Exact finite volume expectation values of conserved currents

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Abstract

The vacuum expectation values of conserved currents play an essential role in the generalized hydrodynamics of integrable quantum field theories. We use analytic continuation to extend these results for the excited state expectation values in a finite volume. Our formulas are valid for diagonally scattering theories and incorporate all finite size corrections.

1 Introduction

Recently there have been interesting developments in calculating expectation values in integrable finite temperature/volume systems. The motivation came from statistical physics [1] as well as from the AdS/CFT duality [2]. In the AdS/CFT duality heavy-heavy light three-point functions can be mapped to expectation values of local operators in finite volume multiparticle states [2, 3, 4, 5]. In statistical physics the recent developments of the generalized hydrodynamics require the knowledge of the finite temperature expectation values of conserved charges and currents as they are the key inputs in formulating the Euler type hydrodynamic evolution [1, 6, 7]. There were interesting direct calculations [8, 9], which expressed the current expectation values in spin chain Bethe states in terms of the charge eigenvalues and the inverse of the Gaudin matrix. These remarkable compact and simple expressions are also valid in quantum field theories for finite volume expectation values in multiparticle states once the exponentially small vacuum polarization effects are neglected. The aim of our paper is to provide a simple derivation of this result and to extend it in order to incorporate all the finite size corrections. As a result we describe exactly the finite volume excited state expectation values of conserved currents. In doing so we continue analytically the structural equations of generalized hydrodynamics [1] and interpret the result in the finite volume setting.

The paper is organized as follows: In the next section we recall the results of the generalized hydrodynamics, which can be interpreted as finite volume vacuum expectation values. We formulate the results in terms of a pairing between functions, which includes the occupation number of the quasiparticles as the integration measure. In section 3 we use analytical continuation to modify the pairing to include also the discrete contributions of physical particles. All formulas remain the same only the pairing has to be exchanged. Finally we perform various tests of our results and conclude.

2 Vacuum expectation values of conserved currents

In the generalized hydrodynamics of integrable models [1] conservation laws

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0 \tag{1}$$

play a crucial role. Local thermal equilibrium can be characterized by temperature like quantities β_i coupled to the infinite family of conserved charges, $Q_i = \int q_i dx$, leading to local averages

$$\langle \mathcal{O} \rangle = Z^{-1} \operatorname{Tr}(\mathcal{O}e^{-\sum_{j}\beta_{j}Q_{j}}) \quad ; \qquad Z = \operatorname{Tr}(e^{-\sum_{j}\beta_{j}Q_{j}}) \tag{2}$$

Here Q_1 is the energy and β_1 is the inverse of the temperature (or volume $\beta_1 = L$ in the finite volume situation). The collection of the β_i "temperatures" can be traded for the expectation values of the conserved charges $\langle q_i \rangle \propto \partial_{\beta_i} \log Z$, implying that the expectation values of the currents $\langle j_i \rangle$ depend on $\langle q_i \rangle$ which is the equation of state $\langle j_i \rangle = F_i(\langle q \rangle)$. Assuming local thermodynamic equilibrium and that these quantities vary slowly in space and time they satisfy the continuity equation $\partial_t \langle q_i \rangle + \partial_x \langle j_i \rangle = \partial_t \langle q_i \rangle + J_{ij} \partial_x \langle q_i \rangle = 0$, an Euler type hydrodynamic equation. Normal fluid modes diagonalize $J_{ij} = \partial_{q_i} F_i$ and propagate as $\partial_t n_i + v_i^{\text{eff}} \partial_x n_i = 0$.

We focus on a relativistic integrable theory of a single particle which scatters on itself with the S-matrix $S(\theta_1 - \theta_2)$, where θ is the rapidity which parametrizes the energy and momentum as $E(\theta) = m \cosh \theta$, $p(\theta) = m \sinh \theta$. In thermal equilibrium the expectation values of charges can be calculated from the densities of quasi-particles $\rho(\theta)$ and the charge eigenvalue on a one-particle state $h_i(\theta)^1$ as

$$\langle q_i \rangle = \int \frac{d\theta}{2\pi} \rho(\theta) h_i(\theta) \tag{3}$$

Here and from now on all integrals go from $-\infty$ to ∞ . Thus the state in a thermal equilibrium can be represented either by β_i or by q_i or alternatively by ρ . The normal modes, however are neither of these, instead they are related to the occupation number n:

$$n(\theta) = \frac{1}{1 + e^{\epsilon(\theta)}} \tag{4}$$

which can be calculated from the Thermodynamic Bethe ansatz (TBA) equation [10]

$$\epsilon(\theta) = \sum_{i} \beta_{i} h_{i}(\theta) - \int \frac{du}{2\pi} \varphi(\theta - u) \log(1 + e^{-\epsilon(u)})$$
(5)

where $\varphi(\theta) = -i\partial_{\theta} \log S(\theta)$. For later generalizations we introduce a pairing including the occupation number as

$$g(\theta) \circ h(\theta) = \int g(\theta)h(\theta)\frac{n(\theta)d\theta}{2\pi}$$
(6)

The TBA equation after integration by parts takes the form

(

$$\epsilon(\theta) = \sum_{i} \beta_{i} h_{i}(\theta) - i \log S(\theta - u) \circ \partial_{u} \epsilon(u)$$
(7)

It is also useful to introduce dressed quantities which satisfy

$$g^{\mathrm{dr}}(\theta) = g(\theta) + \varphi(\theta - u) \circ g^{\mathrm{dr}}(u)$$
(8)

since then the particle density can be written in terms of the occupation number as

$$\rho(\theta) = n(\theta)(p')^{\mathrm{dr}}(\theta) \tag{9}$$

¹Here $h_1(\theta) = m \cosh \theta$, and there are higher spin conserved charges $h_{2n-1}(\theta) \propto \cosh(n\theta)$, $h_{2n}(\theta) \propto \sinh(n\theta)$ for infinitely many odd ns.

where $p'(\theta) = dp(\theta)/d\theta$. This leads to the charge expectation value

$$\langle q_i \rangle = (p')^{\mathrm{dr}}(\theta) \circ h_i(\theta) = p'(\theta) \circ h_i^{\mathrm{dr}}(\theta)$$
(10)

In the second equality we used the fact that the dressing operator $(1 - \varphi \circ)^{-1}$ is symmetric wrt. the pairing. From relativistic invariance it follows [1] that the current expectation values take the form

$$\langle j_i \rangle = E'(\theta) \circ h_i^{\mathrm{dr}}(\theta) = (E')^{\mathrm{dr}}(\theta) \circ h_i(\theta)$$
(11)

Comparing $\langle j_i \rangle$ to $\langle q_i \rangle$ we can extract the effective velocity of the quasi-particles $v^{\text{eff}}(\theta) = (E')^{\text{dr}}/(p')^{\text{dr}}$. These results can also be obtained from the Leclair-Mussardo (LM) formula [11]

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$$\langle \mathcal{O} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} \int \frac{n(\theta_k) d\theta_k}{2\pi} F_{2n,c}^{\mathcal{O}}(\theta_1, \dots, \theta_n)$$
(12)

by taking into account that the connected form factors of the conserved charges and currents are

$$F_{2n,c}^{q_i}(\theta_1,\ldots,\theta_n) = E(\theta_1)\varphi(\theta_1-\theta_2)\ldots\varphi(\theta_{n-1}-\theta_n)h_i(\theta_n) + \text{permutations}$$
(13)

$$F_{2n,c}^{j_i}(\theta_1,\ldots,\theta_n) = p(\theta_1)\varphi(\theta_1-\theta_2)\ldots\varphi(\theta_{n-1}-\theta_n)h_i(\theta_n) + \text{permutations}$$
(14)

Indeed, expanding the dressing operator $(1 - \varphi \circ)^{-1}$ in (10,11) leads to the LM formula.

Using the fact that

$$\partial_{\beta_i} \epsilon(\theta) = h_i^{\rm dr}(\theta) \tag{15}$$

we can express the charge and current expectation values as

$$\langle q_i \rangle = -\partial_{\beta_i} \int \frac{dp(\theta)}{2\pi} \log(1 + e^{-\epsilon(\theta)}) = -\partial_{\beta_i}(p(\theta) \circ \partial_{\theta}\epsilon(\theta))$$
(16)

$$\langle j_i \rangle = -\partial_{\beta_i} \int \frac{dE(\theta)}{2\pi} \log(1 + e^{-\epsilon(\theta)}) = -\partial_{\beta_i}(E(\theta) \circ \partial_{\theta}\epsilon(\theta))$$
(17)

These expectation values are valid in a local thermal equilibrium specified by the "temperatures" β_i .

To make contact with the finite volume description in the crossed channel we need to choose $\beta_1 = L$ to be the volume and put all other β_i to zero². Thus the TBA equation is understood as the generating function of the expectation values of conserved quantities where, after differentiation in (16,17), we have to take $\beta_i = \delta_{1i}L$. In this simplified situation $\partial_{\theta}\epsilon(\theta) = L(E')^{dr}(\theta)$ and we can simplify the current expectation values as

$$\langle j_i \rangle = \frac{1}{L} h_i(\theta) \circ \partial_\theta \epsilon(\theta) = \frac{1}{L} \int \frac{d\theta}{2\pi} h'_i(\theta) \log(1 + e^{-\epsilon(\theta)})$$
(18)

but the same is not true for the charges. From the relativistic invariance we can reformulate the finite temperature partition function and averages in the mirror channel. In the Euclidean version it is obtained by a $\frac{\pi}{2}$ rotation. This is an imaginary Lorentz transformation with rapidity $i\frac{\pi}{2}$: $\theta \to \theta^{\gamma} = \theta + \frac{i\pi}{2}$, for which coordinates transform as $(x, t) \to (it, ix)$, while currents and charges as $(j, q) \to (iq, ij)$, in particular $(p, E) \to (iE, ip)$. This transformation squares to the crossing transformation, which acts as $(j, q) \to -(j, q)$ and changes particles to antiparticles. In the finite volume channel, indicated by a subscript L, the LM formula takes the form

$$\langle 0|\mathcal{O}|0\rangle_L = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=1}^n \int \frac{n(\theta_k)d\theta_k}{2\pi} F_{2n,c}^{\mathcal{O}}(\theta_1^{\gamma}, \dots, \theta_n^{\gamma})$$
(19)

²Keeping β_i nonzero would imply twisted boundary conditions by the conserved charges [12, 13]

which implies

$$\langle 0|q_k|0\rangle_L = i(E')^{\mathrm{dr}}(\theta) \circ h_k(\theta^{\gamma}) = i\langle j_k^{\gamma}\rangle \tag{20}$$

where by j_k^{γ} we mean that we use $h_k^{\gamma}(\theta) = h_k(\theta^{\gamma})$ for the corresponding charge eigenvalue. In particular, the finite volume vacuum expectation value of the conserved charges is

$$\langle 0|q_k|0\rangle_L = \frac{i}{L} \int \frac{d\theta}{2\pi} h'_k(\theta + \frac{i\pi}{2}) \log(1 + e^{-\epsilon(\theta)})$$
(21)

Evaluating this expression for the energy, $h_1(\theta) = m \cosh \theta$, gives

$$L\langle 0|q_1|0\rangle_L = -m \int \frac{d\theta}{2\pi} \cosh\theta \,\log(1+e^{-\epsilon(\theta)}) \tag{22}$$

which agrees with the groundstate energy $E_0(L)$ coming from the saddle point value of the partition function [10].

Similarly, the finite volume vacuum expectation value of the currents can be expressed as

$$\langle 0|j_i|0\rangle_L = i(p')^{\mathrm{dr}}(\theta) \circ h_i(\theta^{\gamma}) = i\langle q_i^{\gamma}\rangle$$
(23)

In the following we generalize these results for finite volume excited states.

3 Excited state expectation values of conserved currents

It was observed in [14] that excited state TBA equations can be obtained from the ground-state one by analytical continuations. The idea is that by doing an analytical continuation in the volume/temperature to complex values a pole singularity of $n(\theta)$ might cross the real integration contour whose residue should be picked up and added as a source term even when the volume is continued back to its physical value. The resulting TBA equation describes excited multiparticle states in the finite volume channel.

In the thermal channel the situation might be interpreted as the presence of some defect lines which correspond to physical particles propagating in the crossed channel. These defects then modify the thermal equilibrium and change the quasi-particle density [15]. As a result we need to use the new densities and occupation numbers to calculate averages in this situation, which we denote by the same symbol as before.

In analyzing the finite volume excited state expectation values in the sinh-Gordon model [16] it turned out that all effects coming from the analytical continuation can be encoded into the pairing. Thus we expect that all formulae remain the same as the groundstate ones except that the pairing has to be replaced with a new pairing:

$$g(\theta) \bullet h(\theta) = \sum_{j} \frac{\eta_{j} i f(\theta_{j}) g(\theta_{j})}{\partial_{\theta} \epsilon(\theta)|_{\theta_{j}}} + \int g(\theta) h(\theta) \frac{n(\theta) d\theta}{2\pi}$$
(24)

Formally we can represent the effect of the continuation with a modified contour as shown on Figure 1. The residues η_i are 1 for poles on the upper and -1 for the lower half-plane.

The rapidities θ_i are determined by $\epsilon(\theta_i) = i\pi(2m_i+1)$ and we have to take $\partial_{\theta}\epsilon(\theta)|_{\theta_j} = \sum_i \beta_i h'_i(\theta_j) + \varphi(\theta_j - u) \bullet \partial_u \epsilon(u)$. In the modified convolution the occupation number is $n = 1/(1 + e^{\epsilon})$, where now ϵ satisfies the excited state TBA equation, which again can be obtained via the new convolution:

$$\epsilon(\theta) = \sum_{k} \beta_{k} h_{k}(\theta) - i \log S(\theta - u) \bullet \partial_{u} \epsilon(u)$$

$$= \sum_{k} \beta_{k} h_{k}(\theta) + \sum_{k} \eta_{k} \log S(\theta - \theta_{k}) - i \log S(\theta - u) \circ \partial_{u} \epsilon(u)$$
(25)



Figure 1: Schematical integration contour for excited states. In the sinh-Gordon model the singularities have imaginary part $i\frac{\pi}{2}$. In the scaling Lee-Yang model they are symmetric for the real line.

The excited state expectation values of the conserved charges and currents are simply

$$\langle q_i \rangle = (p')^{\mathbf{dr}}(\theta) \bullet h_i(\theta) = p'(\theta) \bullet h_i^{\mathbf{dr}}(\theta)$$
(26)

$$\langle j_i \rangle = E'(\theta) \bullet h_i^{\mathbf{dr}}(\theta) = (E')^{\mathbf{dr}}(\theta) \bullet h_i(\theta)$$
(27)

where the dressed quantities, indicated by boldface, are obtained by means of the new convolution

$$g^{\mathbf{dr}}(\theta) = g(\theta) + \varphi(\theta - u) \bullet g^{\mathbf{dr}}(u)$$
(28)

Even simpler expressions can be obtained for the expectation values as

$$\langle q_i \rangle = -\partial_{\beta_i}(p(\theta) \bullet \partial_{\theta} \epsilon(\theta)) \tag{29}$$

$$\langle j_i \rangle = -\partial_{\beta_i} (E(\theta) \bullet \partial_{\theta} \epsilon(\theta)) \tag{30}$$

These are the main results of the paper. Since they were not really derived, merely conjectured based on previous experiences, we perform various consistency checks and elaborate the details.

As a start we separate the contribution of the physical particles and the quasi-particles. In doing so we rewrite the new convolution in terms of the old one.

$$\langle j_i \rangle = -\partial_{\beta_i} (i \sum_k \eta_k E(\theta_k) + E(\theta) \bullet \partial_{\theta} \epsilon(\theta))$$

$$= -i \sum_k \eta_k E'(\theta_k) \partial_{\beta_i} \theta_k - \partial_{\beta_i} (E(\theta) \circ \partial_{\theta} \epsilon(\theta))$$
(31)

In order to calculate $\partial_{\beta_i}\theta_k$ we take the quantization condition $\epsilon(\theta_k) = i\pi(2m_k + 1)$:

$$\epsilon(\theta_k) = \sum_i \beta_i h_i(\theta_k) + \sum_j \eta_j \log S(\theta_k - \theta_j) - i \log S(\theta_k - u) \circ \partial_u \epsilon(u)$$
(32)

and differentiate wrt. β_i . We can do it in two different ways.

In the first we differentiate $\epsilon(\theta)$ by keeping θ_k independent of β_i and then we take into account the β_i dependence of all θ_i s:

$$0 = \partial_{\beta_i} \epsilon(\theta_k) = h_i^{\mathrm{dr}}(\theta_k) + \sum_j \partial_{\theta_j} \epsilon(\theta_k) \partial_{\beta_i} \theta_j$$
(33)

In the second term we recognize the Gaudin matrix

$$G_{jk} = -i\partial_{\theta_j}\epsilon(\theta_k) = -i(\delta_{jk}\partial_{\theta}\epsilon(\theta) + \partial_{\theta_j}\epsilon(\theta))|_{\theta_k} = \delta_{jk}D^{\mathrm{dr}}(\theta_k) - \varphi_j^{\mathrm{dr}}(\theta_k)$$
(34)

where

$$D(\theta) = -i\sum_{l}\beta_{l}h_{l}'(\theta) + \sum_{j}\varphi_{j}(\theta) \quad ; \qquad \varphi_{j}(\theta) = \eta_{j}\varphi(\theta - \theta_{j})$$
(35)

Alternatively, we can separate the β_i -dependence of θ_k in the argument:

$$0 = \partial_{\beta_i} \epsilon(\theta_k) = \partial_{\theta} \epsilon(\theta)|_{\theta_k} \frac{\partial \theta_k}{\partial \beta_i} + \partial_{\beta_i} \epsilon(\theta)|_{\theta_k}$$
(36)

This formula can be used to show that $\partial_{\beta_i} \epsilon(\theta) = h_i^{\mathbf{dr}}(\theta)$. By putting together these contributions we obtain

$$\langle j_i \rangle = \sum_{k,l} \eta_k E'(\theta_k) G_{kl}^{-1} h_i^{\rm dr}(\theta_l) + E'(\theta) \circ h_i^{\rm dr}(\theta)$$
(37)

In order to have a complete separation into physical and quasi-particles we rewrite the new dressing in terms of the old one. In doing so we note that

$$g^{\mathbf{dr}}(\theta) = g(\theta) + \sum_{j} \frac{\varphi_j(\theta)}{D^{\mathrm{dr}}(\theta_j)} g(\theta_j) + \varphi(\theta - u) \circ g^{\mathbf{dr}}(u) = \sum_{n=0}^{\infty} [\varphi(\theta - u)\bullet]^n g(u)$$
(38)

In each term we can use either the discrete or the continuous parts of the convolution. The continuous part dresses up $g(\theta)$ and $\varphi_i(\theta)$ leading to

$$g^{\mathbf{dr}}(\theta) = g^{\mathrm{dr}}(\theta) + \sum_{j} \varphi_{j}^{\mathrm{dr}}(\theta) G_{jk}^{-1} g^{\mathrm{dr}}(\theta_{k})$$
(39)

Here we used that $G_{jk} = D^{dr}(\theta_k)(\delta_{jk} - \varphi_j^{dr}(\theta_k)/D^{dr}(\theta_k))$ in recognizing its inverse. Plugging this back into the current expectation value we obtain a form involving only the old dressing and convolutions:

$$\langle j_i \rangle = \sum_{k,l} \eta_k(E')^{\mathrm{dr}}(\theta_k) G_{kl}^{-1} h_i^{\mathrm{dr}}(\theta_l) + E'(\theta) \circ h_i^{\mathrm{dr}}(\theta)$$
(40)

We note that one can show in general that $f(\theta) \bullet g^{\mathbf{dr}}(\theta) = \sum_{k,l} f^{\mathrm{dr}}(\theta_k) G_{kl}^{-1} g^{\mathrm{dr}}(\theta_l) + f(\theta) \circ g^{\mathrm{dr}}(\theta)$. Writing $h_i^{\mathrm{dr}}(\theta_l) = h_i(\theta_l) + \varphi(\theta_l - u) \circ h_i^{\mathrm{dr}}(u)$ we can observe that the leading part of the result, i.e. the term without any integration, is the same which was obtained in [8, 9] in a more complicated way.

An analogous calculation results in the charge expectation value

$$\langle q_i \rangle = \sum_{k,l} \eta_k(p')^{\mathrm{dr}}(\theta_k) G_{kl}^{-1} h_i^{\mathrm{dr}}(\theta_l) + p'(\theta) \circ h_i^{\mathrm{dr}}(\theta)$$

$$\tag{41}$$

In the following we use these results to calculate the finite volume excited state expectation values. For this reason we again take $\beta_i = L\delta_{i,1}$ and use the relation

$$\partial_{\theta}\epsilon(\theta) = LE'(\theta) + i\sum_{k} \eta_{k}\varphi(\theta - \theta_{k}) + \varphi(\theta - u) \circ \partial_{u}\epsilon(u) = L(E')^{\mathbf{dr}}(\theta)$$
(42)

to obtain

$$\langle j_k \rangle = \frac{1}{L} \partial_\theta \epsilon(\theta) \bullet h_k(\theta) = \frac{1}{L} (i \sum_j \eta_j h_k(\theta_j) + \int \frac{d\theta}{2\pi} h'_k(\theta) \log(1 + e^{-\epsilon(\theta)}))$$
(43)

In the finite volume interpretation the expectation values correspond to excited states diagonal matrix elements

$$\langle \{\theta\} | q_k | \{\theta\} \rangle_L = i(E')^{\mathrm{dr}}(\theta) \bullet h_k(\theta^{\gamma}) = i \langle j_k^{\gamma} \rangle$$
(44)

$$\langle \{\theta\} | j_k | \{\theta\} \rangle_L = i(p')^{\mathrm{dr}}(\theta) \bullet h_k(\theta^{\gamma}) = i\langle q_k^{\gamma} \rangle \tag{45}$$

where $\{\theta\} \equiv \{\theta_1, \ldots, \theta_n\}$ represents the excited state. The parameters θ_i appearing in the formulas above are not the rapidities of the particles, but they are related to them, although in a model-dependent way.

In the following we elaborate further on these results. For the expectation value of the conserved charge we can write

$$L\langle\{\theta\}|q_k|\{\theta\}\rangle_L = -\sum_j \eta_j h_k(\theta_j + \frac{i\pi}{2}) + i \int \frac{d\theta}{2\pi} h'_k(\theta + \frac{i\pi}{2}) \log(1 + e^{-\epsilon(\theta)})$$
(46)

In the sinh-Gordon model $\eta_k = 1$ and $\theta_k = \overline{\theta}_k + \frac{i\pi}{2}$, where $\overline{\theta}_k$ is the rapidity of the particle, thus our formula reproduces the charge eigenvalue correctly, which asymptotically takes the form $Q_i = \sum_k h_i(\overline{\theta}_k)$.

For the current expectation value we have no such a simplification:

$$\langle \{\theta\}|j_k|\{\theta\}\rangle_L = i\sum_{m,l} \eta_m(p')^{\mathrm{dr}}(\theta_m) G_{ml}^{-1} h_k(\theta_l + \frac{i\pi}{2})^{\mathrm{dr}} + ip'(\theta) \circ h_k(\theta + \frac{i\pi}{2})^{\mathrm{dr}}$$
(47)

where in the last terms $h_k(\theta + \frac{i\pi}{2})^{dr}$ means that $h_k(\theta + \frac{i\pi}{2})$ is dressed. This formula is the main result of our paper, which describes the exact finite volume expectation value of conserved currents. It is equivalent to (45) but written in the form where the polynomial and exponential finite size corrections are separated. Indeed, since the convolution kernel n is exponentially small we can forget the dressing operator in each term to obtain the asymptotic results, which in the sinh-Gordon case, reads as

$$\langle \{\theta\} | j_i | \{\theta\} \rangle_L = \sum_{k,l} E'(\bar{\theta}_k) \bar{G}_{kl}^{-1} h_i(\bar{\theta}_l)$$
(48)

Recall that the Gaudin matrix was also the dressed version of its asymptotic form $\bar{G}_{jk} = \delta_{jk}D(\theta_k) - \varphi_j(\theta_k)$. This formula agrees with the recent direct calculations in [8, 9]. We also checked these formulas in the sinh-Gordon theory against the generalization of the LM formula for excited states [17]. In doing so we had to take into account that [17] is valid in the thermal channel for operators with spins. In the finite volume channel the quasi-particle arguments of the connected form factors should be shifted, similarly how (19) is shifted compared to (12), while the discrete rapidities take their physical values.

Let us finally point out that in deriving our result we used the analytical continuation of the charge eigenvalue (3) in the thermal channel and not the current eigenvalue.

4 Conclusions

Using the analytical continuation method for the vacuum expectation values of conserved charges and currents we managed to derive exact excited state expectation values. We performed this calculation both in the thermal and finite volume settings, where the role of the currents and charges are exchanged. In the finite volume situation the charges act diagonally and have simple eigenvalues, while currents act nondiagonally and have more complicated expectation values. In the asymptotic limit, when vacuum polarization effects are neglected the currents expectation values can be expressed in terms of the charge eigenvalues and the inverse of the Gaudin matrix in agreement with previous calculations [8, 9]. Our results provide all the finite size corrections to the asymptotical formulas valid in a diagonally scattering integrable theory with a single species. Multiparticle generalizations for diagonal scatterings are straightforward as well as the extension for flows generated by other conserved charges. It would be very nice to derive similar formulas for non-diagonally scattering theories. The simplest of such results was obtained for the topological current in the sine-Gordon theory in [18].

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