Bi-Hamiltonian structure of a dynamical system introduced by Braden and Hone

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Abstract

We investigate the finite dimensional dynamical system derived by Braden and Hone in 1996 from the solitons of A_{n-1} affine Toda field theory. This system of evolution equations for an $n \times n$ Hermitian matrix L and a real diagonal matrix q with distinct eigenvalues was interpreted as a special case of the spin Ruijsenaars–Schneider models due to Krichever and Zabrodin. A decade later, L.-C. Li re-derived the model from a general framework built on coboundary dynamical Poisson groupoids. This led to a Hamiltonian description of the gauge invariant content of the model, where the gauge transformations act as conjugations of L by diagonal unitary matrices. Here, we point out that the same dynamics can be interpreted also as a special case of the spin Sutherland systems obtained by reducing the free geodesic motion on symmetric spaces, studied by Pusztai and the author in 2006; the relevant symmetric space being $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$. This construction provides an alternative Hamiltonian interpretation of the Braden–Hone dynamics. We prove that the two Poisson brackets are compatible and yield a bi-Hamiltonian description of the standard commuting flows of the model.

Contents

1 Introduction

We are witnesses to intense recent interest in spin extensions [12, 17, 20, 18, 19, 10, 11] of the standard Calogero–Moser–Sutherland and Ruijsenaars–Schneider type many-body models. Current studies [23, 25, 16, 15, 8, 5, 3, 13] are devoted to the mathematical structure and to interesting physical applications of systems of this type. In this paper we take a fresh look at a so far largely neglected, not yet well-understood, aspect of such systems. Namely, we shall uncover a bi-Hamiltonian structure for a remarkable family of examples.

Let L be an $n \times n$ Hermitian matrix and $q = \mathrm{diag}(q_1, q_2, \dots, q_n)$ a real diagonal matrix with distinct eigenvalues. Braden and Hone [2] derived the following evolution equations from the affine Toda solitons:

$$\dot{q}_j = L_{jj}, \qquad \dot{L}_{jj} = 2\sum_{\ell \neq j} |L_{j\ell}|^2 \coth(q_j - q_\ell), \tag{1.1}$$
$$\dot{L}_{jk} = \sum_{\ell \neq j} L_{j\ell} L_{\ell k} \coth(q_j - q_\ell) - \sum_{\ell \neq k} L_{j\ell} L_{\ell k} \coth(q_\ell - q_k), \quad 1 \leq j \neq k \leq n.$$

In their context L has a special form, but the equations make sense for arbitrary L, and here we shall study the system (1.1) in its general form. It will be assumed that q varies in the domain

$$\mathcal{A}^o := \{q \in \mathbb{R}^n \mid q_1 > q_2 > \dots > q_n\}. \tag{1.2}$$

Note that $q \in \mathbb{R}^n$ and the corresponding diagonal matrix are denoted by the same letter. The Braden–Hone equations represent the first member of a hierarchy [19], which is conveniently described utilizing a dynamical r-matrix.

Let us consider the Lie algebra $\mathcal{G} := \mathrm{u}(n)$ and introduce $\mathcal{G}^{\mathbb{C}}_{\mathbb{R}} := \mathrm{gl}(n, \mathbb{C})$. The notation emphasizes that we regard $\mathrm{gl}(n, \mathbb{C})$ as a real Lie algebra. We equip it with the invariant bilinear form

$$\langle X, Y \rangle_{\mathbb{R}} := \Re \mathrm{tr}(XY), \quad \forall X, Y \in \mathcal{G}^{\mathbb{C}}_{\mathbb{R}}. \tag{1.3}$$

This induces the orthogonal vector space decomposition

$$\mathcal{G}^{\mathbb{C}}_{\mathbb{R}} = \mathcal{G} + \mathrm{i}\mathcal{G}, \tag{1.4}$$

which can be further refined as

$$\mathcal{G} = \mathcal{T} + \mathcal{T}^\perp, \quad \mathrm{i}\mathcal{G} = \mathcal{A} + \mathcal{A}^\perp, \tag{1.5}$$

where $\mathcal{T}$ (resp. $\mathcal{A}$) consists of anti-Hermitian (resp. Hermitian) diagonal matrices, and $\mathcal{T}^\perp$ (resp. $\mathcal{A}^\perp$) contains the corresponding off-diagonal matrices. Taking any $w \in \mathcal{H}^o \subset \mathcal{H}$, where

$$\mathcal{H} := \mathcal{A} + \mathcal{T}, \quad \mathcal{H}^o := \mathcal{A}^o + \mathcal{T}, \tag{1.6}$$

we set

$$\mathcal{R}(w)X := 0 \text{ for } X \in (\mathcal{T} + \mathcal{A}), \text{ and } \mathcal{R}(w)X := (\coth \mathrm{ad}_w)(X) \text{ for } X \in (\mathcal{T}^\perp + \mathcal{A}^\perp). \tag{1.7}$$

This gives a well-defined linear operator on $\mathcal{G}^{\mathbb{C}}_{\mathbb{R}}$ that represents a solution of the modified classical dynamical Yang–Baxter equation [4]. By using this dynamical r-matrix, for any $m \in \mathbb{N}$, one can define the following system of evolution equations:

$$\dot{q}_j = (L^m)_{jj}, \quad \dot{L} = [\mathcal{R}(q)L^m, L] \quad \text{for} \quad (q, L) \in \mathcal{A}^o \times \mathrm{i}\mathcal{G}. \tag{1.8}$$

For $m = 1$, this is the Braden–Hone system (1.1).

An important feature of this system is that the evolutional derivations associated with different values of $m \in \mathbb{N}$ *commute after restriction to gauge invariant functions.* By definition, a gauge invariant function F of q and L satisfies

$$F(q, L) = F(q, \eta L \eta^{-1}) \qquad \forall \eta \in \mathbb{T}^n, \tag{1.9}$$

where $\mathbb{T}^n$ is the group of diagonal unitary matrices. We introduce the term 'Braden–Hone hierarchy' to refer to the restrictions of the derivations (1.8) to the gauge invariant functions. The commutativity actually does not hold if we do not restrict to gauge invariant functions. (See Appendix A.) This state of affairs hints that the Braden–Hone hierarchy should result from some suitable Hamiltonian reduction, for which the action of $\mathbb{T}^n$ should represent the gauge transformations on a moment map 'constraint surface'. It turns out that this expectation holds, and can be realized by (at least) two different reduction procedures.

A reduction procedure leading to the Braden–Hone hierarchy was found by L.-C. Li in [19], and another one can be extracted with a little effort from the joint paper [11] by Pusztai and the present author. The purpose of the current work is to show that the Poisson brackets resulting from these two reduction procedures *are compatible, and equip the Braden–Hone hierarchy with a bi-Hamiltonian structure.*

Now we describe the two Poisson brackets and our main result. For any real function $F \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})$ we define its derivatives $\nabla_1 F \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G}, \mathcal{A})$ and $\nabla_2 F \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G}, \mathrm{i}\mathcal{G})$ by requiring that at the point (q, L) we have

$$\langle \delta q, \nabla_1 F \rangle_{\mathbb{R}} = \frac{d}{dt}\bigg|_{t=0} F(q + t\delta q, L) \quad \text{and} \quad \langle \delta L, \nabla_2 F \rangle_{\mathbb{R}} = \frac{d}{dt}\bigg|_{t=0} F(q, L + t\delta L) \tag{1.10}$$

for all $\delta q \in \mathcal{A}$ and $\delta L \in \mathrm{i}\mathcal{G}$. (Here, $F(q + t\delta q, L)$ is well-defined for small t.) For any $m \in \mathbb{N}$, let us define $H_m \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})$ by

$$H_m(q, L) := \frac{1}{m}\mathrm{tr}(L^m). \tag{1.11}$$

Let V_m be the derivation[1] of $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})$ generated by equation (1.8), i.e., by the definition

$$V_m[q_j] := (L^m)_{jj}, \quad V_m[L] := [\mathcal{R}(q)L^m, L]. \tag{1.12}$$

By expanding L in a basis $\{Z^a\}$ of $\mathrm{i}\mathcal{G}$, we have $V_m[L_a Z^a] = V_m[L_a]Z^a$. Note that V_m maps $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$, the ring of gauge invariant functions, to itself.

Theorem 1. *The following formulae define two Poisson brackets on* $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$*:*

$$\{F, H\}_2(q, L) = \langle \nabla_1 F, L\nabla_2 H \rangle_{\mathbb{R}} - \langle \nabla_1 H, L\nabla_2 F \rangle_{\mathbb{R}} - 2\langle \mathcal{R}(q)(L\nabla_2 F), L\nabla_2 H \rangle_{\mathbb{R}} \tag{1.13}$$

and

$$\{F, H\}_1(q, L) = \langle \nabla_1 F, \nabla_2 H \rangle_{\mathbb{R}} - \langle \nabla_1 H, \nabla_2 F \rangle_{\mathbb{R}} + \langle L, [\nabla_2 F, \nabla_2 H]_{\mathcal{R}(q)} \rangle_{\mathbb{R}}, \tag{1.14}$$

where $[X, Y]_{\mathcal{R}(q)} := [\mathcal{R}(q)X, Y] + [X, \mathcal{R}(q)Y]$*, and the* ∇_i *are taken at* (q, L)*. The derivative of* $F \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$ *along the vector field* V_m *(1.12) can be written in Hamiltonian form:*

$$V_m[F] = \{F, H_m\}_2 = \{F, H_{m+1}\}_1. \tag{1.15}$$

[1]The action of V_m on a function F is denoted $V_m[F]$; the components of q and L are evaluation functions.

Moreover, we have $\{H_\ell, H_m\}_2 = \{H_\ell, H_m\}_1 = 0$ *for all* $\ell, m \in \mathbb{N}$.

We shall explain that Theorem 1 follows by elaborating earlier results found in [19, 11]. Let $\mathcal{D}$ be the derivation of the ring $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})$ defined by

$$\mathcal{D}[q_i] := 0, \quad \mathcal{D}[L_{jk}] := \delta_{jk}, \tag{1.16}$$

which preserves the gauge invariant functions. Our main result is then

Theorem 2. *The two Poisson brackets of Theorem 1 satisfy the relations*

$$\{F, H\}_1 = \mathcal{D}[\{F, H\}_2] - \{\mathcal{D}[F], H\}_2 - \{F, \mathcal{D}[H]\}_2, \tag{1.17}$$

$$\mathcal{D}[\{F, H\}_1] - \{\mathcal{D}[F], H\}_1 - \{F, \mathcal{D}[H]\}_1 = 0. \tag{1.18}$$

Consequently, they are compatible and define an exact bi-Hamiltonian structure.

The compatibility of the two Poisson brackets is a consequence of the relation (1.17). For readability, we quote the relevant well-known result together with an indication of its proof.

Lemma 3. *Let* $(\mathfrak{A}, \{\ ,\ \})$ *be a Poisson algebra and* $\mathcal{D}$ *a derivation of the underlying commutative algebra* $\mathfrak{A}$. *Suppose that the bracket* $\{f, h\}^{\mathcal{D}} := \mathcal{D}[\{f, h\}] - \{\mathcal{D}[f], h\} - \{f, \mathcal{D}[h]\}$ *satisfies the Jacobi identity. Then the formula*

$$\{f, h\}_{x,y} = x\{f, h\} + y\{f, h\}^{\mathcal{D}} \tag{1.19}$$

defines a Poisson bracket, for any constant parameters x *and* y.

Proof. For any derivation $\mathcal{D}$, the bracket $\{\ ,\ \}_{x,y}$ is automatically anti-symmetric and verifies the Leibniz property. It is a simple exercise to verify the Jacobi identity by direct calculation. □

The bi-Hamiltonian structures of the form (1.19) are called 'exact' when the application of $\mathcal{D}$ to $\{\ ,\ \}^{\mathcal{D}}$ gives zero, like in (1.18). Equation (1.15) and the compatibility of the two Poisson brackets together show the bi-Hamiltonian character of the Braden–Hone hierarchy. Throughout the paper, we use the standard terminology of bi-Hamiltonian systems, see e.g. [6, 26, 28], although we are mostly dealing with Poisson algebras of gauge invariant functions, and not directly with Poisson manifolds. We proceed in this manner in order to circumvent the complication that the quotient space $(\mathcal{A}^o \times \mathrm{i}\mathcal{G})/\mathbb{T}^n$ is not a smooth manifold. This should not lead to any confusion.

Now we sketch the organization of the text. Section 2 is devoted to a short summary of the construction of the Poisson structure $\{\ ,\ \}_2$ due to L.-C. Li [19]. Our presentation contains some novel elements: the precise connection to the original notations used in [19] is described in Appendix B. In Section 3 we expound the derivation of the Braden–Hone hierarchy with its Poisson structure $\{\ ,\ \}_1$, building on the paper [11]. All results in Section 3 will be obtained relying on analogous results of this reference, despite the fact that there reductions of geodesic motion on *simple* non-compact Lie groups were considered, which formally excludes our present case. Besides explaining Theorem 1, the goal of Section 2 and Section 3 is to prepare the ground for the proof of Theorem 2, which occupies Section 4. Finally, we conclude in Section 5 by pointing out a few open problems for future work.

2 The Poisson structure obtained by L.-C. Li

We tersely summarize those points of the construction of [19], which are directly relevant for us. Some details are relegated to Appendix B.

Let $G_{\mathbb{R}}^{\mathbb{C}}$ denote $\mathrm{GL}(n,\mathbb{C})$ regarded as a real Lie group, and denote by $\mathfrak{H}$ its closed submanifold consisting of the invertible Hermitian matrices of size n. By applying a certain discrete reduction to a dynamical Poisson groupoid structure on $\mathcal{H}^o \times G_{\mathbb{R}}^{\mathbb{C}} \times \mathcal{H}^o$, an interesting Poisson structure on the manifold $\mathcal{H}^o \times \mathfrak{H}$ was obtained. This Poisson structure extends smoothly from the dense open submanifold $\mathcal{H}^o \times \mathfrak{H} \subset \mathcal{H}^o \times \mathrm{i}\mathcal{G}$ to the full of $\mathcal{H}^o \times \mathrm{i}\mathcal{G}$.

Denote the elements of $\mathcal{H}^o \times \mathrm{i}\mathcal{G}$ by pairs (w, L). For any smooth real function $F = F(w, L)$, introduce the derivatives $\nabla_1 F \in C^\infty(\mathcal{H}^o \times \mathrm{i}\mathcal{G}, \mathcal{H})$ and $\nabla_2 F \in C^\infty(\mathcal{H} \times \mathrm{i}\mathcal{G}, \mathrm{i}\mathcal{G})$ in complete analogy with the definition (1.10), using the bilinear form (1.3). Then, as is detailed in Appendix B, the Poisson structure given by equation (5.8) in [19] can be re-written in the following form:

$$\{F, H\}_{\mathrm{Li}}(w, L) = \langle \nabla_1 F, L\nabla_2 H\rangle_{\mathbb{R}} - \langle \nabla_1 H, L\nabla_2 F\rangle_{\mathbb{R}} - 2\langle \mathcal{R}(w)(L\nabla_2 F), L\nabla_2 H\rangle_{\mathbb{R}}. \tag{2.1}$$

Here, F and H are arbitrary elements of $C^\infty(\mathcal{H}^o \times \mathrm{i}\mathcal{G})$, the derivatives are evaluated at (w, L), and we use $\mathcal{R}$ (1.7).

For any $X \in \mathcal{T}$, define $w^X \in C^\infty(\mathcal{H}^o \times \mathrm{i}\mathcal{G})$ by $w^X(w, L) := \langle w, X\rangle_{\mathbb{R}}$. The associated Hamiltonian vector field can be symbolically written as

$$\{w, w^X\}_{\mathrm{Li}} = 0, \qquad \{L, w^X\}_{\mathrm{Li}} = -\frac{1}{2}[X, L]. \tag{2.2}$$

This encodes the Poisson brackets between w^X and the matrix elements of w and L, regarded as functions on $\mathcal{H}^o \times \mathrm{i}\mathcal{G}$. The formula (2.2) has the following important consequence:

- Take $\mathcal{T}$ as the model of its own dual space by means of the trace pairing. Then the map $\phi : (w, L) \mapsto -2\Im(w)$ can be identified as the moment map for the Hamiltonian action of $\mathbb{T}^n$ whereby $\eta \in \mathbb{T}^n$ sends (w, L) to $(w, \eta L \eta^{-1})$.

Notice that the Hamiltonian $H_m \in C^\infty(\mathcal{H}^o \times \mathrm{i}\mathcal{G})$,

$$H_m(w, L) := \frac{1}{m}\mathrm{tr}(L^m), \tag{2.3}$$

is invariant under the above $\mathbb{T}^n$-action. It follows that the Hamiltonian vector field generated by H_m is tangent to the level surfaces of the moment map ϕ. The level surface $\phi = 0$ is the submanifold

$$\mathcal{A}^o \times \mathrm{i}\mathcal{G} \subset \mathcal{H}^o \times \mathrm{i}\mathcal{G}, \tag{2.4}$$

whose elements are denoted by pairs (q, L), like in the Introduction. An easy calculation gives that *the restriction of the Hamiltonian vector field of H_m (2.3) to this level surface is precisely the vector field V_m (1.12).* (It should not lead to any confusion that in equations (1.11) and (1.15) the corresponding restriction of H_m (2.3) is denoted by the same letter.)

One knows from the general reduction theory (the theory of reduction by first class constraints à la Dirac is all what is needed here[2]) that $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$ inherits a Poisson bracket from $(C^\infty(\mathcal{H}^o \times \mathrm{i}\mathcal{G}), \{\ ,\ \}_{\mathrm{Li}})$. Specifically, the induced Poisson bracket of two smooth gauge invariant functions $F, H \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$ can be determined by the standard 'extend–compute–restrict' algorithm. That is, one first extends F and H arbitrarily from the first

[2]We do not consider 'Dirac brackets', since our reductions do not admit globally valid gauge fixings.

class constraint surface, then determines the Poisson bracket of the extended functions, $F^{\rm ext}$ and $H^{\rm ext}$, and finally restricts the result to the $\phi = 0$ constraint surface. This gives a well-defined Poisson bracket. For the theory of Hamiltonian reduction, we recommend the books [14, 21].

Our situation is extremely simple, since by decomposing any $w \in \mathcal{H}^o$ as

$$w = \pi_{\mathcal{A}}(w) + \pi_{\mathcal{T}}(w) \tag{2.5}$$

we can naturally extend any $F \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$ to the phase space $\mathcal{H}^o \times \mathrm{i}\mathcal{G}$ by declaring that

$$F^{\rm ext}(w, L) := F(\pi_{\mathcal{A}}(w), L). \tag{2.6}$$

Here and below, the various projection operators $\pi_{\mathcal{A}}$, $\pi_{\mathcal{T}}$ etc. rely on the decompositions (1.4), (1.5). The function $F^{\rm ext}$ defined in this manner belongs to $C^\infty(\mathcal{H}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$. It follows immediately from (2.1) that the induced Poisson bracket

$$\{F, H\}_2(q, L) := \{F^{\rm ext}, H^{\rm ext}\}_{\rm Li}(q, L), \qquad \forall (q, L) \in \mathcal{A}^o \times \mathrm{i}\mathcal{G}, \tag{2.7}$$

is given by the formula displayed in Theorem 1. A further consequence of the reduction is that we have

$$V_m[F] = \{F, H_m\}_2, \tag{2.8}$$

where H_m is now regarded as a gauge invariant function on the $\phi = 0$ constraint surface. Since $V_m[H_\ell] = 0$, $\{H_\ell, H_m\}_2 = 0$ results as well.

In conclusion, by following [19], we have explained the part of the statements of Theorem 1 pertaining to the Poisson bracket $\{\ ,\ \}_2$.

3 The Braden–Hone system as a spin Sutherland model

The invariant bilinear form $\langle\ ,\ \rangle_{\mathbb{R}}$ on $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ can be used to define a bi-invariant semi-Riemannian metric on the group manifold $G_{\mathbb{R}}^{\mathbb{C}}$, whose geodesics are the orbits of the one-parameter subgroups of $G_{\mathbb{R}}^{\mathbb{C}}$ with respect to right-multiplication (or, equivalently, left-multiplication). Hamiltonian reductions of such 'free geodesic motion' giving rise to spin Sutherland models have been investigated previously, for example in [11]. Building on this reference, we here explain how the Braden–Hone system results from reduction.

We are going to reduce the phase space $T^* G_{\mathbb{R}}^{\mathbb{C}} \times \mathcal{G}^*$, where $\mathcal{G}^*$ is added for technical convenience (akin to the so-called shifting trick of symplectic reduction [21]). We trivialize the cotangent bundle by left-translations, and identify $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ and $\mathcal{G}$ with their own dual spaces by means of the form $\langle\ ,\ \rangle_{\mathbb{R}}$. Thus our unreduced phase space, P, is

$$P := T^* G_{\mathbb{R}}^{\mathbb{C}} \times \mathcal{G}^* \equiv G_{\mathbb{R}}^{\mathbb{C}} \times \mathcal{G}_{\mathbb{R}}^{\mathbb{C}} \times \mathcal{G} = \{(g, J, \xi)\} \tag{3.1}$$

endowed with its standard Poisson structure. For any smooth real functions f and h on P, the Poisson bracket reads

$$\{f, h\}_P(g, J, \xi) = \langle D'_g f, \nabla_J h\rangle_{\mathbb{R}} - \langle D'_g h, \nabla_J f\rangle_{\mathbb{R}} - \langle J, [\nabla_J f, \nabla_J h]\rangle_{\mathbb{R}} + \langle \xi, [\nabla_\xi f, \nabla_\xi h]\rangle_{\mathbb{R}}, \tag{3.2}$$

where $\nabla_J f$ and $\nabla_\xi f$ are $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$-valued and $\mathcal{G}$-valued 'partial gradients' defined by using $\langle\ ,\ \rangle_{\mathbb{R}}$, and $D'_g f(g, J, \xi) \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ is defined by

$$\langle X, D'_g f(g, J, \xi)\rangle_{\mathbb{R}} := \frac{d}{dt}\Big|_{t=0} f(g e^{tX}, J, \xi), \qquad \forall X \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}. \tag{3.3}$$

We single out the 'free Hamiltonians' h_m,

$$h_m(g, J, \xi) := \frac{1}{m} \Re \mathrm{tr}(J^m), \qquad \forall m \in \mathbb{N}, \tag{3.4}$$

which form an Abelian algebra under the Poisson bracket. Denote by $\mathbb{V}_m$ the Hamiltonian vector field generated by h_m. It has the explicit form

$$\mathbb{V}_m[g] = gJ^{m-1}, \quad \mathbb{V}_m[J] = 0, \quad \mathbb{V}_m[\xi] = 0. \tag{3.5}$$

Here, $\mathbb{V}_m[g]$ etc. are understood as derivatives of evaluation functions. This means that $\mathbb{V}_m[g]$ collects the derivatives of (the real and imaginary parts of) the matrix elements of g, which are regarded as functions on P (see also footnote 1).

We reduce relying on the action of the symmetry group $G \times G$ on P, where $G := \mathrm{U}(n)$. We let $(\eta_L, \eta_R) \in G \times G$ act on P by the diffeomorphism Ψ_{η_L,η_R},

$$\Psi_{\eta_L,\eta_R}(g, J, \xi) := (\eta_L g \eta_R^{-1}, \eta_R J \eta_R^{-1}, \eta_L \xi \eta_L^{-1}). \tag{3.6}$$

This is a Hamiltonian action. The corresponding moment map $\Phi : P \to \mathcal{G} \oplus \mathcal{G}$ is given by

$$\Phi(g, J, \xi) = \left(\pi_{\mathcal{G}}(gJg^{-1}) + \xi, -\pi_{\mathcal{G}}(J)\right). \tag{3.7}$$

We impose the constraint $\Phi = 0$, and then divide by the 'gauge transformations' associated with $G \times G$. The gauge invariant functions on $P_0 := \Phi^{-1}(0)$ inherit a Poisson structure, and the vector fields $\mathbb{V}_m$ induce commuting derivations of $C^\infty(P_0)^{G\times G}$.

It is worth noting that the 'partial moment map constraint' $\pi_{\mathcal{G}}(J) = 0$ enforces the reduction of $T^*G_{\mathbb{R}}^{\mathbb{C}}$ to $T^*(G_{\mathbb{R}}^{\mathbb{C}}/G)$, which underlies the link to the approach of the paper [11]. Indeed, one could perform the reduction by $G \times G$ in two steps, and first imposing only $\pi_{\mathcal{G}}(J) = 0$ would lead, in effect, to the starting point of the reduction studied in [11].

From now on we restrict our attention to the dense open submanifold $P^{\mathrm{reg}} \subset P$, which is characterized by the condition that g can be decomposed as

$$g = \eta_L^{-1} e^q \eta_R \quad \text{with} \quad q \in \mathcal{A}^o,\ \eta_L, \eta_R \in G. \tag{3.8}$$

In this decomposition q is unique, while the pair (η_L, η_R) is unique up to the ambiguity of its possible replacement by $(\eta\eta_L, \eta\eta_R)$ with an arbitrary $\eta \in \mathbb{T}^n$. It is plain that every gauge orbit lying in P_0^{reg} has representatives in the following 'gauge slice' $S \subset P_0^{\mathrm{reg}}$:

$$S := \{(e^q, L, \xi) \in P_0 \mid q \in \mathcal{A}^o,\ L \in \mathrm{i}\mathcal{G}\}. \tag{3.9}$$

On the elements of S, the condition $\Phi = 0$ (3.7) translates into the equation

$$\xi + (\sinh \mathrm{ad}_q)(L) = 0, \tag{3.10}$$

and we have a residual gauge action of $\mathbb{T}^n$ on S, given by the maps $\Psi_{\eta,\eta}$:

$$\Psi_{\eta,\eta}(e^q, L, \xi) = (e^q, \eta L \eta^{-1}, \eta \xi \eta^{-1}), \qquad \eta \in \mathbb{T}^n. \tag{3.11}$$

Moreover, we see from the constraint equation (3.10) that

$$\pi_{\mathcal{T}}(\xi) = 0 \quad \text{and} \quad \pi_{\mathcal{T}^\perp}(\xi) = -(\sinh \mathrm{ad}_q)(\pi_{\mathcal{A}^\perp}(L)), \tag{3.12}$$

which provides a parametrization of S by the 'free variables' $(q, L) \in \mathcal{A}^o \times \mathrm{i}\mathcal{G}$. An important consequence is the chain of identifications

$$C^\infty(P_0^{\mathrm{reg}})^{G\times G} \Longleftrightarrow C^\infty(S)^{\mathbb{T}^n} \Longleftrightarrow C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}. \tag{3.13}$$

We note in passing that the linear operator $(\sinh \mathrm{ad}_q)$ is zero on $(\mathcal{A}+\mathcal{T})$, while it maps $\mathcal{T}^\perp$ to $\mathcal{A}^\perp$ and $\mathcal{A}^\perp$ to $\mathcal{T}^\perp$ in an invertible manner, due to the regularity of q (see (1.2) and (1.5)). When below we write $(\sinh \mathrm{ad}_q)^{-1}$, then we mean the unique inverses of these restricted operators.

Now we explain how the commuting vector fields $\mathbb{V}_m$ (3.5) descend to the Braden–Hone hierarchy (1.12). The vector fields $\mathbb{V}_m$ are tangent to P_0, but are not tangent to S. However, since we are interested in the evolution of the gauge invariant 'observables', we can cure this non-tangency by adding a suitable infinitesimal gauge transformation to $\mathbb{V}_m$. The latter is given by a pair of $\mathcal{G}$-valued functions Y^L and Y^R on S, which are required to ensure that the following vector field $\mathbb{V}_m^S$ is tangent to S:

$$\mathbb{V}_m^S[e^q] = e^q L^{m-1} + Y^L e^q - e^q Y^R, \quad \mathbb{V}_m^S[L] = [Y^R, L]. \tag{3.14}$$

The first equation determines the pair (Y^L, Y^R) up to shifts defined by adding (Y, Y), where Y is an arbitrary $\mathcal{T}$-valued function on S. This ambiguity corresponds to the residual gauge transformations acting on S. Indeed, the expression

$$e^{-q}\mathbb{V}_m[e^q] = L^{m-1} - (\sinh \mathrm{ad}_q)(Y^L) + (\cosh \mathrm{ad}_q)(Y^L) - Y^R \tag{3.15}$$

must belong to $\mathcal{A}$, and this condition has the following solution:

$$Y^L = (\sinh \mathrm{ad}_q)^{-1}(\pi_{\mathcal{A}^\perp}(L^{m-1})), \qquad Y^R = \mathcal{R}(q)(L^{m-1}), \tag{3.16}$$

with $\mathcal{R}(q)$ defined in (1.7). We here used that L^{m-1} is Hermitian, i.e., $L^{m-1} \in \mathrm{i}\mathcal{G}$. The reduced dynamics is obtained by substitution of (3.16) into (3.14). The next proposition summarizes the outcome of our line of reasoning.

Proposition 4. *The derivative of the gauge invariant function $F \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$ with respect to the reduction of the Hamiltonian vector field $\mathbb{V}_m$ (3.5) is encoded by the formula*

$$\mathbb{V}_m^S[F] = \langle \mathbb{V}_m^S[q], \nabla_1 F\rangle_{\mathbb{R}} + \langle \mathbb{V}_m^S[L], \nabla_2 F\rangle_{\mathbb{R}}, \tag{3.17}$$

where

$$\mathbb{V}_m^S[q] = \pi_{\mathcal{A}}(L^{m-1}) \quad \text{and} \quad \mathbb{V}_m^S[L] = [\mathcal{R}(q)(L^{m-1}), L]. \tag{3.18}$$

Comparison with equation (1.12) shows that the reduction yields the Braden–Hone hierarchy defined in the Introduction. Specifically, the vector field $\mathbb{V}_{m+1}^S$ reproduces V_m (1.12).

The gauge slice S (3.9) has two distinguished parametrizations. The first one is by the variables $(q, L) \in \mathcal{A}^o \times \mathrm{i}\mathcal{G}$, and the second one is by the variables

$$(q, p, \xi_\perp) \in \mathcal{A}^o \times \mathcal{A} \times \mathcal{T}^\perp, \tag{3.19}$$

which are related to (q, L) by the equation

$$L = p - (\sinh \mathrm{ad}_q)^{-1}(\xi_\perp). \tag{3.20}$$

The correspondence between (q, L) and $(q, p, \xi_\perp)$ is a $\mathbb{T}^n$-equivariant diffeomorphism between $\mathcal{A}^o \times \mathrm{i}\mathcal{G}$ and $\mathcal{A}^o \times \mathcal{A} \times \mathcal{T}^\perp$. Of course, q and p are $\mathbb{T}^n$-invariants, while $\xi_\perp$ transforms by conjugation. This may be used to extend the chain of identifications (3.13) as

$$C^\infty(P_0^{\mathrm{reg}})^{G\times G} \Longleftrightarrow C^\infty(S)^{\mathbb{T}^n} \Longleftrightarrow C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n} \Longleftrightarrow C^\infty(\mathcal{A}^o \times \mathcal{A} \times \mathcal{T}^\perp)^{\mathbb{T}^n}. \tag{3.21}$$

By identifying $\mathcal{A}^*$ with $\mathcal{A}$ using $\langle\ ,\ \rangle_{\mathbb{R}}$, $\mathcal{A}^o \times \mathcal{A}$ can be taken as a model of $\mathbb{T}^*\mathcal{A}^o$, which carries a symplectic form. Moreover, $C^\infty(\mathcal{T}^\perp)^{\mathbb{T}^n}$ is a Poisson algebra, equipped with the reduction of the Lie–Poisson bracket of $\mathcal{G}^* \equiv \mathcal{G}$ defined by the first class constraint $\pi_{\mathcal{T}}(\xi) = 0$.

Proposition 5. *Let us parametrize S (3.9) by the variables $q, p, \xi_\perp$ using (3.20) and consider two gauge invariant functions $F, H \in C^\infty(\mathcal{A}^o \times \mathcal{A} \times \mathcal{T}^\perp)^{\mathbb{T}^n}$. In terms of these functions, the reduced Poisson bracket arising from the Poisson structure (3.2) on P can be written as*

$$\{F, H\}^{\text{red}}_P = \langle \nabla_q F, \nabla_p H\rangle_{\mathbb{R}} - \langle \nabla_q H, \nabla_p F\rangle_{\mathbb{R}} + \langle \xi_\perp, [\nabla_\xi F, \nabla_\xi H]\rangle_{\mathbb{R}}. \tag{3.22}$$

Here, $\nabla_q F$, $\nabla_p F$ are the obvious $\mathcal{A}$-valued gradients, and $\nabla_\xi F$ can be taken from $\mathcal{T}^\perp \subset \mathcal{G}$, applying the definition

$$\langle X, \nabla_\xi F(q, p, \xi_\perp)\rangle_{\mathbb{R}} = \frac{d}{dt}\bigg|_{t=0} F(q, p, \xi_\perp + tX), \qquad \forall X \in \mathcal{T}^\perp. \tag{3.23}$$

The statement of the proposition is basically a special case of more general results proved in [11]. In [11] analogous reductions were studied, but restricting ξ to an arbitrary coadjoint orbit $\mathcal{O}$ of G from the very beginning. A counterpart of the formula (3.22) can be obtained by evaluation of the restriction of the symplectic form of $T^*G^{\mathbb{C}}_{\mathbb{R}} \times \mathcal{O}$ to the gauge slice where $g = e^q$, $q \in \mathcal{A}^o$. This proves the claim in our case, too, since $\mathcal{O} \subset \mathcal{G}^*$ is a symplectic leaf.

We stress that in the formula (3.22) one can also determine $\nabla_\xi F$ utilizing an arbitrary extension of F from $T^*\mathcal{A}^o \times \mathcal{T}^\perp$ to $T^*\mathcal{A}^o \times \mathcal{G}$, computing the $\mathcal{G}$-valued derivative ∇_ξ there, and restricting the result. This leads to an ambiguity regarding the $\mathcal{T}$-components of the derivatives with respect to ξ, which eventually drops out on account of the $\mathbb{T}^n$-invariance of F and H on $T^*\mathcal{A}^o \times \mathcal{T}^\perp$.

Finally, we turn to the description of the reduced Poisson bracket in terms of gauge invariant functions $F, H \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$. For this purpose, we need an auxiliary lemma.

Lemma 6. *Define the map $L : \mathcal{A}^o \times \mathcal{A} \times \mathcal{G} \to \mathrm{i}\mathcal{G}$ by extension of the formula (3.20), i.e., by*

$$L(q, p, \xi) := p - (\sinh \mathrm{ad}_q)^{-1}(\pi_{\mathcal{T}^\perp}(\xi)). \tag{3.24}$$

For any $X \in \mathcal{A}$, $Y \in \mathcal{G}$ and $Z \in \mathrm{i}\mathcal{G}$, define $q^X := \langle X, q\rangle_{\mathbb{R}}$, $\xi^Y := \langle Y, \xi\rangle_{\mathbb{R}}$ and $L^Z := \langle Z, L\rangle_{\mathbb{R}}$. Regarding these as functions on the Poisson manifold $T^*\mathcal{A}^o \times \mathcal{G}^*$, the following formulae hold:

$$\{q^X, L^Z\} = \langle X, Z\rangle_{\mathbb{R}}, \quad \{L^Z, \xi^T\} = L^{[T,Z]} \quad \textit{for all} \quad X \in \mathcal{A},\ T \in \mathcal{T},\ Z \in \mathrm{i}\mathcal{G}, \tag{3.25}$$

and

$$\{L^{Z_1}, L^{Z_2}\} = L^{[Z_1, Z_2]_{\mathcal{R}(q)}} + \langle \pi_{\mathcal{T}}(\xi), [\mathcal{W}(\mathrm{ad}_q)\pi_{\mathcal{A}^\perp}(Z_1), \mathcal{W}(\mathrm{ad}_q)\pi_{\mathcal{A}^\perp}(Z_2)]\rangle_{\mathbb{R}} \tag{3.26}$$

for all $Z_1, Z_2 \in \mathrm{i}\mathcal{G}$, where $\mathcal{W}(\mathrm{ad}_q) : \mathcal{A}^\perp \to \mathcal{T}^\perp$ is given by $\mathcal{W}(\mathrm{ad}_q)Z := (\sinh \mathrm{ad}_q)^{-1}(Z)$. The notation $[Z_1, Z_2]_{\mathcal{R}(q)}$ is defined after (1.14). Of course, the Poisson brackets between any functions of q and $\xi_{\mathcal{T}} = \pi_{\mathcal{T}}(\xi)$ are zero.

The formulae (3.25) are easy consequences of the parametrization of L (3.24), using the canonical Poisson brackets between the components of q, p and the Lie–Poisson bracket

$$\{\xi^{Y_1}, \xi^{Y_2}\} = \xi^{[Y_1, Y_2]}, \qquad \forall Y_1, Y_2 \in \mathcal{G}. \tag{3.27}$$

The verification of equation (3.26) is straightforward, but rather tedious[3] . This is omitted to save place. Note that an analogous result was given in [11], and more recently also in [16].

[3] This calculation was performed by B.G. Pusztai in a more general case during our collaboration in 2006.

Remark 7. The formulae (3.25) and (3.26) can be viewed as the defining relations of a Poisson structure on the manifold

$$\mathcal{A}^o \times \mathrm{i}\mathcal{G} \times \mathcal{T} = \{(q, L, \xi_{\mathcal{T}})\}. \tag{3.28}$$

This is nothing but the Poisson structure of $T^*\mathcal{A}^o \times \mathcal{G}^*$ transferred to $\mathcal{A}^o \times \mathrm{i}\mathcal{G} \times \mathcal{T}$ by means of the invertible change of variables $(q, p, \xi) \leftrightarrow (q, L, \xi_{\mathcal{T}})$ (3.24). Then the Poisson bracket on $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$ can be represented as the reduction of $(C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G} \times \mathcal{T}), \{\ ,\ \})$ defined by the first class constraint $\xi_{\mathcal{T}} = 0$. Indeed, this follows from Proposition 5. The identifications (3.21) give rise to alternative descriptions of the Poisson bracket on $C^\infty(P_0^{\mathrm{reg}})^{G\times G}$, which descends from $\{\ ,\ \}_P$ (3.2).

Proposition 8. *Let us parametrize S (3.9) by the variables q, L and consider two gauge invariant functions $F, H \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$. In terms of these functions, the reduced Poisson bracket descending from the Poisson structure (3.2) on P can be written as*

$$\{F, H\}_P^{\mathrm{red}} = \langle \nabla_1 F, \nabla_2 H\rangle_{\mathbb{R}} - \langle \nabla_1 H, \nabla_2 F\rangle_{\mathbb{R}} + \langle L, [\nabla_2 F, \nabla_2 H]_{\mathcal{R}(q)}\rangle_{\mathbb{R}}, \tag{3.29}$$

which coincides with the Poisson bracket $\{\ ,\ \}_1$ given by equation (1.14) of Theorem 1. Denoting the restriction of h_m (3.4) to S by H_m, we obtain the following consequence of the reduction:

$$\mathbb{V}_m^S[F] = \{F, H_m\}_P^{\mathrm{red}}, \tag{3.30}$$

which implies the second equality in (1.15), since $V_m = \mathbb{V}_{m+1}^S$ by (3.18).

Proof. According to Remark 7, the Poisson bracket on $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$ can be calculated as follows. Regard q, L and $\xi_{\mathcal{T}}$ as independent variables, determine the Poisson brackets of the functions of q and L by utilizing the formulae of Lemma 6, and impose the constraint $\xi_{\mathcal{T}} = 0$ at the end of the calculation. This algorithm proves the claim (3.29). The equality (3.30) is a consequence of the theory of Hamiltonian reduction: $\mathbb{V}_m^S$ represents the reduced Hamiltonian vector field descending from $\mathbb{V}_m$ (3.5) and H_m is the corresponding reduced Hamiltonian (note that $\Re\mathrm{tr}(L^m) = \mathrm{tr}(L^m)$ since L is Hermitian). $\square$

4 Proof of Theorem 2

We begin the proof of Theorem 2 by recapitulating the core points of the preparations.

In equation (2.1) we have introduced the Poisson manifold

$$(\mathcal{H}^o \times \mathrm{i}\mathcal{G}, \{\ ,\ \}_{\mathrm{Li}}). \tag{4.1}$$

The coordinate functions on this manifold can be taken to be the components q_i of $q = \Re(w)$, the components of $\Im(w)$, and the functions $L_a := \langle Z_a, L\rangle_{\mathbb{R}}$ associated with a basis $\{Z_a\}$ of $\mathrm{i}\mathcal{G}$. The Poisson algebra

$$(C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}, \{\ ,\ \}_2) \tag{4.2}$$

results from (4.1) by imposing the first class constraint $\Im(w) = 0$, which is equivalent to the equality $w = q \in \mathcal{A}^o$. The reduced Poisson algebra is completely determined by the Poisson brackets between such gauge invariant functions that depend only on L or only on q.

Let us extend the derivation $\mathcal{D}$ (1.16) to a derivation of $C^\infty(\mathcal{H}^o \times \mathrm{i}\mathcal{G})$ by declaring that the derivatives of all components of $\Im(w)$ are zero. Then introduce the bracket $\{F, H\}_{\mathrm{Li}}^{\mathcal{D}}$ by

$$\{F, H\}_{\mathrm{Li}}^{\mathcal{D}} := \mathcal{D}[\{F, H\}_{\mathrm{Li}}] - \{\mathcal{D}[F], H\}_{\mathrm{Li}} - \{F, \mathcal{D}[H]\}_{\mathrm{Li}}, \tag{4.3}$$

and similarly introduce $\{F,H\}_2^{\mathcal{D}} := \mathcal{D}[\{F,H\}_2] - \{\mathcal{D}[F],H\}_2 - \{F,\mathcal{D}[H]\}_2$. These brackets automatically satisfy the Leibniz and the anti-symmetry properties, but the Jacobi identity is not guaranteed.

Focusing on gauge invariant functions $\mathcal{P}_1$ and $\mathcal{P}_2$ depending only on L, we have

$$\{\mathcal{P}_1,\mathcal{P}_2\}_2(q,L) = \{\mathcal{P}_1,\mathcal{P}_2\}_{\mathrm{Li}}(q,L) = \sum_{a,b}\{L_a,L_b\}_{\mathrm{Li}}(q,L)\left(\frac{\partial\mathcal{P}_1}{\partial L_a}\frac{\partial\mathcal{P}_2}{\partial L_b}\right)(L). \tag{4.4}$$

This is a special case of the formula (2.7). We employ the trivial extension (2.6), and thus we do not need to use a separate notation for the extended functions. For example, a $\mathbb{T}^n$-invariant polynomial formed out of the components of L can be regarded both as a function on $\mathcal{A}^o \times \mathrm{i}\mathcal{G}$ and as a function on $\mathcal{H}^o \times \mathrm{i}\mathcal{G}$. As a consequence of the formula (4.4) and the definition of $\{\ ,\ \}_2^{\mathcal{D}}$, we also have

$$\{\mathcal{P}_1,\mathcal{P}_2\}_2^{\mathcal{D}}(q,L) = \sum_{a,b}\{L_a,L_b\}_{\mathrm{Li}}^{\mathcal{D}}(q,L)\left(\frac{\partial\mathcal{P}_1}{\partial L_a}\frac{\partial\mathcal{P}_2}{\partial L_b}\right)(L). \tag{4.5}$$

This shows that all information about $\{\mathcal{P}_1,\mathcal{P}_2\}_2^{\mathcal{D}}$ is contained in $\{L_a,L_b\}_{\mathrm{Li}}^{\mathcal{D}}$.

On the other hand, as was proved in Section 3, the Poisson algebra

$$(C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}, \{\ ,\ \}_1) \tag{4.6}$$

is a reduction of the Poisson algebra

$$(C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G} \times \mathcal{T}), \{\ ,\ \}), \tag{4.7}$$

where $\{\ ,\ \}$ denotes the Poisson bracket given by Lemma 6 (see also Remark 7). The reduction is defined by the first class constraint $\xi_{\mathcal{T}} = 0$. Accordingly, we have

$$\{\mathcal{P}_1,\mathcal{P}_2\}_1(q,L) = \sum_{a,b}\{L_a,L_b\}(q,L,\xi_{\mathcal{T}}=0)\left(\frac{\partial\mathcal{P}_1}{\partial L_a}\frac{\partial\mathcal{P}_2}{\partial L_b}\right)(L). \tag{4.8}$$

Now, Theorem 2 claims that

$$\{\mathcal{P}_1,\mathcal{P}_2\}_1(q,L) = \{\mathcal{P}_1,\mathcal{P}_2\}_2^{\mathcal{D}}(q,L), \tag{4.9}$$

and we see by comparison of (4.5) and (4.8) that this follows if we can verify that

$$\{L_a,L_b\}(q,L,\xi_{\mathcal{T}}=0) = \{L_a,L_b\}_{\mathrm{Li}}^{\mathcal{D}}(q,L). \tag{4.10}$$

Since $L_a \equiv L^{Z_a}$, Lemma 6 gives

$$\{L_a,L_b\}(q,L,\xi_{\mathcal{T}}=0) = \langle L, [\mathcal{R}(q)Z_a, Z_b] + [Z_a, \mathcal{R}(q)Z_b]\rangle_{\mathbb{R}}. \tag{4.11}$$

We compute the right-hand side of (4.10) from the definition (2.1) noting that by (1.16) $\mathcal{D}[L] = \mathcal{D}[L_a Z^a] := \mathcal{D}[L_a]Z^a = \mathbf{1}_n$ is the unit matrix (the basis $\{Z^a\}$ is dual to $\{Z_a\}$). We find

$$\begin{aligned}\{L_a,L_b\}_{\mathrm{Li}}^{\mathcal{D}}(q,L) &= -2\langle\mathcal{R}(q)(LZ_a), Z_b\rangle_{\mathbb{R}} - 2\langle\mathcal{R}(q)Z_a, LZ_b\rangle_{\mathbb{R}}\\ &= 2\langle L, Z_a\mathcal{R}(q)Z_b - Z_b\mathcal{R}(q)Z_a\rangle_{\mathbb{R}}.\end{aligned} \tag{4.12}$$

Using that $(Z_a\mathcal{R}(q)Z_b)^\dagger = -(\mathcal{R}(q)Z_b)Z_a$, we get

$$\langle L, Z_a\mathcal{R}(q)Z_b\rangle_{\mathbb{R}} = -\langle L, (\mathcal{R}(q)Z_b)Z_a\rangle_{\mathbb{R}}. \tag{4.13}$$

In this way we confirm (4.10), and thus the claim (4.9) holds.

To finish the proof of the claim (1.17), it is sufficient to verify the equalities

$$\{\mathcal{F}, \mathcal{P}\}_1(q, L) = \{\mathcal{F}, \mathcal{P}\}_2^{\mathcal{D}}(q, L) \quad \text{and} \quad \{\mathcal{F}_1, \mathcal{F}_2\}_1(q, L) = \{\mathcal{F}_1, \mathcal{F}_2\}_2^{\mathcal{D}}(q, L) \tag{4.14}$$

for invariant functions $\mathcal{P}$ of L and arbitrary smooth functions $\mathcal{F}$, $\mathcal{F}_i$ depending only on q. These verifications are in principle similar to the above, but are computationally simpler.

Turning to the claim (1.18), now we extend the derivation $\mathcal{D}$ (1.16) to $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G} \times \mathcal{T})$ by setting $\mathcal{D}[\xi_{\mathcal{T}}] := 0$, and then define $\{\ ,\ \}^{\mathcal{D}}$ analogously to (4.3). With $Q := \sum_{i=1}^n q_i$, we notice from Lemma 6 that

$$\mathcal{D}[F] = \{Q, F\}, \qquad \forall F \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G} \times \mathcal{T}). \tag{4.15}$$

This implies that $\{F, H\}^{\mathcal{D}} = 0$ for all functions. From this, referring to Remark 7 and Proposition 8, the claim (1.18) follows.

Since the compatibility of $\{\ ,\ \}_2$ and $\{\ ,\ \}_1 = \{\ ,\ \}_2^{\mathcal{D}}$ is a consequence of Lemma 3 given in the Introduction, the proof of Theorem 2 is now complete.

5 Conclusion

In this paper we combined two approaches to the Braden–Hone hierarchy (1.8), and have shown that together they endow these evolution equations with a bi-Hamiltonian structure.

The approach based on Hamiltonian reduction of free motion on $G_{\mathbb{R}}^{\mathbb{C}}$ enjoys the attractive feature that the initial free flows are complete, and this is automatically inherited by the reduced flows. However, to realize the completeness one might need to drop the restriction to regular elements in the decomposition (3.8), which is valid only on a dense open submanifold. This issue requires further investigation. We note only that the free flow generated by h_m (3.4) reads

$$g(t) = g(0) \exp(t J^{m-1}), \quad J(t) = J(0), \quad \xi(t) = \xi(0), \tag{5.1}$$

from which the flows of the Braden–Hone system result by the standard projection method.

The hyperbolic Braden–Hone hierarchy that we have studied admits a trigonometric version, for which similar results are expected to hold. This, and the question of generalizations to other Lie algebras and twisted cases, will be investigated elsewhere. Previous works relevant to such a future study include [8, 10, 11, 19].

Another very interesting unexplored aspect concerns the integrability (both in Liouville and in non-commutative sense) of the bi-Hamiltonian Braden–Hone hierarchy. In this respect, since the notions of integrability are usually formulated on symplectic manifolds, one should investigate the symplectic leaves of the alternative Poisson structures on $(\mathcal{A}^o \times \mathrm{i}\mathcal{G})/\mathbb{T}^n$. It is also natural to ask what happens to the Poisson bracket $\{\ ,\ \}_2$ if one restricts to a symplectic leaf of $\{\ ,\ \}_1$ (and vice versa)? For the investigation of non-commutative integrability, one may adapt the approach of the papers [22, 23], where non-commutative (degenerate) integrability was proven for a family of trigonometric spin Calogero–Moser–Sutherland systems. It is known (see e.g. [9]) that restriction to a minimal coadjoint orbit $\mathcal{O} \subset \mathcal{G}^*$ in (3.1) leads to the spinless hyperbolic Sutherland model, with its standard Poisson structure arising from $\{\ ,\ \}_1$. There should be a way to recover the spinless hyperbolic Ruijsenaars–Schneider model [24] via restriction to a small symplectic leaf of the structure $\{\ ,\ \}_2$. When studying all these questions, one should of course take note of the fact that $\mathrm{i}\mathcal{G}/\mathbb{T}^n$ is not a smooth manifold, but is a union of smooth strata, since the $\mathbb{T}^n$-action has several different orbit types [21].

One should apply the theory of singular Hamiltonian reduction [21, 27, 29] to uncover the global structure of the reduced system that emerges from the geodesic motion on $G_{\mathbb{R}}^{\mathbb{C}}$.

Finally, it is an open problem if there is any relation between the results of this paper and the previous works [1, 7] devoted to the bi-Hamiltonian structure of the (spinless) rational Calogero–Moser system.

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A Commuting derivations of gauge invariants

In this appendix we show by a direct method that the Braden–Hone hierarchy (1.8) induces commuting derivations of the gauge invariant functions forming $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$.

We start by noting that $\mathcal{R} : \mathcal{H}^o \to \mathrm{End}(\mathcal{G}_{\mathbb{R}}^{\mathbb{C}})$, defined by (1.7), is anti-symmetric with respect to the bilinear form (1.3) and is $\mathcal{H}$-invariant (see (1.6)) in the sense that

$$[\mathrm{ad}_T, \mathcal{R}(w)] = 0, \quad \forall T \in \mathcal{H}, \quad w \in \mathcal{H}^o. \tag{A.1}$$

Let us consider the derivation V_m of $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})$ specified by the rules (1.12). The derivative $V_m[F]$ of $F \in C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})$ reads

$$V_m[F] = \langle V_m[q], \nabla_1 F\rangle_{\mathbb{R}} + \langle V_m[L], \nabla_2 F\rangle_{\mathbb{R}}. \tag{A.2}$$

Using the invariance property of $\mathcal{R}$, it is easily shown that V_m preserves the ring of gauge invariant functions, $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$.

Proposition A1. *The commutator of two derivations V_m and V_ℓ satisfies*

$$(V_m \circ V_\ell - V_\ell \circ V_m)[q] = 0, \qquad (V_m \circ V_\ell - V_\ell \circ V_m)[L] = [T_{m,\ell}(q, L), L] \tag{A.3}$$

with a certain function $T_{m,\ell}(q, L) \in \mathcal{T}$, representing an infinitesimal $\mathbb{T}^n$ gauge transformation. Consequently, the restrictions of the derivations V_m and V_ℓ to $C^\infty(\mathcal{A}^o \times \mathrm{i}\mathcal{G})^{\mathbb{T}^n}$ commute with each other, for any $m, \ell \in \mathbb{N}$.

Proof. We obtain from the definitions

$$(V_m \circ V_\ell - V_\ell \circ V_m)[q] = \left([\mathcal{R}(q)L^m, L^\ell] - [\mathcal{R}(q)L^\ell, L^m]\right)_{\mathcal{A}}, \tag{A.4}$$

where the subscript $\mathcal{A}$ refers to the decomposition (1.5). Taking any $A \in \mathcal{A}$, by using the invariance and anti-symmetry of $\mathcal{R}$, we can write

$$\langle A, [\mathcal{R}(q)L^m, L^\ell]\rangle_{\mathbb{R}} = \langle [A, \mathcal{R}(q)L^m], L^\ell\rangle = \langle \mathcal{R}(q)[A, L^m], L^\ell\rangle_{\mathbb{R}} = -\langle A, [L^m, \mathcal{R}(q)L^\ell]\rangle. \tag{A.5}$$

From this, we get

$$\langle A, (V_m \circ V_\ell - V_\ell \circ V_m)[q]\rangle_{\mathbb{R}} = 0, \qquad \forall A \in \mathcal{A}, \tag{A.6}$$

which is equivalent to the first equality in (A.3).

Next, a simple calculation gives

$$(V_m \circ V_\ell - V_\ell \circ V_m)[L] = [T_{m,\ell}, L] \tag{A.7}$$

with

$$T_{m,\ell} = \mathcal{R}([\mathcal{R}L^m, L^\ell] + [L^m, \mathcal{R}L^\ell]) - [\mathcal{R}L^m, \mathcal{R}L^\ell] + (\nabla_{(L^m)_\mathcal{A}}\mathcal{R})L^\ell - (\nabla_{(L^\ell)_\mathcal{A}}\mathcal{R})L^m. \tag{A.8}$$

We simplified the notation by omitting the argument q of $\mathcal{R}$. For any $T \in \mathcal{H}$, $\nabla_T\mathcal{R}$ denotes the directional derivative of $\mathcal{R}$. In order to show that $T_{m,\ell}$ (A.8) belongs to $\mathcal{T}$, we recall [10] that the modified classical dynamical Yang–Baxter equation, satisfied by $\mathcal{R} : \mathcal{H}^o \to \mathrm{End}(\mathcal{G}^\mathbb{C}_\mathbb{R})$, can be written as follows:

$$\mathcal{R}([X, \mathcal{R}Y] + [\mathcal{R}X, Y]) - [\mathcal{R}X, \mathcal{R}Y] + (\nabla_{X_\mathcal{H}}\mathcal{R})Y - (\nabla_{Y_\mathcal{H}}\mathcal{R})X = [X, Y] + \langle X, (\nabla\mathcal{R})Y\rangle, \tag{A.9}$$

for all $X, Y \in \mathcal{G}^\mathbb{C}_\mathbb{R}$. Using a basis A_i of $\mathcal{A}$ and a basis T_a of $\mathcal{T}$, with corresponding dual bases A^i and T^a with respect to the bilinear form (1.3), we have

$$\langle X, (\nabla\mathcal{R})Y\rangle := \sum_i A^i\langle X, (\nabla_{A_i}\mathcal{R})Y\rangle + \sum_a T^a\langle X, (\nabla_{T_a}\mathcal{R})Y\rangle. \tag{A.10}$$

To determine $T_{m,\ell}$ (A.8) from (A.9), we have to evaluate the expression (A.10) for $X = L^m$ and $Y = L^\ell$, at $q \in \mathcal{A}^o$. Now, for any $T \in (\mathcal{A} + \mathcal{T})$, we find that $(\nabla_T\mathcal{R})(q)$ acts non-trivially on $(\mathcal{A}^\perp + \mathcal{T}^\perp)$ by the operator $\mathrm{ad}_T \circ f(\mathrm{ad}_q)$, where f is the analytic function

$$f(z) = \frac{d\coth(z)}{dz} = -\frac{1}{\sinh^2(z)}. \tag{A.11}$$

This implies that

$$\langle L^m, (\nabla\mathcal{R})(q)L^\ell\rangle = \sum_a T^a\langle L^m, (\nabla_{T_a}\mathcal{R})(q)L^\ell\rangle_\mathbb{R} = [f(\mathrm{ad}_q)((L^\ell)_{\mathcal{A}^\perp}), L^m]_\mathcal{T}. \tag{A.12}$$

The terms $\langle L^m, (\nabla_{A_i}\mathcal{R})(q)L^\ell\rangle_\mathbb{R}$ of (A.10) vanish, because $L^m \in \mathrm{i}\mathcal{G}$ and $(\nabla_{A_i}\mathcal{R})(q)L^\ell \in \mathcal{G}$. □

B Rewriting the Poisson bracket formula of L.-C. Li

In this appendix we recall the Poisson bracket given by the formula (5.8) in [19], and explain its relation to our formula (2.1).

Consider the manifold

$$\mathcal{M} := \mathcal{H}^o \times G^\mathbb{C}_\mathbb{R} \times \mathcal{H}^o = \{(u, g, v)\} \tag{B.1}$$

and its submanifold

$$\mathcal{M}_\mathrm{herm} := \{(u, L, u^\dagger) \mid u \in \mathcal{H}^o,\ L \in \mathfrak{H}\}, \tag{B.2}$$

where $\mathfrak{H}$ is the subset of the Hermitian elements of $G^\mathbb{C}_\mathbb{R} = \mathrm{GL}(n, \mathbb{C})$. Take any real function $f \in C^\infty(\mathcal{M}_\mathrm{herm})$, and extend it arbitrarily to an element $f^\mathrm{ext} \in C^\infty(\mathcal{M})$. Define the $\mathcal{H}$-valued derivatives $\delta_i f^\mathrm{ext}$ ($i = 1, 2$) and the $\mathcal{G}^\mathbb{C}_\mathbb{R}$-valued derivatives Df^ext and $D'f^\mathrm{ext}$ by the requirements

$$\langle \xi, \delta_1 f^\mathrm{ext}(u, g, v)\rangle_\mathbb{R} + \langle \eta, \delta_2 f^\mathrm{ext}(u, g, v)\rangle_\mathbb{R} := \frac{d}{dt}\Big|_{t=0} f^\mathrm{ext}(u + t\xi, g, v + t\eta), \quad \forall \xi, \eta \in \mathcal{H}, \tag{B.3}$$

and

$$\langle X, Df^\mathrm{ext}(u, g, v)\rangle_\mathbb{R} + \langle Y, D'f^\mathrm{ext}(u, g, v)\rangle_\mathbb{R} := \frac{d}{dt}\Big|_{t=0} f^\mathrm{ext}(u, e^{tX}ge^{tY}, v), \quad \forall X, Y \in \mathcal{G}^\mathbb{C}_\mathbb{R}. \tag{B.4}$$

Then, following [19], introduce the notations

$$\delta_1 f(u,L,u^\dagger) := \frac{1}{2}\left(\delta_1 f^{\rm ext}(u,L,u^\dagger) + \left(\delta_2 f^{\rm ext}(u,L,u^\dagger)\right)^\dagger\right), \tag{B.5}$$

and

$$Df(u,L,u^\dagger) := \frac{1}{2}\left(Df^{\rm ext}(u,L,u^\dagger) + \left(D' f^{\rm ext}(u,L,u^\dagger)\right)^\dagger\right). \tag{B.6}$$

These derivatives of f can be checked to be independent of the choice of the extended function.

The Poisson bracket on $C^\infty(\mathcal{M}_{\rm herm})$ given in [19] reads as follows:

$$\{f,h\}_{\mathcal{M}_{\rm herm}} = -2\langle \delta_1 f, Dh\rangle_{\mathbb{R}} + 2\langle \delta_1 h, Df\rangle_{\mathbb{R}} - 2\langle R(u)Df, Dh\rangle_{\mathbb{R}}. \tag{B.7}$$

This is evaluated at the arbitrary point $(u,L,u^\dagger) \in \mathcal{M}_{\rm herm}$, and $R(u) \in {\rm End}(\mathcal{G}_{\mathbb{R}}^{\mathbb{C}})$ acts non-trivially on the orthogonal complement of $(\mathcal{T}+\mathcal{A})$ according to

$$R(u)X = -\left(\frac{1}{2}\coth\frac{1}{2}{\rm ad}_u\right)(X), \quad \forall X \in (\mathcal{T}^\perp + \mathcal{A}^\perp), \tag{B.8}$$

where we use $\mathcal{T}^\perp$ and $\mathcal{A}^\perp$ given in (1.5).

Now, we introduce a one-to-one correspondence between the functions $f \in C^\infty(\mathcal{M}_{\rm herm})$ and the functions $F \in C^\infty(\mathcal{H}^o \times \mathfrak{H})$ by the definition

$$F(w,L) := f(2w, L, 2w^\dagger). \tag{B.9}$$

The factor two is included for convenience (cf. the definitions (B.8) and (1.7)). Since $\mathfrak{H}$ is an open submanifold of $\mathrm{i}\mathcal{G}$, we can take the derivatives $\nabla_1 F \in \mathcal{H}$ and $\nabla_2 F \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ similarly to (1.10). The following simple statement is crucial for us.

Lemma B1. *The derivatives of $f \in C^\infty(\mathcal{M}_{\rm herm})$ given by (B.5) and (B.6) are related to the derivatives of $F \in C^\infty(\mathcal{H}^o \times \mathfrak{H})$ by the identities*

$$Df(2w,L,2w^\dagger) = L\nabla_2 F(w,L), \quad \delta_1 f(2w,L,2w^\dagger) = \frac{1}{4}\nabla_1 F(w,L). \tag{B.10}$$

Proof. For any $X \in \mathcal{G}$, $Y \in \mathrm{i}\mathcal{G}$ and real parameter t, the curves $e^{tX}Le^{-tX}$ and $e^{tY}Le^{tY}$ stay in $\mathfrak{H}$, and thus we have

$$\begin{aligned} F(w, e^{tX}Le^{-tX}) &= f^{\rm ext}(2w, e^{tX}Le^{-tX}, 2w^\dagger), \\ F(w, e^{tY}Le^{tY}) &= f^{\rm ext}(2w, e^{tY}Le^{tY}, 2w^\dagger). \end{aligned} \tag{B.11}$$

Taking the derivative of the first identity gives

$$\langle XL - LX, \nabla_2 F\rangle_{\mathbb{R}} = \langle X, Df^{\rm ext} - D'f^{\rm ext}\rangle_{\mathbb{R}}, \tag{B.12}$$

while the second one gives

$$\langle YL + LY, \nabla_2 F\rangle_{\mathbb{R}} = \langle Y, Df^{\rm ext} + D'f^{\rm ext}\rangle_{\mathbb{R}}, \tag{B.13}$$

where the arguments are as for $t=0$ in (B.11). The relation (B.12) is readily seen to imply

$$2\langle X, L\nabla_2 F\rangle_{\mathbb{R}} = \langle X, Df^{\rm ext} + (D'f^{\rm ext})^\dagger\rangle_{\mathbb{R}}, \tag{B.14}$$

and (B.13) implies

$$2\langle Y, L\nabla_2 F\rangle_{\mathbb{R}} = \langle Y, Df^{\rm ext} + (D'f^{\rm ext})^\dagger\rangle_{\mathbb{R}}. \tag{B.15}$$

By combining these, we obtain the first equality in (B.10).

To obtain the second equality, we note that $f^{\rm ext}$ can be chosen to be independent of v, i.e., in such a way hat

$$f^{\rm ext}(2w, L, v) = f(2w, L) = F(w, L). \tag{B.16}$$

Then $(\delta_1 f)(2w, L) = \frac{1}{2}(\delta_1 f^{\rm ext})(2w, L, 2w^\dagger)$ follows from (B.5). To continue, notice that

$$f^{\rm ext}(2w + 2t\xi, L, v) = F(w + t\xi, L), \quad \forall \xi \in \mathcal{H}, \tag{B.17}$$

entails

$$\langle 2\xi, (\delta_1 f^{\rm ext})(2w, L, 2w^\dagger)\rangle_{\mathbb{R}} = \langle \xi, \nabla_1 F(w, L)\rangle_{\mathbb{R}}. \tag{B.18}$$

Consequently, we get

$$\delta_1 f(2w, L) = \frac{1}{2}\delta_1 f^{\rm ext}(2w, L, 2w^\dagger) = \frac{1}{4}\nabla_1 F(w, L), \tag{B.19}$$

as claimed. □

The final result of the appendix is a direct consequence of the relations given by (B.10).

Proposition B2. *Via the correspondence (B.9), the Poisson bracket (B.7) satisfies*

$$\{f, h\}_{\mathcal{M}_{\rm herm}}(2w, L, 2w^\dagger) = -\frac{1}{2}\{F, H\}_{\rm Li}(w, L), \tag{B.20}$$

where $\{F, H\}_{\rm Li}$ is defined in (2.1). Thus, up to an irrelevant overall constant, which is due to conventions and the change of variable $u = 2w$, $\{\ ,\ \}_{\mathcal{M}_{\rm herm}}$ (B.7) can be identified as the restriction of $\{\ ,\ \}_{\rm Li}$ (2.1) to the dense open submanifold $\mathcal{H}^o \times \mathfrak{H} \subset \mathcal{H}^o \times \mathrm{i}\mathcal{G}$.

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