

ON THE INVARIANCE EQUATION FOR TWO-VARIABLE WEIGHTED NONSYMMETRIC BAJRAKTAREVIĆ MEANS

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Dedicated to the 70th birthday of Professor Karol Baron

ABSTRACT. The purpose of this paper is to investigate the invariance of the arithmetic mean with respect to two weighted Bajraktarević means, i.e., to solve the functional equation

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{tf(x)+sf(y)}{tg(x)+sg(y)}\right) + \left(\frac{h}{k}\right)^{-1}\left(\frac{sh(x)+th(y)}{sk(x)+tk(y)}\right) = x + y \quad (x, y \in I),$$

where $f, g, h, k : I \rightarrow \mathbb{R}$ are unknown continuous functions such that g, k are nowhere zero on I , the ratio functions $f/g, h/k$ are strictly monotone on I , and $t, s \in \mathbb{R}_+$ are constants different from each other. By the main result of this paper, the solutions of the above invariance equation can be expressed either in terms of hyperbolic functions or in terms of trigonometric functions and an additional weight function. For the necessity part of this result, we will assume that $f, g, h, k : I \rightarrow \mathbb{R}$ are four times continuously differentiable.

1. INTRODUCTION

Throughout this paper, the symbols \mathbb{R} , and \mathbb{R}_+ will stand for the sets of real, and positive real numbers, respectively, and I will always denote a nonempty open real interval. The classes of continuous strictly monotone and continuous positive real-valued functions defined on I will be denoted by $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$, respectively.

In the sequel, a function $M : I^2 \rightarrow I$ is called a *two-variable mean* on I if the following so-called mean value property

$$\min(x, y) \leq M(x, y) \leq \max(x, y) \quad (x, y \in I) \quad (1)$$

holds. Also, if both of the inequalities in (1) are strict whenever $x \neq y$, then we say that M is a *strict mean* on I . The *arithmetic* and *geometric* means are well known instances for strict means on \mathbb{R}_+ . More generally, if p is a real number, then the *two-variable Hölder mean* $H_p : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is defined as

$$H_p(x, y) := \begin{cases} \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} & \text{if } p \neq 0, \\ \sqrt{xy} & \text{if } p = 0 \end{cases} \quad (x, y \in \mathbb{R}_+).$$

A classical generalization of Hölder means is the notion of *two-variable quasi-arithmetic mean* (cf. [10]), which is introduced as follows: For a continuous strictly monotone function $f : I \rightarrow \mathbb{R}$, the *two-variable quasi-arithmetic mean* $A_f : I^2 \rightarrow I$ is defined by

$$A_f(x, y) := f^{-1}\left(\frac{f(x) + f(y)}{2}\right) \quad (x, y \in I).$$

Date: September 30, 2018.

2010 Mathematics Subject Classification. 39B12, 39.35, 26E60.

Key words and phrases. Bajraktarević mean, invariant mean, functional equation, invariance equation.

The research of the first author was supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651 and by the EFOP-3.6.1-16-2016-00022 project. This project is co-financed by the European Union and the European Social Fund.

Given two continuous strictly monotone functions $\varphi_1, \varphi_2 : I \rightarrow \mathbb{R}$, the *generalized quasi-arithmetic means* $A_{(\varphi_1, \varphi_2)} : I^2 \rightarrow I$ is defined by

$$A_{(\varphi_1, \varphi_2)}(x, y) := (\varphi_1 + \varphi_2)^{-1} \left(\frac{\varphi_1(x) + \varphi_2(y)}{2} \right) \quad (x, y \in I),$$

which was introduced by Matkowski [22]. For parameters $p, q \in \mathbb{R}$, the *two-variable Gini mean* $G_{p,q} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is defined by

$$G_{p,q}(x, y) := \begin{cases} \left(\frac{x^p + y^p}{x^q + y^q} \right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp \left(\frac{x^p \log(x) + y^p \log(y)}{x^p + y^p} \right) & \text{if } p = q, \end{cases} \quad (x, y \in \mathbb{R}_+).$$

(See [9].) The *two-variable Stolarsky mean* $S_{p,q} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is defined for $x, y \in \mathbb{R}_+$, by

$$S_{p,q}(x, y) := \begin{cases} \left(\frac{q(x^p - y^p)}{p(x^q - y^q)} \right)^{\frac{1}{p-q}} & \text{if } pq(p-q)(x-y) \neq 0, \\ \exp \left(-\frac{1}{p} + \frac{x^p \log(x) - y^p \log(y)}{x^p - y^p} \right) & \text{if } p = q, pq(x-y) \neq 0, \\ \left(\frac{x^p - y^p}{p(\log(x) - \log(y))} \right)^{\frac{1}{p}} & \text{if } q = 0, p(x-y) \neq 0, \\ \left(\frac{x^q - y^q}{q(\log(x) - \log(y))} \right)^{\frac{1}{q}} & \text{if } p = 0, q(x-y) \neq 0, \\ \sqrt{xy} & \text{if } p = q = 0, \\ x & \text{if } x = y. \end{cases}$$

(See [26].)

Given three strict means $M, N, K : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, we say that the triple (M, N, K) satisfies the *invariance equation* if

$$K(M(x, y), N(x, y)) = K(x, y) \quad (x, y \in \mathbb{R}_+) \quad (2)$$

holds. If (2) is valid, then we say that K is invariant with respect to the mean-type mapping (M, N) . The easiest example when the invariance equation is satisfied is the well-known identity

$$\sqrt{xy} = \sqrt{\frac{x+y}{2} \cdot \frac{2xy}{x+y}} \quad (x, y \in \mathbb{R}_+).$$

The last identity means that

$$\mathfrak{G}(x, y) = \mathfrak{G}(\mathcal{A}(x, y), \mathcal{H}(x, y)) \quad (x, y \in \mathbb{R}_+),$$

where \mathcal{A} , \mathfrak{G} , and \mathcal{H} are the two-variable arithmetic, geometric, and harmonic means, respectively. Another invariance equation is the identity

$$\mathcal{A} \otimes \mathfrak{G}(x, y) = \mathcal{A} \otimes \mathfrak{G}(\mathcal{A}(x, y), \mathfrak{G}(x, y)) \quad (x, y \in \mathbb{R}_+),$$

where $\mathcal{A} \otimes \mathcal{G}$ denotes Gauss's arithmetic-geometric mean and defined as follows

$$\mathcal{A} \otimes \mathcal{G}(x, y) = \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{t}{\sqrt{x^2 \cos^2 t + y^2 \sin^2 t}} \right)^{-1} \quad (x, y \in \mathbb{R}_+).$$

The invariance equation in more general classes of means has been studied by several authors in a large number of papers. The invariance equation of Hölder mean solved completely by Daróczy and Páles [8].

Theorem. *Let $p, q, r \in \mathbb{R}$. Then the invariance equation*

$$H_r(H_p(x, y), H_q(x, y)) = H_r(x, y) \quad (x, y \in \mathbb{R}_+)$$

is satisfied if and only if one of the following two possibilities holds:

- (i) $p = q = r$, i.e., all the three means are equal to each other,
- (ii) $p + q = r = 0$, i.e., H_r is the geometric mean and $H_p = H_{-q}$.

The more general invariance equation for quasi-arithmetic means was first solved under infinitely many times differentiability by Sutô [27], [28] and later by Matkowski [19] under twice continuous differentiability. Without imposing unnecessary regularity conditions, this problem was finally solved by Daróczy and Páles [8].

Theorem. *Let $f, g, h : I \rightarrow \mathbb{R}$ be continuous strictly monotone functions. Then the invariance equation*

$$A_f(A_g(x, y), A_h(x, y)) = A_f(x, y) \quad (x, y \in I)$$

holds if and only if there exist $a, b, c, d, p \in \mathbb{R}$ with $ac \neq 0$ such that

$$g = aE_p \circ f + b, \quad h = cE_{-p} \circ f + d,$$

where

$$E_p(t) := \begin{cases} \exp(pt) & \text{if } p \neq 0, \\ t & \text{if } p = 0. \end{cases}$$

Burai [6] and Jarczyk-Matkowski [12] studied the invariance equation involving three weighted arithmetic means. Jarczyk [11] solved this problem without additional regularity assumptions. The invariance of the arithmetic mean with respect to Lagrangian mean has been investigated by Matkowski (cf. [21]). The invariance of the arithmetic, geometric, and harmonic means has been studied by Matkowski [20]. The following result of Baják and Páles [4] describes the invariance of the arithmetic mean with respect to generalized quasi-arithmetic means.

Theorem. *Let $\varphi_1, \varphi_2, \psi_1, \psi_2 : I \rightarrow \mathbb{R}$ be four times continuously differentiable functions such that $\varphi_1' \varphi_2'$ is positive on I . Then the functional equation*

$$(\varphi_1 + \varphi_2)^{-1} \left(\frac{\varphi_1(x) + \varphi_2(y)}{2} \right) + (\psi_1 + \psi_2)^{-1} \left(\frac{\psi_1(x) + \psi_2(y)}{2} \right) = x + y \quad (x, y \in I),$$

holds if and only if

- (i) *either there exist real constants $p, a_1, a_2, c_1, c_2, b_1, b_2, d_1, d_2$ with $p \neq 0, a_1 a_2 > 0, c_1 c_2 > 0$ and $a_1 c_1 = a_2 c_2$ such that for $x \in I$,*

$$\begin{aligned} \varphi_1(x) &= a_1 E_p(x) + b_1, & \varphi_2(x) &= a_2 E_p(x) + b_2, \\ \psi_1(x) &= c_1 E_{-p}(x) + d_1, & \psi_2(x) &= c_2 E_{-p}(x) + d_2; \end{aligned}$$

(ii) or there exist real constants a, b, c, d_1, d_2 with $ac \neq 0$ such that, for $x \in I$,

$$\varphi_1(x) + \varphi_2(x) = ax + b, \quad \psi_1(x) = c\varphi_2(x) + d_1, \quad \text{and} \quad \psi_2(x) = c\varphi_1(x) + d_2.$$

Recently, Baják and Páles [3], [5] have solved the invariance equations of two-variable Gini and Stolarsky means, respectively.

Theorem. *Let $p, q, r, s, u, v \in \mathbb{R}$. Then the invariance equation*

$$G_{p,q}(G_{r,s}(x, y), G_{u,v}(x, y)) = G_{p,q}(x, y) \quad (x, y \in \mathbb{R}_+)$$

holds if and only if one of the following possibilities holds

- (i) $p + q = r + s = u + v = 0$, that is, all the three means are equal to the geometric mean,
- (ii) $\{p, q\} = \{r, s\} = \{u, v\}$, that is, all the three means are equal to each other,
- (iii) $\{r, s\} = \{-u, -v\}$ and $p + q = 0$, that is, $G_{p,q}$ is the geometric mean and $G_{r,s} = G_{-u,-v}$,
- (iv) there exist $a, b \in \mathbb{R}$ such that $\{r, s\} = \{a + b, b\}$, $\{u, v\} = \{a - b, -b\}$, and $\{p, q\} = \{a, 0\}$, in this case $G_{p,q}$ is the power mean,
- (v) there exists $c \in \mathbb{R}$ such that $\{r, s\} = \{3c, c\}$, $u + v = 0$, and $\{p, q\} = \{2c, 0\}$, in this case $G_{p,q}$ is a power mean and $G_{u,v}$ is the geometric mean,
- (vi) there exists $c \in \mathbb{R}$ such that $r + s = 0$, $\{u, v\} = \{3c, c\}$, and $\{p, q\} = \{2c, 0\}$, in this case $G_{p,q}$ is a power mean and $G_{r,s}$ is the geometric mean.

Theorem. *Let $p, q, r, s, u, v \in \mathbb{R}$. Then the invariance equation*

$$S_{u,v}(S_{p,q}(x, y), S_{r,s}(x, y)) = S_{u,v}(x, y) \quad (x, y \in \mathbb{R}_+)$$

holds if and only if one of the following possibilities holds

- (i) $p + q = r + s = u + v = 0$, that is, all the three means are equal to the geometric mean,
- (ii) $\{p, q\} = \{r, s\} = \{u, v\}$, that is, all the three means are equal to each other,
- (iii) $\{p, q\} = \{-r, -s\}$ and $u + v = 0$, that is, $G_{u,v}$ is the geometric mean and $S_{p,q} = S_{-r,-s}$.

Now we recall the notion of weighted two-variable Bajraktarević mean, the class of means where we are going to solve the invariance problem of the arithmetic mean.

Given two continuous functions $f, g : I \rightarrow \mathbb{R}$ such that g is nowhere zero on I and the ratio function f/g is strictly monotone on I , the *weighted two-variable Bajraktarević mean* $B_{f,g} : I^2 \times \mathbb{R}_+^2 \rightarrow I$ is defined by

$$B_{f,g}(x, y; t, s) := \left(\frac{f}{g}\right)^{-1} \left(\frac{tf(x) + sf(y)}{tg(x) + sg(y)} \right) \quad (x, y \in I; t, s \in \mathbb{R}_+).$$

See the paper [2] for the original definition. These means have been extensively investigated by Aczél–Daróczy [1], Daróczy–Losonczi [7], Losonczi [13], [15], [14], [16], Losonczi–Páles [17], [18], Páles [23]. For recent generalizations of Bajraktarević means, the solutions of the comparison, equality and homogeneity problems, we refer to the papers of the authors [25] and [24].

The purpose of this paper is to investigate the invariance of the arithmetic mean with respect to two weighted Bajraktarević means, i.e., to solve

$$B_{f,g}(x, y; t, s) + B_{h,k}(x, y; s, t) = x + y \quad (x, y \in I) \quad (3)$$

or equivalently,

$$\left(\frac{f}{g}\right)^{-1} \left(\frac{tf(x) + sf(y)}{tg(x) + sg(y)} \right) + \left(\frac{h}{k}\right)^{-1} \left(\frac{sh(x) + th(y)}{sk(x) + tk(y)} \right) = x + y,$$

where $f, g, h, k : I \rightarrow \mathbb{R}$ are continuous functions such that g, k are nowhere zero on I , the ratio functions f/g and h/k are strictly monotone on I , and $t, s \in \mathbb{R}_+$ are constants different from each other.

For the sake of convenience and brevity, we introduce certain regularity classes as follows. Let the class $\mathcal{B}_0(I)$ contain all pairs (f, g) such that

- (a) $f, g : I \rightarrow \mathbb{R}$ are continuous functions,
- (b) g is nowhere zero on I , and
- (c) f/g strictly monotone on I .

For $n \geq 1$, let $\mathcal{B}_n(I)$ denote the class of all pairs (f, g) such that

- (+a) $f, g : I \rightarrow \mathbb{R}$ are n times continuously differentiable functions,
- (b) g is nowhere zero on I , and
- (+c) $(f/g)'$ is nowhere zero on I .

Obviously, condition (+a) and (+c) imply (a) and (c), respectively. Therefore, $\mathcal{B}_n(I) \subseteq \mathcal{B}_0(I)$ for all $n \geq 1$ and thus the Bajraktarević mean $B_{f,g}$ is well-defined for $(f, g) \in \mathcal{B}_n(I)$. The following lemma explains the need of the regularity classes $\mathcal{B}_n(I)$.

Lemma 1. *Let $n \in \mathbb{N}$ and $(f, g) \in \mathcal{B}_n(I)$. Then, for every $t, s \in \mathbb{R}_+$, the mapping*

$$(x, y) \mapsto B_{f,g}(x, y; t, s) \quad (x, y \in I)$$

is n -times continuously differentiable on I^2 .

The proof is an easy consequence of standard calculus rules and therefore is left for the reader. In what follows, we establish an implicit equation for the definition of weighted two-variable Bajraktarević means.

Lemma 2. *Let $(f, g) \in \mathcal{B}_0(I)$. Then, for every $x, y \in I$ and $t, s \in \mathbb{R}_+$, the value $z := B_{f,g}(x, y; t, s)$ is the unique solution of the equation*

$$\begin{vmatrix} tf(x) + sf(y) & f(z) \\ tg(x) + sg(y) & g(z) \end{vmatrix} = 0. \quad (4)$$

The proof of this lemma is again elementary, hence it is omitted.

We say that two pairs of functions (f, g) and (h, k) are equivalent (and we write $(f, g) \sim (h, k)$) if there exist constants a, b, c, d with $ad \neq cd$ such that

$$h = af + bg \quad \text{and} \quad k = cf + dg.$$

The equivalence is a key property for characterizing the equality of weighted two-variable Bajraktarević means.

Lemma 3. *Let $(f, g), (h, k) \in \mathcal{B}_0(I)$. Then the equality*

$$B_{f,g}(x, y; t, s) = B_{h,k}(x, y; t, s) \quad (x, y \in I, t, s \in \mathbb{R}_+)$$

holds if and only if $(f, g) \sim (h, k)$.

To obtain another important characterization of the equivalence of pairs $(f, g), (h, k) \in \mathcal{B}_n(I)$ when $n \geq 2$, for $i, j \in \{1, \dots, n\}$ and $(f, g) \in \mathcal{B}_n(I)$, we define

$$W_{f,g}^{i,j} := \begin{vmatrix} f^{(i)} & f^{(j)} \\ g^{(i)} & g^{(j)} \end{vmatrix} \quad \text{and} \quad \Phi_{f,g} := \frac{W_{f,g}^{2,0}}{W_{f,g}^{1,0}}, \quad \Psi_{f,g} := -\frac{W_{f,g}^{2,1}}{W_{f,g}^{1,0}}. \quad (5)$$

Also, it should be noted that, for all $i, j \in \{1, \dots, n\}$,

$$W_{f,g}^{i,j} = -W_{f,g}^{j,i}, \quad W_{f,g}^{i,i} = 0. \quad (6)$$

Lemma 4. *Let $(f, g), (h, k) \in \mathcal{B}_2(I)$. Then $(f, g) \sim (h, k)$ holds if and only if*

$$\Phi_{f,g} = \Phi_{h,k} \quad \text{and} \quad \Psi_{f,g} = \Psi_{h,k}.$$

The following lemma clarifies the importance the auxiliary functions $\Phi_{f,g}$ and $\Psi_{f,g}$.

Lemma 5. *Let $(f, g) \in \mathcal{B}_2(I)$. Then f, g are solutions of the second-order differential equation*

$$y'' = \Phi_{f,g}y' + \Psi_{f,g}y. \quad (7)$$

Proof. Using the definitions of $\Phi_{f,g}, \Psi_{f,g}$ from (5), we can rewrite equation (7) in the following equivalent form

$$y'' = \frac{\begin{vmatrix} f'' & f \\ g'' & g \end{vmatrix}}{\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}}y' - \frac{\begin{vmatrix} f'' & f' \\ g'' & g' \end{vmatrix}}{\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}}y.$$

After multiplying this equation by $\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}$, and rearranging every term to one side of the equation, we infer that (7) is equivalent to

$$\begin{vmatrix} y'' & y' & y \\ f'' & f' & f \\ g'' & g' & g \end{vmatrix} = 0. \quad (8)$$

It is obvious that the functions $y = f$ and $y = g$ are solutions of (8) and therefore they also solve (7). \square

Finally, for a real parameter $p \in \mathbb{R}$, introduce the sine and cosine type functions $S_p, C_p : \mathbb{R} \rightarrow \mathbb{R}$ by

$$S_p(x) := \begin{cases} \sin(\sqrt{-p}x) & \text{if } p < 0, \\ x & \text{if } p = 0, \\ \sinh(\sqrt{p}x) & \text{if } p > 0, \end{cases} \quad \text{and} \quad C_p(x) := \begin{cases} \cos(\sqrt{-p}x) & \text{if } p < 0, \\ 1 & \text{if } p = 0, \\ \cosh(\sqrt{p}x) & \text{if } p > 0. \end{cases}$$

By basic results on second-order linear homogeneous differential equations, it follows that the functions S_p and C_p constructed above form a fundamental system of solutions for the differential equation

$$y'' = py.$$

Using these notations, we are now in the position to formulate the main theorem of this paper.

Theorem 6. *Let $p \in \mathbb{R}$, let $\varphi : I \rightarrow \mathbb{R}_+$ be a positive continuous function and let $(f, g), (h, k) \in \mathcal{B}_0(I)$ such that*

$$(f, g) \sim (S_p/\varphi, C_p/\varphi) \quad \text{and} \quad (h, k) \sim (S_p \cdot \varphi, C_p \cdot \varphi). \quad (9)$$

Then, for all $s, t \in \mathbb{R}_+$, the invariance equation (3) holds.

Conversely, let $(f, g), (h, k) \in \mathcal{B}_4(I)$ and $t, s \in \mathbb{R}_+$ with $t \neq s$ such that the functional equation (3) be valid. Then there exist a positive 4 times continuously differentiable function $\varphi : I \rightarrow \mathbb{R}_+$ and a real parameter $p \in \mathbb{R}$ such that the equivalences (9) are satisfied.

2. FORMULAS FOR GENERALIZED WRONSKI DETERMINANTS

Lemma 7. (i) If $(f, g) \in \mathcal{B}_3(I)$, then

$$\frac{W_{f,g}^{3,0}}{W_{f,g}^{1,0}} = \Phi'_{f,g} + \Phi_{f,g}^2 + \Psi_{f,g} \quad \text{and} \quad \frac{W_{f,g}^{3,1}}{W_{f,g}^{1,0}} = -\Phi_{f,g}\Psi_{f,g} - \Psi'_{f,g}; \quad (10)$$

(ii) If $(f, g) \in \mathcal{B}_4(I)$, then

$$\frac{W_{f,g}^{4,0}}{W_{f,g}^{1,0}} = \Phi''_{f,g} + 3\Phi'_{f,g}\Phi_{f,g} + \Phi_{f,g}^3 + 2\Phi_{f,g}\Psi_{f,g} + 2\Psi'_{f,g}. \quad (11)$$

Proof. Let $(f, g) \in \mathcal{B}_3(I)$. Computing the derivative of $\Phi_{f,g}$ and using $W_{f,g}^{1,1} = 0$, we get

$$\Phi'_{f,g} = \frac{W_{f,g}^{3,0} + W_{f,g}^{2,1}}{W_{f,g}^{1,0}} - \frac{W_{f,g}^{2,0}(W_{f,g}^{2,0} + W_{f,g}^{1,1})}{(W_{f,g}^{1,0})^2} = \frac{W_{f,g}^{3,0}}{W_{f,g}^{1,0}} - \Psi_{f,g} - \Phi_{f,g}^2.$$

Hence the first equality of (10) follows immediately. In order to show the second equality in (10), we differentiate $\Psi_{f,g}$ and use $W_{f,g}^{1,1} = W_{f,g}^{2,2} = 0$ to obtain

$$\Psi'_{f,g} = -\frac{W_{f,g}^{3,1} + W_{f,g}^{2,2}}{W_{f,g}^{1,0}} + \frac{W_{f,g}^{2,1}(W_{f,g}^{2,0} + W_{f,g}^{1,1})}{(W_{f,g}^{1,0})^2} = -\frac{W_{f,g}^{3,1}}{W_{f,g}^{1,0}} - \Phi_{f,g}\Psi_{f,g}.$$

Therefore, the second formula in (10) follows directly.

To prove identity (11), let $(f, g) \in \mathcal{B}_4(I)$. Then by differentiating the both sides of the first equality in (10), we get

$$\frac{W_{f,g}^{4,0} + W_{f,g}^{3,1}}{W_{f,g}^{1,0}} - \frac{W_{f,g}^{3,0}(W_{f,g}^{2,0} + W_{f,g}^{1,1})}{(W_{f,g}^{1,0})^2} = \Phi''_{f,g} + 2\Phi_{f,g}\Phi'_{f,g} + \Psi'_{f,g}.$$

Now, substituting the two formulas of (10) into this equation, equality (11) follows after a simple calculation. \square

The following transformation rules will also be needed.

Lemma 8. If $(f, g) \in \mathcal{B}_2(I)$ and $\varphi : I \rightarrow \mathbb{R}$ is a twice differentiable positive function, then

$$\begin{aligned} \Phi_{\varphi f, \varphi g} &= \Phi_{f,g} + 2\frac{\varphi'}{\varphi}, \\ \Psi_{\varphi f, \varphi g} &= \Psi_{f,g} - \frac{\varphi'}{\varphi}\Phi_{f,g} + \frac{\varphi''}{\varphi} - 2\left(\frac{\varphi'}{\varphi}\right)^2. \end{aligned} \quad (12)$$

Proof. We prove first that, under the assumptions of the lemma, the following identities are valid:

$$\begin{aligned} W_{\varphi f, \varphi g}^{1,0} &= \varphi^2 W_{f,g}^{1,0}, \\ W_{\varphi f, \varphi g}^{2,0} &= \varphi^2 W_{f,g}^{2,0} + 2\varphi'\varphi W_{f,g}^{1,0}, \\ W_{\varphi f, \varphi g}^{2,1} &= \varphi^2 W_{f,g}^{2,1} + \varphi'\varphi W_{f,g}^{2,0} + (2(\varphi')^2 - \varphi''\varphi)W_{f,g}^{1,0}. \end{aligned} \quad (13)$$

Let F denote the vector valued function $\begin{pmatrix} f \\ g \end{pmatrix}$. Then

$$W_{\varphi f, \varphi g}^{1,0} = |(\varphi F)' \ \varphi F| = |\varphi' F + \varphi F' \ \varphi F| = |\varphi' F \ \varphi F| + |\varphi F' \ \varphi F| = \varphi^2 W_{f,g}^{1,0}.$$

Similarly,

$$\begin{aligned} W_{\varphi f, \varphi g}^{2,0} &= |(\varphi F)'' \ \varphi F| = |\varphi'' F + 2\varphi' F' + \varphi F'' \ \varphi F| \\ &= |\varphi'' F \ \varphi F| + 2|\varphi' F' \ \varphi F| + |\varphi F'' \ \varphi F| = \varphi^2 W_{f,g}^{2,0} + 2\varphi' \varphi W_{f,g}^{1,0}. \end{aligned}$$

and

$$\begin{aligned} W_{\varphi f, \varphi g}^{2,1} &= |(\varphi F)'' \ (\varphi F)'| = |\varphi'' F + 2\varphi' F' + \varphi F'' \ \varphi' F + \varphi F'| \\ &= |\varphi'' F \ \varphi' F| + 2|\varphi' F' \ \varphi' F| + |\varphi F'' \ \varphi' F| \\ &\quad + |\varphi'' F \ \varphi F'| + 2|\varphi' F' \ \varphi F'| + |\varphi F'' \ \varphi F'| \\ &= \varphi^2 W_{f,g}^{2,1} + \varphi' \varphi W_{f,g}^{2,0} + (2(\varphi')^2 - \varphi'' \varphi) W_{f,g}^{1,0}. \end{aligned}$$

Dividing the second and third identities by the first one side by side, the formulas stated in (12) follow immediately. \square

3. PARTIAL DERIVATIVES OF WEIGHTED BAJRAKTAREVIĆ MEANS

To solve the invariance equation (3), we need to compute the partial derivatives

$$\partial_1^i B_{f,g}(x, x; t, s) \quad \text{and} \quad \partial_1^i B_{h,k}(x, x; s, t) \quad (i \in \{1, 2, 3, 4\}, x \in I).$$

According to Lemma 2, we can apply implicit differentiation for the identity

$$\left| \begin{array}{cc} tf(x) + sf(y) & f(B_{f,g}(x, y; t, s)) \\ tg(x) + sg(y) & g(B_{f,g}(x, y; t, s)) \end{array} \right| = 0 \quad (x, y \in I), \quad (14)$$

which yields the partial derivative of the mean $B_{f,g}(x, y; t, s)$ and, completely analogously, one can calculate the partial derivatives of $B_{h,k}(x, y; s, t)$.

Theorem 9. *Let $t, s \in \mathbb{R}_+$ be fixed. Then the following formulas are valid for all $x \in I$.*

(i) *If $(f, g) \in \mathcal{B}_1(I)$, then*

$$\partial_1 B_{f,g}(x, x; t, s) = \frac{t}{t+s}; \quad (15)$$

(ii) *If $(f, g) \in \mathcal{B}_2(I)$, then*

$$\partial_1^2 B_{f,g}(x, x; t, s) = \frac{ts}{(t+s)^2} \Phi_{f,g}(x); \quad (16)$$

(iii) *If $(f, g) \in \mathcal{B}_3(I)$, then*

$$\partial_1^3 B_{f,g}(x, x; t, s) = \frac{ts}{(t+s)^3} \left((s-t)(\Phi_{f,g}^2 + \Psi_{f,g}) + (2t+s)\Phi'_{f,g} \right)(x); \quad (17)$$

(iv) *If $(f, g) \in \mathcal{B}_4(I)$, then*

$$\begin{aligned} \partial_1^4 B_{f,g}(x, x; t, s) &= \frac{ts}{(t+s)^4} \left((s^2 + 3ts + 3t^2)\Phi''_{f,g} + (3s^2 + 5ts - 5t^2)\Phi'_{f,g}\Phi_{f,g} \right. \\ &\quad \left. + (s^2 - 4ts + t^2)(\Phi_{f,g}^3 + 2\Phi_{f,g}\Psi_{f,g}) + 2(s^2 + ts - t^2)\Psi'_{f,g} \right)(x). \end{aligned} \quad (18)$$

Proof. For brevity, for $x, y \in I$, denote $M(x, y) := B_{f,g}(x, y; t, s)$ and let F denote the vector valued function $\begin{pmatrix} f \\ g \end{pmatrix}$. Then, with these notations, (14) can be rewritten as

$$\left| tF(x) + sF(y) \quad F(M(x, y)) \right| = 0 \quad (x, y \in I). \quad (19)$$

If $(f, g) \in \mathcal{B}_1(I)$, then differentiating equation (14) with respect to the first variable, we get

$$\left| tF'(x) \quad F(M(x, y)) \right| + \left| tF(x) + sF(y) \quad F'(M(x, y)) \right| \cdot \partial_1 M(x, y) = 0. \quad (20)$$

Substituting $x = y$, it follows that

$$tW_{f,g}^{1,0}(x) + (t + s)W_{f,g}^{0,1}(x) \cdot \partial_1 M(x, x) = 0.$$

Dividing both sides by $(t + s)W_{f,g}^{1,0}(x) \neq 0$, this equality yields

$$\partial_1 B_{f,g}(x, x; t, s) = \partial_1 M(x, x) = \frac{t}{t + s}, \quad (21)$$

which is exactly formula (15).

Let $(f, g) \in \mathcal{B}_2(I)$, then differentiating equation (20) with respect to the first variable, we have

$$\begin{aligned} & \left| tF''(x) \quad F(M(x, y)) \right| + 2 \left| tF'(x) \quad F'(M(x, y)) \right| \partial_1 M(x, y) \\ & + \left| tF(x) + sF(y) \quad F''(M(x, y)) \right| (\partial_1 M(x, y))^2 \\ & + \left| tF(x) + sF(y) \quad F'(M(x, y)) \right| \partial_1^2 M(x, y) = 0. \end{aligned} \quad (22)$$

Now putting $x = y$, we arrive at

$$tW_{f,g}^{2,0}(x) + (t + s)W_{f,g}^{0,2}(x)(\partial_1 M(x, x))^2 + (t + s)W_{f,g}^{0,1}(x) \cdot \partial_1^2 M(x, x) = 0.$$

Substituting the formula for $\partial_1 M(x, x)$ obtained in equation (21), and then dividing the equation side by side by $(t + s)W_{f,g}^{1,0}(x) \neq 0$, it follows that

$$\frac{t}{t + s} \frac{W_{f,g}^{2,0}(x)}{W_{f,g}^{1,0}(x)} + \frac{t^2}{(t + s)^2} \frac{W_{f,g}^{0,2}(x)}{W_{f,g}^{1,0}(x)} = \partial_1^2 M(x, x),$$

which yields that

$$\partial_1^2 B_{f,g}(x, x; t, s) = \partial_1^2 M(x, x) = \frac{t(t + s) - t^2}{(t + s)^2} \cdot \frac{W_{f,g}^{2,0}(x)}{W_{f,g}^{1,0}(x)} = \frac{ts}{(t + s)^2} \Phi_{f,g}(x). \quad (23)$$

To determine the third-order partial derivative $\partial_1^3 B_{f,g}(x, y; t, s)$, assume that $(f, g) \in \mathcal{B}_3(I)$. Then, by computing the derivative of (22) with respect to the first variable, we get

$$\begin{aligned} & \left| tF'''(x) \quad F(M(x, y)) \right| + 3 \left| tF''(x) \quad F'(M(x, y)) \right| \partial_1 M(x, y) \\ & + 3 \left| tF'(x) \quad F'(M(x, y)) \right| \partial_1^2 M(x, y) + 3 \left| tF'(x) \quad F''(M(x, y)) \right| (\partial_1 M(x, y))^2 \\ & + 3 \left| tF(x) + sF(y) \quad F''(M(x, y)) \right| \partial_1 M(x, y) \partial_1^2 M(x, y) \\ & + \left| tF(x) + sF(y) \quad F'''(M(x, y)) \right| (\partial_1 M(x, y))^3 + \left| tF(x) + sF(y) \quad F'(M(x, y)) \right| \partial_1^3 M(x, y) = 0. \end{aligned} \quad (24)$$

Taking $x = y$ and using formulae (21) and (23), hence

$$tW_{f,g}^{3,0}(x) + \frac{3t^2}{t+s}W_{f,g}^{2,1}(x) + \frac{3t^3}{(t+s)^2}W_{f,g}^{1,2}(x) + \frac{3t^2s}{(t+s)^2}W_{f,g}^{0,2}(x)\Phi_{f,g}(x) + \frac{t^3}{(t+s)^2}W_{f,g}^{0,3}(x) \\ + (t+s)W_{f,g}^{0,1}(x) \cdot \partial_1^3 M(x, x) = 0$$

After simple calculations, the last equation reduces to

$$\partial_1^3 M(x, x) = \frac{ts(2t+s)}{(t+s)^3} \cdot \frac{W_{f,g}^{3,0}(x)}{W_{f,g}^{1,0}(x)} + \frac{3t^2s}{(t+s)^3} \cdot \frac{W_{f,g}^{2,1}(x)}{W_{f,g}^{1,0}(x)} - \frac{3t^2s}{(t+s)^3} \frac{W_{f,g}^{2,0}(x)}{W_{f,g}^{1,0}(x)} \Phi_{f,g}(x)$$

Using the first equation in (10) and the the definitions of $\Phi_{f,g}$ and $\Psi_{f,g}$, the following formula follows

$$\partial_1^3 B_{f,g}(x, x; t, s) = \partial_1^3 M(x, x) = \frac{ts}{(t+s)^3} \left((s-t)(\Phi_{f,g}^2 + \Psi_{f,g}) + (2t+s)\Phi'_{f,g} \right) (x). \quad (25)$$

Finally, assume that $(f, g) \in \mathcal{B}_4(I)$. Differentiating equation (24) with respect to x , we get

$$\begin{aligned} & |tF''''(x) \quad F(M(x, y))| + 4 |tF'''(x) \quad F'(M(x, y))| \partial_1 M(x, y) \\ & + 6 |tF''(x) \quad F''(M(x, y))| (\partial_1 M(x, y))^2 + 6 |tF''(x) \quad F'(M(x, y))| \partial_1^2 M(x, y) \\ & + 4 |tF'(x) \quad F'(M(x, y))| \partial_1^3 M(x, y) + 12 |tF'(x) \quad F''(M(x, y))| \partial_1 M(x, y) \partial_1^2 M(x, y) \\ & + 4 |tF'(x) \quad F'''(M(x, y))| (\partial_1 M(x, y))^3 + |tF(x) + sF(y) \quad F''''(M(x, y))| (\partial_1 M(x, y))^4 \\ & + 6 |tF(x) + sF(y) \quad F'''(M(x, y))| (\partial_1 M(x, y))^2 \partial_1^2 M(x, y) \\ & + |tF(x) + sF(y) \quad F''(M(x, y))| \left(3(\partial_1^2 M(x, y))^2 + 4\partial_1 M(x, y) \partial_1^3 M(x, y) \right) \\ & + |tF(x) + sF(y) \quad F'(M(x, y))| \partial_1^4 M(x, y) = 0. \end{aligned}$$

Analogously, putting $x = y$, using (21), (23), and (25), we obtain the following equality

$$tW_{f,g}^{4,0}(x) + \frac{4t^2}{t+s}W_{f,g}^{3,1}(x) + \frac{6t^2s}{(t+s)^2}W_{f,g}^{2,1}(x)\Phi_{f,g}(x) + \frac{12t^3s}{(t+s)^3}W_{f,g}^{1,2}(x)\Phi_{f,g}(x) \\ + \frac{4t^4}{(t+s)^3}W_{f,g}^{1,3}(x) + \frac{t^4}{(t+s)^3}W_{f,g}^{0,4}(x) + \frac{6t^3s}{(t+s)^3}W_{f,g}^{0,3}(x)\Phi_{f,g}(x) + \frac{3t^2s^2}{(t+s)^3}W_{f,g}^{0,2}(x)\Phi_{f,g}^2(x) \\ + \frac{4t^2s}{(t+s)^3}W_{f,g}^{0,2}(x) \left((s-t)(\Phi_{f,g}^2 + \Psi_{f,g}) + (2t+s)\Phi'_{f,g} \right) (x) + (t+s)W_{f,g}^{0,1}(x) \cdot \partial_1^4 M(x, x) = 0.$$

Therefore,

$$\partial_1^4 M(x, x) = \frac{ts}{(t+s)^4} \left((3t^2 + 3st + s^2) \frac{W_{f,g}^{4,0}}{W_{f,g}^{1,0}} + 4t(2t+s) \frac{W_{f,g}^{3,1}}{W_{f,g}^{1,0}} - 6t^2 \frac{W_{f,g}^{3,0}}{W_{f,g}^{1,0}} \Phi_{f,g} \right. \\ \left. + 6t(s-t) \frac{W_{f,g}^{2,1}}{W_{f,g}^{1,0}} \Phi_{f,g} - t \cdot \frac{W_{f,g}^{2,0}}{W_{f,g}^{1,0}} \left((7s-4t)\Phi_{f,g}^2 + 4(s-t)\Psi_{f,g} + 4(2t+s)\Phi'_{f,g} \right) \right) (x).$$

Using the identities in (10) and (11), the desired formula (18) follows immediately. \square

4. NECESSITY

Lemma 10. *Let $t, s \in \mathbb{R}_+$ and let $(f, g), (h, k) \in \mathcal{B}_2(I)$ satisfy the functional equation (3). Then*

$$\Phi_{f,g} + \Phi_{h,k} = 0 \quad (26)$$

Proof. Differentiation equation (3) twice with respect to x , then substituting $y = x$, we get

$$\partial_1^2 B_{f,g}(x, x; t, s) + \partial_1^2 B_{h,k}(x, x; s, t) = 0 \quad (x \in I).$$

Then by the second-order partial derivative formula in (16), we obtain

$$\frac{ts}{(t+s)^2} \Phi_{f,g}(x) + \frac{ts}{(t+s)^2} \Phi_{h,k}(x) = 0 \quad (x \in I),$$

which implies the equality (26). \square

Assuming that $(f, g) \in \mathcal{B}_2(I)$, for simplicity, denote $\Phi_{f,g}$ by Φ . Then, if (3) holds, according to the previous lemma, we have that

$$\Phi_{h,k} = -\Phi. \quad (27)$$

Lemma 11. *Let $t, s \in \mathbb{R}_+$ with $t \neq s$ and let $(f, g), (h, k) \in \mathcal{B}_3(I)$ satisfy the functional equation (3). Then*

$$\Phi' + \Psi_{h,k} - \Psi_{f,g} = 0. \quad (28)$$

Proof. Differentiating (3) three times with respect to the variable x , then substituting $y = x$, we get

$$\partial_1^3 B_{f,g}(x, x; t, s) + \partial_1^3 B_{h,k}(x, x; s, t) = 0 \quad (x \in I).$$

Then the formula for the third-order partial derivatives by (17) implies that

$$((s-t)(\Phi_{f,g}^2 + \Psi_{f,g}) + (2t+s)\Phi'_{f,g}) + ((t-s)(\Phi_{h,k}^2 + \Psi_{h,k}) + (2s+t)\Phi'_{h,k}) = 0.$$

Using $t \neq s$ and (27), the last equation reduces to the desired equality (28). \square

Lemma 12. *Let $t, s \in \mathbb{R}_+$ with $t \neq s$ and let $(f, g), (h, k) \in \mathcal{B}_4(I)$ satisfy the functional equation (3). Then there exists a real constant p such that*

$$\Psi_{f,g} = \frac{1}{2}\Phi' - \frac{1}{4}\Phi^2 + p \quad \text{and} \quad \Psi_{h,k} = -\frac{1}{2}\Phi' - \frac{1}{4}\Phi^2 + p. \quad (29)$$

Proof. Differentiating (3) four times with respect to the variable x , then substituting $y = x$, we arrive at

$$\partial_1^4 B_{f,g}(x, x; t, s) + \partial_1^4 B_{h,k}(x, x; s, t) = 0 \quad (x \in I).$$

Then the formula for the fourth-order partial derivatives by (17) and equality (27) yield that

$$(t^2 - s^2)\Phi'' - (t^2 - 5ts + s^2)\Phi'\Phi + (s^2 - 4ts + t^2)\Phi(\Psi_{f,g} - \Psi_{h,k}) \\ + (s^2 + ts - t^2)\Psi'_{f,g} + (t^2 + ts - s^2)\Psi'_{h,k} = 0.$$

After simple calculations and using formula (28), we get

$$(\Phi^2)' + 2(\Psi'_{h,k} + \Psi'_{f,g}) = 0.$$

By integrating the last equation, we get that there exists a real constant p such that

$$\Phi^2 + 2(\Psi_{h,k} + \Psi_{f,g}) = 4p.$$

This equality, combined with (28), implies the formulas stated in (29). \square

Finally, we can complete the proof of Theorem 6.

Proof of the necessity in Theorem 6. Let $(f, g), (h, k) \in \mathcal{B}_4(I)$ and $t, s \in \mathbb{R}_+$ with $t \neq s$ such that the functional equation (3) be valid. Then, as we have seen it in Lemma 12, there exist a real constant $p \in \mathbb{R}$ such that (29) holds. Define

$$\varphi := \frac{1}{\sqrt{|W_{f,g}^{1,0}|}} = |W_{f,g}^{1,0}|^{-\frac{1}{2}}.$$

Then

$$\frac{\varphi'}{\varphi} = -\frac{(W_{f,g}^{1,0})'}{2W_{f,g}^{1,0}} = -\frac{W_{f,g}^{2,0}}{2W_{f,g}^{1,0}} = -\frac{1}{2}\Phi_{f,g} = -\frac{1}{2}\Phi.$$

Differentiating again, it follows that

$$\frac{\varphi''}{\varphi} = -\frac{1}{2}\Phi' + \frac{1}{4}\Phi^2.$$

Now, using Lemma 8 and the first formula from (29), we get

$$\Phi_{\varphi f, \varphi g} = \Phi_{f,g} + 2\frac{\varphi'}{\varphi} = \Phi + 2\left(-\frac{1}{2}\Phi\right) = 0$$

and

$$\begin{aligned} \Psi_{\varphi f, \varphi g} &= \Psi_{f,g} - \frac{\varphi'}{\varphi}\Phi_{f,g} + \frac{\varphi''}{\varphi} - 2\left(\frac{\varphi'}{\varphi}\right)^2 \\ &= \left(\frac{1}{2}\Phi' - \frac{1}{4}\Phi^2 + p\right) + \frac{1}{2}\Phi^2 - \frac{1}{2}\Phi' + \frac{1}{4}\Phi^2 - 2\left(-\frac{1}{2}\Phi\right)^2 = p. \end{aligned}$$

In view of Lemma 5, these identities imply that φf and φg are solutions of the second-order homogeneous linear differential equation $y'' = py$. Hence, $(\varphi f, \varphi g) \sim (S_p, C_p)$, which implies $(f, g) \sim (S_p/\varphi, C_p/\varphi)$.

A completely analogous argument shows that $(h, k) \sim (S_p \cdot \varphi, C_p \cdot \varphi)$. Thus, we have proved that all the solutions of (3) are of the form stated in Theorem 6. \square

5. SUFFICIENCY

Proof of the sufficiency in Theorem 6. Assume that $p \in \mathbb{R}$ and $\varphi : I \rightarrow \mathbb{R}_+$ is a continuous function and (9) holds on I .

If $p \geq 0$ then C_p is a positive function. Thus, by Lemma 3, we have that $B_{f,g} = B_{S_p/\varphi, C_p/\varphi}$ and $B_{h,k} = B_{S_p \cdot \varphi, C_p \cdot \varphi}$, therefore, it suffices to show that

$$B_{S_p/\varphi, C_p/\varphi}(x, y; t, s) + B_{S_p \cdot \varphi, C_p \cdot \varphi}(x, y; s, t) = x + y \quad (x, y \in I). \quad (30)$$

In the case $p = 0$, this equation can be rewritten as

$$\frac{t\frac{x}{\varphi(x)} + s\frac{y}{\varphi(y)}}{t\frac{1}{\varphi(x)} + s\frac{1}{\varphi(y)}} + \frac{sx\varphi(x) + ty\varphi(y)}{s\varphi(x) + t\varphi(y)} = x + y,$$

which is an easy-to-see identity.

For the case $p > 0$, we recall the addition theorem for the inverse of the tangent hyperbolic function, which reads as follows:

$$\tanh^{-1}(a) + \tanh^{-1}(b) = \tanh^{-1}\left(\frac{a+b}{1+ab}\right) \quad (a, b \in]-1, 1[). \quad (31)$$

We can easily see that

$$B_{S_p/\varphi, C_p/\varphi}(x, y; t, s) = \frac{1}{\sqrt{p}} \tanh^{-1} \left(\frac{t\varphi(y) \sinh(\sqrt{p}x) + s\varphi(x) \sinh(\sqrt{p}y)}{t\varphi(y) \cosh(\sqrt{p}x) + s\varphi(x) \cosh(\sqrt{p}y)} \right),$$

$$B_{S_p \cdot \varphi, C_p \cdot \varphi}(x, y; s, t) = \frac{1}{\sqrt{p}} \tanh^{-1} \left(\frac{s\varphi(x) \sinh(\sqrt{p}x) + t\varphi(y) \sinh(\sqrt{p}y)}{s\varphi(x) \cosh(\sqrt{p}x) + t\varphi(y) \cosh(\sqrt{p}y)} \right).$$

Therefore, with the substitutions $u := \sqrt{p}x$, $v := \sqrt{p}y$, and $T := t\varphi(y)$, $S := s\varphi(x)$, and using the addition formula (31), the left hand side of equation (30) can be rewritten and calculated as

$$\begin{aligned} & \frac{1}{\sqrt{p}} \left(\tanh^{-1} \left(\frac{T \sinh(u) + S \sinh(v)}{T \cosh(u) + S \cosh(v)} \right) + \tanh^{-1} \left(\frac{S \sinh(u) + T \sinh(v)}{S \cosh(u) + T \cosh(v)} \right) \right) \\ &= \frac{1}{\sqrt{p}} \tanh^{-1} \left(\frac{\frac{T \sinh(u) + S \sinh(v)}{T \cosh(u) + S \cosh(v)} + \frac{S \sinh(u) + T \sinh(v)}{S \cosh(u) + T \cosh(v)}}{1 + \frac{T \sinh(u) + S \sinh(v)}{T \cosh(u) + S \cosh(v)} \frac{S \sinh(u) + T \sinh(v)}{S \cosh(u) + T \cosh(v)}} \right) \\ &= \frac{1}{\sqrt{p}} \tanh^{-1} \left(\frac{(T^2 + S^2) \sinh(u + v) + TS(\sinh(2u) + \sinh(2v))}{(T^2 + S^2) \cosh(u + v) + TS(\cosh(2u) + \cosh(2v))} \right) \\ &= \frac{1}{\sqrt{p}} \tanh^{-1} \left(\frac{(T^2 + S^2) \sinh(u + v) + 2TS \sinh(u + v) \cosh(u - v)}{(T^2 + S^2) \cosh(u + v) + 2TS \cosh(u + v) \cosh(u - v)} \right) \\ &= \frac{1}{\sqrt{p}} \tanh^{-1} (\tanh(u + v)) \\ &= \frac{u + v}{\sqrt{p}} = x + y. \end{aligned}$$

This completes the proof of the identity (30).

For the discussion of the case $p < 0$, we recall the folk-addition theorem for the inverse of the tangent function, which reads as follows:

$$\tan^{-1}(a) + \tan^{-1}(b) = \begin{cases} -\pi + \tan^{-1} \left(\frac{a+b}{1-ab} \right) & \text{if } ab > 1, a, b < 0, \\ -\frac{\pi}{2} & \text{if } ab = 1, a, b < 0, \\ \tan^{-1} \left(\frac{a+b}{1-ab} \right) & \text{if } ab < 1, \\ \frac{\pi}{2} & \text{if } ab = 1, a, b > 0, \\ \pi + \tan^{-1} \left(\frac{a+b}{1-ab} \right) & \text{if } ab > 1, a, b > 0. \end{cases} \quad (32)$$

In the case $p < 0$, the functions C_p is not necessarily nonvanishing on I , therefore, we cannot immediately express the means $B_{f,g}$ and $B_{h,k}$ in terms of C_p and S_p . For the sake of brevity, denote $\sqrt{-p}$ by q . First, we shall show that there exist $\alpha \in]-\pi, \pi]$ and $k \in \mathbb{Z}$ such that

$$I \subseteq]\frac{1}{q}(\alpha + (2k - \frac{1}{2})\pi), \frac{1}{q}(\alpha + (2k + \frac{1}{2})\pi)[. \quad (33)$$

In view of the equivalence $(f, g) \sim (S_p/\varphi, C_p/\varphi)$ we have that $(f\varphi, g\varphi) \sim (S_p, C_p)$, hence there exist real constants c, d such that $g\varphi = cS_p + dC_p$ holds on I . The function g is nowhere zero on I , therefore $cS_p + dC_p$ is also nonvanishing on I . We may assume that this function is positive on I (otherwise we

replace (c, d) by $(-c, -d)$). Determine the unique $\alpha \in]-\pi, \pi]$ such that the equalities

$$\sin \alpha = \frac{c}{\sqrt{c^2 + d^2}}, \quad \cos \alpha = \frac{d}{\sqrt{c^2 + d^2}}$$

be valid. Then, we have that $\sin(\alpha)S_p + \cos(\alpha)C_p$ is positive on I , that is, for all $x \in I$,

$$0 < \sin(\alpha)S_p(x) + \cos(\alpha)C_p(x) = \cos(qx - \alpha).$$

With the notation $J := qI - \alpha$ and the substitution $w := qx - \alpha$, this condition means that, for all $w \in J$, $\cos(w) > 0$. Because J is an interval, it follows that there exists $k \in \mathbb{Z}$ such that

$$J \subseteq](2k - \frac{1}{2})\pi, (2k + \frac{1}{2})\pi[.$$

This inclusion directly implies (33).

For the computation of the means $B_{f,g}$ and $B_{h,k}$, observe first that

$$\begin{aligned} (S_p, C_p) &\sim (\cos(\alpha)S_p - \sin(\alpha)C_p, \sin(\alpha)S_p + \cos(\alpha)C_p) \\ &= (\sin(q(\cdot) - \alpha - 2k\pi), \cos(q(\cdot) - \alpha - 2k\pi)). \end{aligned}$$

Therefore, by the equivalences, $(f\varphi, g\varphi) \sim (S_p, C_p)$ and $(h/\varphi, k/\varphi) \sim (S_p, C_p)$, it follows that

$$\begin{aligned} (f, g) &\sim \left(\frac{1}{\varphi} \sin(q(\cdot) - \alpha - 2k\pi), \frac{1}{\varphi} \cos(q(\cdot) - \alpha - 2k\pi)\right), \\ (h, k) &\sim (\varphi \sin(q(\cdot) - \alpha - 2k\pi), \varphi \cos(q(\cdot) - \alpha - 2k\pi)). \end{aligned}$$

Since the function $\cos(q(\cdot) - \alpha - 2k\pi)$ is positive on I , using Lemma 3, we have that

$$\begin{aligned} B_{f,g}(x, y; t, s) &= \frac{1}{q} \left(\tan^{-1} \left(\frac{\frac{t}{\varphi(x)} \sin(qx - \alpha - 2k\pi) + \frac{s}{\varphi(y)} \sin(qy - \alpha - 2k\pi)}{\frac{t}{\varphi(x)} \cos(qx - \alpha - 2k\pi) + \frac{s}{\varphi(y)} \cos(qy - \alpha - 2k\pi)} \right) + \alpha + 2k\pi \right), \\ B_{h,k}(x, y; s, t) &= \frac{1}{q} \left(\tan^{-1} \left(\frac{s\varphi(x) \sin(qx - \alpha - 2k\pi) + t\varphi(y) \sin(qy - \alpha - 2k\pi)}{s\varphi(x) \cos(qx - \alpha - 2k\pi) + t\varphi(y) \cos(qy - \alpha - 2k\pi)} \right) + \alpha + 2k\pi \right). \end{aligned}$$

Now, denote $u := qx - \alpha - 2k\pi$, $v := qy - \alpha - 2k\pi$, and $T := t\varphi(y)$, $S := s\varphi(x)$. Then, by the inclusion (33), we have $u, v \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Hence, it follows that $u + v, u - v \in]-\pi, \pi[$.

With these substitutions, the left hand side of equation (30) can be rewritten as

$$L := \frac{1}{q} \left(\tan^{-1} \left(\frac{T \sin(u) + S \sin(v)}{T \cos(u) + S \cos(v)} \right) + \alpha + 2k\pi + \tan^{-1} \left(\frac{S \sin(u) + T \sin(v)}{S \cos(u) + T \cos(v)} \right) + \alpha + 2k\pi \right).$$

In order to calculate L and to show that $L = x + y$, we have to distinguish several cases.

First we consider the case when $T = S$. Now, observing that $\frac{u+v}{2}, \frac{u-v}{2} \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, we have $\cos(\frac{u-v}{2}) > 0$ and $\tan^{-1}(\tan(\frac{u+v}{2})) = \frac{u+v}{2}$, therefore,

$$\begin{aligned} L &= \frac{2}{q} \tan^{-1} \left(\frac{\sin(u) + \sin(v)}{\cos(u) + \cos(v)} \right) + \frac{2\alpha + 4k\pi}{q} = \frac{2}{q} \tan^{-1} \left(\frac{2 \sin(\frac{u+v}{2}) \cos(\frac{u-v}{2})}{2 \cos(\frac{u+v}{2}) \cos(\frac{u-v}{2})} \right) + \frac{2\alpha + 4k\pi}{q} \\ &= \frac{2}{q} \tan^{-1} \left(\frac{\sin(\frac{u+v}{2})}{\cos(\frac{u+v}{2})} \right) + \frac{2\alpha + 4k\pi}{q} = \frac{2}{q} \cdot \frac{u+v}{2} + \frac{2\alpha + 4k\pi}{q} = \frac{u+v + 2\alpha + 4k\pi}{q} = x + y. \end{aligned}$$

From now on, we may assume that $T \neq S$. In this case,

$$T^2 + S^2 + 2TS \cos(u - v) = (T - S)^2 + 2TS(1 + \cos(u - v)) > 0. \quad (34)$$

Let us denote

$$a := a(u, v) := \frac{T \sin(u) + S \sin(v)}{T \cos(u) + S \cos(v)}, \quad b := b(u, v) := \frac{S \sin(u) + T \sin(v)}{S \cos(u) + T \cos(v)}.$$

With this notation, we can rewrite L in the following form

$$L := \frac{1}{q} (\tan^{-1}(a) + \tan^{-1}(b) + 2\alpha + 4k\pi).$$

In order to apply the addition formula (32), we have to clarify to which subdomain the pair (a, b) belongs to. We shall prove that

$$\begin{aligned} u + v \in] -\pi, -\frac{\pi}{2}[&\implies ab > 1, a, b < 0, \\ u + v = -\frac{\pi}{2} &\implies ab = 1, a, b < 0, \\ u + v \in] -\frac{\pi}{2}, \frac{\pi}{2}[&\implies ab < 1, \\ u + v = \frac{\pi}{2} &\implies ab = 1, a, b > 0, \\ u + v \in] \frac{\pi}{2}, \pi[&\implies ab > 1, a, b > 0. \end{aligned}$$

By $u, v \in] -\frac{\pi}{2}, \frac{\pi}{2}[$, we always have $\cos(u), \cos(v) > 0$. If $u + v \in] -\pi, -\frac{\pi}{2}[$, then $u, v \in] -\frac{\pi}{2}, 0[$, which implies that $\sin(u), \sin(v) < 0$. Therefore, $a, b < 0$. Similarly, if $u + v \in] \frac{\pi}{2}, \pi[$, then $a, b > 0$ holds. A direct computation shows that

$$1 - ab = \frac{(T^2 + S^2 + 2TS \cos(u - v)) \cos(u + v)}{(T \cos(u) + S \cos(v))(S \cos(u) + T \cos(v))}.$$

Therefore, using (34), the sign of $1 - ab$ equals the sign of $\cos(u + v)$. This implies $ab < 1$, $ab = 1$, and $ab > 1$ if and only if $u + v \in] -\frac{\pi}{2}, \frac{\pi}{2}[$, $u + v = \pm\frac{\pi}{2}$, and $u + v \in] -\pi, -\frac{\pi}{2}[\cup] \frac{\pi}{2}, \pi[$, respectively.

In order to complete the proof of sufficiency, we shall calculate L under consideration all cases stated in (32). For brevity, define

$$\sigma := \sigma(u, v) := \begin{cases} -1 & \text{if } u + v \in] -\pi, -\frac{\pi}{2}[, \\ 0 & \text{if } u + v \in] -\frac{\pi}{2}, \frac{\pi}{2}[, \\ 1 & \text{if } u + v \in] \frac{\pi}{2}, \pi[. \end{cases}$$

In the case $u + v \neq \pm \frac{\pi}{2}$ we have that $ab \neq 1$. Therefore, using the addition formula (32), we get

$$\begin{aligned}
L &= \frac{1}{q} \tan^{-1} \left(\frac{a+b}{1-ab} \right) + \frac{2\alpha + 4k\pi + \sigma\pi}{q} \\
&= \frac{1}{q} \tan^{-1} \left(\frac{\frac{T \sin(u)+S \sin(v)}{T \cos(u)+S \cos(v)} + \frac{S \sin(u)+T \sin(v)}{S \cos(u)+T \cos(v)}}{1 - \frac{T \sin(u)+S \sin(v)}{T \cos(u)+S \cos(v)} \frac{S \sin(u)+T \sin(v)}{S \cos(u)+T \cos(v)}} \right) + \frac{2\alpha + 4k\pi + \sigma\pi}{q} \\
&= \frac{1}{q} \tan^{-1} \left(\frac{(T^2 + S^2) \sin(u+v) + TS(\sin(2u) + \sin(2v))}{(T^2 + S^2) \cos(u+v) + TS(\cos(2u) + \cos(2v))} \right) + \frac{2\alpha + 4k\pi + \sigma\pi}{q} \\
&= \frac{1}{q} \tan^{-1} \left(\frac{(T^2 + S^2) \sin(u+v) + 2TS \sin(u+v) \cos(u-v)}{(T^2 + S^2) \cos(u+v) + 2TS \cos(u+v) \cos(u-v)} \right) + \frac{2\alpha + 4k\pi + \sigma\pi}{q} \\
&= \frac{1}{q} \tan^{-1} (\tan(u+v)) + \frac{2\alpha + 4k\pi + \sigma\pi}{q} = \frac{u+v - \sigma\pi}{q} + \frac{2\alpha + 4k\pi + \sigma\pi}{q} \\
&= \frac{u+v + 2\alpha + 4k\pi}{q} = x + y.
\end{aligned}$$

Finally assume that $u + v = \sigma \frac{\pi}{2}$. Then $ab = 1$ such that either $a, b < 0$ or $a, b > 0$. Therefore, by the addition formula (32) again, it results that

$$\tan^{-1}(a) + \tan^{-1}(b) = \sigma \frac{\pi}{2} = u + v.$$

Using this equality, we can see at once that L is equal to $x + y$ also in this case. This completes the proof of the sufficiency in Theorem 6. \square

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