# Exact solutions of the sextic oscillator from the bi-confluent Heun equation 

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#### Abstract

The sextic oscillator is discussed as a potential obtained from the bi-confluent Heun equation after a suitable variable transformation. Following earlier results, the solutions of this differential equation are expressed as a series expansion of Hermite functions with shifted and scaled arguments. The expansion coefficients are obtained from a three-term recurrence relation. It is shown that this construction leads to the known quasi-exactly solvable form of the sextic oscillator when some parameters are chosen in a specific way. By forcing the termination of the recurrence relation, the Hermite functions turn into Hermite polynomials with shifted arguments, and, at the same time, a polynomial expression is obtained for one of the parameters, the roots of which supply the energy eigenvalues. With the $\delta=0$ choice the quartic potential term is cancelled, leading to the reduced sextic oscillator. It was found that the expressions for the energy eigenvalues and the corresponding wave functions of this potential agree with those obtained from the quasi-exactly solvable formalism. Possible generalizations of the method are also presented.


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## I. INTRODUCTION

Exactly solvable models have served as indispensable tools in the exploration of the subatomic world since the dawn of quantum mechanics. After the beginning of the computer era, the solution of the quantum mechanical wave equations, e.g. the Schrödinger equation could be performed with high accuracy using various numerical techniques, nevertheless, exactly solvable models were still employed widely as the starting point of numerical methods. Furthermore, these models are often connected with various symmetries and algebraic structures, so in addition to their practical use, their mathematical beauty also still make them appealing.

In practical situations exact solvability means that the energy eigenvalues and boundstate wave functions, as well as quantities related to scattering (if applicable) can be expressed in closed analytical form. In most cases this is achieved by transforming the Schrödinger equation into the differential equation of some special function $F(z)$ of mathematical physics. With this, the solution of the Schrödinger equation is expressed as $\psi(x)=f(x) F(z(x))$, where the $z(x)$ function represents a variable transformation. The choice of $F(z)$ corresponds to identifying various classes of solvable potentials. When $F(z)$ is the hypergeometric, or confluent hypergeometric function, then the related exactly solvable potentials are referred to as Natanzon [1] or Natanzon confluent [2] potentials that depend on six parameters. In practical applications the bound-state wave functions reduce to Jacobi and generalized Laguerre polynomials [3]. Many of the most well-known textbook examples (harmonic oscillator, Coulomb, Pöschl-Teller, Scarf, Rosen-Morse, etc.) appear as two- or three-parameter members of the shape-invariant [4] subclass of the Natanzon (confluent) potential class. The structure (shape) of these potentials is invariant under a transformation of supersymmetric quantum mechanics (SSQM, for a review, see e.g. Ref. [5]), hence their name.

More recently, a wider class of potentials began to attract much attention. In this case the Schrödinger equation is transformed into various versions of the Heun equation [6 8]. These potentials depend on more parameters, making them more flexible and versatile [9-15]. However, since the solutions of the Heun equation are much less known than the (confluent) hypergeometric functions, the exact analytic treatment of these potentials is usually much more complicated technically than that of the Natanzon (confluent) potentials. In certain
cases the $F(z)$ function is expressed as an expansion of other special functions [15-19]. In Ref. [15], for example, a systematic discussion of potentials obtained from the bi-confluent Heun equation (BHE) is presented, with bound-state wavefunctions expressed as an expansion in terms of Hermite functions. Other examples involving truncated series solutions of the Heun equations in terms of functions of the hypergeometric class are presented in Refs. [20 25]

The concept of solvability has also been extended, revealing further aspects of quantum mechanical potentials problems. In the case of conditionally exactly solvable (CES) potentials, exact solutions are obtained only for potentials in which some of the potential parameters are correlated, or are restricted to constant values. This concept was introduced in Ref. [26] in relation with potentials discussed earlier [27]. However, due to some mathematical inconsistencies pointed out in Ref. [28], the first unquestionable example for conditionally exactly solvability was the Dutt-Khare-Varshni (DKV) potential [29], which has terms with fixed parameters. This potential was later identified [30] as a member of the Natanzon-class, and it was shown that the fixed potential parameters result from the Schwartzian derivative that originates from the variable transformation $z(x)$ (see e.g. [31, 32] for the details and further similar potentials). Another type of CES potentials was obtained as non-trivial supersymetric partners of shape-invariant potentials [33], which are beyond the Natanzon class.

Quasi-exactly solvable (QES) potentials [34-36] typically support infinite number of bound states, of which, however, only the lowest few are discussed exactly. In this case the $F(z)$ function appearing in the bound-state eigenfunctions is expressed as a power series expansion, with coefficients satisfying a three-term recursion relation. For some potential parameters the infinite series can be terminated, and $F(z)$ reduces to a polynomial. Perhaps the most well-known example for QES potentials is the sextic oscillator defined in one dimension or as a radial potential [36]. However, its solutions cannot be obtained for arbitrary potential parameters, only for certain correlated parameter sets. This potential has been applied in realistic calculations describing, for example, shape phase transitions of various nuclei $37-39]$.

Due to the various concepts of solvability, sometimes the same potentials can be discussed within different frameworks. Here we report on a study of this kind: the sextic oscillator, which has been described as a QES potential is analyzed in terms of the BHE approach [15], and the results from the two methods are compared. In the QES approach the parameters
of the sextic oscillator are correlated, while in Ref.[15] this question is not discussed, so the question whether the two methods describe the same potential or not, is raised naturally.

In Sec. II the sextic oscillator is revieweded as a QES potential. The formalism of the BHE approach is outlined in Sec. III, while in Sec. IV the method is used to generate the bound-state solutions and the corresponding energy eigenvalues of the reduced sextic oscillator, i.e. the case containing sextic, quadratic and inverse quadratic terms, but not the quartic one. Finally, the results are summarized in Sec. V, together with further possible applications of the formalism.

## II. THE SEXTIC OSCILLATOR AS A QES POTENTIAL

The most general form of the sextic oscillator is

$$
\begin{equation*}
V(x)=V_{-2} x^{-2}+V_{2} x^{2}+V_{4} x^{4}+V_{6} x^{6} . \tag{1}
\end{equation*}
$$

It is assumed that $V_{6}>0$, so the potential tends to plus infinity, meaning that it has infinitely many bound-state solutions. For $V_{6}=V_{4}=0$, (1) reduces to the radial harmonic oscillator. This potential can be interpreted as the radial component $(x \in[0, \infty))$ of a spherically symmetric potential in $D$ dimension. Here $V_{-2}$ depends on $D$ and on the orbital angular momentum $l$. Setting $V_{-2}=0$ the domain of definition of (1) can be extended to the full $x$ axis, $x \in(-\infty, \infty)$. In this case the bound-state wave functions have definite parity, and the odd solutions that vanish at $x=0$ correspond to the solutions of the radial problem.

Numerous attempts have been made to determine the bound-state energy eigenvalues and the corresponding wave functions of (1) using various methods including continued fractions [40], $1 / N$ expansion [41], perturbation methods [42, 43], asymptotic iteration method 44], etc. It was found that exact analytic solutions are obtained only for specific values of the potential parameters. Even then, only the lowest few of the infinite number of energy eigenstates were obtained analytically. The sextic oscillator has thus been identified as a quasi-exactly solvable (QES) potential [35, 36]. Its usual parametrization is 36]

$$
\begin{equation*}
V(x)=\left(2 s-\frac{1}{2}\right)\left(2 s-\frac{3}{2}\right) x^{-2}+\left[b^{2}-2 a(2 s+1+2 M)\right] x^{2}+2 a b x^{4}+a^{2} x^{6} \tag{2}
\end{equation*}
$$

i.e. the four parameters $V_{i}(i=-2,2,4,6)$ are correlated and reduce to three real parameters $a, b, s$ and a non-negative integer $M$ that determines the number $M+1$ of analytic solutions.

The solutions are written as

$$
\begin{equation*}
\psi(x)=\left(x^{2}\right)^{s-1 / 4} \exp \left(-\frac{a x^{4}}{4}-\frac{b x^{2}}{2}\right) F\left(x^{2}\right) \tag{3}
\end{equation*}
$$

For $s=1 / 4$ and $s=3 / 4$ the centrifugal term in (2) vanishes, and (3) reduces to the even and odd solutions of the one-dimensional problem, respectively. It has to be noted that polynomial potentials with higher order have also been investigated using solutions containing the exponential form of polynomials of $x^{2}$ [45]. It was found that this approach is suitable for potentials with leading terms of the type $x^{4 \nu+2}$. In this context the unique role of the harmonic oscillator $(\nu=0)$ and that of the sextic oscillator $(\nu=1)$ seems natural, and this finding also explains why the quartic oscillator does not belong to the group of QES problems.

If $F\left(x^{2}\right)$ is an $M^{\prime}$ th order polynomial, then a second-order differential equation can be obtained from the Schrödinger equation after the separation of the remaining factors of (3). Matching the appropriate powers on the two sides of the resulting equation, the coefficients of the polynomial can be considered as the components of an $(M+1)$-dimensional vector that satisfy an $(M+1)$-dimensional spectral matrix equation [36]. In a more general approach, $F\left(x^{2}\right)$ can be expressed as a power series in $x^{2}$, and in that case an infinite matrix is obtained. However, with the parametrization used in (2) it is possible to cancel certain offdiagonal matrix elements of the infinite matrix, and thus to separate the ( $M+1$ )-dimensional submatrix appearing in the upper left corner, leading to a quasi-exactly solvable problem.

A more easily tractable problem appears for $b=0$, which cancels the quartic component of (2). In this reduced case of the sextic oscillator the (unnormalized) wave functions and the corresponding energy eigenvalues can be expressed in a relatively simple form for the first few values of $M$ [36]:
$M=0$ In this case the ground-state energy is obtained with $n=0$, and $E_{0}=0$, while the corresponding wave function is

$$
\begin{equation*}
\psi_{0}(x)=\left(x^{2}\right)^{s-1 / 4} \exp \left(-\frac{a x^{4}}{4}\right) \tag{4}
\end{equation*}
$$

$M=1$ Then the ground- and the first excited states are described with $n=0$ and 1 :

$$
\begin{gather*}
E_{n}=(-1)^{n+1}(32 a s)^{1 / 2}  \tag{5}\\
\psi_{n}(x)=\left(x^{2}\right)^{s-1 / 4} \exp \left(-\frac{a x^{4}}{4}\right)\left(a x^{2}-\frac{E_{n}}{4}\right) \tag{6}
\end{gather*}
$$

$\psi_{0}(x)$ has no node for $x>0$, while $\psi_{1}(x)$ has one.
$M=2$ Here the lowest three states are obtained with $n=0,1$ and 2 :

$$
\begin{gather*}
E_{n}=(n-1)[32 a(4 s+1)]^{1 / 2}  \tag{7}\\
\psi_{n}(x)=\left(x^{2}\right)^{s-1 / 4} \exp \left(-\frac{a x^{4}}{4}\right)\left(a x^{4}-\frac{E_{n}}{4} x^{2}+\frac{E_{n}^{2}}{32 a}-2 s-1\right) \tag{8}
\end{gather*}
$$

The quadratic polynomial of $x^{2}$ in (8) has $n$ nodes for $x^{2}>0$, reproducing the expected structure of the the wave functions $\psi_{n}(x)$
$M=3$ The expressions for the first four states are somewhat more involved in this case with $n=0,1,2$ and 3 :

$$
\begin{align*}
& E_{n}=(-1)^{\left[\frac{n}{2}\right]+1}(32 a)^{1 / 2}\left[5\left(s+\frac{1}{2}\right)+(-1)^{\left[\frac{n+1}{2}\right]}\left(25\left(s+\frac{1}{2}\right)^{2}-9 s(s+1)\right)^{1 / 2}\right]^{1 / 2}  \tag{9}\\
& \psi_{n}(x)=\left(x^{2}\right)^{s-1 / 4} \exp \left(-\frac{a x^{4}}{4}\right) \\
& \times\left(a^{2} x^{6}-a \frac{E_{n}}{4} x^{4}+\frac{E_{n}^{2}-96 a(s+1)}{32} x^{2}-\frac{E_{n}^{3}}{384 a}+(7 s+5) \frac{E_{n}}{12}\right) \tag{10}
\end{align*}
$$

The number of nodes of (10) appearing for $x>0$ is again $n$, as expected.
Note that for the sake of simplicity, the wave functions $\psi_{n}(x)$ appear here in an unnormalized form [36], however, the normalization constants can be determined in a straightforward way. Note also that the wave functions vanish at $x=0$ as long as $s \geq 1 / 4$. The properties of the polynomials appearing in these solutions have been discussed in detail in Ref. [46], with special attention to their comparison with other orthogonal polynomials.

## III. POTENTIALS SOLVABLE IN TERMS OF THE BI-CONFLUENT HEUN EQUATION

There has been increased interest in the bi-confluent Heun equation [64 8] recently [10, 21, [23, 47 [51], as a second-order differential equation into which the stationary Schrödinger equation can be transformed. In this Section we combine two approaches that have been applied previously to obtain potentials solvable in terms of certain special functions of mathematical physics, and specify them for the case of the bi-confluent Heun equation. First we apply a variable transformation to identify potentials solved in terms of the
bi-confluent Heun equation, then we review the method of expanding the solutions in terms of Hermite functions [15].

The method based on variable transformations originates from Refs. [52, 53], and it resulted in the systematic classification of shape-invariant potentials [54]. Its generalization has also been used to give a systematic description of the wider Natanzon [1] and Natanzon confluent [2] potentials, i.e potentials with bound-state solutions written in terms of a single hypergeometric or confluent hypergeometric function. Essentially the same method was used in Refs. [9, 12, 15] to identify potentials solved in terms of the bi-confluent Heun function. The method works well for other problems too. For instance, it was applied to construct a variety of analytically solvable quantum two-state models [55], in particular, via reduction of the time-dependent Schrödinger equations to the confluent-hypergeometric and bi-confluent Heun equations [56, 57].

## A. The potentials

The bi-confluent differential equation can be parametrized in terms of five parameters as

$$
\begin{equation*}
\frac{\mathrm{d}^{2} u}{\mathrm{~d} z^{2}}+\left(\frac{\gamma}{z}+\delta+\varepsilon z\right) \frac{\mathrm{d} u}{\mathrm{~d} z}+\frac{\alpha z-q}{z} u . \tag{11}
\end{equation*}
$$

For $\alpha=0$ and $\varepsilon=0$ it essentially reduces to the confluent hypergeometric differential equation [3].

In order to transform Eq. (11) into the stationary Schrödinger equation with units of $2 m=\hbar=1$

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \psi}{\mathrm{~d} x^{2}}=\left[V(x)-E_{n}\right] \psi(x) \tag{12}
\end{equation*}
$$

one may follow the procedure outlined in Ref. 54] and factorize the wave function as $\psi(x)=f(x) u(z(x))$. Substiting this function in (12) and comparing the result with (11), one obtains

$$
\begin{align*}
E-V(x)= & \frac{z^{\prime \prime \prime}}{2 z^{\prime}}-\frac{3}{4}\left(\frac{z^{\prime \prime}}{z^{\prime}}\right)^{2}+\left(z^{\prime}(x)\right)^{2}\left[\left(\frac{\gamma}{2}-\frac{\gamma^{2}}{4}\right) z^{-2}(x)-\left(q+\frac{\gamma \delta}{2}\right) z^{-1}(x)\right. \\
& \left.+\left(\alpha-\frac{\varepsilon}{2}-\frac{\delta^{2}}{4}-\frac{\gamma \delta}{2}\right)-\frac{\delta \varepsilon}{2} z(x)-\frac{\varepsilon^{2}}{4} z^{2}(x)\right] . \tag{13}
\end{align*}
$$

Furthermore, according to Ref. [54], the $f(x)$ function can be expressed as

$$
\begin{equation*}
f(x) \sim\left(z^{\prime}(x)\right)^{-1 / 2}(z(x))^{\frac{\gamma}{2}} \exp \left(\frac{\delta}{2} z(x)+\frac{\varepsilon}{4} z^{2}(x)\right) . \tag{14}
\end{equation*}
$$

In the next step $z(x)$ is determined from the requirement that in (13) the constant $E$ on the left side has to originate from certain terms appearing on the right side in the form $\left(z^{\prime}\right)^{2} z^{k}$, where $k=-2,-1,0,1$ and 1 . Applying this requirement to potentials related to the classical orthogonal polynomials resulted in the classification of shape-invariant potentials [54]. The method presented in Ref. [15] is similar in that individual terms of the right-hand side of (13) correspond to $E$ on the left-hand side, resulting in the differential equation

$$
\begin{equation*}
\frac{\mathrm{d} z}{\mathrm{~d} x}=z^{m} / \sigma \tag{15}
\end{equation*}
$$

where $m=-k / 2$ and $\sigma$ is a technical parameter to be specified later. With this choice, the five terms appearing in the square bracket in (13) yield a constant term, provided that $m$ takes on the values $-1,-1 / 2,0,1 / 2$ and 1 . At the same time, the remaining terms on the right-hand side of (13), the Schwartzian derivative, lead to a term proportional to $z^{2 m-2}$. In general, with the choice (15), Eq. (13) turns into

$$
\begin{align*}
E-V(x)= & \sigma^{-2}\left[-\frac{\gamma-m}{2}\left(\frac{\gamma+m}{2}-1\right) z^{2 m-2}(x)-\left(q+\frac{\gamma \delta}{2}\right) z^{2 m-1}(x)\right. \\
& \left.+\left(\alpha-\frac{\varepsilon}{2}-\frac{\delta^{2}}{4}-\frac{\gamma \varepsilon}{2}\right) z^{2 m}(x)-\frac{\delta \varepsilon}{2} z^{2 m+1}(x)-\frac{\varepsilon^{2}}{4} z^{2 m+2}(x)\right] \tag{16}
\end{align*}
$$

The five choices of $m$ result in the five different potentials listed in Refs. [9, 12, 15]. It is notable that the general discussion of potentials related to the bi-confluent Heun equation in Ref. [10] contain implicitly the same potentials, with the exception of the $m=-1$ case.

The complete solution requires also solving the differential equation (15). It is found that for $m=1$ one gets $z(x)=\exp \left(\left(x+x_{0}\right) / \sigma\right)$, while for the remaining cases $z(x)=$ $\left[(1-m)\left(x+x_{0}\right) / \sigma\right]^{1 /(1-m)}$, where the coordinate shift $x_{0}$ is an integration constant that can be omitted in most cases without the loss of generality. The $\sigma$ parameter is also inessential, as it simply scales the energy. It is also straightforward to see that the terms originating from the Schwartzian derivative contribute to the constant term for $m=1$ and to that with $x^{-2}$ (the centrifugal term, if applicable) in the remaining four cases.

It has to be noted that for $\varepsilon=0$ and $\alpha=0$ (11) reduces to the confluent hypergeometric differential equation. Equation (13) also simplifies, which means that $m$ in that case is restricted only to $1,1 / 2$ and 0 , resulting in the Morse, harmonic oscillator and Coulomb potentials [54], respectively. It is worth mentioning here that more general potentials, the

Natanzon confluent potentials [2] can be obtained by a more flexible choice of $z(x)$. In particular, it is possible to identify combinations of several terms in (13) with the constant $(E)$. As a result, in that case more complicated energy expressions are obtained, and, at the same time, potential terms with fixed coupling coefficients appear, originating from the Schwartzian derivative. The first concrete example of Natanzon confluent potentials is the generalized Coulomb potential [60, 61, which bears the features of both the Coulomb and the harmonic oscillator potentials that can also be obtained from its special limits.

## B. The solutions

The general solution of the Schrödinger equation for the potentials discussed here is written as

$$
\begin{equation*}
\psi(x)=z^{(\gamma-m) / 2}(x) \exp \left(\frac{\delta}{2} z(x)+\frac{\varepsilon}{4} z^{2}(x)\right) H_{B}(\gamma, \delta, \varepsilon ; \alpha, q ; z(x)) \tag{17}
\end{equation*}
$$

where $H_{B}$ denotes the bi-confluent Heun function. Unfortunately, this function is far less well known than those solving other second-order differential equations of mathematical physics, e.g. the (confluent) hypergeometric function. Inspired by previous results [21, 23], the solutions of the bi-confluent Heun equation (11) can be sought for as an expansion in terms of Hermite functions possessing shifted and scaled argument [15]:

$$
\begin{equation*}
H_{B}(z)=\sum_{n} c_{n} u_{n}(z), \quad u_{n}(z)=H_{\alpha_{0}+n}\left(s_{0}\left(z+z_{0}\right)\right) \tag{18}
\end{equation*}
$$

where $\alpha_{0}, s_{0}$ and $z_{0}$ are complex constants. For integer values of $\alpha_{0}$ the Hermite functions reduce to Hermite polynomials [3].

The Hermite functions satisfy the second-order differential equation

$$
\begin{equation*}
\frac{\mathrm{d}^{2} u_{n}}{\mathrm{~d} z^{2}}-2 s_{0}^{2}\left(z+z_{0}\right) \frac{\mathrm{d} u_{n}}{\mathrm{~d} z}+2 s_{0}^{2}\left(\alpha_{0}+n\right) u_{n}=0 \tag{19}
\end{equation*}
$$

As it has been shown in Ref. [15], for specific choice of some parameters, i.e. $s_{0}= \pm(-\varepsilon / 2)^{1 / 2}$ and $z_{0}=\delta / \varepsilon$, a three-term recurrence relation can be obtained for the coefficients $c_{i}$ in (18), after making use of the identities

$$
\begin{align*}
u_{n}^{\prime} & =2 s_{0}\left(\alpha_{0}+n\right) u_{n-1}  \tag{20}\\
s_{0}\left(z+z_{0}\right) u_{n} & =\left(\alpha_{0}+n\right) u_{n-1}+u_{n+1} / 2 \tag{21}
\end{align*}
$$

This relation reads

$$
\begin{align*}
0 & =R_{n} c_{n}+Q_{n-1} c_{n-1}+P_{n-2} c_{n-2}  \tag{22}\\
R_{n} & =\left(-\frac{2}{\varepsilon}\right)^{1 / 2}\left(\alpha_{0}+n\right)\left[\alpha+\left(\alpha_{0}+n-\gamma\right) \varepsilon\right]  \tag{23}\\
Q_{n} & =\mp \frac{1}{\varepsilon}\left[\alpha \delta+\left(q+\left(\alpha_{0}+n\right) \delta\right) \varepsilon\right]  \tag{24}\\
P_{n} & =(-2 \varepsilon)^{-1 / 2}\left[\alpha+\left(\alpha_{0}+n\right) \varepsilon\right] \tag{25}
\end{align*}
$$

where the $\mp$ choice in 24 corresponds to the choices $s_{0}= \pm(-\varepsilon / 2)^{1 / 2}$. The infinite summation in (18) reduces to a finite sum in certain situations specified in Ref. [15]. For this, termination from both sides is required. Termination from below occurs for $\alpha_{0}=0$, in which case the Hermite functions reduce to Hermite polynomials, and also for $\alpha_{0}=\gamma-\alpha / \varepsilon$. Termination from above is possible if $c_{N+1}=0$ is prescribed, which implies that the accessory parameter $q$ has to satisfy an $(N+1)^{\prime}$ 'th degree polynomial equation.

## IV. THE ANALYSIS OF THE REDUCED SEXTIC OSCILLATOR IN TERMS OF THE BHE APPROACH

The formalism is now ready to establish how the general form of the sextic oscillator presented in Ref. [15] is related to the one usually discussed in the QES approach. For this, let us make the $m=1 / 2$ choice in the differential equation (15), and consider its solution with the natural choice $x_{0}=0$ and $\sigma=1$ discussed previously. Then we obtain $z(x)=x^{2} / 4$, with the substitution of which Eqs. (16) and (17) supply the potential

$$
\begin{align*}
V(x)= & \left(\gamma-\frac{1}{2}\right)\left(\gamma-\frac{3}{2}-1\right) x^{-2}+\left(\frac{\delta^{2}}{16}-\frac{\alpha}{4}+\frac{\varepsilon}{8}(\gamma+1)\right) x^{2} \\
& +\frac{\delta \varepsilon}{32} x^{4}+\frac{\varepsilon^{2}}{256} x^{6}, \tag{26}
\end{align*}
$$

the energy eigenvalues

$$
\begin{equation*}
E=-q-\frac{\gamma \delta}{2} \tag{27}
\end{equation*}
$$

and the corresponding bound-state wave functions

$$
\begin{equation*}
\psi(x)=\left(x^{2}\right)^{\frac{\gamma}{2}-\frac{1}{4}} \exp \left(\frac{\varepsilon}{64} x^{4}+\frac{\delta}{8} x^{2}\right) H_{B}\left(\gamma, \delta, \varepsilon ; \alpha, q ; \frac{x^{2}}{4}\right) . \tag{28}
\end{equation*}
$$

On comparing (26) with (2), as well as (28) with (3), one finds that the two problems are equivalent, and the parameters are related as

$$
\begin{equation*}
\gamma=2 s, \quad \delta=-4 b, \quad \varepsilon=-16 a, \quad \alpha=16 a M \tag{29}
\end{equation*}
$$

With the $\delta=0(b=0)$ choice the quartic potential term is cancelled in (26) leading to the reduced sextic oscillator discussed in Section II, while the exponential factor in the boundstate wave function (28) is also simplified. Furthermore, it turns out that with the $\alpha_{0}=0$ choice discussed in Subsection IIIB, the bi-confluent Heun function reduces to a finite sum of Hermite polynomials. The expansion coefficients in the recurrence relation (22) also simplify:

$$
\begin{align*}
R_{n} & =(-2 \varepsilon)^{1 / 2} n(M-n+\gamma)  \tag{30}\\
Q_{n} & =\mp q  \tag{31}\\
P_{n} & =(-\varepsilon / 2)^{1 / 2}(M-n) \tag{32}
\end{align*}
$$

Selecting the upper sign in (31) and with it, also the upper sign in $s_{0}= \pm(-\varepsilon / 2)^{1 / 2}$, as discussed in Subsection III B and in Ref. [15], the bound-state wave functions can be expressed as a finite sum of Hermite polynomials $H_{n}\left((-\varepsilon / 32)^{1 / 2} x^{2}\right)$.

The termination of the series from above is secured by prescribing $c_{N+1}=0$. Expressing this from (22) in terms of $c_{N}$ and $c_{N-1}$ leads to an $(N+1$ 'th degree polynomial in $q$. Since (27) implies $E=-q$, this relation determines the possible energy eigenvalues. The integer parameter $M$ determines the number of energy levels to be discussed. Taking $N=M$, the condition for the termination of the series is found to be

$$
\begin{array}{ll}
M=0: & q=0 \\
M=1: & q^{2}+\gamma \varepsilon=0 \\
M=2: & q^{3}+2 \varepsilon(2 \gamma+1) q=0, \\
M=3: & q^{4}+10 \varepsilon(\gamma+1) q^{2}+9 \gamma(\gamma+2) \varepsilon^{2}=0 \tag{36}
\end{array}
$$

It can be proven in a straightforward way that for these values of $M$ the results of the QES approach outlined in Section II, i.e. Eqs. (4)-(10) are reproduced. In particular, Eqs. (33)-(36) determine the expressions for $E_{n}$, while the bound-state wave functions are constructed using the $c_{i}$ parameters from the recurrence relation (22) with (30)-(32). In order to match these wave functions with those in Eqs. (4) to (10), the explicit form of the Hermite polynomials also has to be resolved. Figures 1 and 2 display the potential, the energy eigenvalues and the bound-state wave functions for some parameters.

It is remarkable that the form of the wave functions (28) that is determined by the procedure summarized in Section III and the choice of some parameters reproduces the


FIG. 1: The sextic potential (26) with $\gamma=2, \delta=0, \varepsilon=-16$ and $\alpha / \varepsilon=-3$ displayed together with the energy eigenvalues located at $E_{0}=-20.926, E_{1}=-6.488, E_{2}=6.488$ and $E_{3}=20.926$.


FIG. 2: Unnormalized wavefunctions $\psi_{n}(x)$ plotted for $n=0,1,2$ and 3 corresponding to the four energy eigenvalues appearing in Fig. 1.
known results obtained through a completely different procedure, the QES approach. A technical difference is that the polynomials of $x^{2}$ in the QES approach are expanded in terms of Hermite polynomials here. Actually, these polynomials have been discussed in detail from the mathematical point of view in Ref. [46], where they have been obtained as the solutions of the reduced sextic oscillator with centrifugal barrier, i.e. potential (2) with $b=0$.

These results establish a direct connection between the formalism of quasi-exactly solvable
potentials [36] and the method of expanding the solutions of the bi-confluent Heun equation in terms of Hermite functions [15].

## V. SUMMARY AND CONCLUSIONS

In the present work we revisited the sextic oscillator potential problem and presented its discussion from a new perspective. We demonstrated that this problem can be solved by transforming the Schrödinger equation into the bi-confluent Heun differential equation. The sextic oscillator is obtained as one of the five potentials identified in Ref. [15], corresponding to a specific $z(x)$ transformation function in (15) with $m=1 / 2$. It is also included implicitly in Ref. [10], where the general expression is discussed for potentials derived from the biconfluent Heun equation, as well as other versions of the Heun equation. However, in these works the parametrization of the sextic oscillator was general, i.e. it was different from the usual form discussed in the quasi-exactly solvable framework, where the parameters are correlated. The question whether the two approaches (i.e. the BHE and the QES) lead to the same results was raised naturally.

Following the formalism of Ref. [15], we sought for the bound-state wave functions in terms of an expansion of Hermite functions with scaled and shifted arguments. It turned out that the sextic oscillator can be obtained after specifying some parameters of the model, in which case the Hermite functions reduce to Hermite polynomials with shifted argument. Cancelling the shift by selecting $\delta=0$, the reduced sextic oscillator is obtained, i.e. the one with vanishing quartic term. The bound-state wave functions then allow expansion in terms of Hermite polynomials with arguments involving $x^{2}$. This expansion is reduced to a finite polynomial terminating with $x^{2 N}$ if the expansion coefficient of the $H_{N+1}$ Hermite polynomial is forced to vanish: $c_{N+1}=0$. This condition implies an $(N+1)$ 'th degree polynomial equation for the parameter $q$, the roots of which supply the bound-state energy eigenvalues.

The explicit construction of the solutions up to $N=3$ recovers the results obtained for the bound-state energy eigenvalues and wave functions derived within the quasi-exactlysolvable formalism [36]. The bound-state wave functions there contain polynomials of $x^{2}$, which are also obtained from the condition of terminating a three-term recurrence relation. The resulting polynomials have been analyzed from the mathematical point of view in Ref.
[46], and were identified as a new set of orthogonal polynomials.
It is remarkable that the energy eigenvalues are located symmetrically with respect to $E=0$. This seems to indicate the specific nature of the reduced sextic oscillator in the sense that its parameters are correlated and do not allow arbitrary independent values for $V_{-2}, V_{2}$ and $V_{6}$ in (11).

The present investigation can be expanded to various directions. First, the general version of the sextic oscillator can be considered with $\delta \neq 0$. In this case the quartic potential term also appears, and the expansion of the wave function contains Hermite functions with shifted arguments. One can expect that forcing the termination of the series expansion also leads to certain conditions and more complex energy spectrum and wave functions. The energy eigenvalues and bound-state wave functions are less well-known in the QES setting, so the comparison of the two approaches needs special care. This will be done in a separate study.

It is also worthwhile to investigate the remaining four potentials identified in Ref. [15]. The case with $m=0$, for example, yields the potential with terms $x^{p}, p=-2,-1,1$ and 2 , i.e. a potential that contains (shifted) harmonic oscillator, Coulombic and centrifugal terms alike. Special versions of this potential have been studied before in terms of numerical 58 and expansion [59] techniques.

The next level of complexity would be considering more general $z(x)$ transformation functions. One possibility is taking the one applied to generate the generalized Coulomb potential [60, 61], which belongs to the Natanzon confluent class. It has both the harmonic oscillator $\left(z(x) \sim x^{2}\right)$ and the Coulomb potential $(z(x) \sim x)$ as a special limit, reproducing the said two potentials there as limiting cases. The same two limits appear in the BHE approach too, corresponding to the sextic oscillator ( $m=1 / 2$ ) and the potential mentioned in the previous paragraph ( $m=0$ ), offering an interesting potential shape. From the technical point of view, this potential would correspond to a specific problem contained implicitly in the general discussion in Ref. [10].

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