k-stretchability of entanglement, and the duality of k-separability and k-producibility

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The notions of k-separability and k-producibility are useful and expressive tools for the characterization of entanglement in multipartite quantum systems, when a more detailed analysis would be infeasible or simply needless. In this work we reveal a partial duality between them, which is valid also for their correlation counterparts. This duality can be seen from a much wider perspective, when we consider the entanglement and correlation properties which are invariant under the permutations of the subsystems. These properties are labeled by Young diagrams, which we endow with a refinement-like partial order, to build up their classification scheme. This general treatment reveals a new property, which we call k-stretchability, being sensitive in a balanced way to both the maximal size of correlated (or entangled) subsystems and the minimal number of subsystems uncorrelated with (or separable from) one another.

Contents

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1   Introduction

The investigation of the *correlations* among the parts of a composite physical system is an essential tool in statistical physics. If the system is described by quantum mechanics, then nonclassical forms of correlations arise, the most notable is *entanglement* [1–4]. It is the main resource of *quantum information theory* [5–7], and its nonclassical properties, playing important role also in many-body physics [8–10], make it influential in the behavior and characterization of *strongly correlated systems* [11–14].

The correlation and entanglement between *two parts* of a system is relatively well-understood [15–17]. At least for pure states, the *convertibility* and the *classification* with respect to (S)LOCC ((stochastic) local operations and classical communication [15, 17–19]) shows a simple structure [20, 21], and there are basically unique correlation and entanglement *measures* [22–24]. The *multipartite* case is much more complicated [19, 25–28]. Even for pure states, the nonexistence of a maximally entangled reference state [29, 30]

and the involved nature of state-transformations in general [31, 32] seem to make the (S)LOCC-based *classification* practically unaccomplishable, and the standard (S)LOCC paradigm less enlightening so less expressive. Taking into account partial entanglement (partial separability) [33–40], or partial correlations [14, 41, 42] only, leads to a combinatoric [43–45], discrete classification (based on the *lattice of set partitions* [44]), endowed naturally with a well-behaving set of correlation and entanglement measures, characterizing the finite number of properties [14, 40].

Even the partial correlation and partial entanglement properties are getting too involved rapidly, with the increasing of the number of subsystems. Singling out particularly expressive properties, we consider the *k-partitionability* and *k-producibility* of correlation and of entanglement. For n subsystems, k ranges from 1 to n, so the number of these properties scales linearly with the number of subsystems, moreover, their structures are the simplest possible ones, chains. (In the case of entanglement, *k*-partitionability is called *k-separability* [35, 37], while *k*-producibility [46–49] is also called *entanglement depth* [50–52]. Here we use both concepts for correlation and also for entanglement, this is why we use the naming "*k*-partitionability of correlation" and "*k*-producibility of correlation", "*k*-partitionability of entanglement" and "*k*-producibility of entanglement" [14, 40, 41].) These characterize the strength of two different (one-parameter-) aspects of multipartite correlation and entanglement: those which cannot be restricted inside at least k parts, and those which cannot be restricted inside parts of size at most k, respectively. The concepts of *k*-partitionability and *k*-producibility of entanglement found application in spin chains [47, 48], appeared in quantum nonlocality [53], leading to device independent certification of them [54, 55], and were also demonstrated in quantum optical experiment [56]. *k*-producibility also plays particularly important role in quantum metrology [57]. *k*-producibly entangled states for larger k lead to higher sensitivity, so better precision in phase estimation, which has been illustrated in experiments [58–60], and which also leads to *k*-producibility entanglement criteria [61].

k-partitionability and *k*-producibility are special cases of properties *invariant under the permutation of the subsystems*. The permutation invariant correlation properties are based on the *integer partitions* [45, 62], also known (represented) as Young diagrams. For the description of the structure of these, we introduce a new order over the integer partitions, called refinement, induced by the refinement order over set partitions, used for the description of the structure of the partial correlation or entanglement properties [40]. The structure of the permutation invariant properties contains the chains of *k*-partitionability and *k*-producibility, and it is simpler than the structure

describing all the properties. The number of these scales still rapidly with the number of subsystems, but slower than the number of all the partial correlation or entanglement properties.

The general treatment of the permutation invariant properties reveals a partial duality between *k*-partitionability and *k*-producibility, which is the manifestation of a duality on a deeper level, relating important properties of Young diagrams, by which *k*-partitionability and *k*-producibility are formulated.

The general treatment of the permutation invariant properties reveals also a particularly expressive new property, which we call *k-stretchability*, leading to the definitions of "*k*-stretchability of correlation" and "*k*-stretchability of entanglement". It combines the advantages of *k*-partitionability and *k*-producibility. Namely, *k*-partitionability is about the number of subsystems uncorrelated with (or separable from) one another, and not sensitive to the size of correlated (or entangled) subsystems [47]; and *k*-producibility is about the size of the largest correlated (or entangled) subsystem, and not sensitive to the number of subsystems uncorrelated with (or separable from) one another; while *k*-stretchability is sensitive to both of these, in a balanced way. The price to pay for this is that we have roughly twice as many *k*-stretchability properties, k goes from $-(n-1)$ to $n-1$. However, *k*-stretchability is linearly ordered with k, while the relations between *k*-partitionability and *k*-producibility are far more complicated.

The organization of this work is as follows. In Section 2, we recall the structure of multipartite correlation and entanglement. In Section 3, we construct the parallel structure for the permutation invariant case. In Section 4, we show another introduction of the permutation invariant classification, which can although be considered simpler, but less transparent. In Section 5, we recall *k*-partitionability and *k*-producibility, introduce *k*-stretchability, and show how these properties are related to each other. In Section 6, summary, remarks and open questions are listed. In Appendix A, we work out the "coarsening" step and some other tools, used in the main text. In Appendix B, we present the proofs of some further propositions given in the main text.

In the course of the presentation, the abstract mathematical structure of the correlation and entanglement properties (given in terms of lattice theoretic constructions) is well-separated from the concrete hierarchies of the state sets and measures. This leads to a transparent construction, which is easy to restrict to the permutation invariant case later. For the convenience of the reader, we provide a short summary on the notations. The abstract correlation and entanglement properties are labeled by Greek letters ξ, υ. On the three levels of the construction, these are typesetted as $\xi, \boldsymbol{\xi}, \underline{\xi}$ for the general case, and as $\hat{\xi}, \hat{\boldsymbol{\xi}}, \hat{\underline{\xi}}$ for the permutation invariant case. On the first two

levels of the construction, there are sets of quantum states of given correlation and entanglement properties, typesetted as $\mathcal{D}_{\xi\text{-unc}}$, $\mathcal{D}_{\xi\text{-sep}}$, $\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$, $\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$ for the general case, and $\mathcal{D}_{\hat{\xi}\text{-unc}}$, $\mathcal{D}_{\hat{\xi}\text{-sep}}$, $\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}}$, $\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-sep}}$ for the permutation invariant case. On the first two levels of the construction, there are LO(CC)-monotonic measures of given correlation and entanglement properties, typesetted as C_ξ, E_ξ, $C_{\boldsymbol{\xi}}$, $E_{\boldsymbol{\xi}}$ for the general case, and $C_{\hat{\xi}}$, $E_{\hat{\xi}}$, $C_{\hat{\boldsymbol{\xi}}}$, $E_{\hat{\boldsymbol{\xi}}}$ for the permutation invariant case. On the third level of the construction, there are classes (disjoint sets) of quantum states of given correlation and entanglement properties, typesetted as $\mathcal{C}_{\boldsymbol{\xi}\text{-unc}}$, $\mathcal{C}_{\boldsymbol{\xi}\text{-sep}}$ for the general case, and $\mathcal{C}_{\hat{\boldsymbol{\xi}}\text{-unc}}$, $\mathcal{C}_{\hat{\boldsymbol{\xi}}\text{-sep}}$ for the permutation invariant case. The abstract correlation and entanglement properties are ordered, which is denoted by $\preceq$ on the three levels of the construction in both the general and the permutation invariant cases. On the first two levels of the construction, thanks to the monotonicity properties, these manifest themselves as $\subseteq$ and $\geq$ for the state sets and for the measures, respectively. On the third level of the construction, these manifest themselves as LO(CC) conversion results, for which we do not introduce notation.

2 Multipartite correlation and entanglement

Here we recall the structure of the classification and quantification of multipartite correlation and entanglement [14, 40, 41]. Our goal is to do this in the way sufficient to see how the permutation invariant properties can be formulated parallel to this in the next section.

2.1 Level 0: subsystems

The classification scheme we present here is rather general. The elementary and composite subsystems can be any discrete finite systems possessing probabilistic description, supposed that the joint systems can be represented by the use of *tensor products*, which is the basic tool in the constructions. Such systems can be distinguishable quantum systems, second quantized bosonic systems, second quantized fermionic systems with fermion number parity superselection rule imposed, or even classical systems, with significant simplification in the structure in the latter case [41].

Let L be the set of the labels of $|L| = n$ *elementary subsystems.* All the (possibly composite) *subsystems* are then labeled by the subsets $X \subseteq L$, the set of which, $P_0 := 2^L$, naturally possesses a Boolean lattice structure with respect to the inclusion $\subseteq$, which is now denoted with $\preceq$. For every elementary subsystem $i \in L$, let the Hilbert space $\mathcal{H}_i$ be associated with it, where $1 < \dim \mathcal{H}_i < \infty$. From these, for ev-

ery subsystem $X \in P_0$, the Hilbert space associated with it is $\mathcal{H}_X = \bigotimes_{i \in X} \mathcal{H}_i$. (For the trivial subsystem $X = \emptyset$, we have the one-dimensional Hilbert space $\mathcal{H}_\emptyset = \text{Span}\{|\psi\rangle\} \cong \mathbb{C}$ [40].) The *states* of the subsystems $X \in P_0$ are given by density operators (positive semidefinite operators of trace 1) acting on $\mathcal{H}_X$, the sets of those are denoted with $\mathcal{D}_X$.

The *mixedness* of a state $\varrho_X \in \mathcal{D}_X$ of a subsystem X can be characterized by the *von Neumann entropy* [7, 63–65]

$$S(\varrho_X) = - \text{Tr}\, \varrho_X \ln \varrho_X; \qquad (1a)$$

and the *distinguishability* of two states $\varrho_X, \sigma_X \in \mathcal{D}_X$ of subsystem X can be characterized by the *Umegaki relative entropy* [6, 7, 64–68]

$$D(\varrho_X \| \sigma_X) = \text{Tr}\, \varrho_X (\ln \varrho_X - \ln \sigma_X). \qquad (1b)$$

The von Neumann entropy is monotone increasing in bistochastic quantum channels, and the relative entropy is monotone decreasing in quantum channels [6, 65, 67] and $S(\varrho_X) = 0 \Leftrightarrow \varrho = |\psi\rangle\langle\psi|$, $D(\varrho_X \| \sigma_X) = 0 \Leftrightarrow \varrho_X = \sigma_X$.

2.2 Level I: set partitions

For handling the different possible splits of a composite system into subsystems, we need to use the mathematical notion of (set) partition [44] of the system L. The *partitions* of L are sets of subsystems,

$$\xi = \{X_1, X_2, \ldots, X_{|\xi|}\} \in \Pi(L), \qquad (2a)$$

where the *parts*, $X \in \xi$, are nonempty disjoint subsystems, for which $\bigcup_{X \in \xi} X = L$. The set of the partitions of L is denoted with

$$P_{\text{I}} := \Pi(L). \qquad (2b)$$

Its size is given by the *Bell numbers* [69], which are rapidly growing with n. There is a natural partial order over the partitions, which is called *refinement* $\preceq$, given as

$$\upsilon \preceq \xi \overset{\text{def.}}{\iff} \forall Y \in \upsilon, \exists X \in \xi \text{ s.t. } Y \subseteq X. \qquad (2c)$$

(For illustrations, see Figure 1.) This grabs our natural intuition of comparing splits of systems. Note that the refinement is a partial order only, there are pairs of partitions which cannot be ordered, for example, $12|3 \npreceq 13|2$ and $12|3 \nsucceq 13|2$, see Figure 1. (We use a simplified notation for the partitions, for example, $12|3 \equiv \{\{1, 2\}, \{3\}\}$, where this does not cause confusion.) Note that P_{I} with the refinement is a lattice, with minimal and maximal elements $\bot = \{\{i\} \mid i \in L\}$ and $\top = \{L\}$, and the least upper and greatest lower bounds, $\vee$ and $\wedge$, can be constructed [44, 45].

With respect to the partitions $\xi \in P_{\text{I}}$, we can define the partial correlation and entanglement properties, as well as the measures quantifying them.

The ξ-*uncorrelated states* are those which are products with respect to the partition ξ,

$$\mathcal{D}_{\xi\text{-unc}} := \left\{ \varrho_L \in \mathcal{D}_L \mid \forall X \in \xi, \exists \varrho_X \in \mathcal{D}_X : \varrho_L = \bigotimes_{X \in \xi} \varrho_X \right\}; \tag{3a}$$

the others are ξ-*correlated states*. With respect to the finest partition $\xi = \bot$, we call $\bot$-uncorrelated states $\bigotimes_{i \in L} \varrho_i$ simply uncorrelated states. The expectation values of all ξ-local observables factorize if and only if the state is ξ-uncorrelated. The ξ-uncorrelated states are exactly those which can be prepared from uncorrelated states by ξ-local operations. (This is abbreviated as ξ-LO. With respect to the finest partition $\xi = \bot$, we write simply LO.) The ξ-*separable states* are convex combinations (or statistical mixtures) of ξ-uncorrelated states,

$$\mathcal{D}_{\xi\text{-sep}} := \text{Conv}\,\mathcal{D}_{\xi\text{-unc}} \equiv \left\{ \sum_j p_j \varrho_j \mid \varrho_j \in \mathcal{D}_{\xi\text{-unc}} \right\}; \tag{3b}$$

the others are ξ-*entangled states*. (Here, and in the entire text, $\{p_j \mid j = 1, 2, \ldots, m\}$ is a probability distribution, $p_j \geq 0$ and $\sum_j p_j = 1$.) The ξ-separable states are exactly those which can be prepared from uncorrelated states by ξ-local operations and classical communication among the parts $X \in \xi$. (This is abbreviated as ξ-LOCC. With respect to the finest partition $\bot$, we write simply LOCC.) Another point of view is that ξ-separable states are exactly those which can be prepared from uncorrelated states by mixtures of ξ-LOs. We also have that $\mathcal{D}_{\xi\text{-unc}}$ is closed under ξ-LO, $\mathcal{D}_{\xi\text{-sep}}$ is closed under ξ-LOCC [40, 41]. Clearly, if a state is product with respect to a partition, then it is also product with respect to any coarser partition, it is always free to forget about some tensor product signs. This means that these properties show the same lattice structure as the partitions [40, 41], P_I, that is,

$$\upsilon \preceq \xi \iff \mathcal{D}_{\upsilon\text{-unc}} \subseteq \mathcal{D}_{\xi\text{-unc}}, \tag{4a}$$

$$\upsilon \preceq \xi \iff \mathcal{D}_{\upsilon\text{-sep}} \subseteq \mathcal{D}_{\xi\text{-sep}}. \tag{4b}$$

One can define the corresponding (information-geometry based) correlation and entanglement measures [14, 40] for all ξ-correlation and ξ-entanglement. These are the most natural generalizations of the *mutual information* [6, 7], the *entanglement entropy* [16], and the *entanglement of formation* [17] for Level I of the multipartite case.

The ξ-*correlation* of a state ϱ is its distinguishability by the relative entropy (1b) from the ξ-uncorrelated states [4, 14, 40, 70],

$$C_\xi(\varrho) := \min_{\sigma \in \mathcal{D}_{\xi\text{-unc}}} D(\varrho\|\sigma) = \sum_{X \in \xi} S(\varrho_X) - S(\varrho), \tag{5a}$$

given in terms of the von Neumann entropies (1a) of the reduced, or marginal states $\varrho_X = \text{Tr}_{L\setminus X}\, \varrho$ of
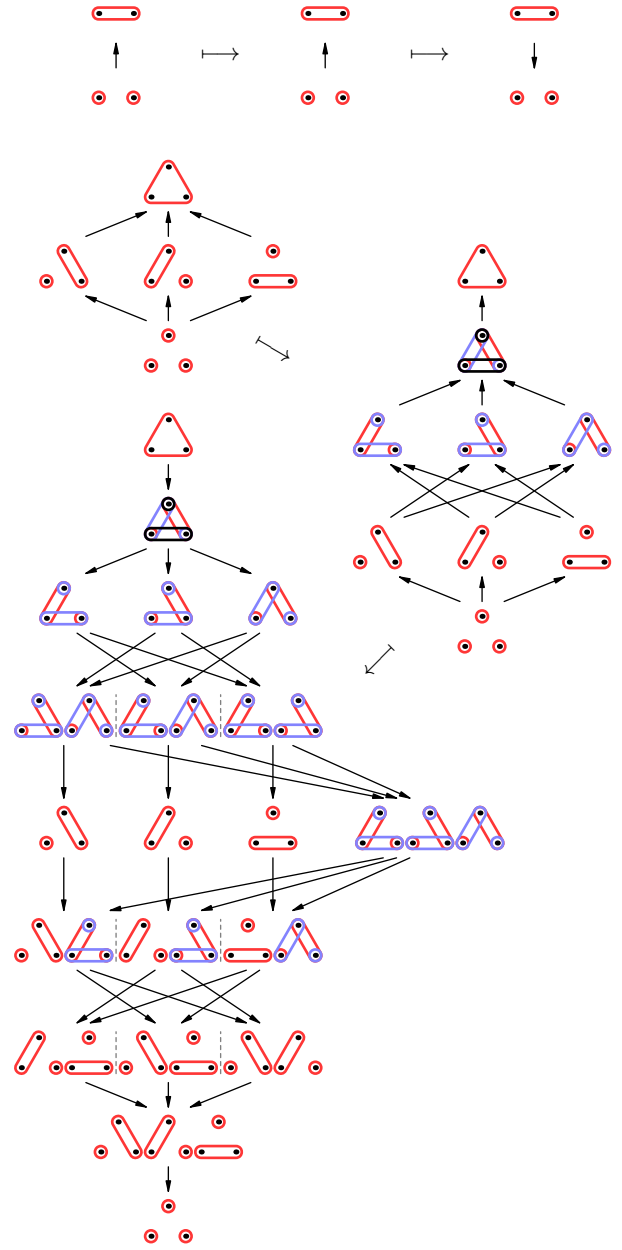

Figure 1: The lattices of the three-level structure of multipartite correlation and entanglement for $n = 2$ and 3. Only the maximal elements of the down-sets of P_I are shown (with different colors) in P_II, while only the minimal elements of the up-sets of P_II are shown (side by side) in P_III. The partial orders (2c), (7c) and (12c) are represented by consecutive arrows.

subsystems $X \in \xi$. Pure states of classical systems are always uncorrelated, the correlation in pure states is of quantum origin, this is what we call entanglement [1–3, 14, 15, 40, 41]. So, for pure states, the measure of entanglement should be that of correlation. The ξ-*entanglement* of a pure state $\pi = |\psi\rangle\langle\psi| \in \mathcal{D}_L$ is its ξ-correlation,

$$E_\xi(\pi) := C_\xi\big|_{\text{pure}}(\pi) = \sum_{X \in \xi} S(\pi_X), \tag{5b}$$

with $\pi_X = \text{Tr}_{L\setminus X}\, \pi$, and for mixed states, one can use,

for example, the convex roof extension [17, 71, 72] to define the ξ-*entanglement of formation*

$$E_\xi(\varrho) := \left(C_\xi\big|_{\text{pure}}\right)^\cup(\varrho)$$
$$\equiv \min\left\{\sum_j p_j C_\xi\big|_{\text{pure}}(\pi_j) \,\Big|\, \sum_j p_j \pi_j = \varrho\right\}, \quad (5c)$$

where $\sum_j p_j \pi_j$ is a pure decomposition of the state [40]. We have that C_ξ is a correlation monotone (not increasing with respect to ξ-LO, for the proof see Appendix B.1), E_ξ is a strong entanglement monotone (convex and not increasing on average with respect to selective ξ-LOCC [22, 23, 40]), and both of these are faithful, $C_\xi(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\xi\text{-unc}}$, $E_\xi(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\xi\text{-sep}}$ [40], moreover, they show the same lattice structure as the partitions, P_{I}, that is,

$$v \preceq \xi \iff C_v \geq C_\xi, \quad (6a)$$
$$v \preceq \xi \iff E_v \geq E_\xi, \quad (6b)$$

which is called *multipartite monotonicity* [14, 40].

Note that, for two subsystems, for the only nontrivial partition $1|2$, we have that $E_{1|2}(\pi) = 2S(\pi_1) = 2S(\pi_2)$ is just two times the usual *entanglement entropy*. However, we have a way of derivation completely different than the usual, based on asymptotic LOCC convertibility from Bell-pairs [16]. The usual way cannot be generalized to the multipartite scenario (there is no "reference state", which was the Bell-pair in the bipartite scenario), while our correlational, or statistical physical approach above could straightforwardly be generalized to the multipartite scenario not only here, but also for the Level II properties in the next subsystem.

2.3 Level II: multiple set partitions

In multipartite entanglement theory, it is necessary to handle mixtures of states uncorrelated with respect to different partitions [35, 37, 40]. For example, there are tripartite states which cannot be written as a mixture of a given kind of, e.g., $12|3$-uncorrelated states, that is, not $12|3$-separable, while can be written as a mixture of $12|3$-uncorrelated and $13|2$-uncorrelated states. Such states should not be considered fully tripartite-entangled, since there is no need for genuine tripartite entangled states in the mixture [35, 37, 39, 40]. Also, if a state is $12|3$-separable and also $13|2$-separable, it is not necessarily $1|2|3$-separable. On the other hand, the order isomorphisms (4a)-(4b) tell us that if we consider states uncorrelated (or separable) with respect to a partition, then we automatically consider states uncorrelated (or separable) with respect to all finer partitions.

To embed these requirements in the labeling of the multipartite correlation and entanglement properties, we use the *nonempty down-sets (nonempty ideals) of partitions* [40], which are sets of partitions

$$\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_{|\boldsymbol{\xi}|}\} \subseteq P_{\text{I}} \quad (7a)$$

which are closed downwards with respect to $\preceq$. That is, if $\xi \in \boldsymbol{\xi}$, then for all partitions $v \preceq \xi$ we have $v \in \boldsymbol{\xi}$. The set of the nonempty partition ideals is denoted with

$$P_{\text{II}} := \mathcal{O}_\downarrow(P_{\text{I}}) \setminus \{\emptyset\}. \quad (7b)$$

This possesses a lattice structure with respect to the standard inclusion as partial order, which we call *refinement* again, and denote with $\preceq$ again,

$$\boldsymbol{v} \preceq \boldsymbol{\xi} \quad \overset{\text{def.}}{\iff} \quad \boldsymbol{v} \subseteq \boldsymbol{\xi}. \quad (7c)$$

(For illustrations, see Figure 1.) The least upper and greatest lower bounds, $\vee$ and $\wedge$, are then the union and the intersection, $\cup$ and $\cap$.

With respect to the partition ideals $\boldsymbol{\xi} \in P_{\text{II}}$, we can define the partial correlation and entanglement properties, as well as the measures quantifying these. The $\boldsymbol{\xi}$-*uncorrelated states* are those which are ξ-uncorrelated (3a) with respect to a $\xi \in \boldsymbol{\xi}$,

$$\mathcal{D}_{\boldsymbol{\xi}\text{-unc}} := \bigcup_{\xi \in \boldsymbol{\xi}} \mathcal{D}_{\xi\text{-unc}}; \quad (8a)$$

the others are $\boldsymbol{\xi}$-*correlated states*. The $\boldsymbol{\xi}$-uncorrelated states are exactly those which can be prepared from uncorrelated states by ξ-LO for a partition $\xi \in \boldsymbol{\xi}$. The $\boldsymbol{\xi}$-*separable states* are convex combinations of $\boldsymbol{\xi}$-uncorrelated states,

$$\mathcal{D}_{\boldsymbol{\xi}\text{-sep}} := \text{Conv}\, \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}; \quad (8b)$$

the others are $\boldsymbol{\xi}$-*entangled states*. The $\boldsymbol{\xi}$-separable states are exactly those which can be prepared from uncorrelated states by mixtures of ξ-LOs for different partitions $\xi \in \boldsymbol{\xi}$. (Note that such transformations do not form a semigroup.) We also have that $\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$ is closed under LO, $\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$ is closed under LOCC [40, 41]. It follows from (4) that these properties show the same lattice structure as the partition ideals [40], P_{II}, that is,

$$\boldsymbol{v} \preceq \boldsymbol{\xi} \iff \mathcal{D}_{\boldsymbol{v}\text{-unc}} \subseteq \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}, \quad (9a)$$
$$\boldsymbol{v} \preceq \boldsymbol{\xi} \iff \mathcal{D}_{\boldsymbol{v}\text{-sep}} \subseteq \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}. \quad (9b)$$

One can define the corresponding (information-geometry based) correlation and entanglement measures [14, 40] for all $\boldsymbol{\xi}$-correlation and $\boldsymbol{\xi}$-entanglement. These are the most natural generalizations of the *mutual information* [6, 7], the *entanglement entropy* [16], and the *entanglement of formation* [17] for Level II of the multipartite case.

The $\boldsymbol{\xi}$-*correlation* of a state ϱ is its distinguishability by the relative entropy (1b) from the $\boldsymbol{\xi}$-uncorrelated states [14, 40],

$$C_{\boldsymbol{\xi}}(\varrho) := \min_{\sigma \in \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}} D(\varrho||\sigma) = \min_{\xi \in \boldsymbol{\xi}} C_\xi(\varrho). \quad (10a)$$

With the same reasoning as in Section 2.2, the $\boldsymbol{\xi}$-*entanglement* of a pure state is

$$E_{\boldsymbol{\xi}}(\pi) := C_{\boldsymbol{\xi}}\big|_{\text{pure}}(\pi), \quad (10b)$$

and for mixed states, one can use the convex roof extension to define the $\boldsymbol{\xi}$-*entanglement of formation*

$$E_{\boldsymbol{\xi}}(\varrho) := \big(C_{\boldsymbol{\xi}}\big|_{\text{pure}}\big)^{\cup}(\varrho). \tag{10c}$$

We have that $C_{\boldsymbol{\xi}}$ is a correlation monotone (for the proof, see Appendix B.1), $E_{\boldsymbol{\xi}}$ is a strong entanglement monotone [40], and both of these are faithful, $C_{\boldsymbol{\xi}}(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$, $E_{\boldsymbol{\xi}}(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$ [40], moreover, they show the same lattice structure as the partition ideals, P_{II}, that is,

$$\boldsymbol{v} \preceq \boldsymbol{\xi} \iff C_{\boldsymbol{v}} \geq C_{\boldsymbol{\xi}}, \tag{11a}$$
$$\boldsymbol{v} \preceq \boldsymbol{\xi} \iff E_{\boldsymbol{v}} \geq E_{\boldsymbol{\xi}}, \tag{11b}$$

which is called *multipartite monotonicity* for Level II [14, 40].

2.4 Level III: classes

The partial correlation and entanglement properties form an inclusion hierarchy (9a)-(9b), for example, if a state is $1|2|3$-separable, then it is also $12|3$-separable. We are interested in the labeling of the strict, or exclusive properties, for example, those states which are $12|3$-separable and not $1|2|3$-separable. In general, we would like to determine all the possible nonempty intersections of the state sets $\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$ and $\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$. We call these intersections *partial correlation* and *partial entanglement classes,* containing states of well-defined partial correlation and partial entanglement properties.

To embed these requirements in the labeling of the strict properties of multipartite correlation and entanglement, we use the *nonempty up-sets (nonempty filters) of nonempty down-sets (nonempty ideals) of partitions* [40], which are sets of down-sets of partitions

$$\underline{\boldsymbol{\xi}} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_{|\underline{\boldsymbol{\xi}}|}\} \subseteq P_{\text{II}} \tag{12a}$$

which are closed upwards with respect to $\preceq$. That is, if $\boldsymbol{\xi} \in \underline{\boldsymbol{\xi}}$, then for all partition ideals $\boldsymbol{v} \succeq \boldsymbol{\xi}$ we have $\boldsymbol{v} \in \underline{\boldsymbol{\xi}}$. The set of the nonempty up-sets of nonempty down-sets of partitions of L is denoted with

$$P_{\text{III}} := \mathcal{O}_{\uparrow}(P_{\text{II}}) \setminus \{\emptyset\}. \tag{12b}$$

This possesses a lattice structure with respect to the standard inclusion as partial order, which we call *refinement* again, and denote with $\preceq$ again,

$$\underline{\boldsymbol{v}} \preceq \underline{\boldsymbol{\xi}} \xleftrightarrow{\text{def.}} \underline{\boldsymbol{v}} \subseteq \underline{\boldsymbol{\xi}}. \tag{12c}$$

(For illustrations, see Figure 1.) Note that here, contrary to Level I and II, we call $\underline{\boldsymbol{\xi}}$ finer and $\underline{\boldsymbol{v}}$ coarser. In the generic case, if the inclusion of sets is described by a poset P, then the possible intersections can be described by $\mathcal{O}_{\uparrow}(P)$ (for the proof, see appendix A.2 in [41]). One may make the classification coarser by selecting a subposet $P_{\text{II*}} \subseteq P_{\text{II}}$, with respect to

which the classification is done, $P_{\text{III*}} := \mathcal{O}_{\uparrow}(P_{\text{II*}}) \setminus \{\emptyset\}$ [40, 41].

With respect to the partition ideal filters $\underline{\boldsymbol{\xi}} \in P_{\text{III*}}$, we can define the strict partial correlation and entanglement properties.

The *strictly $\underline{\boldsymbol{\xi}}$-uncorrelated states* are those which are uncorrelated with respect to all $\boldsymbol{\xi} \in \underline{\boldsymbol{\xi}}$, and correlated with respect to all $\boldsymbol{\xi} \in \overline{\underline{\boldsymbol{\xi}}} = P_{\text{II*}} \setminus \underline{\boldsymbol{\xi}}$, so the *class* of these (*partial correlation class*) is

$$\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-unc}} := \bigcap_{\boldsymbol{\xi} \in \overline{\underline{\boldsymbol{\xi}}}} \overline{\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}} \cap \bigcap_{\boldsymbol{\xi} \in \underline{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}. \tag{13a}$$

The *strictly $\underline{\boldsymbol{\xi}}$-separable states* are those which are separable with respect to all $\boldsymbol{\xi} \in \underline{\boldsymbol{\xi}}$, and entangled with respect to all $\boldsymbol{\xi} \in \overline{\underline{\boldsymbol{\xi}}}$, so the *class* of these (*partial separability class*, or *partial entanglement class*) is

$$\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-sep}} := \bigcap_{\boldsymbol{\xi} \in \overline{\underline{\boldsymbol{\xi}}}} \overline{\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}} \cap \bigcap_{\boldsymbol{\xi} \in \underline{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}. \tag{13b}$$

The meaning of the Level III hierarchy could also be clarified. If there exists a $\varrho \in \mathcal{C}_{\underline{\boldsymbol{v}}\text{-unc}}$ and an LO mapping it into $\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-unc}}$, then $\underline{\boldsymbol{v}} \preceq \underline{\boldsymbol{\xi}}$ [40, 41]; and if there exists a $\varrho \in \mathcal{C}_{\underline{\boldsymbol{v}}\text{-sep}}$ and an LOCC mapping it into $\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-sep}}$, then $\underline{\boldsymbol{v}} \preceq \underline{\boldsymbol{\xi}}$ [40]. In this sense, the Level III hierarchy compares the strength of correlation and entanglement among the classes.

The filters $\underline{\boldsymbol{\xi}}$ are *sufficient* for the description of the nontrivial classes (nonempty intersections) of the correlation and entanglement properties, but they are *not necessary* in general. For the strictly $\underline{\boldsymbol{\xi}}$-uncorrelated states, the structure $P_{\text{III*}}$ simplifies significantly [41]. (For example, for the finest classification $P_{\text{II*}} = P_{\text{II}}$, we have that the structure of the partial correlation classes is $P_{\text{III*}} \cong P_{\text{I}}^{\partial}$, as the unique labeling of the nontrivial partial correlation classes is given as $\underline{\boldsymbol{\xi}} = {\uparrow}\{{\downarrow}\{\xi\}\}$ for the partitions $\xi \in P_{\text{I}}$ [41].) On the other hand, it is still a conjecture that $\underline{\boldsymbol{\xi}}$-separability is nontrivial for all $\underline{\boldsymbol{\xi}} \in P_{\text{III}}$ [40]. The conjecture holds for three subsystems, for which explicit examples were constructed [39, 73].

3 Multipartite correlation and entanglement: permutation invariant properties

Here we build up the structure of the classification and quantification of the permutation invariant multipartite correlation and entanglement properties. We do this from the point of view of the general properties in the previous section. To give a quick guidance

on the construction, we show the diagram

$$
\begin{array}{ccc}
(P_{\mathrm{III}}, \preceq) & \overset{s}{\longmapsto} & (\hat{P}_{\mathrm{III}}, \preceq) \\
\Big\uparrow \mathcal{O}_{\uparrow \setminus \{\emptyset\}} & & \Big\uparrow \mathcal{O}_{\uparrow \setminus \{\emptyset\}} \\
(P_{\mathrm{II}}, \preceq) & \overset{s}{\longmapsto} & (\hat{P}_{\mathrm{II}}, \preceq) \\
\Big\uparrow \mathcal{O}_{\downarrow \setminus \{\emptyset\}} & & \Big\uparrow \mathcal{O}_{\downarrow \setminus \{\emptyset\}} \\
(P_{\mathrm{I}}, \preceq) & \overset{s}{\longmapsto} & (\hat{P}_{\mathrm{I}}, \preceq)
\end{array}
\tag{14}
$$

in advance. This might not seem to be obvious, for the proof, see Corollary 17 in Appendix A.4.

In Section 2, for the description of the structure of multipartite correlations [14, 40, 41], the natural language was that of *set partitions* [44]. For the permutation-invariant case, particularly k-partitionability and k-producibility, it is natural to use the simpler language of *integer partitions* [45, 62].

3.1 Level 0: subsystem sizes

Let us have the map, simply measuring the sizes of the subsystems $X \in P_0$,

$$
s(X) := |X|, \tag{15}
$$

mapping $P_0 \to \hat{P}_0$, which is simply

$$
\hat{P}_0 := s(P_0) = \{0, 1, 2, \ldots, n\}. \tag{16}
$$

This possesses the natural order $\leq$, which is now denoted with $\preceq$, with respect to which s is monotone, $Y \preceq X \Rightarrow s(Y) \preceq s(X)$. We note for the sake of the analogy with the construction in the subsequent levels that one could define $\preceq$ on $\hat{P}_0$ as $y \preceq x$ if there exist $Y \in s^{-1}(y)$ and $X \in s^{-1}(x)$ such that $Y \preceq X$, from which the monotonicity of s follows automatically (see (64) in Appendix A.1).

3.2 Level I: integer partitions

Let the map s in (15) act elementwisely on the partitions $\xi \in P_{\mathrm{I}}$,

$$
s(\xi) := \big\{ s(X) \mid \forall X \in \xi \big\}, \tag{17}
$$

where on the right-hand side a *multiset* stays, that is, a set, which allows multiple instances of its elements. (For example, for $\xi = \{\{1\}, \{2, 4\}, \{3\}\}$, $s(\xi) = \{1, 2, 1\} \equiv \{2, 1, 1\}$.) We call the integer partition $s(\xi)$ the *type* of the set partition ξ. This is then an *integer partition* of n [45, 62], which is denoted as

$$
\hat{\xi} = \{x_1, x_2, \ldots, x_{|\hat{\xi}|}\}, \tag{18a}
$$

where the parts $x \in \hat{\xi}$ are nonzero subsystem sizes, for which $\sum_{x \in \hat{\xi}} x = n$. The set of the integer partitions of n is denoted with

$$
\hat{P}_{\mathrm{I}} := s(P_{\mathrm{I}}). \tag{18b}
$$

Its size is given by the *partition numbers* [74]. Based on the refinement of the set partitions (2c), we define a partial order over the integer partitions, which we call *refinement* again, and denote with $\preceq$ agai again, given as

$$
\hat{v} \preceq \hat{\xi} \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \exists v \in s^{-1}(\hat{v}), \xi \in s^{-1}(\hat{\xi}) \ \text{s.t.} \ v \preceq \xi. \tag{18c}
$$

(For illustrations, see Figures 2 and 3. The integer partitions are represented by their Young diagrams, where the rows of boxes are ordered decreasingly.) This is a proper partial order (for the proof, see Appendix A.4). Note that the refinement is a partial order only, there are pairs of partitions which cannot be ordered, for example, $\{2, 2\} \not\preceq \{3, 1\}$ and $\{2, 2\} \not\succeq \{3, 1\}$, see Figure 2. Note that $\hat{P}_{\mathrm{I}}$ with this partial order is *not* a lattice, since there are no unique least upper and greatest lower bounds with respect to that. (The first example is for $n = 5$, where $\{2, 2, 1\}, \{3, 1, 1\} \preceq \{3, 2\}, \{4, 1\}$, and there is no integer partition between them, see Figure 3.) The refinement also admits a minimal and a maximal element $\perp = \{1, 1, \ldots, 1\}$ and $\top = \{n\}$. By construction, s is monotone with respect to these partial orders, $v \preceq \xi \Rightarrow s(v) \preceq s(\xi)$ (see (64) in Appendix A.1).

With respect to the integer partitions $\hat{\xi} \in \hat{P}_{\mathrm{I}}$, we can define the partial correlation and entanglement properties, as well as the measures quantifying them.

The $\hat{\xi}$-*uncorrelated states* are those which are ξ-uncorrelated (3a) with respect to a partition ξ of type $\hat{\xi}$, which is actually a Level II state set (8a) labeled by the ideal $\boldsymbol{\xi} = \downarrow s^{-1}(\hat{\xi})$,

$$
\mathcal{D}_{\hat{\xi}\text{-unc}} := \bigcup_{\xi : s(\xi) = \hat{\xi}} \mathcal{D}_{\xi\text{-unc}} = \mathcal{D}_{\downarrow s^{-1}(\hat{\xi})\text{-unc}}; \tag{19a}
$$

the others are $\hat{\xi}$-*correlated states*. The $\hat{\xi}$-uncorrelated states are exactly those which can be prepared from uncorrelated states by ξ-LO for a partition ξ of type $\hat{\xi}$. (Note that we consider arbitrary states here, not only permutation invariant ones. The correlation property defined in this way is what permutation invariant is.) The $\hat{\xi}$-*separable states* are convex combinations of $\hat{\xi}$-uncorrelated states, which is actually a Level II state set (8b) labeled by the ideal $\boldsymbol{\xi} = \downarrow s^{-1}(\hat{\xi})$,

$$
\mathcal{D}_{\hat{\xi}\text{-sep}} := \operatorname{Conv} \mathcal{D}_{\hat{\xi}\text{-unc}} = \mathcal{D}_{\downarrow s^{-1}(\hat{\xi})\text{-sep}}; \tag{19b}
$$

the others are $\hat{\xi}$-*entangled states*. The $\hat{\xi}$-separable states are exactly those which can be prepared from uncorrelated states by mixtures of ξ-LOs for different partitions ξ of type $\hat{\xi}$. We also have that $\mathcal{D}_{\hat{\xi}\text{-unc}}$ is closed under LO, $\mathcal{D}_{\hat{\xi}\text{-sep}}$ is closed under LOCC, because these hold for the general Level II state sets $\mathcal{D}_{\downarrow s^{-1}(\hat{\xi})\text{-unc}}$ and $\mathcal{D}_{\downarrow s^{-1}(\hat{\xi})\text{-sep}}$, see in Section 2.3. It also follows that these properties show the same partially ordered structure as the integer partitions, $\hat{P}_{\mathrm{I}}$, that is,

$$
\hat{v} \preceq \hat{\xi} \iff \mathcal{D}_{\hat{v}\text{-unc}} \subseteq \mathcal{D}_{\hat{\xi}\text{-unc}}, \tag{20a}
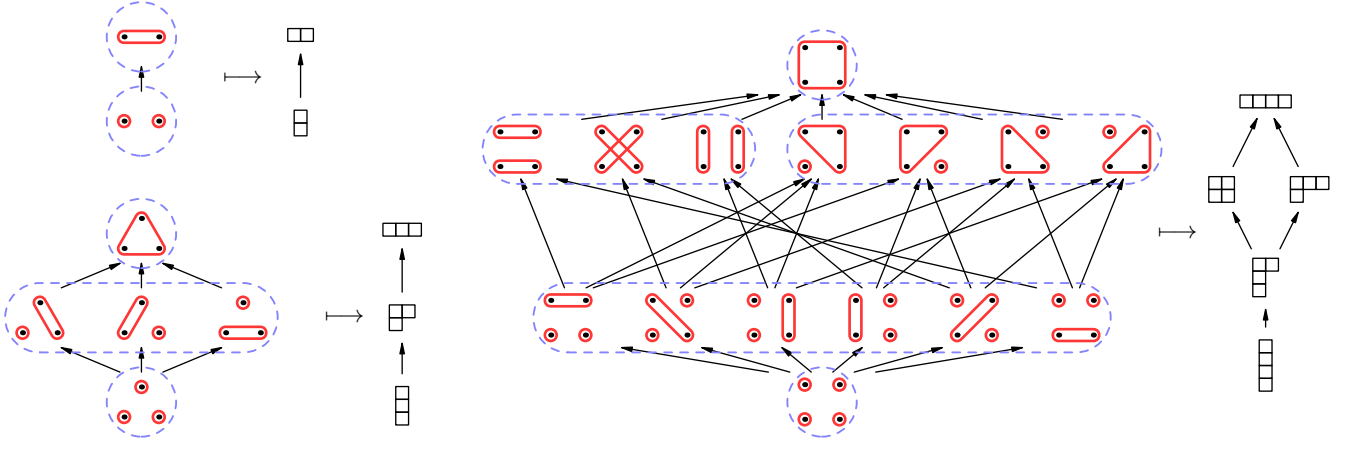$$

Figure 2: The construction of the posets $\hat{P}_\text{I}$ of permutation invariant multipartite correlation and entanglement properties from the lattices P_I for $n = 2, 3$ and 4. The integer partitions (17) are represented by their Young diagrams, the partial order (18c) is represented by consecutive arrows.
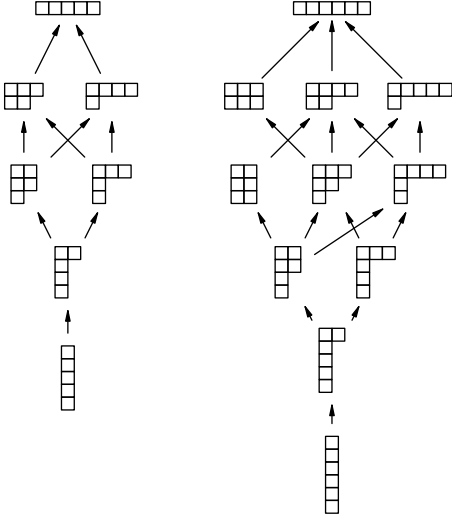

Figure 3: The posets $\hat{P}_\text{I}$ of permutation invariant multipartite correlation and entanglement properties for $n = 5$ and 6. The notation is the same as in Figure 2.

$$\hat{v} \preceq \hat{\xi} \iff \mathcal{D}_{\hat{v}\text{-sep}} \subseteq \mathcal{D}_{\hat{\xi}\text{-sep}}. \tag{20b}$$

(These come by using (74a) and (73a) in Appendix A.4, then (9) and (19).)

In accordance with the general case in Section 2.2, one can define the corresponding (information-geometry based) correlation and entanglement measures for all $\hat{\xi}$-correlation and $\hat{\xi}$-entanglement. These are the most natural generalizations of the *mutual information* [6, 7], the *entanglement entropy* [16], and the *entanglement of formation* [17] for Level I of the permutation invariant multipartite case.

The $\hat{\xi}$-*correlation* of a state ϱ is its distinguishability by the relative entropy (1b) from the $\hat{\xi}$-uncorrelated states, which is actually a Level II measure (10a) labeled by the ideal $\boldsymbol{\xi} = \,\downarrow s^{-1}(\hat{\xi})$,

$$C_{\hat{\xi}}(\varrho) := \min_{\sigma \in \mathcal{D}_{\hat{\xi}\text{-unc}}} D(\varrho||\sigma) = C_{\downarrow s^{-1}(\hat{\xi})}(\varrho). \tag{21a}$$

With the same reasoning as in Section 2.2, the $\hat{\xi}$-*entanglement* of a pure state is

$$E_{\hat{\xi}}(\pi) := C_{\hat{\xi}}\big|_{\text{pure}}(\pi) = E_{\downarrow s^{-1}(\hat{\xi})}(\pi), \tag{21b}$$

and for mixed states, one can use the convex roof extension, which is actually a Level II measure (10c) labeled by the ideal $\boldsymbol{\xi} = \,\downarrow s^{-1}(\hat{\xi})$,

$$E_{\hat{\xi}}(\varrho) := \big(C_{\hat{\xi}}\big|_{\text{pure}}\big)^{\cup}(\varrho) = E_{\downarrow s^{-1}(\hat{\xi})}(\varrho). \tag{21c}$$

These inherit the properties of the general case in Section 2.3. So we have that $C_{\hat{\xi}}$ is a correlation monotone, $E_{\hat{\xi}}$ is a strong entanglement monotone, and both of these are faithful, $C_{\hat{\xi}}(\varrho) = 0 \Leftrightarrow \varrho \in \mathcal{D}_{\hat{\xi}\text{-unc}}$, $E_{\hat{\xi}}(\varrho) = 0 \Leftrightarrow \varrho \in \mathcal{D}_{\hat{\xi}\text{-sep}}$, moreover, they show the same partially ordered structure as the integer partitions, $\hat{P}_\text{I}$, that is,

$$\hat{v} \preceq \hat{\xi} \iff C_{\hat{v}} \geq C_{\hat{\xi}}, \tag{22a}$$

$$\hat{v} \preceq \hat{\xi} \iff E_{\hat{v}} \geq E_{\hat{\xi}}, \tag{22b}$$

which we call *multipartite monotonicity* for Level I of the permutation invariant case. (These come by using (74a) and (73a) in Appendix A.4, then (11) and (21).)

3.3 Level II: multiple integer partitions

Let the map s in (17) act elementwise on the partition ideals $\boldsymbol{\xi} \in P_\text{II}$,

$$s(\boldsymbol{\xi}) := \big\{ s(\xi) \mid \forall \xi \in \boldsymbol{\xi} \big\}, \tag{23}$$

where on the right-hand side a *set* stays. We call $s(\boldsymbol{\xi})$ the *type* of the set partition ideal $\boldsymbol{\xi}$. This is a set of integer partitions, which is denoted as

$$\hat{\boldsymbol{\xi}} = \big\{ \hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_{|\hat{\boldsymbol{\xi}}|} \big\} \subseteq \hat{P}_\text{I}, \tag{24a}$$

which is a nonempty ideal of integer partitions, the set of which is denoted as

$$\hat{P}_\text{II} := s(P_\text{II}) = \mathcal{O}_{\downarrow}(\hat{P}_\text{I}) \setminus \{\emptyset\}. \tag{24b}$$

This might not seem to be obvious, this is given in (14), proven in Corollary 17 in Appendix A.4. In particular, all $\xi \in \boldsymbol{\xi}$ is of type $\hat{\xi}$ for a $\hat{\xi} \in s(\boldsymbol{\xi})$, on the other hand, all $\hat{\xi} \in s(\boldsymbol{\xi})$ has representative $\xi \in \boldsymbol{\xi}$ of type $\hat{\xi}$. Based on the refinement of the set partition ideals (7c), we define a partial order over the integer partition ideals, which we call *refinement* again, and denote with $\preceq$ again, given as

$$\hat{\boldsymbol{v}} \preceq \hat{\boldsymbol{\xi}} \overset{\text{def.}}{\iff} \exists \boldsymbol{v} \in s^{-1}(\hat{\boldsymbol{v}}), \boldsymbol{\xi} \in s^{-1}(\hat{\boldsymbol{\xi}}) \text{ s.t. } \boldsymbol{v} \preceq \boldsymbol{\xi}$$
$$\iff \hat{\boldsymbol{v}} \subseteq \hat{\boldsymbol{\xi}}, \tag{24c}$$

which turns out to be the inclusion, being the natural partial order for ideals. This might not seem to be obvious, this is given in (14), proven in Corollary 17 in Appendix A.4. (For illustrations, see Figure 4.) Because of these, $\hat{P}_{\text{II}}$ with this partial order *is* a lattice. By construction, s is monotone with respect to these partial orders, $\boldsymbol{v} \preceq \boldsymbol{\xi} \Rightarrow s(\boldsymbol{v}) \preceq s(\boldsymbol{\xi})$ (see (64) in Appendix A.1).

With respect to the integer partition ideals $\hat{\boldsymbol{\xi}} \in \hat{P}_{\text{II}}$, we can define the partial correlation and entanglement properties, as well as the measures quantifying these.

The $\hat{\boldsymbol{\xi}}$-*uncorrelated states* are those which are ξ-uncorrelated (3a) with respect to a partition ξ of type $\hat{\xi}$ for a $\hat{\xi} \in \hat{\boldsymbol{\xi}}$, which is actually a Level II state set (8a) labeled by the ideal $\boldsymbol{\xi} = \vee s^{-1}(\hat{\boldsymbol{\xi}}) = \bigvee_{\hat{\xi} \in \hat{\boldsymbol{\xi}}} \downarrow s^{-1}(\hat{\xi})$,

$$\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}} := \bigcup_{\boldsymbol{\xi}: s(\boldsymbol{\xi}) = \hat{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-unc}} = \mathcal{D}_{\vee s^{-1}(\hat{\boldsymbol{\xi}})\text{-unc}}$$
$$= \bigcup_{\boldsymbol{\xi}: s(\boldsymbol{\xi}) \in \hat{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-unc}} = \mathcal{D}_{\bigvee_{\hat{\xi} \in \hat{\boldsymbol{\xi}}} \downarrow s^{-1}(\hat{\xi})\text{-unc}}, \tag{25a}$$

(for the proof, see (73b) in Appendix A.4); the others are $\hat{\boldsymbol{\xi}}$-*correlated states*. (We use the notation $\vee A := \bigvee_{a \in A} a$ for any subset A of a lattice.) The $\hat{\boldsymbol{\xi}}$-uncorrelated states are exactly those which can be prepared from uncorrelated states by ξ-LO for a partition ξ of a type $\hat{\xi} \in \hat{\boldsymbol{\xi}}$. The $\hat{\boldsymbol{\xi}}$-*separable states* are convex combinations of $\hat{\boldsymbol{\xi}}$-uncorrelated states, which is actually a Level II state set (8b) labeled by the ideal $\boldsymbol{\xi} = \vee s^{-1}(\hat{\boldsymbol{\xi}}) = \bigvee_{\hat{\xi} \in \hat{\boldsymbol{\xi}}} \downarrow s^{-1}(\hat{\xi})$,

$$\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-sep}} := \text{Conv}\, \mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}} = \mathcal{D}_{\vee s^{-1}(\hat{\boldsymbol{\xi}})\text{-sep}}, \tag{25b}$$

the others are $\hat{\boldsymbol{\xi}}$-*entangled states*. The $\hat{\boldsymbol{\xi}}$-separable states are exactly those which can be prepared from uncorrelated states by mixtures of ξ-LOs for different partitions ξ of different types $\hat{\xi} \in \hat{\boldsymbol{\xi}}$. We also have that $\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}}$ is closed under LO, $\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-sep}}$ is closed under LOCC, because these hold for the general Level II state sets $\mathcal{D}_{\vee s^{-1}(\hat{\boldsymbol{\xi}})}$ and $\mathcal{D}_{\vee s^{-1}(\hat{\boldsymbol{\xi}})}$, see in Section 2.3. It also follows that these properties show the same lattice structure as the integer partition ideals, $\hat{P}_{\text{II}}$, that is,

$$\hat{\boldsymbol{v}} \preceq \hat{\boldsymbol{\xi}} \iff \mathcal{D}_{\hat{\boldsymbol{v}}\text{-unc}} \subseteq \mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}}, \tag{26a}$$

$$\hat{\boldsymbol{v}} \preceq \hat{\boldsymbol{\xi}} \iff \mathcal{D}_{\hat{\boldsymbol{v}}\text{-sep}} \subseteq \mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-sep}}. \tag{26b}$$

(These come by using (74b) and (73b) in Appendix A.4, then (9) and (25).)

In accordance with the general case in Section 2.3, one can define the corresponding (information-geometry based) correlation and entanglement measures for all $\hat{\boldsymbol{\xi}}$-correlation and $\hat{\boldsymbol{\xi}}$-entanglement. These are the most natural generalizations of the *mutual information* [6, 7], the *entanglement entropy* [16], and the *entanglement of formation* [17] for Level II of the permutation invariant multipartite case.

The $\hat{\boldsymbol{\xi}}$-*correlation* of a state ϱ is its distinguishability by the relative entropy (1b) from the $\hat{\boldsymbol{\xi}}$-uncorrelated states, which is actually a Level II measure (10a) labeled by the ideal $\boldsymbol{\xi} = \vee s^{-1}(\hat{\boldsymbol{\xi}})$,

$$C_{\hat{\boldsymbol{\xi}}}(\varrho) := \min_{\sigma \in \mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}}} D(\varrho \| \sigma) = C_{\vee s^{-1}(\hat{\boldsymbol{\xi}})}(\varrho). \tag{27a}$$

With the same reasoning as in Section 2.3, the $\hat{\boldsymbol{\xi}}$-*entanglement* of a pure state is

$$E_{\hat{\boldsymbol{\xi}}}(\pi) := C_{\hat{\boldsymbol{\xi}}}\big|_{\text{pure}}(\pi) = E_{\vee s^{-1}(\hat{\boldsymbol{\xi}})}(\pi), \tag{27b}$$

and for mixed states, one can use the convex roof extension, which is actually a Level II measure (10c) labeled by the ideal $\boldsymbol{\xi} = \vee s^{-1}(\hat{\boldsymbol{\xi}})$,

$$E_{\hat{\boldsymbol{\xi}}}(\varrho) := \big(C_{\hat{\boldsymbol{\xi}}}\big|_{\text{pure}}\big)^{\cup}(\varrho) = E_{\vee s^{-1}(\hat{\boldsymbol{\xi}})}(\varrho). \tag{27c}$$

These inherit the properties of of the general case in Section 2.3. So we have that $C_{\hat{\boldsymbol{\xi}}}$ is a correlation monotone, $E_{\hat{\boldsymbol{\xi}}}$ is a strong entanglement monotone, and both of these are faithful, $C_{\hat{\boldsymbol{\xi}}}(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}}$, $E_{\hat{\boldsymbol{\xi}}}(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-sep}}$, moreover, they show the same lattice structure as the integer partition ideals, $\hat{P}_{\text{II}}$, that is,

$$\hat{\boldsymbol{v}} \preceq \hat{\boldsymbol{\xi}} \iff C_{\hat{\boldsymbol{v}}} \geq C_{\hat{\boldsymbol{\xi}}}, \tag{28a}$$

$$\hat{\boldsymbol{v}} \preceq \hat{\boldsymbol{\xi}} \iff E_{\hat{\boldsymbol{v}}} \geq E_{\hat{\boldsymbol{\xi}}}, \tag{28b}$$

which we call *multipartite monotonicity* for Level II of the permutation invariant case. (These come by using (74b) and (73b) in Appendix A.4, then (11) and (27).)

3.4 Level III: classes

Let the map s in (23) act elementwisely on the partition ideal filters $\underline{\boldsymbol{\xi}} \in P_{\text{III}}$,

$$s(\underline{\boldsymbol{\xi}}) := \big\{ s(\boldsymbol{\xi}) \mid \forall \boldsymbol{\xi} \in \underline{\boldsymbol{\xi}} \big\}, \tag{29}$$

where on the right-hand side a *set* stays. We call $s(\underline{\boldsymbol{\xi}})$ the *type* of the set partition ideal filter $\underline{\boldsymbol{\xi}}$. This is a set of integer partition filters, which is denoted as

$$\hat{\underline{\boldsymbol{\xi}}} = \big\{ \hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2, \ldots, \hat{\boldsymbol{\xi}}_{|\hat{\underline{\boldsymbol{\xi}}}|} \big\} \subseteq \hat{P}_{\text{II}}, \tag{30a}$$

which is a nonempty filter of ideals of integer partitions, the set of which is denoted as

$$\hat{P}_{\mathrm{III}} := s(P_{\mathrm{III}}) = \mathcal{O}_\uparrow(\hat{P}_{\mathrm{II}}) \setminus \{\emptyset\}. \qquad (30\mathrm{b})$$

This might not seem to be obvious, this is given in (14), proven in Corollary 17 in Appendix A.4. Based on the refinement of the set partition ideal filters (12c), we define a partial order over the integer partition ideal filters, which we call *refinement* again, and denote with $\preceq$ again, given as

$$\hat{\underline{\boldsymbol{v}}} \preceq \hat{\underline{\boldsymbol{\xi}}} \quad \overset{\mathrm{def.}}{\Longleftrightarrow} \quad \exists \underline{\boldsymbol{v}} \in s^{-1}(\hat{\underline{\boldsymbol{v}}}), \underline{\boldsymbol{\xi}} \in s^{-1}(\hat{\underline{\boldsymbol{\xi}}}) \text{ s.t. } \underline{\boldsymbol{v}} \preceq \underline{\boldsymbol{\xi}}$$
$$\Longleftrightarrow \quad \hat{\underline{\boldsymbol{v}}} \subseteq \hat{\underline{\boldsymbol{\xi}}}, \qquad (30\mathrm{c})$$

which turns out to be the inclusion, being the natural partial order for filters. This might not seem to be obvious, this is given in (14), proven in Corollary 17 in Appendix A.4. (For illustrations, see Figure 4.) Note that here, contrary to Level I and II, we call $\hat{\underline{\boldsymbol{\xi}}}$ finer and $\hat{\underline{\boldsymbol{v}}}$ coarser. Because of these, $\hat{P}_{\mathrm{III}}$ with this partial order *is* a lattice. By construction, s is monotone with respect to these partial orders, $\underline{\boldsymbol{v}} \preceq \underline{\boldsymbol{\xi}} \Rightarrow s(\underline{\boldsymbol{v}}) \preceq s(\underline{\boldsymbol{\xi}})$ (see (64) in Appendix A.1).

With respect to the integer partition ideal filters $\hat{\underline{\boldsymbol{\xi}}} \in \hat{P}_{\mathrm{III}}$, we can define the strict partial correlation and entanglement properties.

The *strictly $\hat{\underline{\boldsymbol{\xi}}}$-uncorrelated states* are those which are uncorrelated with respect to all $\hat{\boldsymbol{\xi}} \in \hat{\underline{\boldsymbol{\xi}}}$, and correlated with respect to all $\hat{\boldsymbol{\xi}} \in \overline{\hat{\underline{\boldsymbol{\xi}}}} = \hat{P}_{\mathrm{II}} \setminus \hat{\underline{\boldsymbol{\xi}}}$, so the *class* of these (*permutation invariant partial correlation class*) is

$$\mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}\text{-unc}} := \bigcap_{\hat{\boldsymbol{\xi}} \in \overline{\hat{\underline{\boldsymbol{\xi}}}}} \overline{\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}}} \cap \bigcap_{\hat{\boldsymbol{\xi}} \in \hat{\underline{\boldsymbol{\xi}}}} \mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}}. \qquad (31\mathrm{a})$$

The *strictly $\hat{\underline{\boldsymbol{\xi}}}$-separable states* are those which are separable with respect to all $\hat{\boldsymbol{\xi}} \in \hat{\underline{\boldsymbol{\xi}}}$, and entangled with respect to all $\hat{\boldsymbol{\xi}} \in \overline{\hat{\underline{\boldsymbol{\xi}}}}$, so the *class* of these (*permutation invariant partial separability class*, or *permutation invariant partial entanglement class*) is

$$\mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}\text{-sep}} := \bigcap_{\hat{\boldsymbol{\xi}} \in \overline{\hat{\underline{\boldsymbol{\xi}}}}} \overline{\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-sep}}} \cap \bigcap_{\hat{\boldsymbol{\xi}} \in \hat{\underline{\boldsymbol{\xi}}}} \mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-sep}}. \qquad (31\mathrm{b})$$

The meaning of the permutation invariant Level III hierarchy can also be clarified. If there exists a $\varrho \in \mathcal{C}_{\hat{\underline{\boldsymbol{v}}}\text{-unc}}$ and an LO mapping it into $\mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}\text{-unc}}$, then $\hat{\underline{\boldsymbol{v}}} \preceq \hat{\underline{\boldsymbol{\xi}}}$; and if there exists a $\varrho \in \mathcal{C}_{\hat{\underline{\boldsymbol{v}}}\text{-sep}}$ and an LOCC mapping it into $\mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}\text{-sep}}$, then $\hat{\underline{\boldsymbol{v}}} \preceq \hat{\underline{\boldsymbol{\xi}}}$. In this sense, the permutation invariant Level III hierarchy compares the strength of correlation and entanglement among the permutation invariant classes. (These come by the analogue result for the general case in Section 2.4, for the coarsened classification based on the permutation invariant properties $P_{\mathrm{II}*} = \{\vee s^{-1}(\hat{\underline{\boldsymbol{\xi}}}) \mid \hat{\underline{\boldsymbol{\xi}}} \in \hat{P}_{\mathrm{II}}\}$, see in the next section.)
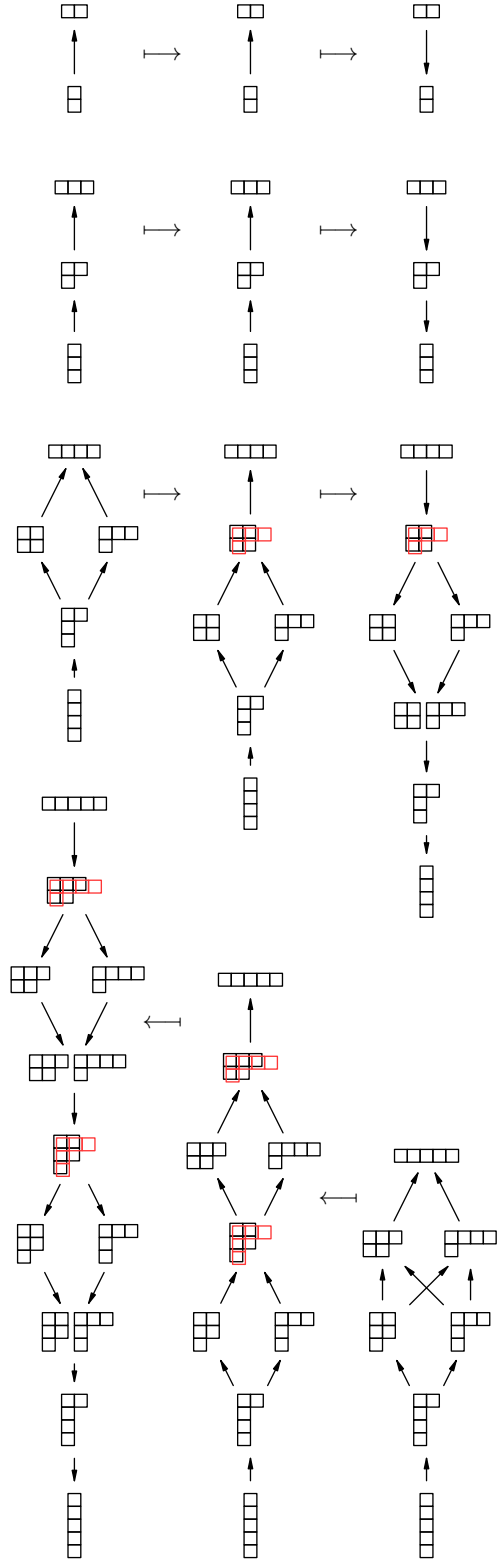

Figure 4: The posets of the three-level structure of permutation invariant multipartite correlation and entanglement for $n = 2, 3, 4$ and 5. Only the maximal elements of the downsets of $\hat{P}_{\mathrm{I}}$ are shown (with different colors) in $\hat{P}_{\mathrm{II}}$, while only the minimal elements of the up-sets of $\hat{P}_{\mathrm{II}}$ are shown (side by side) in $\hat{P}_{\mathrm{III}}$. The partial orders (18c), (24c) and (30c) are represented by consecutive arrows.

4 An alternative way of introducing permutation invariance

The Level II structure P_{II} encodes the different partial correlation and entanglement properties. In Sections 2 and 3, we have built up the structure of the classification, parallel for the general, and for the permutation invariant cases. Here we show a more compact, but less transparent treatment of the same structure, by embedding both constructions into P_{II}.

The three-level building of the correlation and entanglement classification was given by the construction

$$P_{\mathrm{I}} \;\mapsto\; P_{\mathrm{II}} = \mathcal{O}_\downarrow(P_{\mathrm{I}}) \setminus \{\emptyset\} \;\mapsto\; P_{\mathrm{III}} = \mathcal{O}_\uparrow(P_{\mathrm{II}}) \setminus \{\emptyset\}$$

in Section 2. Because of (4), one may embed P_{I} into P_{II}, using the *principal ideals* in P_{II},

$$P_{\mathrm{I}} \cong P_{\mathrm{II\ pr.}} := \big\{ {\downarrow}\{\xi\} \mid \xi \in P_{\mathrm{I}} \big\} \subseteq P_{\mathrm{II}}, \qquad (32a)$$

by noting that

$$v \preceq \xi \iff {\downarrow}\{v\} \preceq {\downarrow}\{\xi\}. \qquad (32b)$$

In this way, P_{I} is a sublattice of P_{II}.

The same can be done for the permutation invariant correlation and entanglement classification given by the construction

$$\hat{P}_{\mathrm{I}} \;\mapsto\; \hat{P}_{\mathrm{II}} = \mathcal{O}_\downarrow(\hat{P}_{\mathrm{I}}) \setminus \{\emptyset\} \;\mapsto\; \hat{P}_{\mathrm{III}} = \mathcal{O}_\uparrow(\hat{P}_{\mathrm{II}}) \setminus \{\emptyset\}$$

in Section 3. In the same way, because of (20), one may embed $\hat{P}_{\mathrm{I}}$ into $\hat{P}_{\mathrm{II}}$, using the *principal ideals* in $\hat{P}_{\mathrm{II}}$,

$$\hat{P}_{\mathrm{I}} \cong \hat{P}_{\mathrm{II\ pr.}} := \big\{ {\downarrow}\{\hat{\xi}\} \mid \hat{\xi} \in \hat{P}_{\mathrm{I}} \big\} \subseteq \hat{P}_{\mathrm{II}}, \qquad (33a)$$

by noting that

$$\hat{v} \preceq \hat{\xi} \iff {\downarrow}\{\hat{v}\} \preceq {\downarrow}\{\hat{\xi}\}. \qquad (33b)$$

In this way, $\hat{P}_{\mathrm{I}}$ is a sublattice of $\hat{P}_{\mathrm{II}}$.

The third point here is that, noting that $\downarrow s^{-1}(\hat{\xi}) = \vee s^{-1}(\downarrow\{\hat{\xi}\})$ (for the proof, see (73a) in Appendix A.4), one may also embed $\hat{P}_{\mathrm{II}}$ (thus also $\hat{P}_{\mathrm{I}}$) into P_{II}, using

$$\hat{P}_{\mathrm{I}} \cong \big\{ \vee s^{-1}(\downarrow\{\hat{\xi}\}) \mid \hat{\xi} \in \hat{P}_{\mathrm{I}} \big\} \subset P_{\mathrm{II}}, \qquad (34a)$$

$$\hat{P}_{\mathrm{II}} \cong \big\{ \vee s^{-1}(\hat{\boldsymbol{\xi}}) \mid \hat{\boldsymbol{\xi}} \in \hat{P}_{\mathrm{II}} \big\} =: P_{\mathrm{II\ pinv.}} \subset P_{\mathrm{II}}, \qquad (34b)$$

by noting that

$$\hat{v} \preceq \hat{\xi} \iff \vee s^{-1}(\downarrow\{\hat{v}\}) \preceq \vee s^{-1}(\downarrow\{\hat{\xi}\}), \qquad (34c)$$

$$\hat{\boldsymbol{v}} \preceq \hat{\boldsymbol{\xi}} \iff \vee s^{-1}(\hat{\boldsymbol{v}}) \preceq \vee s^{-1}(\hat{\boldsymbol{\xi}}). \qquad (34d)$$

(For the proof, see (74a) and (74b) in Appendix A.4.) In this way, $\hat{P}_{\mathrm{I}}$ and $\hat{P}_{\mathrm{II}}$ are sublattices of P_{II}.

A $\boldsymbol{\xi} \in P_{\mathrm{II}}$ is permutation invariant, if and only if $\vee s^{-1}(s(\boldsymbol{\xi})) = \boldsymbol{\xi}$. (Then $\boldsymbol{\xi} \in P_{\mathrm{II}}$ describes the same property as $\hat{\boldsymbol{\xi}} := s(\boldsymbol{\xi}) \in \hat{P}_{\mathrm{II}}$.) So $P_{\mathrm{II\ pinv.}}$ can be given directly as

$$P_{\mathrm{II\ pinv.}} = \big\{ \boldsymbol{\xi} \in \hat{P}_{\mathrm{II}} \mid \vee s^{-1}(s(\boldsymbol{\xi})) = \boldsymbol{\xi} \big\}, \qquad (35)$$

and the permutation invariant classification can be described as a coarsened classification $\hat{P}_{\mathrm{III}} \cong P_{\mathrm{III*}} = \mathcal{O}_\uparrow(P_{\mathrm{II*}}) \setminus \{\emptyset\}$ with respect to the permutation invariant properties $P_{\mathrm{II*}} = P_{\mathrm{II\ pinv.}}$.

We note that P_{II} could also be embedded into P_{III} by principal filters (and similarly $\hat{P}_{\mathrm{II}}$ into $\hat{P}_{\mathrm{III}}$), however, the construction has led to a different direction.

5 k-partitionability, k-producibility and k-stretchability

In this section, we consider k-partitionability and k-producibility as particular cases of permutation invariant properties of partial correlation and entanglement, and elaborate a duality between them. Our point of view makes possible to reveal a new property, which we call k-stretchability, combining some advantages of k-partitionability and k-producibility. We also investigate the relations among these three properties.

5.1 k-partitionability and k-producibility of correlation and entanglement

k-partitionability and k-producibility are permutation invariant Level II properties. A partition is k-partitionable, if the number of its parts is at least k, while it is k-producible, if the sizes of its parts are at most k. (For illustration, see Figure 5.) These are encoded by the ideals of k-*partitionable* and k-*producible partitions*,

$$\boldsymbol{\xi}_{k\text{-part}} := \big\{ \xi \in P_{\mathrm{I}} \mid |\xi| \geq k \big\} \in P_{\mathrm{II}}, \qquad (36a)$$

$$\boldsymbol{\xi}_{k\text{-prod}} := \big\{ \xi \in P_{\mathrm{I}} \mid \forall X \in \xi : |X| \leq k \big\} \in P_{\mathrm{II}}, \quad (36b)$$

for $k = 1, 2, \ldots, n$, forming chains in the lattice P_{II},

$$\boldsymbol{\xi}_{l\text{-part}} \preceq \boldsymbol{\xi}_{k\text{-part}} \iff l \geq k, \qquad (37a)$$

$$\boldsymbol{\xi}_{l\text{-prod}} \preceq \boldsymbol{\xi}_{k\text{-prod}} \iff l \leq k. \qquad (37b)$$

Since k-partitionability and k-producibility are permutation invariant properties, it is enough to consider their types (23)

$$\hat{\boldsymbol{\xi}}_{k\text{-part}} := s(\boldsymbol{\xi}_{k\text{-part}})$$
$$= \big\{ \hat{\xi} \in \hat{P}_{\mathrm{I}} \mid |\hat{\xi}| \geq k \big\} \in \hat{P}_{\mathrm{II}}, \qquad (38a)$$

$$\hat{\boldsymbol{\xi}}_{k\text{-prod}} := s(\boldsymbol{\xi}_{k\text{-prod}})$$
$$= \big\{ \hat{\xi} \in \hat{P}_{\mathrm{I}} \mid \forall x \in \hat{\xi} : x \leq k \big\} \in \hat{P}_{\mathrm{II}}, \qquad (38b)$$

forming chains in the lattice $\hat{P}_{\mathrm{II}}$,

$$\hat{\boldsymbol{\xi}}_{l\text{-part}} \preceq \hat{\boldsymbol{\xi}}_{k\text{-part}} \iff l \geq k, \qquad (39a)$$
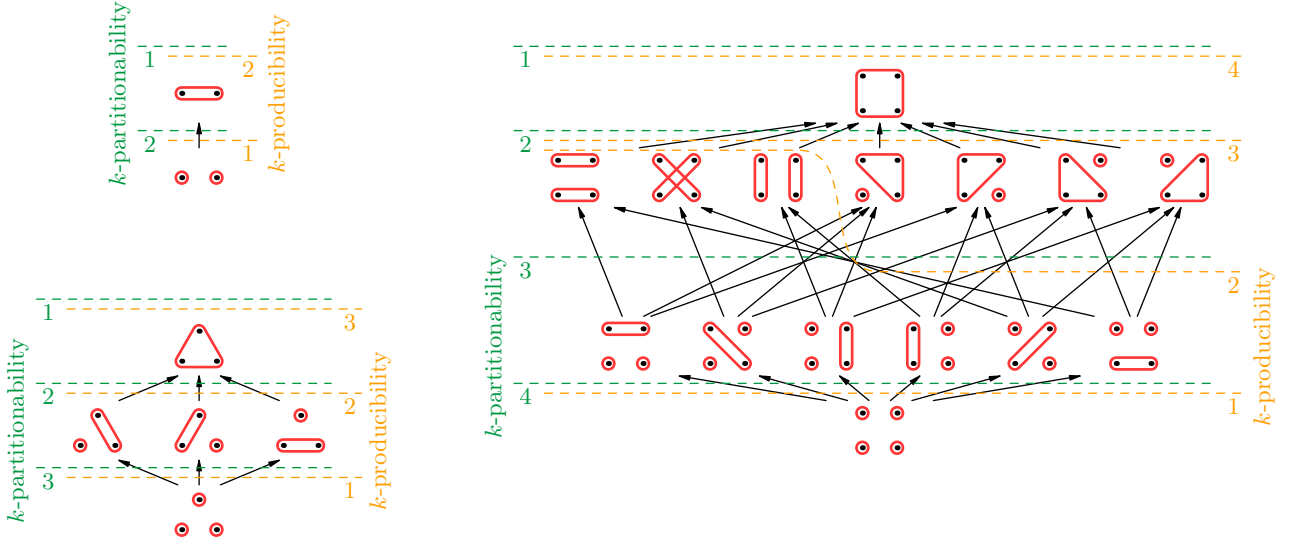
Figure 5: The down-sets corresponding to k-partitionability and k-producibility, illustrated on the lattices P_I for $n = 2, 3$ and 4. (The two kinds of down-sets contain the elements below the specific green and yellow dashed lines.)

$$\hat{\boldsymbol{\xi}}_{l\text{-part}} \preceq \hat{\boldsymbol{\xi}}_{k\text{-prod}} \iff l \leq k. \qquad (39\text{b})$$

The corresponding *k-partitionably uncorrelated* and *k-producibly uncorrelated* states (25a) are

$$\mathcal{D}_{k\text{-part unc}} := \mathcal{D}_{\boldsymbol{\xi}_{k\text{-part}}\text{-unc}} = \mathcal{D}_{\hat{\boldsymbol{\xi}}_{k\text{-part}}\text{-unc}}$$
$$= \left\{ \bigotimes_{X \in \xi} \varrho_X \;\middle|\; \forall \xi \in P_\text{I} \text{ s.t. } |\xi| \geq k \right\}, \qquad (40\text{a})$$

$$\mathcal{D}_{k\text{-prod unc}} := \mathcal{D}_{\boldsymbol{\xi}_{k\text{-prod}}\text{-unc}} = \mathcal{D}_{\hat{\boldsymbol{\xi}}_{k\text{-prod}}\text{-unc}}$$
$$= \left\{ \bigotimes_{X \in \xi} \varrho_X \;\middle|\; \forall \xi \in P_\text{I} \text{ s.t. } \forall X \in \xi : |X| \leq k \right\}, \qquad (40\text{b})$$

(with $\varrho_X \in \mathcal{D}_X$), which are products of density operators of at least k subsystems, and of density operators of subsystems containing at most k elementary subsystems, respectively. The corresponding *k-partitionably separable* (also called *k-separable* [35, 37, 47]) and *k-producibly separable* (also called *k-producible* [46, 47, 49]) states (25b) are

$$\mathcal{D}_{k\text{-part sep}} := \mathcal{D}_{\boldsymbol{\xi}_{k\text{-part}}\text{-sep}} = \mathcal{D}_{\hat{\boldsymbol{\xi}}_{k\text{-part}}\text{-sep}}$$
$$= \text{Conv } \mathcal{D}_{k\text{-part unc}}$$
$$= \left\{ \sum_j p_j \bigotimes_{X \in \xi_j} \varrho_{X,j} \;\middle|\; \forall j, \forall \xi_j \in P_\text{I} \text{ s.t. } |\xi_j| \geq k \right\}, \qquad (40\text{c})$$

$$\mathcal{D}_{k\text{-prod sep}} := \mathcal{D}_{\boldsymbol{\xi}_{k\text{-prod}}\text{-sep}} = \mathcal{D}_{\hat{\boldsymbol{\xi}}_{k\text{-prod}}\text{-sep}}$$
$$= \text{Conv } \mathcal{D}_{k\text{-prod unc}}$$
$$= \left\{ \sum_j p_j \bigotimes_{X \in \xi_j} \varrho_{X,j} \;\middle|\; \forall j, \right.$$
$$\left. \forall \xi_j \in P_\text{I} \text{ s.t. } \forall X \in \xi_j : |X| \leq k \right\}, \qquad (40\text{d})$$

which can be decomposed into k-partitionably, or k-producibly uncorrelated states, respectively. Because

of (26a), these properties show the same chain structure as the chains of the corresponding partition ideals (39), that is,

$$l \geq k \iff \mathcal{D}_{l\text{-part unc}} \subseteq \mathcal{D}_{k\text{-part unc}}, \qquad (41\text{a})$$
$$l \leq k \iff \mathcal{D}_{l\text{-prod unc}} \subseteq \mathcal{D}_{k\text{-prod unc}}, \qquad (41\text{b})$$
$$l \geq k \iff \mathcal{D}_{l\text{-part sep}} \subseteq \mathcal{D}_{k\text{-part sep}}, \qquad (41\text{c})$$
$$l \leq k \iff \mathcal{D}_{l\text{-prod sep}} \subseteq \mathcal{D}_{k\text{-prod sep}}, \qquad (41\text{d})$$

that is, if a state is l-partitionably uncorrelated (separable) then it is also k-partitionably uncorrelated (separable) for all $l \geq k$, and if a state is l-producibly uncorrelated (separable) then it is also k-producibly uncorrelated (separable) for all $l \leq k$.

The corresponding *k-partitionability correlation* and *k-producibility correlation* (27a) are [14]

$$C_{k\text{-part}} := C_{\boldsymbol{\xi}_{k\text{-part}}} = C_{\hat{\boldsymbol{\xi}}_{k\text{-part}}} = \min_{\xi : |\xi| \geq k} C_\xi, \qquad (42\text{a})$$

$$C_{k\text{-prod}} := C_{\boldsymbol{\xi}_{k\text{-prod}}} = C_{\hat{\boldsymbol{\xi}}_{k\text{-prod}}} = \min_{\xi : \forall X \in \xi : |X| \leq k} C_\xi. \qquad (42\text{b})$$

The corresponding *k-partitionability entanglement* and *k-producibility entanglement* (27c) are [40]

$$E_{k\text{-part}} := E_{\boldsymbol{\xi}_{k\text{-part}}} = E_{\hat{\boldsymbol{\xi}}_{k\text{-part}}}, \qquad (42\text{c})$$

$$E_{k\text{-prod}} := E_{\boldsymbol{\xi}_{k\text{-prod}}} = E_{\hat{\boldsymbol{\xi}}_{k\text{-prod}}}. \qquad (42\text{d})$$

Because of (28), these measures show the same chain structure as the chains of the corresponding partition ideals (39), that is,

$$l \geq k \iff C_{l\text{-part}} \geq C_{k\text{-part}}, \qquad (43\text{a})$$
$$l \leq k \iff C_{l\text{-prod}} \geq C_{k\text{-prod}}, \qquad (43\text{b})$$
$$l \geq k \iff E_{l\text{-part}} \geq E_{k\text{-part}}, \qquad (43\text{c})$$
$$l \leq k \iff E_{l\text{-prod}} \geq E_{k\text{-prod}}, \qquad (43\text{d})$$

that is, l-partitionability correlation (entanglement) is always stronger than k-partitionability correlation (entanglement) for all $l \geq k$, and l-producibility correlation (entanglement) is always stronger than k-producibility correlation (entanglement) for all $l \leq k$, which is the multipartite monotonicity of these measures.

For the labeling of the strict partitionability and producibility properties, we have the ideal filters

$$\hat{\underline{\boldsymbol{\xi}}}_{k\text{-part}} := \uparrow\{\hat{\boldsymbol{\xi}}_{k\text{-part}}\} = \{\hat{\boldsymbol{\xi}}_{l\text{-part}} \mid l \leq k\}, \quad (44a)$$

$$\hat{\underline{\boldsymbol{\xi}}}_{k\text{-prod}} := \uparrow\{\hat{\boldsymbol{\xi}}_{k\text{-prod}}\} = \{\hat{\boldsymbol{\xi}}_{l\text{-prod}} \mid l \geq k\}, \quad (44b)$$

by (36) and (37). The corresponding *strictly k-partitionably uncorrelated* and *strictly k-producibly uncorrelated* states (31a) are

$$\mathcal{C}_{k\text{-part unc}} := \mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}_{k\text{-part}}\text{-unc}} \\ = \mathcal{D}_{k\text{-part unc}} \setminus \mathcal{D}_{(k+1)\text{-part unc}}, \quad (45a)$$

$$\mathcal{C}_{k\text{-prod unc}} := \mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}_{k\text{-prod}}\text{-unc}} \\ = \mathcal{D}_{k\text{-prod unc}} \setminus \mathcal{D}_{(k-1)\text{-prod unc}}, \quad (45b)$$

which are products of nonproduct density operators of exactly k subsystems, and of nonproduct density operators of subsystems containing exactly k elementary subsystems, respectively. The corresponding *strictly k-partitionably separable* and *strictly k-producibly separable states* (31b) are

$$\mathcal{C}_{k\text{-part sep}} := \mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}_{k\text{-part}}\text{-sep}} \\ = \mathcal{D}_{k\text{-part sep}} \setminus \mathcal{D}_{(k+1)\text{-part sep}}, \quad (45c)$$

$$\mathcal{C}_{k\text{-prod sep}} := \mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}_{k\text{-part}}\text{-sep}} \\ = \mathcal{D}_{k\text{-prod sep}} \setminus \mathcal{D}_{(k-1)\text{-prod sep}}, \quad (45d)$$

which can be decomposed into k-partitionably but not $(k+1)$-partitionably, and k-producibly but not $(k-1)$-producibly uncorrelated states, respectively. ($\mathcal{D}_{(n+1)\text{-part unc/sep}} = \emptyset$ and $\mathcal{D}_{0\text{-prod unc/sep}} = \emptyset$ are understood.)

5.2 Duality by conjugation

Looking at the k-partitionability and k-producibility properties in Figure 5, and also at their definitions (36a) and (36b), it is not clear how these are related to each other.

In Section 3, the permutation invariant correlation properties were described by the use of integer partitions, represented by Young diagrams in the figures. Important properties of Young diagrams are their *height*, *width* and *rank*, given for an integer partition $\hat{\xi} \in \hat{P}_{\mathrm{I}}$ as

$$h(\hat{\xi}) := |\hat{\xi}|, \quad (46a)$$

$$w(\hat{\xi}) := \max \hat{\xi}, \quad (46b)$$

$$r(\hat{\xi}) := w(\hat{\xi}) - h(\hat{\xi}), \quad (46c)$$

and also for set partition $\xi \in P_{\mathrm{I}}$ as $h(\xi) := h(s(\xi)) = |\xi|$, $w(\xi) := w(s(\xi)) = \max_{X \in \xi} |X|$, $r(\xi) := r(s(\xi)) = w(\xi) - h(\xi)$. It is enlightening now to arrange the poset $\hat{P}_{\mathrm{I}}$ according to the height and width of the partitions, in the way as can be seen in Figure 6, i.e., the height is increasing downwards, the width is increasing to the right, then the rank is increasing up-right. It is easy to see that the height is strictly decreasing, the width is increasing, and the rank is strictly increasing monotone with respect to the refinement (18c),

$$\hat{v} \prec \hat{\xi} \implies \\ h(\hat{v}) > h(\hat{\xi}), \quad w(\hat{v}) \leq w(\hat{\xi}), \quad r(\hat{v}) < r(\hat{\xi}). \quad (47)$$

(For illustration, see Figure 6: the arrows point always upwards, and possibly to the up-right, never to the up-left, horizontally or downwards.)

Now the partitionability and producibility properties (36a) and (36b) can be formulated with the height and width as

$$\hat{\boldsymbol{\xi}}_{k\text{-part}} = \{\hat{\xi} \in \hat{P}_{\mathrm{I}} \mid h(\hat{\xi}) \geq k\}, \quad (48a)$$

$$\hat{\boldsymbol{\xi}}_{k\text{-prod}} = \{\hat{\xi} \in \hat{P}_{\mathrm{I}} \mid w(\hat{\xi}) \leq k\}, \quad (48b)$$

so k-partitionability is the minimal height, k-producibility is the maximal width of the partition. In the height-width based arrangement of $\hat{P}_{\mathrm{I}}$ in Figure 6, the k-partitionable partitions (48a) are below the k-th rows (counting from the top), and k-producible partitions (48b) are to the left from the k-th column (counting from the left). We can also introduce another property, called k-*stretchability*, as

$$\hat{\boldsymbol{\xi}}_{k\text{-str}} = \{\hat{\xi} \in \hat{P}_{\mathrm{I}} \mid r(\hat{\xi}) \leq k\}, \quad (48c)$$

for $k = -(n-1), -(n-2), \ldots, n-2, n-1$, so k-stretchability is the maximal rank of the partition. It is easy to see that $\hat{\boldsymbol{\xi}}_{k\text{-str}} \in \hat{P}_{\mathrm{II}}$, that is, it is a down-set for all k, by the third inequality in (47). We elaborate on the arising correlation and entanglement properties later, in Section 5.3.

On the set of Young diagrams, an involution arises naturally, called *conjugation*, being the reflection of the diagram with respect to its "diagonal" [45]. Considering the integer partitions themselves, the conjugation is given as follows. For decreasingly ordered values $(x_1, x_2, \ldots, x_{|\hat{\xi}|})$ of the elements of $\hat{\xi}$, the number of elements x' that equal to i is $x_i - x_{i+1}$ in the conjugated partition (setting $x_{i+1} = 0$ for $i = h(\hat{\xi})$). With multisets, the conjugation is given as the map $\hat{P}_{\mathrm{I}} \to \hat{P}_{\mathrm{I}}$,

$$\hat{\xi} \mapsto \hat{\xi}^{\dagger} = \{|\{x \in \hat{\xi} \mid x \geq i\}| \mid i = 1, 2, \ldots, w(\hat{\xi})\}, \quad (49)$$

where both sets on the right-hand side are multisets.

The partial order (18c) does not show nice properties with respect to the conjugation. (For illustration,
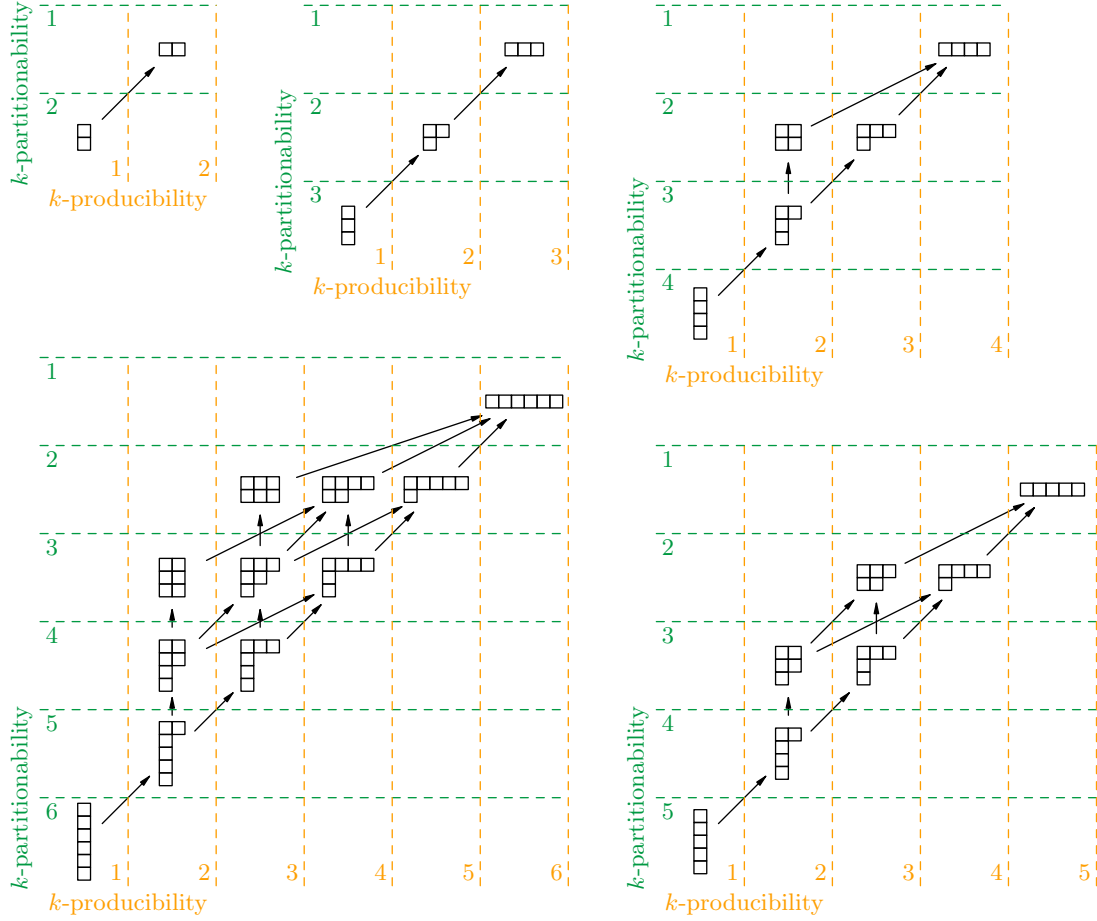
Figure 6: The down-sets corresponding to k-partitionability and k-producibility, illustrated on the lattices $\hat{P}_\mathrm{I}$ for $n = 2, 3, 4, 5$ and 6. (The two kinds of down-sets contain the elements below the green and to the left from the yellow dashed lines.) For the first three cases, compare with P_I in Figure 5.

see the height-width based arrangement in Figure 6.) The conjugation is not an anti-automorphism, if $\hat{\upsilon} \preceq \hat{\xi}$ then it does not follow that $\hat{\upsilon}^\dagger \succeq \hat{\xi}^\dagger$ (the first example is for $n = 4$, where we have $\{2, 1, 1\} \preceq \{2, 2\}$ and $\{2, 1, 1\}^\dagger = \{3, 1\} \not\succeq \{2, 2\} = \{2, 2\}^\dagger$). The conjugation is neither an automorphism of course, if $\hat{\upsilon} \preceq \hat{\xi}$ then it does not follow that $\hat{\upsilon}^\dagger \preceq \hat{\xi}^\dagger$ (the first example is for $n = 2$, where we have $\{1, 1\} \preceq \{2\}$ and $\{1, 1\}^\dagger = \{2\} \not\preceq \{1, 1\} = \{2\}^\dagger$). An integer partition $\hat{\xi} \in \hat{P}_\mathrm{I}$ and its conjugate cannot be ordered by refinement (the first example is for $n = 6$, where we have $\{2, 2, 2\}$ and $\{3, 3\}$, which are conjugates of each other, and cannot be ordered).

The conjugation interchanges the height and width, and multiplies the rank with -1,

$$h(\hat{\xi}^\dagger) = w(\hat{\xi}), \quad w(\hat{\xi}^\dagger) = h(\hat{\xi}), \quad r(\hat{\xi}^\dagger) = -r(\hat{\xi}). \quad (50)$$

(For illustration, see the height-width based arrangement in Figure 6: the conjugation brings to the position mirrored with respect to the diagonal $w(\hat{\xi}) = h(\hat{\xi})$.) Note that, although the conjugation interchanges the height and width, it does not interchange the partitionability (48a) and producibility (48b) properties, since these are given in different ways by height and width, namely, by lower and upper bounds.

5.3 k-stretchability of correlation and entanglement

In Section 5.2, we introduced the permutation invariant property k-stretchability (48c) for $k = -(n - 1), -(n - 2), \ldots, n - 2, n - 1$. Because of the third inequality in (47), these form a chain in the lattice $\hat{P}_\mathrm{II}$,

$$\hat{\boldsymbol{\xi}}_{l\text{-str}} \preceq \hat{\boldsymbol{\xi}}_{k\text{-str}} \iff l \leq k. \quad (51)$$

(For illustration, see the height-width based arrangement in Figure 6.)

The corresponding k-stretchably uncorrelated states (25a) are

$$\mathcal{D}_{k\text{-str unc}} := \mathcal{D}_{\hat{\boldsymbol{\xi}}_{k\text{-str}}\text{-unc}}$$
$$= \Big\{ \bigotimes_{X \in \xi} \varrho_X \ \Big| \ \forall \xi \in P_\mathrm{I} \ \text{s.t.} \ \max_{X \in \xi} |X| - |\xi| \leq k \Big\}, \quad (52a)$$

which are products of density operators of as few subsystems as possible, of size as large as possible,

of difference upper-bounded by k. In this sense, this is a combination of k-partitionability and k-producibility. The corresponding *k-stretchably separable states* (25b) are

$$\mathcal{D}_{k\text{-str sep}} := \mathcal{D}_{\hat{\boldsymbol{\xi}}_{k\text{-str}}\text{-sep}} = \text{Conv}\, \mathcal{D}_{k\text{-str unc}}$$
$$= \Big\{ \sum_j p_j \bigotimes_{X \in \xi_j} \varrho_{X,j} \,\Big|\, \forall j, \tag{52b}$$
$$\forall \xi_j \in P_{\mathrm{I}} \text{ s.t. } \max_{X \in \xi_j} |X| - |\xi_j| \le k \Big\},$$

which can be decomposed into k-stretchably uncorrelated states. Because of (26a), these properties show the same chain structure as the chains of the corresponding partition ideals (51), that is,

$$l \le k \iff \mathcal{D}_{l\text{-str unc}} \subseteq \mathcal{D}_{k\text{-str unc}}, \tag{53a}$$
$$l \le k \iff \mathcal{D}_{l\text{-str sep}} \subseteq \mathcal{D}_{k\text{-str sep}}, \tag{53b}$$

that is, if a state is l-stretchably uncorrelated (separable) then it is also k-stretchably uncorrelated (separable) for all $l \le k$.

The corresponding *k-stretchability correlation* (27a) is

$$C_{k\text{-str}} := C_{\boldsymbol{\xi}_{k\text{-str}}} = C_{\hat{\boldsymbol{\xi}}_{k\text{-str}}} = \min_{\substack{\xi : \max\limits_{X \in \xi} |X| - |\xi| \le k}} C_\xi. \tag{54a}$$

The corresponding *k-stretchability entanglement* (27c) is

$$E_{k\text{-str}} := E_{\boldsymbol{\xi}_{k\text{-str}}} = E_{\hat{\boldsymbol{\xi}}_{k\text{-str}}}. \tag{54b}$$

Because of (28), these measures show the same chain structure as the chain of the corresponding partition ideals (51), that is,

$$l \le k \iff C_{l\text{-str}} \ge C_{k\text{-str}}, \tag{55a}$$
$$l \le k \iff E_{l\text{-str}} \ge E_{k\text{-str}}, \tag{55b}$$

that is, l-stretchability correlation (entanglement) is always stronger than k-stretchability correlation (entanglement) for all $l \le k$, which is the multipartite monotonicity of these measures.

For the labeling of the strict stretchability properties, we have the ideal filters

$$\hat{\underline{\boldsymbol{\xi}}}_{k\text{-str}} := \uparrow\{\hat{\boldsymbol{\xi}}_{k\text{-str}}\} = \{\hat{\boldsymbol{\xi}}_{l\text{-str}} \,\big|\, l \ge k\}, \tag{56}$$

by (48c) and (51). The corresponding *strictly k-stretchably uncorrelated states* (31a) are

$$\mathcal{C}_{k\text{-str unc}} := \mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}_{k\text{-str}}\text{-unc}} = \mathcal{D}_{k\text{-str unc}} \setminus \mathcal{D}_{(k-1)\text{-str unc}}, \tag{57a}$$

which are products of nonproduct density operators of as few subsystems as possible, of size as large as possible, of difference exactly k. The corresponding *strictly k-stretchably separable states* (31b) are

$$\mathcal{C}_{k\text{-str sep}} := \mathcal{C}_{\hat{\underline{\boldsymbol{\xi}}}_{k\text{-str}}\text{-sep}} = \mathcal{D}_{k\text{-str sep}} \setminus \mathcal{D}_{(k-1)\text{-str sep}}, \tag{57b}$$

which can be decomposed into k-stretchably but not $(k-1)$-stretchably uncorrelated states. ($\mathcal{D}_{(-n)\text{-str unc/sep}} = \emptyset$ is understood.)

k-partitionability (48a), k-producibility (48b) and k-stretchability (48c) give, of course, a rather coarsened description of permutation invariant correlation or entanglement properties. Because of this, they must show some disadvantages. First, k-partitionability (48a) takes into account only the number of subsystems uncorrelated with (or separable from) one another, and does not distinguish between the cases when subsystems of roughly equal sizes are uncorrelated (or separable), and when some elementary subsystems are uncorrelated with (or separable from) the rest of a large system, although these two cases represent highly different situations form a resource-theoretical point of view. (For illustration, see the rows in Figure 6.) Second, k-producibility (48b) takes into account only the size of the largest subsystem uncorrelated with (or separable from) the other part of the system, and does not distinguish between the cases when a large subsystem is uncorrelated with (or separable from) a slightly smaller subsystem, or many elementary subsystems, although these two cases represent again highly different situations form a resource-theoretical point of view. (For illustration, see the columns in Figure 6.) Third, k-stretchability (48c) combines the advantages of the previous two properties in a balanced way, it takes into account a kind of "difference" of them, and does not distinguish among cases with rather different types. (For illustration, see the lines parallel to the diagonal in Figure 6.)

5.4 Relations among k-partitionability, k-producibility and k-stretchability

The height $h(\hat{\xi})$, width $w(\hat{\xi})$ and rank $r(\hat{\xi})$ (46) of the integer partitions $\hat{\xi} \in \hat{P}_{\mathrm{I}}$ of $n \in \mathbb{N}$ are bounded by one another as

$$n/w \le h \le n + 1 - w, \tag{58a}$$
$$n/h \le w \le n + 1 - h, \tag{58b}$$
$$n/h - h \le r \le n + 1 - 2h, \tag{58c}$$
$$-(n+1) + 2w \le r \le w - n/w, \tag{58d}$$
$$\frac{1}{2}\big(\sqrt{r^2 + 4n} - r\big) \le h \le \frac{1}{2}(n + 1 - r), \tag{58e}$$
$$\frac{1}{2}\big(\sqrt{r^2 + 4n} + r\big) \le w \le \frac{1}{2}(n + 1 + r). \tag{58f}$$

The first inequality in (58a) can be seen by noting that $n \le hw$, because the right-hand side is the area of the smallest rectangle into which the Young diagram of n boxes fits. The second inequality in (58a) can be seen by noting that $h + w \le n + 1$, because the left-hand side is the half circumference of the smallest rectangle into which the Young diagram of n boxes fits. The inequalities in (58b) come by the same reasoning, or

 15

by conjugation (50) in (58a). The inequalities in (58c) come by subtracting h from (58b). The inequalities in (58d) come by an analogous reasoning, or by conjugation (50) in (58c). The inequalities in (58e) come by solving the respective inequalities in (58c) for h. The inequalities in (58f) come by an analogous reasoning, or by conjugation (50) in (58e).

The first inequality in (58d) and the second inequalities in the others in (58) are saturated for the partition $\hat{\xi} = \{m, 1, 1, \ldots, 1\}$, where the integer 1 occurs $n - m$ times. The second inequality in (58d) and the first inequalities in the others in (58) are saturated for the partition $\hat{\xi} = \{m, m, \ldots, m, n - (\lceil n/m \rceil - 1)m\}$, where the integer m occurs $\lceil n/m \rceil - 1$ times. In the cases where noninteger value stays on one side of an inequality, saturation is understood for the inequality strengthened by the ceiling or floor functions $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$ or $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$; that is, if $q \leq m \in \mathbb{N}$, then also $\lceil q \rceil \leq m$, and if $q \geq m \in \mathbb{N}$, then also $\lfloor q \rfloor \geq m$. (For illustration, see Figure 6. The second inequalities in (58a) and (58b) express that the arrangement of Young diagrams in Figure 6 is "skew upper triangular", while the first inequalities in (58a) and (58b) describe the "hyperbolic" shape of the upper boundary. The other inequalities in (58) give the boundaries in the rank-height or width-rank plane.)

Since the partitionability is related to lower-bounding the height (48a), the producibility is related to upper-bounding the width (48b), and the stretchability is related to their difference (48c), the bounds in (58) lead to the following relations between these properties

$$\hat{\boldsymbol{\xi}}_{k\text{-part}} \preceq \hat{\boldsymbol{\xi}}_{(n+1-k)\text{-prod}}, \tag{59a}$$

$$\hat{\boldsymbol{\xi}}_{k\text{-part}} \preceq \hat{\boldsymbol{\xi}}_{(n+1-2k)\text{-str}}, \tag{59b}$$

$$\hat{\boldsymbol{\xi}}_{k\text{-prod}} \preceq \hat{\boldsymbol{\xi}}_{(\lceil n/k \rceil)\text{-part}}, \tag{59c}$$

$$\hat{\boldsymbol{\xi}}_{k\text{-prod}} \preceq \hat{\boldsymbol{\xi}}_{(k-\lceil n/k \rceil)\text{-str}}, \tag{59d}$$

$$\hat{\boldsymbol{\xi}}_{k\text{-str}} \preceq \hat{\boldsymbol{\xi}}_{\frac{1}{2}(\lceil \sqrt{k^2+4n} \rceil - k)\text{-part}}, \tag{59e}$$

$$\hat{\boldsymbol{\xi}}_{k\text{-str}} \preceq \hat{\boldsymbol{\xi}}_{\frac{1}{2}(n+1+k)\text{-prod}}. \tag{59f}$$

The relations in (59a) and (59b) can be seen by using the second inequality in (58b) and the second inequality in (58c), respectively, with (48a). The relations in (59c) and (59d) can be seen by using the first inequality in (58a) and the second inequality in (58d), respectively, with (48b). (For noninteger values, strengthening by the ceiling function was also exploited, as before.) The relations in (59e) and (59f) can be seen by using the first inequality in (58e) and the second inequality in (58f), respectively, with (48c).

From the relations (59), for the inclusion of the state spaces (40), we have

$$\mathcal{D}_{k\text{-part unc}} \subseteq \mathcal{D}_{(n+1-k)\text{-prod unc}}, \tag{60a}$$

$$\mathcal{D}_{k\text{-part unc}} \subseteq \mathcal{D}_{(n+1-2k)\text{-str unc}}, \tag{60b}$$

$$\mathcal{D}_{k\text{-prod unc}} \subseteq \mathcal{D}_{\lceil n/k \rceil\text{-part unc}}, \tag{60c}$$

$$\mathcal{D}_{k\text{-prod unc}} \subseteq \mathcal{D}_{(k-\lceil n/k \rceil)\text{-str unc}}, \tag{60d}$$

$$\mathcal{D}_{k\text{-str unc}} \subseteq \mathcal{D}_{\frac{1}{2}(\lceil \sqrt{k^2+4n} \rceil - k)\text{-part unc}}, \tag{60e}$$

$$\mathcal{D}_{k\text{-str unc}} \subseteq \mathcal{D}_{\frac{1}{2}(n+1+k)\text{-prod unc}}, \tag{60f}$$

$$\mathcal{D}_{k\text{-part sep}} \subseteq \mathcal{D}_{(n+1-k)\text{-prod sep}}, \tag{60g}$$

$$\mathcal{D}_{k\text{-part sep}} \subseteq \mathcal{D}_{(n+1-2k)\text{-str sep}}, \tag{60h}$$

$$\mathcal{D}_{k\text{-prod sep}} \subseteq \mathcal{D}_{\lceil n/k \rceil\text{-part sep}}, \tag{60i}$$

$$\mathcal{D}_{k\text{-prod sep}} \subseteq \mathcal{D}_{(k-\lceil n/k \rceil)\text{-str sep}}, \tag{60j}$$

$$\mathcal{D}_{k\text{-str sep}} \subseteq \mathcal{D}_{\frac{1}{2}(\lceil \sqrt{k^2+4n} \rceil - k)\text{-part sep}}, \tag{60k}$$

$$\mathcal{D}_{k\text{-str sep}} \subseteq \mathcal{D}_{\frac{1}{2}(n+1+k)\text{-prod sep}}, \tag{60l}$$

by the order isomorphisms (26); and for the bounds of the measures (42), we have

$$C_{k\text{-part}} \geq C_{(n+1-k)\text{-prod}}, \tag{61a}$$

$$C_{k\text{-part}} \geq C_{(n+1-2k)\text{-str}}, \tag{61b}$$

$$C_{k\text{-prod}} \geq C_{\lceil n/k \rceil\text{-part}}, \tag{61c}$$

$$C_{k\text{-prod}} \geq C_{(k-\lceil n/k \rceil)\text{-str}}, \tag{61d}$$

$$C_{k\text{-str}} \geq C_{\frac{1}{2}(\lceil \sqrt{k^2+4n} \rceil - k)\text{-part}}, \tag{61e}$$

$$C_{k\text{-str}} \geq C_{\frac{1}{2}(n+1+k)\text{-prod}}, \tag{61f}$$

$$E_{k\text{-part}} \geq E_{(n+1-k)\text{-prod}}, \tag{61g}$$

$$E_{k\text{-part}} \geq E_{(n+1-2k)\text{-str}}, \tag{61h}$$

$$E_{k\text{-prod}} \geq E_{\lceil n/k \rceil\text{-part}}, \tag{61i}$$

$$E_{k\text{-prod}} \geq E_{(k-\lceil n/k \rceil)\text{-str}}, \tag{61j}$$

$$E_{k\text{-str}} \geq E_{\frac{1}{2}(\lceil \sqrt{k^2+4n} \rceil - k)\text{-part}}, \tag{61k}$$

$$E_{k\text{-str}} \geq E_{\frac{1}{2}(n+1+k)\text{-prod}}, \tag{61l}$$

by the multipartite monotonicity (28).

6 Summary, remarks and open questions

In this work we investigated the *partial correlation and entanglement properties* which are *invariant under the permutations of the subsystems.* The set partition based three-level structure, describing the classification of partial correlation and entanglement (Section 2) was mapped to a parallel, integer partition based three-level structure, describing the permutation invariant case (Section 3). This mapping is easy to understand on Level I of the construction, however, to see that it is working well through the whole construction, a formal proof was given (Appendix A). The construction can be made more compact, although less transparent (Section 4), which can be used as a starting point of more advanced investigations.

We also investigated k-partitionability and k-producibility, fitting naturally into the structure of permutation invariant properties. A kind of combination of these two gives k-stretchability, which is sensitive in a balanced way to both the maximal size of correlated (or entangled) subsystems and the minimal

number of subsystems uncorrelated with (or separable from) one another. We studied their relations, and a duality, connecting the former two (Section 5).

In the following, we list some remarks and open questions.

The first point to note is that we have followed a treatment of entanglement, which is somewhat different than the standard LOCC paradigm [15]. The LOCC paradigm grabs the essence of entanglement as a correlation which cannot be created or increased by classical communication (classical interaction). This point of view leads to a classification too detailed and practically unaccomplishable for the multipartite scenario. Our treatment is rooted more in statistics, by noticing that (i) *pure states of classical systems are always uncorrelated, so in pure states, correlations are of quantum origin, and this is what we call entanglement [1]*; and (ii) *mixed states of classical systems can always be formed by forgetting about the identity of pure (hence uncorrelated) states, so in mixed states, correlations which cannot be described in this way are of quantum origin, and this is what we call entanglement.* These principles are working painlessly in the multipartite scenario, entangled states and correlation based definitions of entanglement measures were defined based on these in the first two levels of the structure of multipartite entanglement. In particular, from (i) it follows that entanglement in pure states should be measured by correlation (see (5b) and (10b), and the corresponding quantities for the permutation invariant case, and the notes in the end of Section 2.2). We emphasize that this statistical point of view is fully compatible with LOCC, and has led to a much coarser, finite classification.

Being uncertain (or forgetting) about the identity of the state of the system is a guiding principle in the definition of the different aspects of multipartite entanglement in Section 2 (see also in point (ii) in Section VII.A. in [40]).
States in $\mathcal{D}_L$: We are uncertain about the (pure) state, by which the system is described (Section 2.1).
States in $\mathcal{D}_{\xi\text{-}unc}$ ($\mathcal{D}_{\xi\text{-}sep}$): We are uncertain about the pure state, by which the system is described, but we are certain about the partition with respect to which the state is uncorrelated (separable) (Section 2.2).
States in $\mathcal{D}_{\boldsymbol{\xi}\text{-}unc}$ ($\mathcal{D}_{\boldsymbol{\xi}\text{-}sep}$): We are uncertain about the pure state, by which the system is described, and we are also uncertain about the partition with respect to which the state is uncorrelated (separable), but we are certain about the possible partitions with respect to which the state is uncorrelated (separable) (Section 2.3).
States in $\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-}unc}$ ($\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-}sep}$): We are uncertain about the pure state, by which the system is described, and we are also uncertain about the partition with respect to which the state is uncorrelated (separable), but we are certain about the possible partitions with respect to which the state is uncorrelated (separable), and

we are also certain about the possible partitions with respect to which the state is correlated (entangled) (Section 2.4).

We also mention here that for the state sets of given multipartite correlation (entanglement) properties $\mathcal{D}_{\boldsymbol{\xi}\text{-}unc}$ ($\mathcal{D}_{\boldsymbol{\xi}\text{-}sep}$), except from the fully uncorrelated (or fully separable) case $\boldsymbol{\xi} = \{\bot\}$, we cannot formulate semigroups of quantum channels which could play the role of "free operations" for these "free state" sets in a usual resource-theoretical scenario [75]. Instead of this, being uncertain (or forgetting) about the identity of the *maps* is the guiding principle here (see the characterizations in Sections 2.2 and 2.3, as well as in the permutation invariant case in Sections 3.2 and 3.3). Note that this is physical: only the fully uncorrelated (fully separable) states are for free, the others are resource states, possibly of different "value."

Note that, after the general case (Section 2), we considered the permutation invariant (correlation or entanglement) *properties* (from Section 3). The use of these is not restricted to permutation invariant *states*: these are simply the permutation invariant properties of also non-permutation-invariant states. If the states considered are permutation invariant (for example, in the first quantization of bosonic or fermionic systems), then these are the only relevant properties.

The permutation invariant properties were described by the use of *integer partitions,* for which a refinement-like order was given, which is just the coarsening of the refinement order of *set partitions.* In the literature, there are several partial orders constructed for integer partitions, such as the *dominance order* [76] (based on majorization, leading to a lattice), the *Young order* [45, 62] (based on diagram containment, given for partitions of all integers, leading to *Young's lattice*, here partitions of the same integer cannot be ordered, and the order is invariant to the conjugation), or the *reverse lexicographic order* (leading to a chain). The *refinement order* for integer partitions, introduced in (18c) through the *refinement order* for set partitions (2c), is different from the above orders, and, to our knowledge [45, 62], was not considered upon its merits in the literature before. Although Birkhoff mentioned this partial order in his book [77] as an example, it was not used for any reasonable purpose. The reason for this might be that the resulting poset shows properties not so nice or powerful as those shown by the others. However, this order is what needed is in the classification problem of permutation invariant properties, that is reflected also by the monotonicities (47).

k-partitionability is about the natural gradation of the lattice of set partitions P_I or of the poset of integer partitions $\hat{P}_\mathrm{I}$, but it was not clear, how k-producibility can be understood. The conjugation of integer partitions has explained the role of the latter, by establishing a kind of duality between the two properties. Note, however, that this duality is only par-

tial: although the conjugation of integer partitions (49) interchanges the height and the width (50), by which the k-partitionability and k-producibility properties are given, but the latter properties are given in an opposite way by height and width, see (48a) and (48b). So the conjugation does not interchange k-partitionability and k-producibility, it establishes a connection on the deeper level of integer partitions only. This is also reflected in the relations among the different k-partitionability and k-producibility properties, given in (59a) and (59c).

For $n \leq 6$, the height and width determine $\hat{\xi}$ uniquely. For $n = 7$, we have $\{3, 2, 2\}$ and $\{3, 3, 1\}$, being of height 3 and width 3. Note that two different integer partitions of the same height and width can never be ordered. (Ordered pairs are of different height, because of the strict monotonicity of the height (47).)

The property of k-stretchability (48c) was introduced by the rank of the partition. We note that the *rank* of an integer partition (46c) was defined and used originally in a very different context in number theory and combinatorics. It was introduced by Freeman Dyson [78] in his investigations of Ramanujan's congruences in the partition function [74].

For n subsystems, the number of nontrivial k-partitionability and k-producibility properties is $n - 1$ in both cases (1-partitionability and n-producibility are trivial), while the number of nontrivial k-stretchability properties is $2(n-1)$ ($(n-1)$-stretchability is trivial). All of these three properties form chains, see (39a), (39b) and (51), and the relations among them are given in (59). Note that k-stretchability combines the advantages of k-partitionability and k-producibility, in the sense that it rewards large correlated (or entangled) subsystems, while it punishes the larger number of subsystems uncorrelated (or separable) with one another. Also, while the relations between k-partitionability and k-producibility are highly nontrivial (59a), (59c), the k-stretchability properties form a chain (51). The price to pay for this is that sometimes there are rather different properties not distinguished by stretchability, for example, $\{3, 3\}$ and $\{4, 1, 1\}$ are both 1-stretchable. Taking stretchability seriously leads to a highly nontrivial, balanced comparison of permutation invariant multipartite correlation or entanglement properties.

Acknowledgments

A   On the structure of the classification of permutation invariant correlations

A.1   Coarsening a poset

Let us have a *finite set* P endowed with a *binary relation* $\preceq$, and a *function* f, mapping P to $P' := f(P)$. (In the appendix we consider finite sets only, even if it is not mentioned explicitly.) The same function transforms also the relation naturally as follows. A binary relation can be considered as a set of ordered pairs in P, written as $\preceq \equiv \{(b, a) \in P \times P \mid b \preceq a\}$, then $\preceq' := (f \times f)(\preceq) = \{(f(b), f(a)) \mid \forall (b, a) \in \preceq\}$ by the elementwise action of $f \times f$. We use the shorthand notation $\preceq' = f(\preceq)$ for this. The meaning of this is that

$$\forall a', b' \in P', \quad b' \preceq' a' \iff \\ \exists a \in f^{-1}(a'), \exists b \in f^{-1}(b') \text{ s.t. } b \preceq a. \tag{62}$$

(Here $f^{-1}(a') = \{a \in P \mid f(a) = a'\} \subseteq P$ denotes the inverse image of the singleton $\{a'\}$.)

If the binary relation $\preceq$ is a *partial order* [43, 45], the transformed one $\preceq'$ is not necessarily that. To make it a partial order, we need some constraints imposed on f. Let us introduce the following three conditions:

$$\forall a', b' \in P', \text{ if } b' \preceq' a' \text{ then} \\ \forall a \in f^{-1}(a'), \exists b \in f^{-1}(b'), \text{ s.t. } b \preceq a; \tag{63a}$$

$$\forall a', b' \in P', \text{ if } b' \preceq' a' \text{ then} \\ \forall b \in f^{-1}(b'), \exists a \in f^{-1}(a'), \text{ s.t. } b \preceq a; \tag{63b}$$

$$\forall a, b, c \in P, \text{ if } c \preceq b \preceq a \\ \text{and } f(c) = f(a) \text{ then } f(b) = f(a). \tag{63c}$$

These conditions, although being rather strong, hold in the construction in which we need them (see Appendix A.4, and Lemma 16). They turn out to be sufficient for $\preceq'$ to be a partial order, as the following lemma states.

Lemma 1. *Let $(P, \preceq)$ be a poset, then $(P', \preceq') = (f(P), f(\preceq))$ is a poset if (63a) and (63c) hold, or if (63b) and (63c) hold.*

Proof. We need to prove that $\preceq'$ is a partial order in these cases.
(i) *Reflexivity* ($\forall a' \in P', a' \preceq' a'$): for all $a' \in P'$ we have $f^{-1}(a') \neq \emptyset$, since f is surjective, and we need $b \in f^{-1}(a')$ and $a \in f^{-1}(a')$ for which $b \preceq a$ by (62); this holds for the choice $b = a$, since the partial order $\preceq$ is reflexive, $a \preceq a$. (This proof does not use any of

the constraints (63).)

(ii) *Antisymmetry* ($\forall a', b' \in P'$, if $a' \preceq' b'$ and $b' \preceq' a'$, then $a' = b'$): let $b' \preceq' a'$, which means that there exist $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$ such that $b \preceq a$, by (62). Fix such a pair b and a. Also, let $a' \preceq' b'$, applying (63a) for b, we have that there exists $c \in f^{-1}(a')$, such that $c \preceq b$. Then, applying (63c) to $c \preceq b \preceq a$, we have that $b' = f(b) = f(a) = a'$. (This proof uses (63a) and (63c). A similar proof can be given by using (63b) and (63c).)

(iii) *Transitivity* ($\forall a', b', c' \in P'$, if $c' \preceq' b'$ and $b' \preceq' a'$, then $c' \preceq' a'$): let $b' \preceq' a'$, which means that there exist $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$ such that $b \preceq a$, by (62). Fix such a pair b and a. Also, let $c' \preceq' b'$, applying (63a) for b, we have that there exists $c \in f^{-1}(c')$, such that $c \preceq b$. Then, since the partial order $\preceq$ is transitive, we have that $c \preceq a$ for an $a \in f^{-1}(a')$ and $c \in f^{-1}(c')$, which means that $c' \preceq' a'$ by (62). (This proof uses (63a). A similar proof can be given by using (63b).) □

The following construction would be much simpler if the conditions (63) would also be necessary. Note that this is not the case: if $\preceq'$ is a partial order, then (63c) holds, but (63a) and (63b) do not hold. For example, for the poset $P = \{a_1, a_2, b\}$ with the only arrow $b \prec a_1$, and the function given as $f(a_1) = f(a_2) = a' \neq f(b) = b'$, we have $b' \prec' a'$, but (63a) does not hold.

If (63a) or (63b), and (63c) hold, then the new poset $(P', \preceq')$ can be considered as a coarsening of $(P, \preceq)$ by f. ($f : P \to P'$ is surjective by definition. If it is also injective, then $(P, \preceq)$ and $(P', \preceq')$ are isomorphic, and the coarsening is trivial.) Note that, because of the construction, $f : P \to P'$ is automatically monotone,

$$b \preceq a \implies f(b) \preceq' f(a). \qquad (64)$$

(Indeed, $a \in f^{-1}(f(a))$ and $b \in f^{-1}(f(b))$, so (62) holds.) Note also that, if the poset $(P, \preceq)$ is a lattice, the transformed one $(P', \preceq')$ is not necessarily that. To make it a lattice, further conditions have to be imposed; this problem is not addressed here.

A.2 Down-sets and up-sets

In a poset P, a *down-set* (*order ideal*) is a subset $\mathbf{a} \subseteq P$, which is closed downwards [43, 45],

$$\mathbf{a} \in \mathcal{O}_\downarrow(P) \quad \overset{\text{def.}}{\iff} \quad \forall a \in \mathbf{a}, \, \forall b \preceq a : \, b \in \mathbf{a}; \quad (65a)$$

while an *up-set* (*order filter*) is a subset $\mathbf{a} \subseteq P$, which is closed upwards [43, 45],

$$\mathbf{a} \in \mathcal{O}_\uparrow(P) \quad \overset{\text{def.}}{\iff} \quad \forall a \in \mathbf{a}, \, \forall b \succeq a : \, b \in \mathbf{a}. \quad (65b)$$

These form lattices with respect to the inclusion $\subseteq$, with the join $\vee$ (least upper bound), being the union $\cup$, and the meet $\wedge$ (greatest lower bound), being the intersection $\cap$.

As in the previous section, let us have the poset $(P, \preceq)$, the function f, by which $(P', \preceq') := (f(P), f(\preceq))$, and for which (63a) or (63b), and (63c) hold. Additionally, let P have *bottom* and *top elements*. (Then also P' has *bottom* and *top elements*. Indeed, this is because the bottom and top elements in P are mapped to the bottom and top elements in P', because of the (64) monotonicity of f.) Now let us form from the posets $(P, \preceq)$ and $(P', \preceq')$ the lattices of nonempty down-sets

$$(Q, \sqsubseteq) := (\mathcal{O}_\downarrow(P) \setminus \{\emptyset\}, \subseteq), \qquad (66a)$$

$$(Q', \sqsubseteq') := (\mathcal{O}_\downarrow(P') \setminus \{\emptyset\}, \subseteq). \qquad (66b)$$

(The existence of the bottom elements in P and P' ensures that not only the down-sets but also the nonempty down-sets form lattices in both cases [40].) Then, by denoting the elementwise action of f with $g : 2^P \to 2^{P'}$, that is, $g(\mathbf{a}) = \{f(a) \mid a \in \mathbf{a}\}$, in the following lemmas we will show that $(Q', \sqsubseteq') = (g(Q), g(\sqsubseteq))$, that is, the diagram

$$
\begin{array}{ccc}
(Q, \sqsubseteq) & \overset{g}{\longmapsto} & (Q', \sqsubseteq') \\
\Big\uparrow {\scriptstyle \mathcal{O}_\downarrow \setminus \{\emptyset\}} & & \Big\uparrow {\scriptstyle \mathcal{O}_\downarrow \setminus \{\emptyset\}} \\
(P, \preceq) & \overset{f}{\longmapsto} & (P', \preceq')
\end{array} \qquad (67)
$$

commutes. Here $g(\sqsubseteq)$ is the partial order transformed by g in the same way as $\preceq$ was transformed by f in (62), and we also have automatically that g is monotone for $\sqsubseteq$ and $g(\sqsubseteq)$, in the same way as in (64).

The following two technical lemmas concerning the sets $\bigcup_{a' \in \mathbf{a}'} f^{-1}(a') \subseteq P$ will be used several times later.

Lemma 2. *In the above setting (and assuming (63)), for all $\mathbf{a}' \in Q'$, we have $g\big(\bigcup_{a' \in \mathbf{a}'} f^{-1}(a')\big) = \mathbf{a}'$.*

Proof. This is because g is the elementwise action of f, so $g\big(\bigcup_{a' \in \mathbf{a}'} f^{-1}(a')\big) = g\big(\bigcup_{a' \in \mathbf{a}'}\{a \in P \mid f(a) = a'\}\big) = \bigcup_{a' \in \mathbf{a}'} g\big(\{a \in P \mid f(a) = a'\}\big) = \bigcup_{a' \in \mathbf{a}'}\{a'\} = \mathbf{a}'$. □

Lemma 3. *In the above setting (and assuming (63)), for all $\mathbf{a}' \in Q'$, we have $\bigcup_{a' \in \mathbf{a}'} f^{-1}(a') \in Q$, that is, it is a nonempty down-set.*

Proof. First, denote $\mathbf{a} := \bigcup_{a' \in \mathbf{a}'} f^{-1}(a')$. On the one hand, $\mathbf{a} \neq \emptyset$, since $\mathbf{a}' \neq \emptyset$ and f is surjective. On the other hand, if $b \preceq a$ for an $a \in \mathbf{a}$, then $f(b) \preceq' f(a)$ because of the monotonicity (64). Since $f(a) \in \mathbf{a}'$ by the construction of $\mathbf{a}$, and $\mathbf{a}'$ is a down-set (65a), we have that $f(b) \in \mathbf{a}'$. Then $b \in f^{-1}(f(b)) \subseteq \mathbf{a}$ by the construction of $\mathbf{a} = f^{-1}(f(b)) \cup \bigcup_{a' \in \mathbf{a}', a' \neq f(b)} f^{-1}(a')$, and then $b \in \mathbf{a}$, so $\mathbf{a}$ is a down-set. Altogether we have that $\mathbf{a} \in Q$. □

The following two lemmas show (67).

Lemma 4. *In the above setting (and assuming* (63)*), we have* $g(Q) = Q'$.

Proof. We need to prove both inclusions.
(i) We need that $g(Q) \subseteq Q'$. Let $\mathbf{a} \in Q$ and $\mathbf{a}' := g(\mathbf{a})$. Then for all $a' \in \mathbf{a}'$ let us have $a \in f^{-1}(a') \cap \mathbf{a}$. (Note that $f^{-1}(a') \cap \mathbf{a} \neq \emptyset$ by construction.) Then let $b' \preceq' a'$, and applying (63a) for a, we have that there exists $b \in f^{-1}(b')$ such that $b \preceq a$. Since $a \in \mathbf{a}$, and $\mathbf{a}$ is a down-set (65a), we have that $b \in \mathbf{a}$, and then $b' \in \mathbf{a}'$, so $\mathbf{a}'$ is a down-set.
(ii) We also need that $g(Q) \supseteq Q'$. Let $\mathbf{a}' \in Q'$, then we construct an $\mathbf{a} \in Q$, for which $\mathbf{a}' = g(\mathbf{a})$. The element $\mathbf{a} := \bigcup_{a' \in \mathbf{a}'} f^{-1}(a')$ fulfills these criteria, we have $\mathbf{a}' = g(\mathbf{a})$ by Lemma 2, and $\mathbf{a} \in Q$ by Lemma 3. $\qquad\square$

Lemma 5. *In the above setting (and assuming* (63)*), we have* $g(\sqsubseteq) = \sqsubseteq'$.

Proof. We need to prove both directions.
(i) We need that for all $\mathbf{a}', \mathbf{b}' \in Q'$, if $\mathbf{b}'g(\sqsubseteq)\mathbf{a}'$ then $\mathbf{b}' \sqsubseteq' \mathbf{a}'$. $\mathbf{b}'g(\sqsubseteq)\mathbf{a}'$ means that $\exists \mathbf{b} \in g^{-1}(\mathbf{b}')$ and $\exists \mathbf{a} \in g^{-1}(\mathbf{a}')$ such that $\mathbf{b} \sqsubseteq \mathbf{a}$. (Again, $g^{-1}(\mathbf{a}') = \{\mathbf{a} \in Q \mid g(\mathbf{a}) = \mathbf{a}'\}$ is the inverse image of the singleton $\{\mathbf{a}'\}$.) In this case, $\mathbf{b}' = g(\mathbf{b}) = \{f(b) \mid b \in \mathbf{b}\} \subseteq \{f(a) \mid a \in \mathbf{a}\} = g(\mathbf{a}) = \mathbf{a}'$, so $\mathbf{b}' \sqsubseteq' \mathbf{a}'$.
(ii) We also need that for all $\mathbf{a}', \mathbf{b}' \in Q'$, if $\mathbf{b}' \sqsubseteq' \mathbf{a}'$ then $\mathbf{b}'g(\sqsubseteq)\mathbf{a}'$. From $\mathbf{b}'$ and $\mathbf{a}'$, let us form $\mathbf{b} := \bigcup_{b' \in \mathbf{b}'} f^{-1}(b')$ and $\mathbf{a} := \bigcup_{a' \in \mathbf{a}'} f^{-1}(a')$. We have that $\mathbf{b} \in Q$ and $\mathbf{a} \in Q$ (they are nonempty down-sets) by Lemma 3, also $g(\mathbf{b}) = \mathbf{b}'$ and $g(\mathbf{a}) = \mathbf{a}'$ by Lemma 2, and if $\mathbf{b}' \sqsubseteq' \mathbf{a}'$ then $\mathbf{b} \sqsubseteq \mathbf{a}$ by construction. That is, we have constructed $\mathbf{b} \in g^{-1}(\mathbf{b}')$ and $\mathbf{a} \in g^{-1}(\mathbf{a}')$, for which $\mathbf{b} \sqsubseteq \mathbf{a}$, so we have $\mathbf{b}'g(\sqsubseteq)\mathbf{a}'$. $\qquad\square$

Summing up, Lemma 4 and Lemma 5 together state that (67) commutes. Later we also need that the properties (63a), (63b) and (63c) are inherited.

Lemma 6. *In the above setting (and assuming* (63)*), we have*

$$\forall \mathbf{a}', \mathbf{b}' \in Q', \;\; \text{if } \mathbf{b}' \sqsubseteq' \mathbf{a}' \;\; \text{then} \\ \forall \mathbf{a} \in g^{-1}(\mathbf{a}') \; \exists \mathbf{b} \in g^{-1}(\mathbf{b}') \;\; s.t. \;\; \mathbf{b} \sqsubseteq \mathbf{a}; \tag{68a}$$

$$\forall \mathbf{a}', \mathbf{b}' \in Q', \;\; \text{if } \mathbf{b}' \sqsubseteq' \mathbf{a}' \;\; \text{then} \\ \forall \mathbf{b} \in g^{-1}(\mathbf{b}') \; \exists \mathbf{a} \in g^{-1}(\mathbf{a}') \;\; s.t. \;\; \mathbf{b} \sqsubseteq \mathbf{a}; \tag{68b}$$

$$\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in Q, \;\; \text{if } \mathbf{c} \sqsubseteq \mathbf{b} \sqsubseteq \mathbf{a} \\ \text{and } g(\mathbf{c}) = g(\mathbf{a}), \;\; \text{then } g(\mathbf{b}) = g(\mathbf{a}). \tag{68c}$$

Proof. (68a) can be proven by the explicit construction of $\mathbf{b}$. Let $\mathbf{b}' \sqsubseteq' \mathbf{a}'$, and $\mathbf{a} \in g^{-1}(\mathbf{a}')$, then $\mathbf{b} := \left(\bigcup_{b' \in \mathbf{b}'} f^{-1}(b') \right) \cap \mathbf{a}$. By construction, $\mathbf{b} \sqsubseteq \mathbf{a}$. Also, $\mathbf{b} \in Q$, that is, it is a nonempty down-set, since it is the intersection (meet) of two nonempty down-sets: $\mathbf{a} \in Q$, and $\bigcup_{b' \in \mathbf{b}'} f^{-1}(b') \in Q$ by Lemma 3, and Q is a lattice. In addition, we need that $\mathbf{b} \in g^{-1}(\mathbf{b}')$, that is, $g(\mathbf{b}) = \mathbf{b}'$. Since $g(\mathbf{a}) = \mathbf{a}'$, we have that $\forall a' \in \mathbf{a}', \exists a \in \mathbf{a}$ such that $f(a) = a'$. Then also $\forall b' \in \mathbf{b}' \sqsubseteq' \mathbf{a}', \exists a \in \mathbf{a}$ such that $f(a) = b'$, that is,

$a \in f^{-1}(b')$, and also $a \in \mathbf{a}$, so $a \in \mathbf{b}$ by construction of $\mathbf{b}$.
(68b) can be proven analogously.
(68c) is a simple consequence of the properties of $\sqsubseteq$. Let $\mathbf{c} \sqsubseteq \mathbf{b} \sqsubseteq \mathbf{a}$, then $g(\mathbf{c}) \sqsubseteq' g(\mathbf{b}) \sqsubseteq' g(\mathbf{a})$ by the monotonicity of g, so by the transitivity and antisymmetry of the partial order $\sqsubseteq'$, if $g(\mathbf{c}) = g(\mathbf{a})$ then $g(\mathbf{b}) = g(\mathbf{a})$.
Note that the conditions (63) were not used explicitly in these proofs, they are assumed only for establishing the partial order $\preceq'$ for the set P', see Lemma 1. $\qquad\square$

Lemma 7. *Lemmas* 2, 3, 4, 5 *and* 6, *hold also if upsets are used instead of down-sets in the construction* (67).

Proof. All the proofs can be repeated with slight modifications, as interchanging the roles of (63a) and (63b), down-closures and up-closures, and reversing orders. $\qquad\square$

Thanks to Lemma 4, Lemma 5, Lemma 6 and Lemma 7, the construction (67) can be repeated arbitrary times, with either up- or down-sets, if the conditions (63) hold for the first level. Note that $\sqsubseteq$ and $\sqsubseteq'$ are partial orders by construction anyway. The inheritance of (63), shown in Lemma 6 is needed not for that, but also for, e.g., Lemma 4, when applied for the next level.

A.3 Embedding

Now, with the definition (66), let the function $g' : Q' \to Q$ be given as

$$g'(\mathbf{a}') := \vee g^{-1}(\mathbf{a}'), \tag{69}$$

and its image $Q'' := g'(Q') \subseteq Q$. We will show that this is a subposet, and $(Q'', \sqsubseteq) \cong (Q', \sqsubseteq')$, and g' is an embedding of Q' into Q. (We use the notation $\vee A := \bigvee_{a \in A} a$ for any subset A of a lattice.)

The following two technical lemmas concerning the images $g'(\mathbf{a}')$ will be used several times later.

Lemma 8. *In the above setting (and assuming* (63)*), for all* $\mathbf{a}' \in Q'$*, we have*

$$g'(\mathbf{a}') = \{a \in P \mid f(a) \in \mathbf{a}'\}, \tag{70a}$$

$$g'(\mathbf{a}') = \bigcup_{a' \in \mathbf{a}'} f^{-1}(a'). \tag{70b}$$

Proof. (70a) is by definition $g'(\mathbf{a}') \equiv \vee\{\mathbf{a} \in Q \mid g(\mathbf{a}) = \mathbf{a}'\} = \{a \in P \mid f(a) \in \mathbf{a}'\}$, and we need to prove both inclusions. First, $\vee\{\mathbf{a} \in Q \mid g(\mathbf{a}) = \mathbf{a}'\} \subseteq \{a \in P \mid f(a) \in \mathbf{a}'\}$ holds, since g is the elementwise action of f. Second, for $\vee\{\mathbf{a} \in Q \mid g(\mathbf{a}) = \mathbf{a}'\} \supseteq \{a \in P \mid f(a) \in \mathbf{a}'\}$, we need to see that for all $a \in P$ such that $f(a) \in \mathbf{a}'$ there exists an $\mathbf{a} \in Q$ for which $a \in \mathbf{a}$, while $g(\mathbf{a}) = \mathbf{a}'$. The element $\mathbf{a} := \bigcup_{a' \in \mathbf{a}'} f^{-1}(a')$ fulfills these criteria, we

have $\mathbf{a}' = g(\mathbf{a})$ by Lemma 2, and $\mathbf{a} \in Q$ by Lemma 3; while $a \in \mathbf{a}$, since $f(a) \in \mathbf{a}'$, so it appears in the union $\bigcup_{a' \in \mathbf{a}'} f^{-1}(a') = f^{-1}(f(a)) \cup \bigcup_{a' \in \mathbf{a}', a' \neq f(a)} f^{-1}(a')$. (70b) can be proven by noting that for the left-hand side, we have by (70a) that $g'(\mathbf{a}') = \{a \in P \mid f(a) \in \mathbf{a}'\} \equiv \{a \in P \mid \exists a' \in \mathbf{a}' \text{ s.t. } f(a) = a'\}$, which is the same as the right-hand side $\bigcup_{a' \in \mathbf{a}'} \{a \in P \mid f(a) = a'\}$. $\square$

Lemma 9. *In the above setting (and assuming* (63)*), for all* $\mathbf{a}' \in Q'$, *we have* $g(g'(\mathbf{a}')) = \mathbf{a}'$.

Proof. Using (70b), we have that $g(g'(\mathbf{a}')) = g\left(\bigcup_{a' \in \mathbf{a}'} f^{-1}(a')\right)$, then Lemma 2 leads to the claim. $\square$

The following lemma shows that $g' : Q' \to Q'' \subseteq Q$ is an embedding of Q' into Q.

Lemma 10. *In the above setting (and assuming* (63)*),* $(Q'', \sqsubseteq) \cong (Q', \sqsubseteq')$.

Proof. We need to prove that for all $\mathbf{a}', \mathbf{b}' \in Q'$, $\mathbf{b}' \sqsubseteq' \mathbf{a}' \Leftrightarrow g'(\mathbf{b}') \sqsubseteq g'(\mathbf{a}')$, then, from the antisymmetry of the partial order, we have that g' is bijective. To see the *"if"* direction, we have $g'(\mathbf{b}') \sqsubseteq g'(\mathbf{a}')$ then $g(g'(\mathbf{b}')) \sqsubseteq' g(g'(\mathbf{a}'))$ by the monotonicity of g, similarly to (64), then $\mathbf{b}' \sqsubseteq' \mathbf{a}'$ by Lemma 9. To see the *"only if"* direction, we have $\mathbf{b}' \sqsubseteq' \mathbf{a}'$, then $\{b \in P \mid f(b) \in \mathbf{b}'\} \sqsubseteq \{a \in P \mid f(a) \in \mathbf{a}'\}$, then $g'(\mathbf{b}') \sqsubseteq g'(\mathbf{a}')$ by (70a). $\square$

In the following four lemmas, we provide some simple tools, concerning principal ideals.

Lemma 11. *In the above setting (and assuming* (63)*), for all* $a \in P$, *we have* $g(\downarrow\{a\}) = \downarrow\{f(a)\}$.

Proof. For the left-hand side we have $g(\downarrow\{a\}) = g(\{b \in P \mid b \preceq a\}) = \{b' \in P' \mid \exists b \in P \text{ s.t. } f(b) = b', b \preceq a\} = \{b' \in P' \mid \exists b \in f^{-1}(b') \text{ s.t. } b \preceq a\}$, while for the right-hand side we have $\downarrow\{f(a)\} = \{b' \in P' \mid b' \preceq' f(a)\}$, by definitions. So we need that for all $a \in P$ and $b' \in P'$, $(\exists b \in f^{-1}(b') \text{ s.t. } b \preceq a) \Leftrightarrow b' \preceq' f(a)$. To see the *"if"* direction, we have that if $b' \preceq' f(a)$ then for $a \in f^{-1}(f(a))$ there exists $b \in f^{-1}(b')$ such that $b \preceq a$ by (63a). To see the *"only if"* direction, we have that if $\exists b \in f^{-1}(b')$ such that $b \preceq a$, then $f(b) \preceq' f(a)$ by the monotonicity (64) of f, then $b' \preceq' f(a)$, since $f(b) = b'$ by the assumption $b \in f^{-1}(b')$. $\square$

Lemma 12. *In the above setting (and assuming* (63)*), for all* $a' \in P'$, *we have* $g'(\downarrow\{a'\}) = \downarrow f^{-1}(a')$.

Proof. For the left-hand side we have $g'(\downarrow\{a'\}) = \{b \in P \mid f(b) \in \downarrow\{a'\}\} = \{b \in P \mid f(b) \preceq a'\}$ by applying (70a), while for the right-hand side we have $\downarrow f^{-1}(a') = \downarrow\{a \in P \mid f(a) = a'\} = \{b \in P \mid \exists a \in P \text{ s.t. } b \preceq a, f(a) = a'\} = \{b \in P \mid \exists a \in f^{-1}(a') \text{ s.t. } b \preceq a\}$, by definitions. So we need that

for all $a' \in P'$ and $b \in P$ ($\exists a \in f^{-1}(a')$ s.t. $b \preceq a$) $\Leftrightarrow f(b) \preceq' a'$.

To see the *"if"* direction, we have that if $f(b) \preceq' a'$ then for $b \in f^{-1}(f(b))$ there exists $a \in f^{-1}(a')$ such that $b \preceq a$ by (63b). To see the *"only if"* direction, we have that if $\exists a \in f^{-1}(a')$ such that $b \preceq a$, then $f(b) \preceq' f(a)$ by the monotonicity (64) of f, then $f(b) \preceq' a'$, since $f(a) = a'$ by the assumption $a \in f^{-1}(a')$. $\square$

Lemma 13. *In the above setting (and assuming* (63)*), for all* $a', b' \in P'$, *we have* $b' \preceq' a' \Leftrightarrow g'(\downarrow\{b'\}) \sqsubseteq g'(\downarrow\{a'\})$.

Proof. First, we have $b' \preceq' a' \Leftrightarrow \downarrow\{b'\} \sqsubseteq' \downarrow\{a'\}$ obviously. Second, we have $\downarrow\{b'\} \sqsubseteq' \downarrow\{a'\} \Leftrightarrow g'(\downarrow\{b'\}) \sqsubseteq g'(\downarrow\{a'\})$, which is a special case of Lemma 10, for principal ideals in Q'. $\square$

Lemma 14. *In the above setting (and assuming* (63)*), for all* $\mathbf{a}' \in Q'$, *we have*

$$g'(\mathbf{a}') = \bigvee_{a' \in \mathbf{a}'} \downarrow f^{-1}(a'). \qquad (71)$$

Proof. We have $\bigvee_{a' \in \mathbf{a}'} \downarrow f^{-1}(a') = \bigvee_{a' \in \mathbf{a}'} g'(\downarrow\{a'\}) = \bigvee_{a' \in \mathbf{a}'} \bigcup_{b' \in \downarrow\{a'\}} f^{-1}(b') = \bigcup_{a' \in \mathbf{a}', b' \preceq a'} f^{-1}(b') = \bigcup_{a' \in \mathbf{a}'} f^{-1}(b') = g'(\mathbf{a}')$, by using Lemma 12, (70b), exploiting that $\mathbf{a}'$ is a down-set (65a), then (70b) again, respectively. $\square$

Although it will not be used, the following lemma shows that the whole construction in this section works also if up-sets are used instead of down-sets.

Lemma 15. *Lemmas 8, 9, 10, 11, 12, 13 and 14 hold also if up-sets are used instead of down-sets in the construction* (67)*.*

Proof. All the proofs can be repeated with slight modifications. $\square$

A.4 Application to the set and integer partitions

Now we turn to the case of the main text, for which the machinery developed in the previous sections is applied. That is, for the roles of $(P, \preceq)$ and $(P', \preceq')$ we have the poset $(P_\text{I}, \preceq)$ of set partitions (2b) with the refinement relation (2c), and the poset $(\hat{P}_\text{I}, \preceq)$ of integer partitions (18b) with the refinement relation (18c).

First, we have the $\sigma \in \text{S}(L)$ permutations of the elementary subsystems L, acting naturally on the partitions $\xi = \{X_1, X_2, \ldots, X_{|\xi|}\} \in P_\text{I}$ as $\sigma(\xi) = \{\{\sigma(i) \mid i \in X\} \mid X \in \xi\} \in P_\text{I}$. We will make use of the obvious observations that all the permutations of a partition $\xi \in P_\text{I}$ are of the same type, $s(\sigma(\xi)) = s(\xi) \in \hat{P}_\text{I}$; and all partitions of a given type $\hat{\xi} \in \hat{P}_\text{I}$ can be transformed into one another by suitable permutations, that is, for all $\xi_1, \xi_2 \in s^{-1}(\hat{\xi})$,

there exists at least one $\sigma \in \mathrm{S}(L)$ by which $\xi_2 = \sigma(\xi_1)$. Also, for all $\sigma \in \mathrm{S}(L)$ permutations, $\upsilon \preceq \xi$ if and only if $\sigma(\upsilon) \preceq \sigma(\xi)$. Using these, we can show that (63)-like conditions hold for this case.

Lemma 16. *In the setting of the main text, the following conditions hold*

$$\forall \hat{\xi}, \hat{\upsilon} \in \hat{P}_I, \ \ if \ \hat{\upsilon} \preceq \hat{\xi} \ then$$
$$\forall \xi \in s^{-1}(\hat{\xi}), \ \exists \upsilon \in s^{-1}(\hat{\upsilon}), \ s.t. \ \upsilon \preceq \xi; \tag{72a}$$

$$\forall \hat{\xi}, \hat{\upsilon} \in \hat{P}_I, \ \ if \ \hat{\upsilon} \preceq \hat{\xi} \ then$$
$$\forall \upsilon \in s^{-1}(\hat{\upsilon}), \ \exists \xi \in s^{-1}(\hat{\xi}), \ s.t. \ \upsilon \preceq \xi; \tag{72b}$$

$$\forall \xi, \upsilon, \zeta \in P_I, \ \ if \ \zeta \preceq \upsilon \preceq \xi$$
$$and \ s(\zeta) = s(\xi), \ then \ s(\upsilon) = s(\xi). \tag{72c}$$

Proof. Recall that $\hat{\upsilon} \preceq \hat{\xi}$ by definition if there exist $\upsilon_1 \in s^{-1}(\hat{\upsilon})$ and $\xi_1 \in s^{-1}(\hat{\xi})$, such that $\upsilon_1 \preceq \xi_1$, see (18c), in accordance with (62). Then, for all $\xi_2 \in s^{-1}(\hat{\xi})$ we can construct υ_2 which is $\upsilon_2 \preceq \xi_2$, simply as $\upsilon_2 := \sigma(\upsilon_1)$ by any of the permutations implementing $\xi_2 = \sigma(\xi_1)$, leading to (72a).
(72b) can be proven analogously.
(72c) can be proven by noting that for all $\xi, \upsilon \in P$ we have that if $\upsilon \preceq \xi$ then $|\upsilon| \leq |\xi|$, by (2c); and $|\xi| = |s(\xi)|$, by (17). Then $\zeta \preceq \upsilon \preceq \xi$ and $s(\zeta) = s(\xi)$ lead to $|\zeta| \leq |\upsilon| \leq |\xi|$ with $|\zeta| = |\xi|$, form which we have $|\upsilon| = |\xi|$. Now we need that $|\upsilon| = |\xi|$ and $\upsilon \preceq \xi$ lead to $\upsilon = \xi$ (then $s(\upsilon) = s(\xi)$), which holds, because (2c) must be fulfilled with the same number of parts: let us number them accordingly as $Y_i \subseteq X_i$ for $\xi = \{X_1, X_2, \ldots, X_k\}$ and $\upsilon = \{Y_1, Y_2, \ldots, Y_k\}$, with the constraints of being disjoint and $\bigcup_{i=1}^{k} X_i = \bigcup_{i=1}^{k} Y_i = L$, which leads to $Y_i = X_i$. $\square$

Recall that Lemma 1 states that if a poset $(P, \preceq)$ is transformed by a function f as given in (62), then $(P', \preceq') = (f(P), f(\preceq))$ is a poset if the conditions (63) hold. Now Lemma 16 shows that for the poset $(P_I, \preceq)$, if transformed by s as given in (18b) and (18c) in the main text, the corresponding conditions (72) hold, so $(\hat{P}_I, \preceq) = (s(P), s(\preceq))$ is a poset.

Recall also that if we form the down-set lattices (66), then Lemma 4 and Lemma 5 together state that (67) commutes, Lemma 6 states that the conditions in (63) are inherited. Then with Lemma 7, we have that in the main text, the three-level construction is well-defined as follows.

Corollary 17. *The diagram* (14) *in the main text commutes.*

With Lemma 12 and (71), we also have that for the first two levels of the construction in the main text, the following identities concerning the principal ideals hold.

Corollary 18. *In the setting of the main text, for all $\hat{\xi} \in \hat{P}_I$ and $\hat{\boldsymbol{\xi}} \in \hat{P}_{II}$, we have*

$$\vee s^{-1}(\downarrow\{\hat{\xi}\}) = \downarrow s^{-1}(\hat{\xi}), \tag{73a}$$

$$\vee s^{-1}(\hat{\boldsymbol{\xi}}) = \bigvee_{\hat{\xi} \in \hat{\boldsymbol{\xi}}} \downarrow s^{-1}(\hat{\xi}). \tag{73b}$$

With Lemma 13 and Lemma 10, we also have that for the first two levels of the construction in the main text, the following order isomorphisms hold.

Corollary 19. *In the setting of the main text, for all $\hat{\xi}, \hat{\upsilon} \in \hat{P}_I$ and $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\upsilon}} \in \hat{P}_{II}$, we have*

$$\hat{\upsilon} \preceq \hat{\xi} \iff \vee s^{-1}(\downarrow\{\hat{\upsilon}\}) \preceq \vee s^{-1}(\downarrow\{\hat{\xi}\}), \tag{74a}$$

$$\hat{\boldsymbol{\upsilon}} \preceq \hat{\boldsymbol{\xi}} \iff \vee s^{-1}(\hat{\boldsymbol{\upsilon}}) \preceq \vee s^{-1}(\hat{\boldsymbol{\xi}}). \tag{74b}$$

B Miscellaneous proofs

B.1 Monotonicity of correlation measures

Here we show the ξ-LO monotonicity of the ξ-correlation (5a), from which the LO monotonicity of the $\boldsymbol{\xi}$-correlation (10a) follows, because the latter one is a minimum of some of the former ones. A quantum channel (completely positive trace preserving map [6, 7]) Φ is ξ-LO, if it can be written as $\Phi = \bigotimes_{X \in \xi} \Phi_X$ with quantum channels $\Phi_X : \mathcal{D}_X \to \mathcal{D}_X$ for all $X \in \xi$. With this,

$$
\begin{aligned}
C_\xi\big(\Phi(\varrho)\big) &\overset{(5a)}{=} \min_{\sigma \in \mathcal{D}_{\xi\text{-unc}}} D\big(\Phi(\varrho)\big\|\sigma\big) \\
&\overset{(75)}{\leq} \min_{\sigma' \in \Phi(\mathcal{D}_{\xi\text{-unc}})} D\big(\Phi(\varrho)\big\|\sigma'\big) \\
&= \min_{\sigma \in \mathcal{D}_{\xi\text{-unc}}} D\big(\Phi(\varrho)\big\|\Phi(\sigma)\big) \\
&\overset{(76)}{\leq} \min_{\sigma \in \mathcal{D}_{\xi\text{-unc}}} D\big(\varrho\big\|\sigma\big) \\
&\overset{(5a)}{=} C_\xi(\varrho),
\end{aligned}
$$

where the first inequality comes from

$$\Phi(\mathcal{D}_{\xi\text{-unc}}) \subseteq \mathcal{D}_{\xi\text{-unc}}, \tag{75}$$

which holds for ξ-LOs; and the second inequality comes from the monotonicity of the relative entropy (1b),

$$D\big(\Phi(\varrho)\big\|\Phi(\sigma)\big) \leq D(\varrho\|\sigma), \tag{76}$$

which holds for all quantum channels [6, 7].

Note that the monotonicity shown here is a particular case of a much more general property of monotone distance based *geometric measures* [65] in resource theories [75]: the monotonicity with respect to free maps, by which the set of free states is mapped onto itself.

References

[1] Ervin Schrödinger. Die gegenwärtige Situation in der Quantenmechanik. *Naturwissenschaften*, 23: 807, 1935. DOI: 10.1007/BF01491987.

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22

[2] Ervin Schrödinger. Discussion of probability relations between separated systems. *Math. Proc. Camb. Phil. Soc.*, 31:555, 1935. DOI: 10.1017/S0305004100013554.

[3] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Rev. Mod. Phys.*, 81(2):865–942, Jun 2009. DOI: 10.1103/RevModPhys.81.865.

[4] Kavan Modi, Tomasz Paterek, Wonmin Son, Vlatko Vedral, and Mark Williamson. Unified view of quantum and classical correlations. *Phys. Rev. Lett.*, 104:080501, Feb 2010. DOI: 10.1103/PhysRevLett.104.080501.

[5] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press, 1 edition, October 2000. ISBN 0521635039. DOI: 10.1017/CBO9780511976667.

[6] Dénes Petz. *Quantum Information Theory and Quantum Statistics.* Springer, 2008. DOI: 10.1007/978-3-540-74636-2.

[7] Mark M. Wilde. *Quantum Information Theory.* Cambridge University Press, 2013. DOI: 10.1017/CBO9781139525343.

[8] Valerie Coffman, Joydip Kundu, and William K. Wootters. Distributed entanglement. *Phys. Rev. A*, 61:052306, Apr 2000. DOI: 10.1103/PhysRevA.61.052306.

[9] Masato Koashi and Andreas Winter. Monogamy of quantum entanglement and other correlations. *Phys. Rev. A*, 69:022309, Feb 2004. DOI: 10.1103/PhysRevA.69.022309.

[10] Jens Eisert, Marcus Cramer, and Martin B. Plenio. Colloquium: Area laws for the entanglement entropy. *Rev. Mod. Phys.*, 82:277–306, Feb 2010. DOI: 10.1103/RevModPhys.82.277.

[11] Luigi Amico, Rosario Fazio, Andreas Osterloh, and Vlatko Vedral. Entanglement in many-body systems. *Rev. Mod. Phys.*, 80:517–576, May 2008. DOI: 10.1103/RevModPhys.80.517.

[12] Örs Legeza and Jenő Sólyom. Quantum data compression, quantum information generation, and the density-matrix renormalization-group method. *Phys. Rev. B*, 70:205118, Nov 2004. DOI: 10.1103/PhysRevB.70.205118.

[13] Szilárd Szalay, Max Pfeffer, Valentin Murg, Gergely Barcza, Frank Verstraete, Reinhold Schneider, and Örs Legeza. Tensor product methods and entanglement optimization for ab initio quantum chemistry. *Int. J. Quantum Chem.*, 115(19):1342–1391, 2015. ISSN 1097-461X. DOI: 10.1002/qua.24898.

[14] Szilárd Szalay, Gergely Barcza, Tibor Szilvási, Libor Veis, and Örs Legeza. The correlation theory of the chemical bond. *Scientific Reports*, 7:2237, May 2017. DOI: 10.1038/s41598-017-02447-z.

[15] Reinhard F. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, 40(8):4277–4281, Oct 1989. DOI: 10.1103/PhysRevA.40.4277.

[16] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher. Concentrating partial entanglement by local operations. *Phys. Rev. A*, 53:2046–2052, Apr 1996. DOI: 10.1103/PhysRevA.53.2046.

[17] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. *Phys. Rev. A*, 54:3824–3851, Nov 1996. DOI: 10.1103/PhysRevA.54.3824.

[18] Eric Chitambar, Debbie Leung, Laura Mancinska, Maris Ozols, and Andreas Winter. Everything you always wanted to know about LOCC (but were afraid to ask). *Commun. Math. Phys.*, 328(1):303–326, 2014. ISSN 0010-3616. DOI: 10.1007/s00220-014-1953-9.

[19] Wolfgang Dür, Guifre Vidal, and J. Ignacio Cirac. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A*, 62:062314, Nov 2000. DOI: 10.1103/PhysRevA.62.062314.

[20] Michael A. Nielsen. Conditions for a class of entanglement transformations. *Phys. Rev. Lett.*, 83:436–439, Jul 1999. DOI: 10.1103/PhysRevLett.83.436.

[21] Guifré Vidal. Entanglement of pure states for a single copy. *Phys. Rev. Lett.*, 83:1046–1049, Aug 1999. DOI: 10.1103/PhysRevLett.83.1046.

[22] Guifre Vidal. Entanglement monotones. *J. Mod. Opt*, 47(2):355–376, Feb 2000. DOI: 10.1080/09500340008244048.

[23] Michał Horodecki. Entanglement measures. *Quant. Inf. Comp.*, 1:3, May 2001. DOI: 10.26421/QIC1.1.

[24] Martin B. Plenio and Shashank Virmani. An introduction to entanglement measures. *Quant. Inf. Comp.*, 7:1, Jan 2007. DOI: 10.26421/QIC7.1-2.

[25] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde. Four qubits can be entangled in nine different ways. *Phys. Rev. A*, 65:052112, Apr 2002. DOI: 10.1103/PhysRevA.65.052112.

[26] Jean-Gabriel Luque and Jean-Yves Thibon. Polynomial invariants of four qubits. *Phys. Rev. A*, 67:042303, Apr 2003. DOI: 10.1103/PhysRevA.67.042303.

[27] Christopher Eltschka and Jens Siewert. Quantifying entanglement resources. *Journal of Physics A: Mathematical and Theoretical*, 47(42):424005, 2014. DOI: 10.1088/1751-8113/47/42/424005.

[28] Hayata Yamasaki, Alexander Pirker, Mio Murao, Wolfgang Dür, and Barbara Kraus. Multipartite entanglement outperforming bipartite entanglement under limited quantum system sizes.

Phys. Rev. A, 98:052313, Nov 2018. DOI: 10.1103/PhysRevA.98.052313.

[29] Martin. Hebenstreit, Cornelia. Spee, and Barbara Kraus. Maximally entangled set of tripartite qutrit states and pure state separable transformations which are not possible via local operations and classical communication. *Phys. Rev. A*, 93:012339, Jan 2016. DOI: 10.1103/PhysRevA.93.012339.

[30] Cornelia Spee, Julio I. de Vicente, and Barbara Kraus. The maximally entangled set of 4-qubit states. *Journal of Mathematical Physics*, 57(5): 052201, 2016. DOI: 10.1063/1.4946895.

[31] Gilad Gour, Barbara Kraus, and Nolan R. Wallach. Almost all multipartite qubit quantum states have trivial stabilizer. *Journal of Mathematical Physics*, 58(9):092204, 2017. DOI: 10.1063/1.5003015.

[32] David Sauerwein, Nolan R. Wallach, Gilad Gour, and Barbara Kraus. Transformations among pure multipartite entangled states via local operations are almost never possible. *Phys. Rev. X*, 8:031020, Jul 2018. DOI: 10.1103/PhysRevX.8.031020.

[33] Wolfgang Dür, J. Ignacio Cirac, and Rolf Tarrach. Separability and distillability of multiparticle quantum systems. *Phys. Rev. Lett.*, 83:3562–3565, Oct 1999. DOI: 10.1103/PhysRevLett.83.3562.

[34] Wolfgang Dür and J. Ignacio Cirac. Classification of multiqubit mixed states: Separability and distillability properties. *Phys. Rev. A*, 61:042314, Mar 2000. DOI: 10.1103/PhysRevA.61.042314.

[35] Antonio Acín, Dagmar Bruß, Maciej Lewenstein, and Anna Sanpera. Classification of mixed three-qubit states. *Phys. Rev. Lett.*, 87:040401, Jul 2001. DOI: 10.1103/PhysRevLett.87.040401.

[36] Koji Nagata, Masato Koashi, and Nobuyuki Imoto. Configuration of separability and tests for multipartite entanglement in Bell-type experiments. *Phys. Rev. Lett.*, 89:260401, Dec 2002. DOI: 10.1103/PhysRevLett.89.260401.

[37] Michael Seevinck and Jos Uffink. Partial separability and entanglement criteria for multiqubit quantum states. *Phys. Rev. A*, 78(3):032101, Sep 2008. DOI: 10.1103/PhysRevA.78.032101.

[38] Szilárd Szalay. Separability criteria for mixed three-qubit states. *Phys. Rev. A*, 83:062337, Jun 2011. DOI: 10.1103/PhysRevA.83.062337.

[39] Szilárd Szalay and Zoltán Kökényesi. Partial separability revisited: Necessary and sufficient criteria. *Phys. Rev. A*, 86:032341, Sep 2012. DOI: 10.1103/PhysRevA.86.032341.

[40] Szilárd Szalay. Multipartite entanglement measures. *Phys. Rev. A*, 92:042329, Oct 2015. DOI: 10.1103/PhysRevA.92.042329.

[41] Szilárd Szalay. The classification of multipartite quantum correlation. *Journal of Physics A: Mathematical and Theoretical*, 51(48): 485302, 2018. DOI: dx.doi.org/10.1088/1751-8121/aae971.

[42] Jan Brandejs, Libor Veis, Szilárd Szalay, Jiří Pittner, and Örs Legeza. Quantum information-based analysis of electron-deficient bonds. *The Journal of Chemical Physics*, 150(20):204117, 2019. DOI: 10.1063/1.5093497.

[43] Brian A. Davey and Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002. ISBN 9780521784511. DOI: 10.1017/CBO9780511809088.

[44] Steven Roman. *Lattices and Ordered Sets*. Springer, first edition, 2008. ISBN 978-0-387-78900-2. DOI: 10.1007/978-0-387-78901-9.

[45] Richard P. Stanley. *Enumerative Combinatorics, Volume 1*. Cambridge University Press, second edition, 2012. ISBN 9781107602625. DOI: 10.1017/CBO9781139058520.

[46] Michael Seevinck and Jos Uffink. Sufficient conditions for three-particle entanglement and their tests in recent experiments. *Phys. Rev. A*, 65:012107, Dec 2001. DOI: 10.1103/PhysRevA.65.012107.

[47] Otfried Gühne, Géza Tóth, and Hans J. Briegel. Multipartite entanglement in spin chains. *New J. Phys.*, 7(1):229, 2005. DOI: 10.1088/1367-2630/7/1/229.

[48] Otfried Gühne and Géza Tóth. Energy and multipartite entanglement in multidimensional and frustrated spin models. *Phys. Rev. A*, 73:052319, May 2006. DOI: 10.1103/PhysRevA.73.052319.

[49] Géza Tóth and Otfried Gühne. Separability criteria and entanglement witnesses for symmetric quantum states. *Appl. Phys. B*, 98(4):617–622, 2010. ISSN 0946-2171. DOI: 10.1007/s00340-009-3839-7.

[50] Anders S. Sørensen and Klaus Mølmer. Entanglement and extreme spin squeezing. *Phys. Rev. Lett.*, 86:4431–4434, May 2001. DOI: 10.1103/PhysRevLett.86.4431.

[51] Bernd Lücke, Jan Peise, Giuseppe Vitagliano, Jan Arlt, Luis Santos, Géza Tóth, and Carsten Klempt. Detecting multiparticle entanglement of dicke states. *Phys. Rev. Lett.*, 112:155304, Apr 2014. DOI: 10.1103/PhysRevLett.112.155304.

[52] Ji-Yao Chen, Zhengfeng Ji, Nengkun Yu, and Bei Zeng. Entanglement depth for symmetric states. *Phys. Rev. A*, 94:042333, Oct 2016. DOI: 10.1103/PhysRevA.94.042333.

[53] Florian John Curchod, Nicolas Gisin, and Yeong-Cherng Liang. Quantifying multipartite nonlocality via the size of the resource. *Phys. Rev. A*, 91:012121, Jan 2015. DOI: 10.1103/PhysRevA.91.012121.

[54] Yeong-Cherng Liang, Denis Rosset, Jean-Daniel Bancal, Gilles Pütz, Tomer Jack Barnea, and

Nicolas Gisin. Family of bell-like inequalities as device-independent witnesses for entanglement depth. *Phys. Rev. Lett.*, 114:190401, May 2015. DOI: 10.1103/PhysRevLett.114.190401.

[55] Pei-Sheng Lin, Jui-Chen Hung, Ching-Hsu Chen, and Yeong-Cherng Liang. Exploring bell inequalities for the device-independent certification of multipartite entanglement depth. *Phys. Rev. A*, 99:062338, Jun 2019. DOI: 10.1103/PhysRevA.99.062338.

[56] He Lu, Qi Zhao, Zheng-Da Li, Xu-Fei Yin, Xiao Yuan, Jui-Chen Hung, Luo-Kan Chen, Li Li, Nai-Le Liu, Cheng-Zhi Peng, Yeong-Cherng Liang, Xiongfeng Ma, Yu-Ao Chen, and Jian-Wei Pan. Entanglement structure: Entanglement partitioning in multipartite systems and its experimental detection using optimizable witnesses. *Phys. Rev. X*, 8:021072, Jun 2018. DOI: 10.1103/PhysRevX.8.021072.

[57] Géza Tóth and Iagoba Apellaniz. Quantum metrology from a quantum information science perspective. *Journal of Physics A: Mathematical and Theoretical*, 47(42):424006, oct 2014. DOI: 10.1088/1751-8113/47/42/424006.

[58] Philipp Hyllus, Wiesław Laskowski, Roland Krischek, Christian Schwemmer, Witlef Wieczorek, Harald Weinfurter, Luca Pezzé, and Augusto Smerzi. Fisher information and multiparticle entanglement. *Phys. Rev. A*, 85:022321, Feb 2012. DOI: 10.1103/PhysRevA.85.022321.

[59] Manuel Gessner, Luca Pezzè, and Augusto Smerzi. Sensitivity bounds for multiparameter quantum metrology. *Phys. Rev. Lett.*, 121:130503, Sep 2018. DOI: 10.1103/PhysRevLett.121.130503.

[60] Zhongzhong Qin, Manuel Gessner, Zhihong Ren, Xiaowei Deng, Dongmei Han, Weidong Li, Xiaolong Su, Augusto Smerzi, and Kunchi Peng. Characterizing the multipartite continuous-variable entanglement structure from squeezing coefficients and the fisher information. *npj Quantum Information*, 5(3), 2019. DOI: 10.1038/s41534-018-0119-6.

[61] Géza Tóth. Multipartite entanglement and high-precision metrology. *Phys. Rev. A*, 85:022322, Feb 2012. DOI: 10.1103/PhysRevA.85.022322.

[62] George E. Andrews. *The Theory of Partitions*. Cambridge University Press, 1984. DOI: 10.1017/CBO9780511608650.

[63] John von Neumann. Thermodynamik quantenmechanischer Gesamtheiten. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1927: 273–291, 1927. URL http://eudml.org/doc/59231.

[64] Masanori Ohya and Dénes Petz. *Quantum Entropy and Its Use*. Springer Verlag, 1 edition, October 1993. ISBN 978-3-540-20806-8.

[65] Ingemar Bengtsson and Karol Życzkowski. *Geometry of Quantum States: An Introduction to Quantum Entanglement*. Cambridge University Press, 2006. ISBN 0521814510. DOI: 10.1017/CBO9780511535048.

[66] Hisaharu Umegaki. Conditional expectation in an operator algebra. iv. entropy and information. *Kodai Math. Sem. Rep.*, 14(2):59–85, 1962. DOI: 10.2996/kmj/1138844604.

[67] Göran Lindblad. Entropy, information and quantum measurements. *Communications in Mathematical Physics*, 33(4):305–322, Dec 1973. ISSN 1432-0916. DOI: 10.1007/BF01646743.

[68] Fumio Hiai and Dénes Petz. The proper formula for relative entropy and its asymptotics in quantum probability. *Communications in Mathematical Physics*, 143(1):99–114, 1991. ISSN 0010-3616. DOI: 10.1007/BF02100287.

[69] The On-Line Encyclopedia of Integer Sequences. A000110: Bell or exponential numbers: ways of placing n labeled balls into n indistinguishable boxes, . URL http://oeis.org/A000110.

[70] Fedor Herbut. On mutual information in multipartite quantum states and equality in strong subadditivity of entropy. *Journal of Physics A: Mathematical and General*, 37(10):3535, 2004. DOI: 10.1088/0305-4470/37/10/016.

[71] Armin Uhlmann. Entropy and optimal decompositions of states relative to a maximal commutative subalgebra. *Open Sys. Inf. Dyn.*, 5:209–228, 1998. ISSN 1230-1612. DOI: 10.1023/A:1009664331611.

[72] Armin Uhlmann. Roofs and convexity. *Entropy*, 12:1799, July 2010. DOI: 10.3390/e12071799.

[73] Kyung Hoon Han and Seung-Hyeok Kye. Construction of three-qubit biseparable states distinguishing kinds of entanglement in a partial separability classification. *Phys. Rev. A*, 99:032304, Mar 2019. DOI: 10.1103/PhysRevA.99.032304.

[74] The On-Line Encyclopedia of Integer Sequences. A000041: Number of partitions of n: ways of placing n unlabelled balls into n indistinguishable boxes, . URL http://oeis.org/A000041.

[75] Eric Chitambar and Gilad Gour. Quantum resource theories. *Rev. Mod. Phys.*, 91:025001, Apr 2019. DOI: 10.1103/RevModPhys.91.025001.

[76] Thomas Brylawski. The lattice of integer partitions. *Discrete Mathematics*, 6:201, 1973. DOI: 10.1016/0012-365X(73)90094-0.

[77] Garrett Birkhoff. *Lattice Theory*. American Mathematical Society, New York, 3rd edition, 1973.

[78] Freeman J. Dyson. Some guesses in the theory of partitions. *Eureka*, 8:10–15, 1944.