Diagonal finite volume matrix elements in the sinh-Gordon model

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Abstract

Using the fermionic basis we conjecture exact expressions for diagonal finite volume matrix elements of exponential operators and their descendants in the sinh-Gordon theory. Our expressions sum up the LeClair-Mussardo type infinite series generalized by Pozsgay for excited state expectation values. We checked our formulae against the Liouville three-point functions for small, while against Pozsgay's expansion for large volumes and found complete agreement.

1 Introduction

Integrable models are ideal testing grounds of various methods and ideas in quantum field theories. The simplest interacting model of this type is the sinh-Gordon theory, which has a single particle type and the full finite volume energy spectrum can be calculated from the scattering phase of these particles. There is a hope that similar exact results can be obtained also for finite volume matrix elements.

The finite volume matrix elements of local operators are essentially the building blocks of finite volume correlations functions, which are relevant in statistical and solid state systems [1, 2]. Their non-local counterparts can be used in the AdS/CFT correspondence to describe three-point functions in the gauge theory and the string field theory vertex in string theory [3, 4, 5, 6, 7]. Diagonal matrix elements play a special role there, as they describe the HHL type correlation functions [8, 9, 10, 11, 12].

There were two alternative approaches for the calculation of finite volume matrix elements. For generic operators and theories one can try to use the infinite volume form factors [13, 14] and the scattering matrix [15] to develop a systematic large volume expansion. Polynomial volume corrections originate from momentum quantization [16], while exponentially small finite size corrections from the presence of virtual particles [17]. The LeClair-Mussardo formula [18] provides an infinite series for the exact finite volume one-point function, where each term contains the contribution of a given number of virtual particles in terms of their infinite volume connected form factors and a weight function, which is related to the Thermodynamic Bethe ansatz (TBA) densities of these particles [19]. This formula was then generalized by analytical continuation for diagonal matrix elements, which replaces

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the ground-state TBA densities with the excited state ones and contains additional factors, which can be interpreted as partial density of states [20, 21].

Alternatively, there is an other approach which focuses on specific theories and exploits their hidden (Grassmann) structure to provide compact expressions for finite volume matrix elements [22]. These specific continuum models arise as limits of integrable lattice models and the most studied examples are the sinh-Gordon and sine-Gordon models. There have been active work and relevant progress in deriving finite volume one-point functions for the exponential operators and their descendants in these theories [23, 24]. These results were then extended for diagonal matrix elements in the sine-Gordon theory [25, 26, 27] and the aim of our paper is to provide similar expressions in the sinh-Gordon theory.

The paper is organized as follows: Section 2 reviews the description of the finite size energy spectrum of the sinh-Gordon theory. A multi-particle state for large volumes can be labelled by momentum quantum numbers, which we relate at small volume to the spectrum of the Liouville conformal field theory by matching the eigenvalues of the conserved charges. In Section 3 we formulate our main conjecture for the finite volume exceptions values in the fermionic basis. The novelty compared to the vacuum expectation values is the discrete part of the convolutions, which carries information on the particles' rapidities. We check this conjecture for large volumes in Section 4. The discrete part of the convolution contains the polynomial, while the continuous part the exponentially small corrections in the volume. In Section 5 we compare our conjecture with Liouville three-point functions for low lying states including non-degenerate and degenerate L_0 subspaces. All the checks performed confirm our conjecture, thus we close the paper with conclusions in Section 6.

2 Energy spectrum

In this section we summarize the exact description of the finite volume energy spectrum together with its large and small volume formulations [28].

The sinh-Gordon theory is defined by the Lagrangian:

$$\mathcal{L} = \frac{1}{4\pi}(\partial\phi)^2 + \frac{2\mu^2}{\sin\pi b^2}\cosh(b\varphi)\,.$$

In the literature there is an abundance of notations for the parameters of this model. We decided to follow the paper [29] by introducing the background charge of the related Liouville model and the renormalized coupling constant as

$$Q = b + b^{-1}\,, \qquad p = \frac{b^2}{1+b^2}\,.$$

The sinh-Gordon model is the simplest integrable interacting two dimensional quantum field theory. It has one single particle of mass m with the corresponding two particle scattering matrix

$$S(\theta) = \frac{\sinh\theta - i\sin(\pi p)}{\sinh\theta + i\sin(\pi p)}\,.$$

The finite size energy spectrum, in a volume R, can be formulated in terms of the $\mathcal{Q}$ function, which satisfies the following functional relations:

$$\mathcal{Q}(\theta + \tfrac{i\pi}{2})\mathcal{Q}(\theta - \tfrac{i\pi}{2}) = 1 + \mathcal{Q}(\theta + \tfrac{i\pi}{2}(1-2p))\mathcal{Q}(\theta - \tfrac{i\pi}{2}(1-2p)) \equiv 1 + e^{-\epsilon(\theta)}\,,$$

where we introduced the TBA pseudo-energy ϵ. Excited states can be labeled by the zeros of $\mathcal{Q}$ as: $\{\theta_1,\dots,\theta_N\}$. With the prescribed large θ asymptotics, $\log\mathcal{Q}(\theta) \simeq -\frac{r}{2}\frac{\cosh\theta}{\sin\pi p}$ there is a unique solution

$$\mathcal{Q}(\theta) = \prod_{k=1}^{N}\tanh\Big(\frac{\theta-\theta_k}{2}\Big)\exp\Big(-\frac{r\cosh(\theta)}{2\sin(\pi p)} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\cosh(\theta-\theta')}\log(1+e^{-\epsilon(\theta')})d\theta'\Big)\,.$$

Here $r = mR$ is the dimensionless volume. Thanks to the functional relation $\epsilon(\theta)$ can be fixed from the following TBA equation

$$\epsilon(\theta) = r\cosh\theta + \sum_{k=1}^{N} \log S(\theta - \theta_k - \tfrac{\pi i}{2}) - \int_{-\infty}^{\infty} K(\theta - \theta') \log(1 + e^{-\epsilon(\theta')}) d\theta' \,. \tag{2.1}$$

where the kernel is related to the scattering matrix as

$$K(\theta) = \frac{1}{2\pi i}\frac{d}{d\theta}\log S(\theta) = \frac{1}{2\pi i}\Big(\frac{1}{\sinh(\theta - \pi i p)} - \frac{1}{\sinh(\theta + \pi i p)}\Big) \,.$$

The finite size spectrum can be characterized by a set of integers $\{N_k\}$, denoted by $\mathcal{N}$, via the zeros of the $\mathcal{Q}$ function, written equivalently as

$$f(\theta_k) = \pi N_k \,, \tag{2.2}$$

where

$$f(\theta) = r\sinh\theta + \sum_{k=1}^{N} \arg(-S(\theta - \theta_k)) - \int_{-\infty}^{\infty} K(\theta - \theta' + \tfrac{\pi i}{2}) \log(1 + e^{-\epsilon(\theta')}) d\theta' \,.$$

coincides at the positions θ_k with the analytical continuation of $-i\epsilon(\theta + i\pi/2)$ (with certain choice of the branches of logarithms). Here we use $-S(\theta - \theta_k)$ under arg for computational convenience (notice that $-S(0) = 1$).

These equations are called the Bethe Ansatz (BA) equations and can be interpreted as the momentum quantization equations of the particles with rapidity θ_k. The energy of the multi-particle state with rapidities $\{\theta_1, \dots, \theta_N\}$ can be written as

$$E_{\mathcal{N}} = \sum_{i=1}^{N} m\cosh\theta_i - m\int_{-\infty}^{\infty} \cosh\theta \log(1 + e^{-\epsilon(\theta)})\frac{d\theta}{2\pi} \,. \tag{2.3}$$

2.1 Large volume expansion

Since in the large volume limit the TBA pseudo-energy behaves as $\epsilon = r\cosh\theta + O(1)$ the integral terms are of order $O(e^{-r})$ and can be neglected. This results in the large volume limit of the BA equations

$$r\sinh\theta_j + \sum_{k=1}^{N} \arg(-S(\theta_j - \theta_k)) = \pi N_j \,. \tag{2.4}$$

Let us assume that rapidities are labeled such that $\{\theta_1 > \theta_2 > \cdots > \theta_m\}$. We recall that in [28] it was proven that for any given set of integers $\{n_1, \dots, n_m\}$ the equations

$$r\sinh\theta_j - \sum_{k=1}^{j-1} \arg(S(\theta_k - \theta_j)) + \sum_{k=j+1}^{N} \arg(S(\theta_j - \theta_k)) = 2\pi n_j \,, \tag{2.5}$$

have a unique solution[1]. The idea of the proof was to introduce $P(\theta) = \int_0^\theta \arg(S(v))dv$ and to show that the rapidities $\{\theta_j\}$ minimize the positive definite Yang-Yang functional

$$\sum_j (r\cosh\theta_j - 2\pi n_j\theta_j) + \sum_{j<k} P(\theta_j - \theta_k) \,.$$

[1] Note that, although here the quantum numbers $\{n_j\}$ can be equal, the solutions for the rapidities $\{\theta_j\}$ can not, so the system is nevertheless fermionic and not bosonic type.

In order to compare eq. (2.4) to eq. (2.5) we recall that for positive arguments

$$\arg(S(\theta)) = \arg(-S(\theta)) - i\pi \quad ; \qquad \theta > 0\,.$$

This leads to the following relation between the quantum numbers $\{N_j\}$ and $\{n_j\}$:

$$N_j = 2n_j - N - 1 + 2j\,.$$

In particular, the state labelled by $\{0, \dots, 0\}$ in eq. (2.5) will be mapped to $\{-M+1, -M+3, \dots, M-3, M-1\}$, with all quantum numbers being distinct. This also shows that states with even number of particles are labelled by odd quantum numbers, while states with odd number of particles with even quantum numbers in $\mathcal{N}$. Once the equations (2.4) are solved for the rapidities the large volume energy is

$$E_{\mathcal{N}} = \sum_{i=1}^{N} m \cosh\theta_i\,.$$

In the following we analyze the small volume limit of the energies.

2.2 Small volume limit

In the small volume limit we compare the energy eigenvalues with the spectrum of the Liouville theory [30]. In this description the sinh-Gordon theory is understood as the perturbation of the Liouville theory with the operator $e^{-b\varphi}$:

$$\mathcal{L} = \mathcal{L}_{CFT} + \frac{\mu^2}{\sin\pi b^2} e^{-b\varphi}\,.$$

There are infinitely many conserved charges and the energy is related to the first two as $E = -\frac{\pi}{12R}(I_1 + \bar{I}_1)$, where

$$I_1 = -\frac{6r}{\pi}\Big(\sum_{k=1}^{N} e^{\theta_k} - \int_{-\infty}^{\infty} e^{\theta}\log(1+e^{-\epsilon(\theta)})\frac{d\theta}{2\pi}\Big), \quad \bar{I}_1 = -\frac{6r}{\pi}\Big(\sum_{k=1}^{N} e^{-\theta_k} - \int_{-\infty}^{\infty} e^{-\theta}\log(1+e^{-\epsilon(\theta)})\frac{d\theta}{2\pi}\Big).$$

The Liouville theory is a conformal field theory with a continuous spectrum. Its Hilbert space is built up from the non-compact zero mode and the oscillators. The zero mode determines the dimension of the primary fields, while the oscillators create descendants. Once the perturbation is introduced the spectrum of primary fields can be approximated by the quantization of the zero mode [30]:

$$4P_L(r)Q\log\Big(Z(p)rb^{\frac{b^2-1}{b^2+1}}\Big) = -\pi L + \frac{1}{i}\log\frac{\Gamma(1+2iP_L(r)b)\Gamma(1+2iP_L(r)/b)}{\Gamma(1-2iP_L(r)b)\Gamma(1-2iP_L(r)/b)}\,. \tag{2.6}$$

with $L = 1, 2, \dots$. Here and later we use the mass scale

$$Z(p) = \frac{1}{16Q\pi^{3/2}}\Gamma\Big(\frac{p}{2}\Big)\Gamma\Big(\frac{1-p}{2}\Big).$$

The eigenvalues of the conserved charges in the CFT can be written as

$$I_1^{\rm CFT} = P_L(r)^2 - \frac{1}{24} + M\,, \quad \bar{I}_1^{\rm CFT} = P_L(r)^2 - \frac{1}{24} + \overline{M}\,,$$

where $M, \overline{M}$ are levels of descendants for the two chiralities. By comparing the energies in the TBA and the perturbed Liouville descriptions we can relate the quantum numbers $\mathcal{N}$ to $\{L, M, \bar{M}\}$.

To give an example, we claim that $\mathcal{N} = \{-2, 0, 2\}$ corresponds to the primary field with $L = 4$. Indeed, by numerically solving the TBA and BA equations (2.1,2.2) on the one hand and the zero mode quantization (2.6) on the other we found for $r = .001$ the ratios:

$$\frac{I_1}{I_1^{\rm CFT}} = 1.00003\,, \quad \frac{\bar{I}_1}{\bar{I}_1^{\rm CFT}} = 0.999989\,.$$

In this way we obtain the following correspondence, which we present both at the language of $\mathcal{N}$ and that of $\{n_j\}$ with $n_j = (N_j + M + 1)/2 - j$:

$\mathcal{N}$	$\{n_i\}$	L	M	$\bar{M}$
{ }	{}	1	0	0
{0}	{0}	2	0	0
{-1,1}	{0,0}	3	0	0
{-2,0,2}	{0,0,0}	4	0	0
{2}	{1}	1	1	0
{-1,3}	{0,1}	2	1	0
{-2,0,4}	{0,0,1}	3	1	0
{-3,3}	{-1,1}	1	1	1
{4}	{2}	1	2	0
{1,3}	{1,1}	1	2	0
{-1,5}	{0,2}	2	2	0
{-2,2,4}	{0,1,1}	2	2	0
{-3,5}	{-1,2}	1	2	1
{-5,5}	{-2,2}	1	2	2
{1,5}	{1,2}	1	3	0
{0,2,4}	{1,1,1}	1	3	0
{-3,-1,1,5}	{0,0,0,1}	4	1	0

(2.7)

Clearly in the parametrization $\{n_j\}$ the number of zeros is $L - 1$, while the sum of positive/negative numbers is $M/\bar{M}$, in agreement with [28]. Starting from $M = 2$ the spectrum of L_0 is degenerate. The degeneracy can be lifted using the second integral of motion as we demonstrate in Section 5.2. We will use all these states later to compare our form factor conjecture to the Liouville three-point functions in the small volume limit.

3 Finite volume expectation values

In the following we provide formulas for the expectation values

$$\langle \theta_1, \ldots, \theta_m | \mathcal{O} | \theta_m, \ldots, \theta_1 \rangle_R \,,$$

where $|\theta_m, \ldots, \theta_1\rangle_R$ is a normalized finite volume energy eigenstate (2.3) and $\mathcal{O}$ is a local operator. Expectation values of local operators obtained by commuting with a conserved charge, $[I_n, \mathcal{O}]$, vanish, thus we consider only the quotient space, where these operators are factored out.

Local operators in massive perturbed conformal field theories are in one-to-one correspondence with the states of the conformal Hilbert space of the unperturbed model. The sinh-Gordon theory can be considered either as the perturbation of the free massless boson with $\cosh(b\varphi)$ or as the perturbation of the Liouville theory with the operator $e^{-b\varphi}$. Local operators are the exponentials $\Phi_\alpha = e^{\frac{Q\alpha}{2}\varphi}(0)$ together with their descendants $\mathcal{O}_\alpha$, which can be generated in two different ways [31, 23, 24]. If the modes of the free massless boson are used the operators are called *Heisenberg descendants* and the expectation values of the corresponding operators in the quotient space have the $\sigma_1 : \alpha \to -\alpha$ symmetry. In the perturbed Liouville scheme *Virasoro descendants* are generated by the Virasoro modes and the expectation values have the symmetry $\sigma_2 : \alpha \to 2 - \alpha$. Relating these two descriptions should provide a basis of the CFT adapted to the integrable perturbation. Direct attempt to find such a basis failed for level higher than 2 because it requires solving a rather complicated Riemann-Hilbert problem. The solution came from a rather distant study of lattice integrable models which lead to the discovery of the fermionic basis. The latter provides in the scaling limit the fermionic basis for the sine-Gordon model [32, 22]. As has been shown in [23] this fermionic basis brings the Riemann-Hilbert problem in question to the diagonal form.

3.1 Fermionic basis

The definition of the fermionic basis in the CFT case can be considered as a purely algebraic one, that is why it is equally suitable for the sinh-Gordon case. Analytical advantage of using the fermionic basis in the sine-Gordon model is due to the fact that the expectation values of the elements of the fermionic basis are expressed as determinants. We do not know how to prove similar fact for the sinh-Gordon case, so, like in [24] we shall formulate it as a conjecture and then perform numerous checks.

The fermionic basis is created by the anti-commutative operators β^*, γ^* (and $\bar{\beta}^*, \bar{\gamma}^*$ for the other chirality). They can be used to generate the quotient space as

$$\beta^*_M \gamma^*_N \bar{\beta}^*_{\bar{M}} \bar{\gamma}^*_{\bar{N}} \Phi_\alpha = \beta^*_{m_1} \dots \beta^*_{m_k} \gamma^*_{n_1} \dots \gamma^*_{n_k} \bar{\beta}^*_{\bar{m}_1} \dots \bar{\beta}^*_{\bar{m}_{\bar{k}}} \bar{\gamma}^*_{\bar{n}_1} \dots \bar{\gamma}^*_{\bar{n}_{\bar{k}}} \Phi_\alpha \,,$$

with all modes being odd and positive. Later we shall have these operators for negative indices. By definition they are related to annihilation operators $\beta^*_{-j} = \gamma_j$, $\gamma^*_{-j} = \beta_j$ (together with similar relations for the other chirality) such that their anti-commutator is

$$\{\beta_m, \beta^*_n\} = \{\bar{\gamma}_m, \bar{\gamma}^*_n\} = -t_m(\alpha)\delta_{m,n} \quad ; \qquad t_n(\alpha) = \frac{1}{2\sin\pi(\alpha - np)} \,.$$

The relation between the fermionic basis and the Heisenberg or Virasoro basis is a very complicated problem and requires a case by case study. Later we shall have examples.

What is particularly nice about the fermionic basis is that the finite volume expectation values take a very simple determinant form. Indeed, the main result of our paper is a conjecture of the form

$$\frac{\langle \theta_1, \dots, \theta_m | \beta^*_M \gamma^*_N \bar{\beta}^*_{\bar{M}} \bar{\gamma}^*_{\bar{N}} \Phi_\alpha | \theta_m, \dots, \theta_1 \rangle_R}{\langle \theta_1, \dots, \theta_m | \Phi_\alpha | \theta_m, \dots, \theta_1 \rangle_R} = \mathcal{D}(\{M \cup (-\bar{M})\} | \{N \cup (-\bar{N})\} | \alpha) \,.$$

where for the index sets $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ the determinant is

$$\mathcal{D}(A|B|\alpha) = \prod_{j=1} \frac{\mathrm{sgn}(a_j)\mathrm{sgn}(b_j)}{\pi} \mathrm{Det}_{j,k}\left(\Omega_{a_j, b_k}\right) \quad ; \quad \Omega_{n,m} = \omega_{n,m} - \pi \mathrm{sgn}(n) \delta_{n,-m} t_n(\alpha) \,.$$

The construction of the matrix $||\omega_{m,n}||$ is explained in the next section.

3.2 The matrix $\omega_{m,n}$

The matrix $\omega_{m,n}$ is built via a deformation of a linear operator involved in the linearization of the TBA equations. We start by explaining this linearization. Consider the variation of the TBA equations (2.1),(2.2) with respect to r. The functions $\epsilon(\theta)$ and $f(\theta)$ depend actually on θ and r, while the points of the discrete spectrum θ_k depends on r. We have

$$\partial_r \epsilon(\theta) = \cosh\theta - 2\pi i \sum_{k=1}^{N} K(\theta - \theta_k + \tfrac{\pi i}{2}) \frac{d\theta_k}{dr} + \int_{-\infty}^{\infty} K(\theta - \theta') \partial_r \epsilon(\theta') \frac{1}{1 + e^{\epsilon(\theta')}} d\theta' \tag{3.1}$$

$$= \cosh\theta + 2\pi i \sum_{k=1}^{N} K(\theta - \theta_k + \tfrac{\pi i}{2}) \frac{1}{\partial_\theta f(\theta_k)} \partial_r f(\theta_k) + \int_{-\infty}^{\infty} K(\theta - \theta') \partial_r \epsilon(\theta') \frac{1}{1 + e^{\epsilon(\theta')}} d\theta' \,,$$

where we used that

$$\partial_r f(\theta_k) + \partial_\theta f(\theta_k) \frac{d\theta_k}{dr} = 0 \,,$$

following from (2.2).

Consider functions on discrete and continuous spectra $G = \{g_1, \cdots, g_k, g(\theta)\}$. Motivated by (3.1) we introduce paring for two such functions

$$G * H = 2\pi i \sum \frac{1}{\partial_\theta f(\theta_k)} g_k h_k + \int_{-\infty}^{\infty} g(\theta)h(\theta) \frac{d\theta}{1+e^{\epsilon(\theta)}} \,. \tag{3.2}$$

By using this convolution the matrix element $\omega_{n,m}$ entering in our conjecture can be written as

$$\omega_{n,m} = e_n * (1 + \mathcal{K}_\alpha + \mathcal{K}_\alpha * \mathcal{K}_\alpha + \dots) * e_m \equiv e_n * (1 + \mathcal{R}_{\text{dress},\alpha}) * e_m \,,$$

where $e_n = \{e^{n(\theta_1 + \frac{\pi i}{2})}, \dots, e^{n(\theta_m + \frac{\pi i}{2})}, e^{n\theta}\}$ and $\mathcal{K}_\alpha$ has a matrix structure

$$\mathcal{K}_\alpha = \begin{pmatrix} K_\alpha(\theta_k - \theta_l) & K_\alpha(\theta_k - \theta + \frac{\pi i}{2}) \\ K_\alpha(\theta - \theta_l - \frac{\pi i}{2}) & K_\alpha(\theta - \theta') \end{pmatrix} .$$

reflecting the fact that the convolution has a discrete and the continuous part. Here K_α is the deformation of the TBA kernel

$$K_\alpha(\theta) = \frac{1}{2\pi i}\Big(\frac{e^{-i\pi\alpha}}{\sinh(\theta - \pi i p)} - \frac{e^{i\pi\alpha}}{\sinh(\theta + \pi i p)}\Big) ,$$

which satisfy $K_{\alpha+2}(\theta) = K_\alpha(\theta)$.

Similar determinant expression to ours was proposed and tested for vacuum expectation values in [24]. Our formulae are the extensions of VEVs for excited states and the novel complication is the discrete part of the convolutions. In the next section we explain how to work with these expressions.

There was a nice observation in [22] that one might relax the condition that the number of β^* and γ^* are the same, but in the same time maintain the determinant form. By this way operators with different sectors can be connected as

$$\beta^*_M \gamma^*_N \bar{\beta}^*_{\bar{M}} \bar{\gamma}^*_{\bar{N}} \Phi_{\alpha+2mp} = \frac{C_m(\alpha)}{\prod_{j=1}^m t_{2j-1}(\alpha)} \beta^*_{M+2m} \gamma^*_{N-2m} \bar{\beta}^*_{\bar{M}-2m} \bar{\gamma}^*_{\bar{N}+2m} \beta^*_{\{m\}} \bar{\gamma}^*_{\{m\}} \Phi_\alpha \,.$$

where $\{m\} = 1, 3, \dots, 2m-1$ and $C_m(\alpha)$ is the ratio of the infinite volume vacuum expectation values [31]:

$$C_m(\alpha) = \frac{\langle \Phi_{\alpha-2mp} \rangle_\infty}{\langle \Phi_\alpha \rangle_\infty} \,.$$

The simplest of these relations is

$$\frac{\Phi_{\alpha-2p}}{\langle \Phi_{\alpha-2p} \rangle_\infty} = \frac{1}{t_1(\alpha)} \beta^*_1 \bar{\gamma}^*_1 \frac{\Phi_\alpha}{\langle \Phi_\alpha \rangle_\infty} \,.$$

This relation is understood in the weak sense, i.e. for matrix elements. In the next section we take diagonal matrix elements of this relation and compare its large volume expansion with Pozsgay's result [20].

4 Large volume checks

In this section we make some IR checks of our formulae for the diagonal finite volume matrix elements, which we normalize as

$$F(\theta_1, \dots, \theta_m | \alpha) = \frac{\langle \theta_1, \dots, \theta_m | \Phi_\alpha | \theta_m, \dots, \theta_1 \rangle_R}{\langle \Phi_\alpha \rangle_\infty} \,.$$

Reflection properties with σ_1 and σ_2 ensure the invariance under the $\alpha \to \alpha+2$ shift. The finite volume state $|\theta_m,\dots,\theta_1\rangle_R$ is symmetric in the rapidity variables, which satisfy the BA equations $f(\theta_k)=\pi N_k$. These states can be labelled either by the discrete quantum numbers N_k or by the rapidities θ_k and are naturally normalized to Kronecker delta functions. In the following we investigate the simplest non-trivial example

$$\frac{F(\theta_1,\dots,\theta_m|\alpha-2p)}{F(\theta_1,\dots,\theta_m|\alpha)}=1+\frac{2}{\pi}\sin\pi(p-\alpha)(e_1*e_{-1}+e_1*\mathcal{R}_{\text{dress},\alpha}*e_{-1})\,. \tag{4.1}$$

where e_n is related to $e^{n\theta}$. For each function g we have a discrete and a continues part: $(g_1,\dots,g_m,g(\theta))$ with $g_j=g(\theta_j+i\frac{\pi}{2})$ and the convolution is understood as in (3.2). We would like to compare these formulae with the available results in literature which we recall now.

4.1 Form factor expansion of the diagonal finite volume matrix elements

A *finite* volume diagonal form factor can be expressed in terms of the *infinite* volume connected form factors, which are defined to be the finite (ϵ-independent) part in the crossed expression

$$F_{2n}^{\mathcal{O}}(\theta_1+i\pi+\epsilon_1,\dots,\theta_m+i\pi+\epsilon_m,\theta_m,\dots,\theta_1)=\frac{O(\epsilon^m)}{\epsilon_1\dots\epsilon_m}+F_{2n,c}^{\mathcal{O}}(\theta_1,\dots,\theta_m)+O(\epsilon)\,.$$

These connected form factors have interesting properties [21]. For the exponential operators, normalized by the VEV's, $\Phi_\alpha/\langle\Phi_\alpha\rangle_\infty$, the first two connected form factors read as [18, 24]:

$$F_{2,c}^{\alpha}=4\sin(\pi p)[k_\alpha]^2\,,$$
$$F_{4,c}^{\alpha}(\theta_1,\theta_2)=4\pi F_{2,c}^{\alpha}K(\theta_1-\theta_2)\Big(\cosh(\theta_1-\theta_2)[k_\alpha]^2-\frac{[k_\alpha-1][k_\alpha+1]}{\cosh(\theta_1-\theta_2)}\Big)\,,$$

where

$$[k]=\frac{\sin(\pi pk)}{\sin(\pi p)}\quad;\qquad k_\alpha=\frac{\alpha}{2p}\,.$$

The six particle connected form factor would fill a half page and there is no closed form available for the general case. In principle such expressions could be obtained from the determinant representation of form factors [33] using the limiting behavior of the symmetric polynomials [21]. But this procedure is quite cumbersome and the results do not seem to have any nice structure, thus we restrict our investigations for these first two form factors only.

The exact formula for the finite volume diagonal matrix elements was conjectured in [20] based on carefully evaluating the contour deformation trick in the LM formula for 1 and 2 particle states. For m particles the conjecture takes the form

$$F(\theta_1,\dots,\theta_m|\alpha)=\frac{\sum_{I\subseteq M}\mathcal{F}_{m-|I|}^{\alpha}(M\setminus I)\rho_{|I|}(I)}{\rho_m(M)}\,, \tag{4.2}$$

where the full index set is denoted by $M=\{1,2,\dots,m\}$, and an index set $I=\{i_1,\dots,i_k\}$ in the argument abbreviates the set of rapidities $\theta_{i_1},\dots,\theta_{i_k}$. The appearing densities $\rho_{|I|}(I)$ are defined to be the determinant

$$\rho_k(\theta_{i_1},\dots,\theta_{i_k})=\det_{j,l}\Big|\partial_{\theta_{i_j}}f(\theta_{i_l})\Big|\,.$$

For $I=M$ this is simply the density of the finite volume m-particle states. The quantity $\mathcal{F}_k^{\alpha}$ is the generalization of the LM expansion for the connected form factor $F_{2k,c}^{\alpha}$:

$$\mathcal{F}_k^{\alpha}(\theta_1,\dots,\theta_k)=\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int\frac{dm(v_j)}{2\pi}F_{2(k+n),c}^{\alpha}(\theta_1+\tfrac{\pi i}{2},\dots,\theta_k+\tfrac{\pi i}{2},v_1,\dots,v_n)\,,$$

where $dm(v)=\frac{dv}{1+e^{\epsilon(v)}}$.

4.2 Checks at polynomial order

The finite volume diagonal form factor at any polynomial order in the inverse of the volume can be obtained from (4.2) by neglecting the integral terms both in f and also in $\mathcal{F}$. At this order

$$\mathcal{F}_k^\alpha(\theta_1,\dots,\theta_k)=F_{2k,c}^\alpha(\theta_1,\dots,\theta_k)+O(e^{-r})\,,$$

and

$$\rho_k(\theta_{i_1},\dots,\theta_{i_k})=\det_{j,l}\left|\delta_{j,l}(r\cosh\theta_j+2\pi\sum_{n=1}^m K(\theta_{i_j}-\theta_n))-2\pi K(\theta_{i_j}-\theta_{i_l})\right|+O(e^{-r})\,,$$

This asymptotic expression was conjectured in [34] based on form factor perturbation theory and later proved in [35], moreover, it leads to the proof of the LM series, whose analytic continuation provided the exact conjecture (4.2). In the following we recover these asymptotic results from our fermionic expression (4.1). We proceed in the particle number.

4.2.1 1-particle case

We need to check the relation

$$\frac{F(\theta_1|\alpha-2p)}{F(\theta_1|\alpha)}=\frac{F_{2,c}^{\alpha-2p}+r\cosh\theta_1}{F_{2,c}^\alpha+r\cosh\theta_1}+O(e^{-r})= \qquad (4.3)$$
$$=1+\frac{2}{\pi}\sin\pi(p-\alpha)(e_1*e_{-1}+e_1*\mathcal{R}_{\mathrm{dress},\alpha}*e_{-1})\,,$$

where e_1 represents one discrete particle with θ_1 and the function e^{v_1} as $(e^{\theta_1+\frac{i\pi}{2}},e^{v_1})$. Thus the first convolution in (3.2) explicitly reads as

$$e_1*e_{-1}=\frac{2\pi}{f'(\theta_1)}+\int dm(v_1)=\frac{2\pi}{r\cosh\theta_1+2\pi K(0)}+O(e^{-r})\,, \qquad (4.4)$$

where we evaluated it neglecting exponentially small terms. Clearly, each convolution in the discrete part introduces a polynomial suppression factor r^{-1} while in the continuous part an exponential one e^{-r}. In order to obtain all polynomial volume corrections we need to sum up the iterated series in the discrete part

$$e_1*e_{-1}+e_1*\mathcal{R}_{\mathrm{dress},\alpha}*e_{-1}=e_1*(1+K_\alpha(0)+K_\alpha(0)*K_\alpha(0)+\dots)*e_{-1} \qquad (4.5)$$
$$=\frac{2\pi}{f'(\theta_1)}\left(1+\sum_{n=1}^\infty\frac{(2\pi K_\alpha(0))^n}{f'(\theta_1)^n}\right)=\frac{2\pi}{f'(\theta_1)-2\pi K_\alpha(0)}$$
$$=\frac{2\pi}{r\cosh\theta_1+2\pi(K(0)-K_\alpha(0))}+O(e^{-r})\,.$$

The relation $2\pi(K(0)-K_\alpha(0))=F_{2,c}^\alpha$ together with $F_{2,c}^{\alpha-2p}-F_{2,c}^\alpha=4\sin(\pi(p-\alpha))$ imply that the two forms (4.3) and (4.5) are indeed equivalent.

4.2.2 2-particle case

For two particles the form factor expression, neglecting exponential corrections, gives

$$\frac{F(\theta_1,\theta_2|\alpha-2p)}{F(\theta_1,\theta_2|\alpha)}=\frac{F_{4,c}^{\alpha-2p}(\theta_1,\theta_2)+F_{2,c}^{\alpha-2p}(\rho_1(\theta_1)+\rho_1(\theta_2))+\rho_2(\theta_1,\theta_2)}{F_{4,c}^{\alpha}(\theta_1,\theta_2)+F_{2,c}^{\alpha}(\rho_1(\theta_1)+\rho_1(\theta_2))+\rho_2(\theta_1,\theta_2)}\,. \qquad (4.6)$$

In the fermionic formulation we consider only the discrete part, thus in the convolution e_1 represents $i(e^{\theta_1},e^{\theta_2})$, while e_{-1} is nothing but $-i(e^{-\theta_1},e^{-\theta_2})$. The kernel is a 2×2 matrix: $(\hat{K}_\alpha)_{ij}\equiv K_\alpha(\theta_i-\theta_j)$.

The convolution in the discrete part can be traded for ordinary matrix multiplication by introducing an extra matrix factor[2] $(\hat{f})_{ij}=\delta_{ij}f'(\theta_j)$. As a result we obtain

$$\frac{F(\theta_1,\theta_2|\alpha-2p)}{F(\theta_1,\theta_2|\alpha)}=1+4\sin\pi(p-\alpha)(e^{\theta_1},e^{\theta_2})(\hat{f}-2\pi\hat{K}_\alpha)^{-1}\begin{pmatrix} e^{-\theta_1} \\ e^{-\theta_2}\end{pmatrix}, \tag{4.7}$$

where exponential corrections are neglected, but all polynomial corrections are summed up. We have checked explicitly that this result (4.7) agrees with the form factor description (4.6).

4.2.3 m-particle case

Similar calculation can be repeated for the generic m-particle case. Now $\hat{K}_\alpha$ and $\hat{f}$ are $m\times m$ matrices with entries

$$(\hat{K}_\alpha)_{ij}=K_\alpha(\theta_i-\theta_j)\quad;\qquad (\hat{f})_{ij}=\delta_{ij}(r\cosh\theta_j+2\pi\sum_{k=1}^{m}K(\theta_j-\theta_k)),$$

leading to the analogous formula

$$\frac{F(\theta_1,\dots,\theta_m|\alpha-2p)}{F(\theta_1,\dots,\theta_m|\alpha)}=1+4\sin\pi(p-\alpha)(e^{\theta_1},\dots,e^{\theta_m})(\hat{f}-2\pi\hat{K}_\alpha)^{-1}\begin{pmatrix} e^{-\theta_1} \\ \vdots \\ e^{-\theta_m}\end{pmatrix}.$$

Since higher than two-particle connected form factors are very complicated we did not check explicitly this result, although we have no doubts about its correctness. However, we would like to point out that substituting $r=0$ in the formula provides a very compact and simple expression for the ratios of diagonal matrix elements. These are actually nothing but the symmetric evaluations of the form factors [34]. We believe that this observation could be used to find some nice parametrization of these form factors in the generic case.

4.3 Checks at the leading exponential order

We now check the leading exponential correction for the simplest 1-particle form factor

$$\frac{F(\theta_1|\alpha-2p)}{F(\theta_1|\alpha)}=\frac{F_{2,c}^{\alpha-2p}+\rho_1(\theta_1)+\int\frac{dm(v_1)}{2\pi}(F_{4,c}^{\alpha-2p}(\theta_1+\frac{i\pi}{2},v_1)+\rho_1(\theta_1)F_{2,c}^{\alpha-2p})}{F_{2,c}^{\alpha}+\rho_1(\theta_1)+\int\frac{dm(v_1)}{2\pi}(F_{4,c}^{\alpha}(\theta_1+\frac{i\pi}{2},v_1)+\rho_1(\theta_1)F_{2,c}^{\alpha})}+O(e^{-2r}), \tag{4.8}$$

where we also need to expand $\rho_1(\theta_1)$. In doing so we recall that

$$\rho_1(\theta_1)=\partial_{\theta_1}f(\theta_1)=r\cosh\theta_1-i\int dm(\theta)K(\theta_1+i\frac{\pi}{2}-\theta)\Big(\frac{\partial\epsilon(\theta)}{\partial\theta}+\frac{\partial\epsilon(\theta)}{\partial\theta_1}\Big).$$

By differentiating the TBA equation wrt. to both θ_1 and θ we obtain linear integral equations with solutions

$$\frac{\partial\epsilon(\theta)}{\partial\theta}=r\sinh\theta+2\pi iK(\theta-\frac{i\pi}{2}-\theta_1)+\int dm(v)R_{\rm dress}(\theta-v)(r\sinh v+2\pi iK(v-\frac{i\pi}{2}-\theta_1)), \tag{4.9}$$

$$\frac{\partial\epsilon(\theta)}{\partial\theta_1}=-2\pi iK(\theta-\frac{i\pi}{2}-\theta_1)-2\pi i\int dm(v)R_{\rm dress}(\theta-v)K(v-\frac{i\pi}{2}-\theta_1)), \tag{4.10}$$

[2] Note that $f'(\theta_j)-\partial_{\theta_j}f(\theta_j)=2\pi K(0)+O(e^{-r})$.

where the resolvent $R_{\rm dress}$ satisfies the equation

$$R_{\rm dress}(\theta)-\int dm(v)R_{\rm dress}(\theta-v)K(v)=K(\theta)\,.$$

Thus at the leading exponential order

$$\rho_1(\theta_1)=r\Big(\cosh\theta_1-i\int dm(\theta)K(\theta_1+i\frac{\pi}{2}-\theta)\sinh\theta\Big)+O(e^{-2r})\,.$$

This allows us to expand the denominator and keep only the leading exponential piece in order to compare with the formula coming from the fermionic description (4.1).

Evaluating the leading piece in the fermionic formula provides (4.4). To get the remaining terms we sum up the iterative terms. Keeping in mind that e_1 represents the discrete and the continuous parts (ie^{θ_1},e^{v_1}) the k^{th} convolution gives

$$\begin{aligned}e_1*\mathcal{K}_\alpha*\cdots*\mathcal{K}_\alpha e_{-1}=&\frac{2\pi}{f'(\theta_1)}\frac{(2\pi K_\alpha(0))^{k-2}}{f'(\theta_1)^{k-2}}\Bigg(\frac{(2\pi K_\alpha(0))^2}{f'(\theta_1)^2}+\frac{2\pi K_\alpha(0))}{f'(\theta_1)}\times\\&i\int dm(v_1)(K_\alpha(\theta_1-v_1+\frac{i\pi}{2})e^{\theta_1-v_1}-K_\alpha(v_1-\theta_1-\frac{i\pi}{2})e^{v_1-\theta_1})\\&+(k-1)\frac{(2\pi)}{f'(\theta_1)}\int dm(v_1)K_\alpha(\theta_1-v_1+\frac{i\pi}{2})K_\alpha(v_1-\theta_1-\frac{i\pi}{2})\Bigg)\,,\end{aligned}\tag{4.11}$$

where we kept only terms with at most one continuous convolution. We need to sum the first line from $k=0$, the second from $k=1$, while the last from $k=2$ to infinity. Also there is one more convolution from (4.4). Let us recall that

$$f'(\theta_1)=r\cosh\theta_1+2\pi K(0)-i\int dm(\theta)K(\theta_1+\frac{i\pi}{2}-\theta)\frac{\partial\epsilon(\theta)}{\partial\theta}\,,$$

with the solution given by (4.9). Expanding this formula up to the leading exponential order and plugging back to the expressions summed up agrees with (4.8). Let us emphasize that to obtain the leading exponential contribution we need to sum up infinitely many terms in the discrete parts. Thus the agreement found is a highly non-trivial test of our approach.

5 Small volume checks

For small volume we compare the ratios of the expectation values to the ratios of three-point functions in the Liouville conformal field theory in a cylindrical geometry shown on Figure 1.

The general three-point function in the CFT takes the form

$$\langle\Delta_+|\mathcal{O}_\alpha(0)|\Delta_-\rangle=\langle\Delta|L_{n_1}\dots L_{n_i}(\mathbf{l}_{-m_1}\dots\mathbf{l}_{-m_j}\Phi_\alpha)L_{-p_1}\dots L_{-p_r}|\Delta\rangle\,,\tag{5.1}$$

where two different Virasoro modes are introduced. Both are related to the same energy momentum tensor, but expanded around different points.

Let us introduce a complex coordinate on the cylinder as $z=x+iy$ with $y\equiv y+2\pi$. By expanding $T(z)$ around the origin we can act and change the operator, which is inserted. This action is called the *local* action:

$$T(z)=\sum_{n=-\infty}^{\infty}\mathbf{l}_nz^{-n-2}\quad;\qquad\mathbf{l}_n\Phi_\alpha=\oint\frac{dz}{2\pi i}z^{n+1}T(z)\Phi_\alpha\,.$$

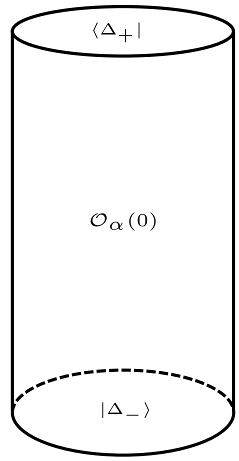

Figure 1: Cylindrical geometry for the conformal three-point functions.

For diagonal matrix elements, i.e. for expectation values, only even mode numbers are used to generate the quotient space, where the action of the conserved charges is factored out.

By expanding $T(z)$ at $z \to \pm\infty$ we obtain the *global* action of the Virasoro algebra which can alter the initial and final states:

$$T(z) = \sum_{n=-\infty}^{\infty} L_n e^{nz} - \frac{c}{24}\,.$$

In order to relate the three-point function of the descendants (5.1) to that of the primary $\langle\Delta|\Phi_\alpha|\Delta\rangle \equiv \langle\Phi_\alpha\rangle_\Delta$ we use the cylinder conformal Ward identities:

$$\begin{aligned}\langle T(z_k)\cdots T(z_1)\Phi_\alpha\rangle_\Delta = &-\frac{c}{12}\sum_{j=2}^{k}\chi'''(z_1-z_j)\langle T(z_k)\cdots \overset{j}{\wedge}\cdots T(z_2)\Phi_\alpha\rangle_\Delta \\ &+\Big\{\sum_{j=2}^{k}\big(-2\chi'(z_1-z_j)+(\chi(z_1-z_j)-\chi(z_1))\frac{\partial}{\partial z_j}\big) \\ &\qquad -\Delta_\alpha\chi'(z_1)+\Delta-\frac{c}{24}\Big\}\langle T(z_k)\cdots T(z_2)\Phi_\alpha\rangle_\Delta\end{aligned} \tag{5.2}$$

where $\chi(z) = \frac{1}{2}\coth\left(\frac{z}{2}\right)$.

In calculating (5.1) we follow the prescription of [36]: we first take $k = i + j + r$ and send $z_1, \dots, z_i$ to $-\infty$, $z_{i+1}, \dots, z_{i+j}$ to 0, while $z_{i+j+1}, \dots, z_k$ to ∞. By picking up the coefficient of the appropriate power of $e^{\pm z}$ at $\mp\infty$ and z around 0 the three-point function (5.1) can be calculated. In the following we first analyze non-degenerate L_0 subspaces, i.e. highest weight states and their first descendants, and then level 2 states.

5.1 Non-degenerate L_0 eigenspaces

We perform this analysis for the low lying operators and states with a non-degenerate L_0. This includes the state $|\Delta\rangle$ and $|\Delta + 1\rangle \equiv L_{-1}|\Delta\rangle$, thus from the table (2.7) we take all rows with $L = 1, 2, 3, 4$ and

$M = 0, 1, \bar{M} = 0$. The computation using Ward identities provides

$$\frac{\langle \mathbf{l}_{-2}\Phi_\alpha\rangle_\Delta}{\langle\Phi_\alpha\rangle_\Delta} = \Delta - \frac{c}{24} - \frac{\Delta_\alpha}{12}\,, \tag{5.3}$$
$$\frac{\langle \mathbf{l}_{-4}\Phi_\alpha\rangle_\Delta}{\langle\Phi_\alpha\rangle_\Delta} = \frac{\Delta_\alpha}{240}\,,$$
$$\frac{\langle \mathbf{l}_{-2}^2\Phi_\alpha\rangle_\Delta}{\langle\Phi_\alpha\rangle_\Delta} = \Delta^2 - \Delta\frac{2\Delta_\alpha + c + 2}{12} + \frac{20\Delta_\alpha^2 + 56\Delta_\alpha + 20c\Delta_\alpha + 5c^2 + 22c}{2880}\,.$$

$$\frac{\langle\Phi_\alpha\rangle_{\Delta+1}}{\langle\Phi_\alpha\rangle_\Delta} = 2\Delta + \Delta_\alpha^2 - \Delta_\alpha\,, \tag{5.4}$$
$$\frac{\langle \mathbf{l}_{-2}\Phi_\alpha\rangle_{\Delta+1}}{\langle\Phi_\alpha\rangle_\Delta} = 2\Delta^2 + \Delta\frac{12\Delta_\alpha^2 + 34\Delta_\alpha + 24 - c}{12} - \frac{(\Delta_\alpha - 1)\Delta_\alpha(2\Delta_\alpha - 24 + c)}{24}\,,$$
$$\frac{\langle \mathbf{l}_{-4}\Phi_\alpha\rangle_{\Delta+1}}{\langle\Phi_\alpha\rangle_\Delta} = \Delta\frac{241\Delta_\alpha}{120} - \frac{\Delta_\alpha^2}{240} + \frac{\Delta_\alpha^3}{240}\,,$$
$$\frac{\langle \mathbf{l}_{-2}^2\Phi_\alpha\rangle_{\Delta+1}}{\langle\Phi_\alpha\rangle_\Delta} = 2\Delta^3 + \Delta^2\frac{70 - c + 40\Delta_\alpha + 6\Delta_\alpha^2}{6}$$
$$+ \Delta\frac{2400 - 218c + 5c^2 + 4616\Delta_\alpha - 340c\Delta_\alpha + 1940\Delta_\alpha^2 - 120c\Delta_\alpha^2 - 240\Delta_\alpha^3}{1440}$$
$$+ \frac{(\Delta_\alpha - 1)\Delta_\alpha(2400 - 218c + 5c^2 - 424\Delta_\alpha + 20c\Delta_\alpha + 20\Delta_\alpha^2)}{2880}\,.$$

where the central charge and the scaling dimensions of the operator and of the asymptotical state are

$$c = 1 + 6Q^2, \quad \Delta_\alpha = \frac{Q^2}{4}\alpha(2 - \alpha), \quad \Delta = \frac{P^2}{2} + \frac{Q^2}{4}\,. \tag{5.5}$$

We also need the ratio of the three-point functions for the primary fields:

$$\frac{\langle\Phi_{\alpha-2p}\rangle_\Delta}{\langle\Phi_\alpha\rangle_\Delta} = \frac{\gamma^2(ab - b^2)}{\gamma(2ab - 2b^2)\gamma(2ab - b^2)}\gamma(ab - b^2 - 2ibP)\gamma(ab - b^2 + 2ibP)\,; \quad a = \frac{\alpha Q}{2}\,. \tag{5.6}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$, and we used notations close to [30] in order to simplify comparison.

Using the results of [32, 22] we can relate the fermionic basis to the low lying Virasoro descendants as

$$\Omega_{1,1} \simeq r^{-2} D_1(\alpha, p) D_1(2 - \alpha, p)\frac{\langle \mathbf{l}_{-2}\Phi_\alpha\rangle}{\langle\Phi_\alpha\rangle}\,, \tag{5.7}$$
$$\Omega_{3,1} \simeq r^{-4}\frac{1}{2} D_3(\alpha, p) D_1(2 - \alpha, p)\Big\{\frac{\langle \mathbf{l}_{-2}^2\Phi_\alpha\rangle}{\langle\Phi_\alpha\rangle} + \Big(\frac{2c - 32}{9} + \frac{2}{3}d(\alpha, p)\Big)\frac{\langle \mathbf{l}_{-4}\Phi_\alpha\rangle}{\langle\Phi_\alpha\rangle}\Big\}\,, \tag{5.8}$$
$$\Omega_{1,3} \simeq r^{-4}\frac{1}{2} D_1(\alpha, p) D_3(2 - \alpha, p)\Big\{\frac{\langle \mathbf{l}_{-2}^2\Phi_\alpha\rangle}{\langle\Phi_\alpha\rangle} + \Big(\frac{2c - 32}{9} - \frac{2}{3}d(\alpha, p)\Big)\frac{\langle \mathbf{l}_{-4}\Phi_\alpha\rangle}{\langle\Phi_\alpha\rangle}\Big\}\,, \tag{5.9}$$
$$\Omega_{1,-1} \simeq r^{2(\Delta_\alpha - \Delta_{\alpha-b})} t_1(\alpha) F(\alpha, p)\frac{\langle\Phi_{\alpha-2p}\rangle}{\langle\Phi_\alpha\rangle}\,, \tag{5.10}$$

where

$$d(\alpha, p) = \frac{2p - 1}{p(p - 1)}(\alpha - 1)\,,$$

and the expectation values are taken in the finite volume eigenstate of the conserved charges. The appearing coefficients for descendants originate from the normalization of the fermionic operators

$$D_m(\alpha, p) = \frac{1}{2i\sqrt{\pi}} Z(p)^{-m}\Gamma\Big(\frac{\alpha + mp}{2}\Big)\Gamma\Big(\frac{\alpha + m(1 - p)}{2}\Big)\,, \tag{5.11}$$

while for primaries they are essentially the ratio of two Lukyanov-Zamolodchikov one-point functions

$$F(\alpha,p)=Z(p)^{2(\Delta_\alpha-\Delta_{\alpha-2p})}\frac{2}{1-p}\,\gamma\Big(\frac{\alpha+1-p}{2}\Big)\gamma\Big(\frac{2-\alpha+p}{2}\Big)\gamma\Big(\frac{\alpha-p}{1-p}\Big)\,, \tag{5.12}$$

For asymptotical states we consider either primary fields parametrized by the quantum number L or their first descendants. For the primary fields the formulae above are taken literally. For the descendants we have to use the formulae (5.3) carefully as, for instance,

$$\begin{aligned}\frac{\langle \mathbf{l}_{-2}\Phi_\alpha\rangle_{\Delta+1}}{\langle\Phi_\alpha\rangle_{\Delta+1}} &= \frac{\langle \mathbf{l}_{-2}\Phi_\alpha\rangle_{\Delta+1}}{\langle\Phi_\alpha\rangle_{\Delta}}\frac{\langle\Phi_\alpha\rangle_{\Delta}}{\langle\Phi_\alpha\rangle_{\Delta+1}}\\ &= \frac{48\Delta^2+2\Delta(12\Delta_\alpha^2+34\Delta_\alpha+24-c)-(\Delta_\alpha-1)\Delta_\alpha(2\Delta_\alpha-24+c)}{24(2\Delta+\Delta_\alpha^2-\Delta_\alpha)}\,.\end{aligned} \tag{5.13}$$

In the Table 1 we compare the numerical values of $\Omega_{i,j}$ obtained for

$$r=.001,\quad a=\frac{87}{80},\quad b=\frac{2}{5}\,, \tag{5.14}$$

to their CFT limits.

state	$M=0$	$L=1$	$M=1$	$L=1$
	numerical	CFT	numerical	CFT
$\Omega_{1,1}$	$3.85677\cdot 10^6$	$3.85677\cdot 10^6$	$-6.60202\cdot 10^7$	$-6.60203\cdot 10^7$
$\Omega_{3,1}$	$1.00405\cdot 10^{14}$	$1.00405\cdot 10^{14}$	$1.07476\cdot 10^{16}$	$1.07475\cdot 10^{16}$
$\Omega_{1,3}$	$1.04361\cdot 10^{14}$	$1.04361\cdot 10^{14}$	$1.05988\cdot 10^{16}$	$1.05987\cdot 10^{16}$
$\Omega_{1,-1}$	-0.0028363	-0.0028363	-0.00231607	-0.00231668

state	$M=0$	$L=2$	$M=1$	$L=2$
	numerical	CFT	numerical	CFT
$\Omega_{1,1}$	$3.79053\cdot 10^6$	$3.79053\cdot 10^6$	$-6.61159\cdot 10^7$	$-6.61132\cdot 10^7$
$\Omega_{3,1}$	$9.93725\cdot 10^{13}$	$9.93725\cdot 10^{13}$	$1.0771\cdot 10^{16}$	$1.07703\cdot 10^{16}$
$\Omega_{1,3}$	$1.03188\cdot 10^{14}$	$1.03188\cdot 10^{14}$	$1.06245\cdot 10^{16}$	$1.06237\cdot 10^{16}$
$\Omega_{1,-1}$	-0.00276414	-0.00276451	-0.00225609	-0.00225862

state	$M=0$	$L=3$	$M=1$	$L=3$
	numerical	CFT	numerical	CFT
$\Omega_{1,1}$	$3.68203\cdot 10^6$	$3.68197\cdot 10^6$	$-6.62725\cdot 10^7$	$-6.62653\cdot 10^7$
$\Omega_{3,1}$	$9.77084\cdot 10^{13}$	$9.77074\cdot 10^{13}$	$1.08094\cdot 10^{16}$	$1.08076\cdot 10^{16}$
$\Omega_{1,3}$	$1.01297\cdot 10^{14}$	$1.01296\cdot 10^{14}$	$1.06668\cdot 10^{16}$	$1.06648\cdot 10^{16}$
$\Omega_{1,-1}$	-0.00265524	-0.00265529	-0.00216529	-0.0021703

state	$M=0$	$L=4$	$M=1$	$L=4$
	numerical	CFT	numerical	CFT
$\Omega_{1,1}$	$3.53306\cdot 10^6$	$3.53306\cdot 10^6$	$-6.64869\cdot 10^7$	$-6.64737\cdot 10^7$
$\Omega_{3,1}$	$9.54791\cdot 10^{13}$	$9.5479\cdot 10^{13}$	$1.08622\cdot 10^{16}$	$1.08589\cdot 10^{16}$
$\Omega_{1,3}$	$9.87646\cdot 10^{13}$	$9.87645\cdot 10^{14}$	$1.07248\cdot 10^{16}$	$1.07212\cdot 10^{16}$
$\Omega_{1,-1}$	-0.00251988	-0.00252043	-0.00205411	-0.00206125

Table 1: We calculate numerically $\Omega_{i,j}$ for various states and compare them to their exact conformal counterparts.

5.2 Checks with degenerate L_0 spaces

Here we consider the simplest case of degeneracy: level 2. We work in the basis:

$$L_{-2}|\Delta\rangle, \quad L_{-1}^2|\Delta\rangle\,. \tag{5.15}$$

There are two eigenvectors of the local integrals of motion. Since $I_1 = L_0 - c/24$ does not distinguish between them we consider the next conserved charge:

$$I_3 = 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880}\,. \tag{5.16}$$

This integral of motion is a 2×2 matrix in the basis above, with eigenvalues

$$\lambda_\pm(\Delta) = \frac{17}{3} + \frac{c(5c+982)}{2880} - \frac{c-142}{12}\Delta + \Delta^2 \pm \frac{1}{2}\sqrt{288\Delta + (c-4)^2}\,, \tag{5.17}$$

and eigenvectors

$$\psi_\pm = \begin{pmatrix} \frac{1}{12}c - 4 \pm \sqrt{288\Delta + (c-4)^2} \\ 1 \end{pmatrix} \tag{5.18}$$

For simplicity we consider $L = 1$. In table (2.7) we present two cases with $L = 1, M = 2, \bar{M} = 0$: $\{1,3\}$, $\{4\}$. We first identify which one corresponds to λ_+ and which one to λ_-. In doing so we recall the general eigenvalue of the local integral of motion:

$$I_n(r) = \frac{1}{C_n(p)}\Big(-\frac{1}{n}\sum_{j=1}^{m} e^{n\theta_k} + (-1)^{\frac{n-1}{2}}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{n\theta}\log\Big(1+e^{-\epsilon(\theta)}\Big)d\theta\Big)\,, \tag{5.19}$$

where

$$C_n(p) = -\frac{Z(p)^{-n}}{4\sqrt{\pi}Q\frac{n+1}{2}!}\Gamma(np)\Gamma(n(1-p))\,. \tag{5.20}$$

The normalized eigenvalue $\tilde{I}_n(r) = r^n I_n(R)$ should approach the CFT limit. For $r = 10^{-3}$ we obtained the following numerical results:

$$\begin{aligned} \mathcal{N}_- &= \{1,3\}\,, \quad \tilde{I}_3(r) = 21.3773\,, \quad \lambda_-(\Delta(r)) = 21.3767\,, \\ \mathcal{N}_+ &= \{4\}\,, \quad\;\; \tilde{I}_3(r) = 74.8405\,, \quad \lambda_+(\Delta(r)) = 74.8399\,. \end{aligned} \tag{5.21}$$

which establishes the required correspondence.

For any local operator $\mathcal{O}$ we introduce a 2×2 matrix, which contains its matrix elements in the basis (5.15). We denote this matrix by $\frac{\langle \mathbf{l}_{-2}O\rangle_{\Delta+2}}{\langle\Phi_\alpha\rangle_\Delta}$. We need the following two cases

$$\frac{\langle\Phi_\alpha\rangle_{\Delta+2}}{\langle\Phi_\alpha\rangle_\Delta} = \begin{pmatrix} 4\Delta - 4\Delta_\alpha + 4\Delta_\alpha^2 + \frac{c}{2} & 2(3\Delta - \Delta_\alpha + \Delta_\alpha^3) \\ 2(3\Delta - \Delta_\alpha + \Delta_\alpha^3) & 8\Delta^2 + \Delta(4 - 8\Delta_\alpha + 8\Delta_\alpha^2) - 2\Delta_\alpha + 3\Delta_\alpha^2 - 2\Delta_\alpha^3 + \Delta_\alpha^4 \end{pmatrix}. \tag{5.22}$$

and

$$\frac{\langle \mathbf{l}_{-2}\Phi_\alpha\rangle_{\Delta+2}}{\langle\Phi_\alpha\rangle_\Delta} = \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{1,2} & M_{2,2}\end{pmatrix},$$

with entries

$$M_{1,1} = \frac{1}{48}\big(48c - c^2 - 672\Delta_\alpha + 102c\Delta_\alpha + 976\Delta_\alpha^2 - 8c\Delta_\alpha^2 - 16\Delta_\alpha^3 + \Delta(384 + 16c + 560\Delta_\alpha + 192\Delta_\alpha^2) + 192\Delta^2\big)\,,$$

$$M_{1,2} = \frac{1}{12}\big(-72\Delta_\alpha + 7c\Delta_\alpha + 14\Delta_\alpha^2 + 6c\Delta_\alpha^2 + 84\Delta_\alpha^3 - c\Delta_\alpha^3 - 2\Delta_\alpha^4 + \Delta(144 - 3c + 258\Delta_\alpha + 144\Delta_\alpha^2 + 24\Delta_\alpha^3) + 72\Delta^2\big)\,,$$

$$M_{2,2} = \frac{1}{24}\big(-96\Delta_\alpha + 2c\Delta_\alpha + 52\Delta_\alpha^2 - 3c\Delta_\alpha^2 - 6\Delta_\alpha^3 + 2c\Delta_\alpha^3 + 52\Delta_\alpha^4 - c\Delta_\alpha^4 - 2\Delta_\alpha^5 + \Delta(192 - 4c + 232\Delta_\alpha + 8c\Delta_\alpha + 568\Delta_\alpha^2 - 8c\Delta_\alpha^2 + 128\Delta_\alpha^3 + 24\Delta_\alpha^4) + \Delta^2(480 - 8c + 560\Delta_\alpha + 192\Delta_\alpha^2) + 192\Delta^3\big)\,.$$

We now rewrite the general formulae (5.7), (5.10) for the present case

$$\Omega^{\pm}_{1,1} \simeq r^{-2} D_1(\alpha,p) D_1(2-\alpha,p) \frac{\psi^t_\pm \cdot \langle \mathbf{l}_{-2}\Phi_\alpha\rangle_{\Delta+2} \cdot \psi_\pm}{\psi^t_\pm \cdot \langle \Phi_\alpha\rangle_{\Delta+2} \cdot \psi_\pm}\,,$$

$$\Omega^{\pm}_{1,-1} \simeq r^{2(\Delta_\alpha - \Delta_{\alpha-2p})} t_1(a,b) F(\alpha,p) \frac{\psi^t_\pm \cdot \langle \Phi_{\alpha-2p}\rangle_{\Delta+2} \cdot \psi_\pm}{\psi^t_\pm \cdot \langle \Phi_\alpha\rangle_{\Delta+2} \cdot \psi_\pm}\,.$$

We compute these quantities at the numerical values (5.14). The results are summarized in the table

eigenvalue	Ω^-	CFT	Ω^+	CFT
$\Omega_{1,1}$	$-1.08278 \cdot 10^8$	$-1.08276 \cdot 10^8$	$-1.88722 \cdot 10^8$	$-1.88716 \cdot 10^8$
$\Omega_{1,-1}$	-0.00210992	-0.00211103	-0.00245252	-0.00245289

Thus we see that our procedure works well in the case with degenerate L_0, too. This completes the small volume check of our conjecture.

6 Conclusions

We conjectured compact expressions for the finite volume diagonal matrix elements of exponential operators and their descendants in the sinh-Gordon theory. By using the fermionic basis to create the descendant operators we could relate their finite volume expectation values to that of the primaries in terms of a determinant with entries, which satisfies a linear integral equation. Careful choice of the fermionic creation operators can relate the matrix elements of two different exponential operators allowing, in principle, they complete determination. The linear integral equation contains a measure, which is built up from the pseudo-energy of the excited state TBA equations and a kernel, which is a deformation of the TBA kernel. Excited states are characterized by the discrete rapidities of the particles and the continuous pseudo-energy and the two parts are connected by the TBA and BA equations. They both appear in the linear integral equations, which can be solved by iterations. The discrete part is responsible for the polynomial finite size corrections, while the continuous part for the exponentially small ones. We checked for low number of particles that summing up all the polynomial corrections the asymptotic diagonal finite volume form factors can be recovered. We also checked the leading exponential correction against Pozsgay's formula and found complete agreement. The integral equation can also be solved numerically. The small volume limit of the solution allows us to map multi-particle states to the spectrum of the Liouville conformal field theory and compare our conjecture to the CFT three-point functions providing ample evidence for its correctness.

In calculating the asymptotic expressions for the finite volume form factors we used a deformation of the TBA kernel, which is the logarithmic derivative of the scattering matrix. We believe that this alternative form for the connected and symmetric form factors can be used to find a compact and closed expression for them and we initiate a study into this direction.

It would be very nice to extend our exact finite volume results for non-diagonal form factors. These results then could be tested for large volumes against the leading exponential correction of form factors [37].

Finally, the knowledge of all form factors could give rise to the determination of finite volume correlation functions relevant both in statistical and solid state physics.

It is an interesting question whether the very nice structure we obtained for the sinh-Gordon model extends to other integrable models such as O(N) models or the AdS/CFT correspondence.

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