, we define (truncated) debt forgiveness of a creditor as the difference between the (truncated) liability towards him and the payment he receives. Formally, let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and $x \in \mathbb{R}^N_+$ be an allocation. The *debt forgiveness* of creditor $i \in C$ is given by $\ell_i - x_i$. The *truncated debt forgiveness* by creditor $i \in C$ is given by $\ell_i^A - x_i = \min\{A, \ell_i\} - x_i$.

Example 10. Consider a generic liability problem with two creditors, so $N = \{0, 1, 2\}$ and $A \le \ell_1 + \ell_2$. The induced liability game v is the following:

S	$\{0\}$	$\{1\}$	$\{2\}$	$\{0,1\}$	$\{0,2\}$	$\{1,2\}$	$\{0,1,2\}$
$v(S)$	0	$A-\ell_2^A$	$A-\ell_1^A$	ℓ_1^A	ℓ_2^A	A	A

We can compute the Shapley allocation from sharing system (9) derived for 3-player constant-sum games. This format is very useful for studying various properties of the Shapley rule.

$S \ni 0$	$\{0\}$	$\{0,1\}$	$\{0,2\}$	N
$v(S)$	0	ℓ_1^A	ℓ_2^A	A
ϕ_0	2/3	1/3	1/3	−1/3
ϕ_1	−1/3	1/3	−2/3	2/3
ϕ_2	−1/3	−2/3	1/3	2/3

(11)

The Shapley payments are obtained by multiplying row $[v(S)]$ of the coalition values by row $[\phi_k]$ of the shares for player $k \in N$. We can derive the following formulas and bounds (from $A \le \ell_1 + \ell_2$ implying $A \le \ell_1^A + \ell_2^A \le 2A$).

For the firm,

$$0 \le \phi_0 = \frac{\ell_1^A + \ell_2^A - A}{3} \le \frac{A}{3}.$$

Clearly both bounds are sharp. Notice that at the Shapley allocation, an insolvent firm ends up with a strictly positive payoff.

For creditor $i \ne j \in C$, since $0 \le A - \ell_j^A \le \ell_i^A$,

$$\frac{\ell_i^A}{3} \le \phi_i = \frac{\ell_i^A - 2\ell_j^A + 2A}{3} = \ell_i^A - 2\phi_0 \le \ell_i^A.$$

It is easily seen that both bounds are sharp. For the debt forgiveness and for the truncated debt forgiveness of creditor $i \in C$, we immediately get the following sharp bounds:

$$\ell_i - \ell_i^A \le \ell_i - \phi_i \le \ell_i - \frac{\ell_i^A}{3}, \qquad 0 \le \ell_i^A - \phi_i = 2\phi_0 \le \frac{2\ell_i^A}{3}.$$

Observe that both creditors give the same truncated debt forgiveness ($2\phi_0$) to the firm.

It also follows from the above formulas that if $\ell_i \le \ell_j$, hence also $\ell_i^A \le \ell_j^A$, then $\phi_i \le \phi_j$ and $\ell_i - \phi_i \le \ell_j - \phi_j$. That is, at the Shapley allocation, the creditor with higher claim gets higher payment, but it also gives an at least as high debt forgiveness.

4. Properties of the Shapley liability rule

In this section, we generalize the observations we made on the Shapley allocations for 2-creditor liability problems in Example 10 and investigate further properties of the Shapley rule.

As observed in Proposition 9, the Shapley rule satisfies efficiency, non-negativity and (truncated) liabilities boundedness, hence it is a liability rule. As noticed in Remark 8, the Shapley rule (as any rule induced by a solution of an associated TU game) *ignores excessive parts of claims*, that is, $\phi(A, \ell) = \phi(A, \ell^A)$. It is also easily seen that the Shapley rule *respects minimal rights of creditors*, that is, it satisfies $\phi_i \ge \max\{0, A - \ell(C \setminus i)\}$ for any $i \in C$. Indeed, the minimal right of creditor i is precisely his value $v(i)$ in the associated liability game, which is superadditive, and the Shapley value is well-known to prescribe individually acceptable payoffs in superadditive games.

Since liability games are constant-sum, from sharing table (8), taken into account that $v(0 \cup S) = \ell_S^A$ for coalitions of the form $0 \cup S$ with $S \subseteq C$, we get that for liability problem (A, ℓ) the Shapley rule prescribes the following payments.

$$\phi_0(A, \ell) = -A + 2\sum_{S\subseteq C} \gamma_n(s)\ell_S^A, \tag{12}$$

$$\phi_i(A, \ell) = 2\sum_{S\subseteq C\setminus i} \gamma_n(s+1)(\ell_{S\cup i}^A - \ell_S^A), \qquad (i \in C) \tag{13}$$

where $s = |S|$ and $\ell_S^A = \min\{A, \sum_{i\in S} \ell_i\}$.

4.1. Bounds on the Shapley payments

First, we establish lower and upper bounds for the Shapley payment of the firm.

Proposition 11. *Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and let v be the induced liability game on N. Then for the Shapley payment of the firm ϕ_0 we have that*

$$0 \le \frac{n-2}{n} \min\{A, \min_{i\in C} \ell_i, \ell_C - A\} \le \phi_0(A, \ell) \le \frac{n-2}{n} A. \tag{14}$$

Proof. Since $v(0)=\ell_\emptyset^A=0$, $v(N)=\ell_C^A=A$, and $\gamma_n(S)=\gamma_n(C\setminus S)$ for $S\subseteq C$,

$$\phi_0(A,\ell) \quad = \quad \sum_{\emptyset\neq S\neq C}\gamma_n(s)(\ell_S^A+\ell_{C\setminus S}^A)+\frac{2-n}{n}A. \tag{15}$$

If $n=2$ then the summation in (15) is over the empty set, thus $\phi_0(A,\ell)=0$. It means that the Shapley rule allocates the full asset value to the single creditor. In contrast, if $c\geq 2$, then the firm has some implicit bargaining leverage by threatening to form a coalition with the other creditors and compensate them first up to their full liabilities or the asset value. From $\ell_S^A+\ell_{C\setminus S}^A=\min\{2A, A+\ell_S, A+\ell_{C\setminus S}, \ell_C\}=A+\min\{A,\ell_S,\ell_{C\setminus S},\ell_C-A\}$ and $\sum_{\emptyset\neq S\neq C}\gamma_n(s)=\frac{n-2}{n}$, where $s=|S|$, we get $\phi_0=\sum_{\emptyset\neq S\neq C}\gamma_n(s)\min\{A,\ell_S,\ell_{C\setminus S},\ell_C-A\}$. Eq. (14) now follows. □

In the insolvent (non-degenerate) case, that is, if $\ell_C>A$ (and $A>0$ and $\min_{i\in C}\ell_i>0$), the lower bound is positive, that is, the firm ends up with positive payoff. The lower bound in (14) is sharp if and only if $\ell_C-A\leq A$ and $\ell_C-A\leq\min_{i\in C}\ell_i$, that is, the deficiency of the firm does not exceed any of the individual liabilities and the asset value. The upper bound in (14) is sharp if and only if $A\leq\min_{i\in C}\ell_i$ (that implies $\ell_C-A\geq A$ for $c\geq 2$), that is, all creditors claim the full asset value so each one is willing to forgive some of its debt to stay a partner of the firm and receive some positive payment. Note that in this case as the number of creditors increases, the firm can keep almost all the asset value.

Second, we establish lower and upper bounds for the Shapley payments of the creditors.

Proposition 12. *Let $(A,\ell)\in\mathcal{L}^N$ be a liability problem and let v be the induced liability game on N. Then for any $i\in C$ for the Shapley payment of the creditor ϕ_i have that*

$$\frac{2}{n(n-1)}\ell_i^A\leq\phi_i(A,\ell)\leq\left(\frac{2}{n(n-1)}+\frac{(n-2)(n+1)}{n(n-1)}\right)\ell_i^A=\ell_i^A. \tag{16}$$

Proof. Since $v(0)=\ell_\emptyset^A=0$ and $\gamma_n(1)=\frac{1}{n(n-1)}$, from (13) we get for $i\in C$,

$$\phi_i(A,\ell)=\frac{2}{n(n-1)}\ell_i^A+2\sum_{\emptyset\neq S\subseteq C\setminus i}\gamma_n(s+1)(\ell_{S\cup i}^A-\ell_S^A). \tag{17}$$

If $n=2$, that is, $C=\{1\}$, then the summation in (17) is over the empty set, thus $\phi_1(A,\ell)=\ell_i^A$. It means that the Shapley rule allocates the full asset value to the single creditor. In contrast, if $c\geq 2$ then the summation in (17) is clearly non-negative, and it is zero if and only if $A\leq\ell_i^A$ for all $i\in C$. On the other side, $\ell_{S\cup i}^A-\ell_S^A=\min\{A-\ell_S^A,\ell_i^A\}\leq\ell_i^A$ in case of $A>\ell_S^A$. It follows from $\sum_{\emptyset\neq S\subseteq C\setminus i}\gamma_n(s+1)=\sum_{s=1}^{n-2}\binom{n-2}{s}\gamma_n(s+1)=\sum_{s=1}^{n-2}\frac{(n-2)!}{s!(n-2-s)!}\frac{1}{n}\frac{(s+1)!(n-2-s)!}{(n-1)!}=\sum_{s=1}^{n-2}\frac{s+1}{n(n-1)}=\frac{(n-2)(n+1)}{2n(n-1)}$ that the summation in (17) is at most $\frac{(n-2)(n+1)}{n(n-1)}\ell_i^A$, and equality holds if and only if $A\geq\ell_C$ (that implies $A\geq\ell_{S\cup i}$ for all $S\subseteq C\setminus i$). □

Both bounds are sharp in (16). The lower bound is attained when all creditors claim the full asset value, hence considerably weaken each other's bargaining position. On the other side, the creditors can be fully compensated if and only if the firm is solvent.

4.2. Order preservation and monotonicity properties

First, we show that creditors with higher claims get higher Shapley payments, a property called *order preservation* in the review article on bankruptcy rules by Thomson (2015). We also show that creditors with higher claims also give higher (truncated) debt forgiveness.

Proposition 13. *Let $(A,\ell)\in\mathcal{L}^N$ be a liability problem and v the induced liability game. Let $i,j\in C$ be such that $\ell_i\leq\ell_j$. At the Shapley value it holds that $\phi_i\leq\phi_j$, $\ell_i-\phi_i\leq\ell_j-\phi_j$ and $\ell_i^A-\phi_i\leq\ell_j^A-\phi_j$.*

Proof. Let $i,j\in C$ be two creditors with $\ell_i\leq\ell_j$, hence also $\ell_i^A\leq\ell_j^A$. Since liability games are constant-sum games, we use formula (2) to show $0\leq\phi_j-\phi_i\leq\ell_j^A-\ell_i^A\leq\ell_j-\ell_i$.

When taking the difference $\phi_j-\phi_i$ the terms $v(S\cup i\cup j)$, $S\subseteq N\setminus\{i,j\}$, containing both players cancel out, so we get

$$\phi_j(v)-\phi_i(v)=\frac{2}{n}\sum_{s=0}^{n-1}\frac{1}{\binom{n-1}{s}}\sum_{S\subseteq N\setminus\{i,j\}:|S|=s}(v(S\cup j)-v(S\cup i)). \tag{18}$$

It is easily checked from the definition of v that $0\leq v(S\cup j)-v(S\cup i)\leq\ell_j^A-\ell_i^A\leq\ell_j-\ell_i$ for all $S\subseteq N\setminus\{i,j\}$. Substituting each term in (18) with these non-negative constant bounds gives

$$0\leq\phi_j(v)-\phi_i(v)\leq(\ell_j^A-\ell_i^A)\cdot\frac{2}{n}\sum_{s=0}^{n-1}\frac{1}{\binom{n-1}{s}}\binom{n-2}{s}, \tag{19}$$

since there are $\binom{n-2}{s}$ coalitions $S\subseteq N\setminus\{i,j\}$ of cardinality s. From $\frac{2}{n}\sum_{s=0}^{n-1}\frac{1}{\binom{n-1}{s}}\binom{n-2}{s}=\frac{2}{n}\sum_{s=0}^{n-1}(1-\frac{s}{n-1})=\frac{2}{n}(n-\frac{1}{n-1}\sum_{s=0}^{n-1}s)=\frac{2}{n}(n-\frac{n}{2})=1$, and the obvious $\ell_j^A-\ell_i^A\leq\ell_j-\ell_i$, the claim follows. □

Note that order preservation in Proposition 13 obviously implies *equal treatment of equal creditors*, that is, if two creditors have the same claims, then they should get the same compensations. From Proposition 13 we readily get that the Shapley rule treats creditors with equal (truncated) liabilities in the same way.

Corollary 14. *Let $(A,\ell)\in\mathcal{L}^N$ be a liability problem and v the induced liability game. Let $i,j\in C$ be such that $\ell_i=\ell_j$. At the Shapley value it holds that $\phi_i=\phi_j$, $\ell_i-\phi_i=\ell_j-\phi_j$ and $\ell_i^A-\phi_i=\ell_j^A-\phi_j$.*

Next, we discuss three basic monotonicity properties of liability rules. The question is how changes in certain parameters of a liability problem influence the payments of the agents.

Definition 15. Liability rule $f:\mathcal{L}^N\to\mathbb{R}_+^N$ is said to be

1. *liability monotonic* if for any creditor $i\in C$ and liability problems (A,ℓ), (A,ℓ') such that $\ell_i'>\ell_i$ and $\ell_k'=\ell_k$ for all $k\in C\setminus i$, it holds that $f_i(A,\ell')\geq f_i(A,\ell)$.
2. *asset monotonic for creditors* if for any creditor $i\in C$ and liability problems (A,ℓ), (A',ℓ) such that $\ell(C)\geq A'>A$, it holds that $f_i(A',\ell)\geq f_i(A,\ell)$.
3. *super-modular for creditors* if for any two creditors $i,j\in C$ with $\ell_i\geq\ell_j$ and liability problems (A,ℓ), (A',ℓ) such that $\ell(C)\geq A'>A$, it holds that $f_i(A',\ell)-f_i(A,\ell)\geq f_j(A',\ell)-f_j(A,\ell)$.

In the following three theorems we prove that the Shapley rule satisfies these three monotonicity properties. We also make some observations on the changes in the firm's payment.

First, we show that the Shapley rule is liability monotonic. It means that the payment of a creditor can only increase if his liability increases, but every other parameter of the problem stays put. Moreover, we show that also the firm can only benefit from the increase of a liability.

Proposition 16. *Let liability problems* (A,ℓ) *and* (A,ℓ') *be such that* $\ell'_i > \ell_i$ *for* $i \in C$, *and* $\ell'_k = \ell_k$ *for all* $k \in C \setminus i$. *Then*

$$\phi_i(A,\ell') \geq \phi_i(A,\ell) + \frac{2}{n(n-1)}\min\{\ell'_i - \ell_i, A - \ell_i^A\}.$$

Moreover, $\phi_0(A,\ell') \geq \phi_0(A,\ell)$.

Proof. Let liability problems (A,ℓ) and (A,ℓ') be such that $\ell'_i > \ell_i$ for $i \in C$, and $\ell'_k = \ell_k$ for all $k \in C \setminus i$. Clearly, $\ell'^A_{S\cup i} \geq \ell^A_{S\cup i}$ and $\ell'^A_S = \ell^A_S$ whenever $S \subseteq C \setminus i$. From formula (17) we get

$$\phi_i(A,\ell') - \phi_i(A,\ell) = \frac{2}{n(n-1)}(\ell'^A_i - \ell^A_i) + 2\sum_{\emptyset \neq S \subseteq C\setminus i}\gamma_n(s+1)(\ell'^A_{S\cup i} - \ell^A_{S\cup i}). \tag{20}$$

Since the summation term in (20) is non-negative, and $\ell'^A_i - \ell^A_i = \min\{\ell'_i - \ell_i, A - \ell^A_i\}$, the inequality for $\phi_i(A,\ell)$ follows.

From formula (12) we get

$$\phi_0(A,\ell') - \phi_0(A,\ell) = 2\sum_{S \subseteq C\setminus i}\gamma_n(s+1)(\ell'^A_{S\cup i} - \ell^A_{S\cup i}) + 2\sum_{S \subseteq C\setminus i}\gamma_n(s)(\ell'^A_S - \ell^A_S). \tag{21}$$

Since each term in the first summation is non-negative, and zero in the second one, we conclude that the payment to the firm can only increase if a liability increases. □

Second, we show that the Shapley rule is asset monotonic for creditors. It means that the payments to the creditors can only increase if the asset value increases, but all liabilities remain the same. Moreover, we observe that the firm can end up with smaller or with higher payoff.

Proposition 17. *Let liability problems* (A,ℓ) *and* (A',ℓ) *be such that* $\ell(C) \geq A' > A$. *Then for any creditor* $i \in C$,

$$0 \leq \phi_i(A',\ell) - \phi_i(A,\ell) \leq \min\{A' - A, \ell_i\},$$

and for the firm,

$$\frac{2-n}{n}(A'-A) \leq \phi_0(A',\ell) - \phi_0(A,\ell) \leq \frac{n-2}{n}(A'-A).$$

Moreover, for $c = |C| \geq 2$, *all bounds are sharp.*

In case of a single creditor, $C = \{1\}$, $\phi_1(A',\ell) - \phi_1(A,\ell) = A' - A$ *and* $\phi_0(A',\ell) = \phi_0(A,\ell)$.

Proof. Let liability problems (A,ℓ) and (A',ℓ) be such that $\ell(C) \geq A' > A$. From formula (13) we get for any $i \in C$,

$$\phi_i(A',\ell) - \phi_i(A,\ell) = 2\sum_{S \subseteq C\setminus i}\gamma_n(s+1)\left[(\ell^{A'}_{S\cup i} - \ell^A_{S\cup i}) - (\ell^{A'}_S - \ell^A_S)\right]. \tag{22}$$

First of all, since $\left[(\ell^{A'}_{S\cup i} - \ell^A_{S\cup i}) - (\ell^{A'}_S - \ell^A_S)\right] = \left[(\ell^{A'}_{S\cup i} - \ell^{A'}_S) - (\ell^A_{S\cup i} - \ell^A_S)\right]$ and the difference $\ell^A_{S\cup i} - \ell^A_S = \min\{\ell_i, A - \ell^A_S\}$ where $A - \ell^A_S = \max\{A - \ell_S, 0\}$ is clearly non-decreasing in A, we get that the difference in the bracket in each term is non-negative, implying asset monotonicity for creditor $i \in C$.

Let us assume $c \geq 2$. Then there are at least two different terms in (22). One is the term for $S = \emptyset$. It equals $\frac{2}{n(n-1)}\left[(\ell^{A'}_i - \ell^A_i) - (0-0)\right]$. The difference in the bracket can range from 0 (attained, if $\ell_i \leq A < A'$) to $\min\{A'-A, \ell_i\}$ (attained, if $A < A' \leq \ell_i$). The other term is for $S = C\setminus i \neq \emptyset$. It equals $\frac{2}{n}\left[(\ell^{A'}_C - \ell^A_C) - (\ell^{A'}_{C\setminus i} - \ell^A_{C\setminus i})\right] = \frac{2}{n}\left[(A'-A) - (\ell^{A'}_{C\setminus i} - \ell^A_{C\setminus i})\right]$. Again, the difference in the bracket can range from 0 (attained, if $A < A' \leq \ell_{C\setminus i}$) to $(A'-A)$ (attained, if $\ell_{C\setminus i} \leq A < A'$)). Likewise, if $\ell_i \leq A < A'$ but $A < A' \leq \ell_j$ for any other creditor $j \neq i$, then all terms in (22) are zero, implying that the zero lower bound is indeed sharp. In contrast, if $A < A' \leq \ell_i$ but $\ell_{C\setminus i} \leq A < A'$ (implying $\ell_j \leq A < A'$ for any other creditor $j \neq i$), then the differences in all brackets in (22) are equal to $\min\{A'-A, \ell_i\}$. In light of $2\sum_{S\subseteq C\setminus i}\gamma_n(s+1) = 1$, the claimed upper bound is also sharp.

For the change in the Shapley payment to the firm, taken into account that $\ell^A_\emptyset = 0$ and $\ell^A_C = A$, from formula (12) we get

$$\phi_0(A',\ell) - \phi_0(A,\ell) = -(A'-A) + 2\sum_{\emptyset\neq S\subsetneq C}\gamma_n(s)(\ell^{A'}_S - \ell^A_S) + \frac{2}{n}(A'-A). \tag{23}$$

Since the difference $\ell^{A'}_S - \ell^A_S$ is clearly non-negative but cannot exceed $A'-A$, from $\sum_{\emptyset\neq S\subsetneq C}\gamma_n(s) = 1 - \frac{2}{n}$, the claimed inequalities for the difference $\phi_0(A',\ell) - \phi_0(A,\ell)$ follow. The negative lower bound is attained if $\ell_S \leq A$ for every non-empty set of creditors $S \neq C$ implying $\ell^{A'}_S - \ell^A_S = 0$. The positive upper bound is attained if $\ell_i \geq A'$ for all creditors $i \in C$ implying $\ell_S \geq A'$ and $\ell^{A'}_S - \ell^A_S = A' - A$ for every non-empty set of creditors $S \neq C$.

Finally, in case of a single creditor $C = \{1\}$, Equation (22) simplifies to $\phi_1(A',\ell) - \phi_1(A,\ell) = \frac{2}{2(2-1)}\left[(\ell^{A'}_i - \ell^A_i) - (0-0)\right] = A' - A$, reconfirming that the Shapley rule gives everything to the single creditor. By efficiency, the firm ends up with nothing, thus, $\phi_0(A',\ell) - \phi_0(A,\ell) = 0 - 0 = 0$. Notice that for $n = 2$, the summation in (23) is over the empty set, and the claimed lower and upper bounds coincide at zero. □

Finally, we show that the Shapley rule is super-modular for creditors. It means that creditors with higher liabilities receive more from the increment in the asset value. This property is a kind of combination of order preservation (when the payments to two creditors in the same problem are compared) and asset monotonicity (when the payments to the same creditor in two related problems are compared).

Proposition 18. *Let liability problems* (A,ℓ) *and* (A',ℓ) *be such that* $\ell(C) \geq A' > A$. *If* $\ell_i \geq \ell_j$ *for creditors* $i, j \in C$ *then*

$$0 \leq \left(\phi_i(A',\ell) - \phi_i(A,\ell)\right) - \left(\phi_j(A',\ell) - \phi_j(A,\ell)\right) \leq \min\{\ell_i - \ell_j; A' - A\}. \tag{24}$$

Proof. Given two creditors $i, j \in C$, a set of creditors $S \subseteq C$ can be one of four types: S contains both i and j; contains i but not j; contains j but not i; contains neither i nor j. For brevity, we represent $S \subseteq C$ respectively as Rij, Ri, Rj, R with a generic $R \subseteq C \setminus \{i,j\}$. From the formula in (13) we get that $\phi_i(A',\ell) - \phi_i(A,\ell) =$

$$2\sum_R\left\{\gamma_n(r+1)\left[\ell^{A'}_{Ri} - \ell^A_{Ri} - \ell^{A'}_R + \ell^A_R\right] + \gamma_n(r+2)\left[\ell^{A'}_{Rji} - \ell^A_{Rji} - \ell^{A'}_{Rj} + \ell^A_{Rj}\right]\right\}. \tag{25}$$

Exchanging i and j gives $\phi_j(A',\ell) - \phi_j(A,\ell) =$

$$2\sum_R\left\{\gamma_n(r+1)\left[\ell^{A'}_{Rj} - \ell^A_{Rj} - \ell^{A'}_R + \ell^A_R\right] + \gamma_n(r+2)\left[\ell^{A'}_{Rij} - \ell^A_{Rij} - \ell^{A'}_{Ri} + \ell^A_{Ri}\right]\right\}. \tag{26}$$

Subtracting (26) from (25) gives $(\phi_i(A',\ell) - \phi_i(A,\ell)) - (\phi_j(A',\ell) - \phi_j(A,\ell)) =$

$$2\sum_R[\gamma_n(r+1) + \gamma_n(r+2)]\left[(\ell^{A'}_{Ri} - \ell^A_{Ri}) - (\ell^{A'}_{Rj} - \ell^A_{Rj})\right]. \tag{27}$$

Suppose $\ell_i \geq \ell_j$, implying $\ell_{Ri} \geq \ell_{Rj}$. It is easily checked that

$$(\ell^{A'}_{Ri} - \ell^A_{Ri}) - (\ell^{A'}_{Rj} - \ell^A_{Rj}) = \begin{cases} 0, & \text{if } \ell_{Rj} \leq \ell_{Ri} \leq A \leq A', \\ \ell_{Ri} - A, & \text{if } \ell_{Rj} \leq A \leq \ell_{Ri} \leq A', \\ A' - A, & \text{if } \ell_{Rj} \leq A \leq A' \leq \ell_{Ri}, \\ \ell_{Ri} - \ell_{Rj}, & \text{if } A \leq \ell_{Rj} \leq \ell_{Ri} \leq A', \\ A' - \ell_{Rj}, & \text{if } A \leq \ell_{Rj} \leq A' \leq \ell_{Ri}, \\ 0, & \text{if } A \leq A' \leq \ell_{Rj} \leq \ell_{Ri}. \end{cases}$$

It follows that

$0 \le (\ell_{Ri}^{A'} - \ell_{Ri}^{A}) - (\ell_{Rj}^{A'} - \ell_{Rj}^{A}) \le \min\{\ell_{Ri} - \ell_{Rj} = \ell_i - \ell_j; A' - A\}.$

Taken into account that

$$\sum_{R\subseteq C\setminus ij} [\gamma_n(r+1) + \gamma_n(r+2)] = \sum_{R\subseteq C\setminus ij} \gamma_n(r+1) + \sum_{j\in Q\subseteq C\setminus i} \gamma_n(q+1)$$
$$= \sum_{S\subseteq C\setminus i} \gamma_n(s+1) = 1/2,$$

where $q = |Q|$ and $s = |S|$, from (27) we get the claimed inequalities in (24). □

A straightforward corollary of Proposition 18 is that if $\ell_i = \ell_j$ for creditors $i, j \in C$ then $\phi_i(A', \ell) - \phi_i(A, \ell) = \phi_j(A', \ell) - \phi_j(A, \ell)$. Clearly, this also follows from the equal treatment property of the Shapley rule (Corollary 14).

5. Complexity of computing the Shapley value

Even though liability games are constant-sum games and we showed in (12) and (13) that the Shapley value of a liability game can be directly calculated from the parameters of the underlying liability problem, now we prove that calculating the Shapley payoff to the firm is NP-hard[3].

Theorem 19. *Given two liability problems and the induced liability games, it is NP-hard to verify whether the firm has the same Shapley value in both games.*

Proof. Recall the NP-complete subset sum problem SUBSUM (See for instance Garey & Johnson, 1979): given $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ and $K \in \mathbb{Z}$ we ask whether there exists a subset $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ such that $\sum a_{i_j} = K$. Here we consider a special case of this problem, HALFSUM: given positive integers $a_1, a_2, \ldots, a_n$ we ask whether there exists a subset $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ such that $\sum a_{i_j} = \frac{\sum a_i}{2}$. It is very easy to show by the following steps that HALFSUM is still NP-complete.

- It is trivial to show that SUBSUM is NP-complete if we restrict it to even numbers, so we can assume that $\sum a_i$ is even.
- We get an equivalent instance of SUBSUM if we replace K by $\sum a_i - K$. Using this observation, it is clear that we can assume that $K \le \frac{\sum a_i}{2}$.
- This special form of SUBSUM can be reduced to HALFSUM by adding an extra number $a_{n+1} = \frac{\sum a_i}{2} - K$ to the set.

We reduce HALFSUM to the Shapley value calculation. Let $HS = (a_1, a_2, \ldots, a_n)$ be an instance of the HALFSUM problem. Consider the liability problems (A, ℓ) and $(A-1, \ell)$, where $\ell = (\ell_1, \ell_2, \ldots, \ell_n) = (a_1, a_2, \ldots, a_n)$ and $A = \frac{\sum a_i}{2}$. Let v and v_2 be the liability games corresponding to (A, ℓ) and $(A-1, \ell)$, respectively. We show that the defaulting firm has a different Shapley value in v and v_2 if and only if the instance of the HALFSUM problem has a solution.

Given a subset of creditors $S \subseteq C$, let $\mathrm{mc}(S) = v(S \cup \{0\}) - v(S)$ be the marginal contribution of player 0 in the liability game v, corresponding to the first liability problem. We claim that

$$\mathrm{mc}(S) = \begin{cases} \ell(S), & \text{if } \ell(S) \le A, \\ \ell(C \setminus S), & \text{if } \ell(S) \ge A. \end{cases} \tag{28}$$

To prove (28), recall that the value of the assets A is exactly half of the sum of liabilities. Notice that creditors in S can be paid if and only if creditors in $C \setminus S$ cannot be paid. If $\ell(S) \le A$, then $v(S) = 0$, however, in this case $v(S \cup \{0\}) = \ell(S)$. If $\ell(S) \ge A$, then $v(S) = A - \ell(C \setminus S)$ and $v(S \cup \{0\}) = A$.

[3] Aziz (2013) shows that for regular bankruptcy problems, the computation of the Shapley value is #P-complete.

Let ϕ_0 be the Shapley value of player 0 in v. We have that

$$n!\phi_0 = \sum_{S\subseteq C} |S|!(n-|S|-1)!\mathrm{mc}(S) = \sum_{\ell(S)<A} |S|!(n-|S|-1)!\ell(S)$$
$$+ \sum_{\ell(S)=A} |S|!(n-|S|-1)!A + \sum_{\ell(S)>A} |S|!(n-|S|-1)!\ell(C \setminus S). \tag{29}$$

Now consider the game v_2, that is, decrease the asset value A by 1. Let $\mathrm{mc}_2(S) = v_2(S \cup \{0\}) - v_2(S)$.

If S is a coalition such that $\ell(S) < A$, then $\ell(S) \le A - 1$, so the liabilities in S can still be paid in v_2 and $\ell(C \setminus S) > A > A - 1$, liabilities in $C \setminus S$ obviously cannot be paid with less asset value. It follows that $v_2(S) = 0$ and $v_2(S \cup \{0\}) = \ell(S)$. (Recall that ℓ is the same in both problems.) Now let's consider a coalition of creditors $S \subset C$ such that $\ell(S) > A$. In this case $\ell(C \setminus S) < A$, that is, $\ell(C \setminus S) \le A - 1$. Liabilities in S cannot be paid and liabilities in $C \setminus S$ can be paid not only in game v but also in game v_2. This means that $v_2(S) = A - 1 - \ell(C \setminus S)$ and $v_2(S \cup \{0\}) = A - 1$, so $\mathrm{mc}(S) = (A-1) - (A - 1 - \ell(C \setminus S) = \ell(C \setminus S)$.

It follows that in (29), the first and the last term do not change in v_2, implying that if HS is a FALSE instance of problem HALFSUM, then the sum of these terms does not change when we decrease the value of assets by 1. In this case, the second term is empty.

On the other hand, let's consider a coalition where $\ell(S) = A$ exactly. In this case, $v(S) = 0$ and $v(S \cup \{0\}) = \mathrm{mc}(S) = A$ in the first game. However, in the second game, $v_2(S) = v(S) = 0$ but $v_2(S \cup \{0\}) = \mathrm{mc}_2(S) = A - 1$. If HS is a TRUE instance of the HALFSUM problem, then the Shapley value of player 0 decreased in game v_2 compared to game v. □

6. Concluding remarks

Liability games are constant-sum transferable utility games, generalizing bankruptcy games by treating the estate (firm) as a player. We investigate the Shapley value of liability games. We propose a basis for the linear vector space of constant-sum games that provides a specialized formula for the Shapley payoff to a player in a constant-sum game. We show that the Shapley value can also be used as a liability allocation rule, that is, it allocates the asset value non-negatively among the creditors and the firm in such a way that no creditor gets more than his liability. We establish lower and upper bounds for the Shapley payments to the creditors as well as to the firm. On top of proving order preservation, we establish three main monotonicity properties of the Shapley rule: liability monotonicity, asset monotonicity for creditors, and super-modularity for creditors. Finally, we show that in liability problems calculating the Shapley payment to the insolvent firm is NP-hard.

Structurally, liability games and bankruptcy games are intimately connected: the subgame of a liability game restricted to the set of creditors is a bankruptcy game (as defined by O'Neill, 1982) and the other "half-game" (on the coalitions containing the firm) is the dual game of that bankruptcy subgame. One may wonder if there is any relation between the Shapley value in a liability game and the Shapley value in its bankruptcy subgame or in another "naturally associated" bankruptcy game. Based on Example 20, we do not believe that any "simple structural" relation could be found.

Example 20. We take two-creditor liability games (the simplest non-trivial type) and associate with each one two two-creditor bankruptcy games:

- Type A: we only reduce the asset value with the Shapley payment to the firm, but keep the original claims of the creditors;
- Type B: we reduce the asset value with the Shapley payment to the firm, and we also reduce the claims by half of

the firm's payment (assuming that the creditors accept this amount as a fixed loss, split it equally, and reduce their claims accordingly).

Consider the following three instances, differing only in the asset value.

1. The Shapley payments to the creditors in the liability situation are different from the Shapley payments in both bankruptcy situations:

liability	$A = 36,\ \ell_1 = 34,\ \ell_2 = 32$	$\varphi_0 = 10,$	$\varphi_1 = 14,\ \varphi_2 = 12$
bankruptcy A	$E = 26,\ c_1 = 34,\ c_2 = 32$		$\varphi_1^A = 13,\ \varphi_2^A = 13$
bankruptcy B	$E = 26,\ c_1 = 29,\ c_2 = 27$		$\varphi_1^B = 13,\ \varphi_2^B = 13$

2. The creditor's Shapley payments in the liability situation are different from the Shapley payments in the type A bankruptcy situation, but coincide with those in the type B bankruptcy situation:

liability	$A = 39,\ \ell_1 = 34,\ \ell_2 = 32$	$\varphi_0 = 9,$	$\varphi_1 = 16,\ \varphi_2 = 14$
bankruptcy A	$E = 30,\ c_1 = 34,\ c_2 = 32$		$\varphi_1^A = 15,\ \varphi_2^A = 15$
bankruptcy B	$E = 30,\ c_1 = 59/2,\ c_2 = 55/2$		$\varphi_1^B = 16,\ \varphi_2^B = 14$

3. The Shapley payments to the creditors in the liability situation are the same as the Shapley payments in both associated bankruptcy situations:

liability	$A = 42,\ \ell_1 = 34,\ \ell_2 = 32$	$\varphi_0 = 8,$	$\varphi_1 = 18,\ \varphi_2 = 16$
bankruptcy A	$E = 34,\ c_1 = 34,\ c_2 = 32$		$\varphi_1^A = 18,\ \varphi_2^A = 16$
bankruptcy B	$E = 34,\ c_1 = 30,\ c_2 = 28$		$\varphi_1^B = 18,\ \varphi_2^B = 16$

However, there are many possibilities for further research. One could investigate the analogues of the various other monotonicity properties discussed in the rich literature on bankruptcy problems, see the book (Thomson, 2019) for a detailed treatment. This, together with our basis, could help to get a new characterization of the Shapley value on the class of liability or constant-sum games. There are also other important values to be considered in a liability game, for instance the Banzhaf value (Banzhaf, 1965) or the solidarity value (Nowak & Radzik, 1994). Liability games could be generalized to a setting with nontransferable utility, when players have individual utility functions over their monetary payoffs. Analyzing generalizations of solutions concepts in such a setting is also very promising.

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