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Global asymptotic stability of a scalar delay Nicholson's blowflies equation in periodic environment

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Abstract. This paper is considered with a scalar delay Nicholson's blowflies equation in periodic environment. By taking advantage of some novel differential inequality techniques and the fluctuation lemma, we set up the sharp condition to characterize the global asymptotic stability of positive periodic solutions on the addressed equation. The obtained results improve and supplement some existing ones in recent literature, and then give a more perfect answer to an open problem proposed by Berezansky et al. in [*Appl. Math. Model.* **34**(2010), 1405–1417]. In particular, two numerical examples are provided to verify the reliability and feasibility of the theoretical findings.

Keywords: positive periodic solution, global asymptotic stability, scalar delay Nicholson's blowflies equation, sharp condition.

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1 Introduction

Classical population dynamics model

$$x'(t) = -\delta x(t) + \beta x(t-\tau)e^{-x(t-\tau)}, \qquad \delta, \beta, \tau \in (0,\ +\infty), \tag{1.1}$$

is known as Nicholson's blowflies equation [1, 8, 14]. Here $x(t)$ stands for the population of blowflies at time t, δ represents the average daily mortality of adult blowflies, β describes the maximum average daily egg laying rate, and τ denotes mature time delay. Over the past 40 years, plenty of research results have been obtained on the qualitative behaviour and stability of (1.1) (see [1, 3, 4, 10, 11] and their references). In particular, it has been successively shown in [3, 4, 10, 11] that the zero equilibrium point of equation (1.1) possesses global asymptotic stability when $\frac{\beta}{\delta} \leq 1$ and its positive equilibrium point has global asymptotic stability under $1 < \frac{\beta}{\delta} < e^2$. Meanwhile, it was proved in [15] that the positive equilibrium point of (1.1)

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possesses global attractivity when the delay τ is small and $1 < \frac{\beta}{\delta}$. Recently, references [2] and [19] substantiate that the positive equilibrium point of (1.1) is globally asymptotically stable when the condition

$$1 < \frac{\beta}{\delta} \le e^2, \tag{1.2}$$

holds. It is worth pointing out that Yang and So [18] demonstrate the instability of the positive equilibrium and the existence of a Hopf bifurcation when $\frac{\beta}{\delta} > e^2$ and the delay τ is large. This implies that (1.2) is the sharp stability condition on the positive equilibrium points of the autonomous delay Nicholson's blowflies model (1.1).

In general, the external environment of actual organisms often vary periodically with seasonal changes and climate. Therefore, (1.1) can be normally generalized to the following non-autonomous equation:

$$x'(t) = -\delta(t)x(t) + \beta(t)x(t-\tau(t))e^{-x(t-\tau(t))}, \tag{1.3}$$

where $t \ge t_0$, $\delta(t) > 0$, $\beta(t) > 0$ and $\tau(t) \ge 0$ are continuous ω-periodic functions ($\omega > 0$). As we all know, the periodic population dynamics model often generates a globally stable positive periodic solution. Based on this, the authors of [1] proposed an open problem: Establish global asymptotic stability findings on positive periodic solutions of non-autonomous delay Nicholson's blowflies equation. Subsequently, the global attractivity of the positive periodic solutions of (1.3) is established in [12] when the following condition

$$\kappa \approx 0.7215355, \quad \frac{1-\kappa}{e^\kappa} = \frac{1}{e^2} \quad \text{and} \quad e^\kappa < \min_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} \le \max_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} < e^2 \tag{1.4}$$

is obeyed, which gives an answer for the above open problem. Recently, [6] studied the periodicity of the delay Nicholson's blowflies system accompanying patch structure, where the main results involving the periodic scalar Nicholson's blowflies case can be described as follows.

Theorem 1.1. *Suppose m is a nonnegative integer, and*

$$\tau(t) \equiv m\omega, \quad \text{and} \quad 1 < \min_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} \le \max_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} < e^2, \tag{1.5}$$

then the positive periodic solution of the scalar Nicholson's blowflies equation (1.3) is globally attractive.

As in [12], the author of [6] have neither analysed the local stability of positive periodic solutions, nor have they given opinions about the sharp conditions which ensure the global asymptotic stability of positive periodic solutions of (1.3). Therefore, a notable problem naturally arises: What is the sharp condition guaranteeing the globally asymptotic stability of the positive periodic solutions of (1.3)? Because (1.2) is the sharp stability condition on the positive equilibrium points of the autonomous delay Nicholson's blowflies model (1.1), it is reasonable to assert:

$$1 < \min_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} \le \max_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} \le e^2, \tag{1.6}$$

is the sharp condition ensuring the globally asymptotic stability on the periodic solutions of (1.3). To prove this assertion, the ultimate intention of this work is to develop a new strategy to gain the existence and globally asymptotic stability of positive periodic solutions of equation (1.3) under the assumption (1.6) without any other conditions. Meanwhile, we will establish some completely new results on periodic stability of (1.3) without assuming $\tau = m\omega$, and then a more complete answer is given to the open problem on the global periodic stability conditions of Nicholson's blowflies equation in [1].

2 Some lemmas

For convenience, denote

$$\bar{\tau} = \max_{t\in\in[t_0,\ t_0+\omega]} \tau(t), \qquad B = C([-\bar{\tau}, 0],\ \mathbb{R}), \qquad B_+ = \{\varphi \in B | \varphi(\theta) \geq 0, \forall \theta \in [-\bar{\tau},\ 0]\},$$

and let $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ be the solution of (1.3) satisfying the admissible initial conditions:

$$x_{t_0} = \varphi, \qquad \varphi \in B_+ \quad \text{with} \quad \varphi(0) > 0. \tag{2.1}$$

According to the conclusions of Example 2.8 in [7], we have

Lemma 2.1. *If* $1 < \min_{t\in[t_0,t_0+\omega]} \frac{\beta(t)}{\delta(t)}$, *then* $x(t; t_0, \varphi)$ *exists and possesses uniqueness on* $[t_0, +\infty)$. *Moreover,* $x(t; t_0, \varphi)$ *has positiveness and persistence.*

Lemma 2.2 ([5, Lemma 2.3])**.** *Assume* $a \in (0, 2]$, *then*

$$\left| be^{-b} - ae^{-a} \right| < e^{-a}|b - a| \quad \textit{for all}\ \ b > 0 \textit{ and } b \neq a.$$

Lemma 2.3 ([6, Corollary 3.1])**.** *If* $1 < \min_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)}$, *then equation* (1.3) *has a positive* ω*-periodic solution* $x^*(t)$.

Lemma 2.4. *Suppose that* (1.6) *holds, and equation* (1.3) *has a positive* ω*-periodic solution* $x^*(t)$ *satisfying* $\max_{t\in[t_0,\ t_0+\omega]} x^*(t) \leq 2$, *then* $x^*(t)$ *is globally asymptotically stable.*

Proof. Obviously,

$$0 < k^* := \min_{t\in[t_0,\ t_0+\omega]} x^*(t) \leq \max_{t\in[t_0,\ t_0+\omega]} x^*(t) \leq 2. \tag{2.2}$$

For all $t \in [t_0 - \bar{\tau},\ +\infty)$, let us introduce the notations

$$x(t) = x(t; t_0, \varphi) \quad \text{and} \quad w(t) = \frac{x(t)}{x^*(t)} - 1.$$

Then, for all $t \geq t_0$,

$$\begin{aligned} w'(t) = \frac{\beta(t)}{x^*(t)} \Big\{ & - x^*(t - \tau(t))e^{-x^*(t-\tau(t))}w(t) \\ & + \big[x^*(t - \tau(t))(w(t - \tau(t)) + 1)e^{-x^*(t-\tau(t))(w(t-\tau(t))+1)} \\ & - x^*(t - \tau(t))e^{-x^*(t-\tau(t))}\big]\Big\}. \end{aligned} \tag{2.3}$$

Now, we prove the local stability of $x^*(t)$.

For arbitrary $\varepsilon > 0$, let $H = \frac{k^*\varepsilon}{2}$ and $\|\varphi - x^*\| < H$ with $\|\cdot\|$ denoting the supremum norm, we shall reveal $|x(t) - x^*(t)| < \varepsilon$ for all $t \in [t_0 - \bar{\tau}, +\infty)$. Noting that

$$|w(t)| = \left|\frac{\varphi(t) - x^*(t)}{x^*(t)}\right| < \frac{H}{x^*(t)} \leq \frac{H}{k^*} \quad \text{for arbitrary } t \in [t_0 - \bar{\tau},\ t_0],$$

we assert that

$$|w(t)| < \frac{H}{k^*} \quad \text{for arbitrary } t > t_0. \tag{2.4}$$

Otherwise, there exists $S_1 > t_0$ such that either

$$w(S_1) = \frac{H}{k^*} \quad \text{and} \quad |w(t)| < \frac{H}{k^*} \quad \text{for arbitrary } t \in [t_0 - \bar{\tau},\ S_1) \tag{2.5}$$

or

$$w(S_1) = -\frac{H}{k^*} \quad \text{and} \quad |w(t)| < \frac{H}{k^*} \quad \text{for arbitrary } t \in [t_0 - \bar{\tau},\ S_1) \tag{2.6}$$

holds.

Assuming that (2.5) holds, from Lemma 2.2, we acquire that for $w(S_1 - \tau(S_1)) \neq 0$,

$$\begin{aligned}
0 &\leq w'(S_1) \\
&\leq \frac{\beta(S_1)}{x^*(S_1)}\Big\{ - x^*(S_1 - \tau(S_1))e^{-x^*(S_1-\tau(S_1))}w(S_1) + |x^*(S_1 - \tau(S_1))(w(S_1 - \tau(S_1)) + 1) \\
&\qquad\qquad \times e^{-x^*(S_1-\tau(S_1))(w(S_1-\tau(S_1))+1)} - x^*(S_1 - \tau(S_1))e^{-x^*(S_1-\tau(S_1))}|\Big\} \\
&< \frac{\beta(S_1)}{x^*(S_1)}\Big\{ - x^*(S_1 - \tau(S_1))e^{-x^*(S_1-\tau(S_1))}\frac{H}{k^*} + x^*(S_1 - \tau(S_1))\,|w(S_1 - \tau(S_1))|\, e^{-x^*(S_1-\tau(S_1))}\Big\} \\
&= \frac{\beta(S_1)}{x^*(S_1)}x^*(S_1 - \tau(S_1))e^{-x^*(S_1-\tau(S_1))}\left[-\frac{H}{k^*} + |w(S_1 - \tau(S_1))|\right] \\
&\leq 0,
\end{aligned}$$

which is an obvious contradiction. Similarly, one can derive a contradiction from the situation (2.6). Moreover, when $w(S_1 - \tau(S_1)) = 0$, one can also derive the above contradiction. Thus, the assertion (2.4) is true and

$$|x(t) - x^*(t)| < x^*(t)\frac{H}{k^*} \leq \varepsilon \quad \text{for all } \ t \in [t_0,\ +\infty),$$

which follows that $x^*(t)$ is locally stable.

Next, we demonstrate the global attractivity of $x^*(t)$. Let

$$\mu = \limsup_{t\to+\infty} \omega(t) \quad \text{and} \quad \lambda = \liminf_{t\to+\infty} \omega(t).$$

Clearly, the global attractivity of $x^*(t)$ is equivalent to show $\max\{|\mu|, |\lambda|\} = 0$. In order to obtain a contradiction, we just assume $\max\{|\mu|, |\lambda|\} = \mu > 0$ (the situation of $\max\{|\mu|, |\lambda|\} = -\lambda > 0$ is similar). According to the fluctuation lemma [16, Lemma A.1.], one can find a sequence $\{s_k\}_{k=1}^{+\infty}$ obeying

$$\lim_{k\to+\infty} s_k = +\infty, \qquad \lim_{k\to+\infty} w(s_k) = \mu, \quad \text{and} \quad \lim_{k\to+\infty} w'(s_k) = 0.$$

Without any loss of generality, we may also assume that $\lim_{k\to+\infty} \beta(s_k)$, $\lim_{k\to+\infty} \delta(s_k)$, $\lim_{k\to+\infty} w(s_k - \tau(s_k))$, $\lim_{k\to+\infty} x^*(s_k - \tau(s_k))$ and $\lim_{k\to+\infty} x^*(s_k)$ exist. It follows from (2.3) and Lemma 2.2 that for $\lim_{k\to+\infty} w(s_k - \tau(s_k)) \neq 0$,

$$\begin{aligned}
0 &= \lim_{k\to+\infty} w'(s_k) \\
&= \frac{\lim_{k\to+\infty} \beta(s_k)}{\lim_{k\to+\infty} x^*(s_k)}\Big\{ - e^{-\lim_{k\to+\infty} x^*(s_k-\tau(s_k))} \lim_{k\to+\infty} x^*(s_k - \tau(s_k)) \lim_{k\to+\infty} w(s_k) \\
&\quad + \Big[\lim_{k\to+\infty} x^*(s_k - \tau(s_k))(1 + \lim_{k\to+\infty} w(s_k - \tau(s_k)))e^{-\lim_{k\to+\infty} x^*(s_k-\tau(s_k))(1+\lim_{k\to+\infty} w(s_k-\tau(s_k)))} \\
&\quad - \lim_{k\to+\infty} x^*(s_k - \tau(s_k))e^{-\lim_{k\to+\infty} x^*(s_k-\tau(s_k))}\Big]\Big\}
\end{aligned}$$

$$< \frac{\lim_{k\to+\infty}\beta(s_k)}{\lim_{k\to+\infty}x^*(s_k)}\Big\{ -e^{-\lim_{k\to+\infty}x^*(s_k-\tau(s_k))}\lim_{k\to+\infty}x^*(s_k-\tau(s_k))\lim_{k\to+\infty}w(s_k)$$
$$+e^{-\lim_{k\to+\infty}x^*(s_k-\tau(s_k))}\lim_{k\to+\infty}x^*(s_k-\tau(s_k))\lim_{k\to+\infty}|w(s_k-\tau(s_k))|\Big\}$$
$$\leq 0,$$

which leads to a contradiction. Especially, if $\lim_{k\to+\infty} w(s_k-\tau(s_k))=0$, the above contradiction is obvious. This yields $\max\{|\mu|,|\lambda|\}=0$, and the proof of Lemma 2.4 is finished. □

3 Globally asymptotic stability of positive periodic solutions

Theorem 3.1. *Let* (1.6) *be satisfied, and m be a nonnegative integer obeying $\tau(t)\equiv m\omega$. Then, equation* (1.3) *possesses a globally asymptotically stable positive periodic solution.*

Proof. On account of Lemma 2.3, one can discover that equation (1.3) possesses a positive ω-periodic solution $x^*(t)$. In view of Lemma 2.4, to finish the proof of Theorem 3.1, we only need to reveal that $\max_{t\in[t_0,\,t_0+\omega]}x^*(t)\leq 2$. For this purpose, let

$$\sigma\in(0,1),\quad \sigma e^{-\sigma}=2e^{-2},\quad \text{and}\quad \eta=\sup\left\{\rho\mid \beta(t)e^{-\rho}>\delta(t),\ t\in[0,\omega],\ \rho>0\right\}. \tag{3.1}$$

We claim that

$$k^{**}:=\min\{\sigma,\eta\}\leq x^*(t)\leq 2\quad \text{for arbitrary } t\in\mathbb{R}.$$

In fact, let $t_1,t_2\in[\omega,2\omega]$ such that

$$x^*(t_1)=\max_{t\in\mathbb{R}}x^*(t)\quad\text{and}\quad x^*(t_2)=\min_{t\in\mathbb{R}}x^*(t),$$

then

$$0=-\delta(t_i)\,x^*(t_i)+\beta(t_i)\,x^*(t_i)\,e^{-x^*(t_i)}\qquad (i=1,2).$$

Hence, from (1.6) and (3.1), we acquire

$$\begin{aligned} e^{x^*(t_1)}&=\frac{\beta(t_1)}{\delta(t_1)}\leq e^2\quad\text{with}\quad x^*(t_1)\leq 2,\quad\text{and}\\ \delta(t_2)&=\beta(t_2)\,e^{-x^*(t_2)}\quad\text{with}\quad x^*(t_2)\geq\eta\geq k^{**}, \end{aligned} \tag{3.2}$$

which finishes the proof. □

Remark 3.2. Evidently, Theorem 1.1 as a main conclusion in [6] is a direct corollary of Theorem 3.1 in this present paper, and the proof of our conclusion is only established under sharp condition (1.6). Meanwhile, we present a detailed proof of the local stability of positive periodic solutions, which is not involved in the existing literature [6,12]. Therefore, the conclusion of this paper improves and generalizes the corresponding ones of the above literature, which provides a more perfect answer to the open problem in [1] which has been mentioned in the Introduction section of this article.

Theorem 3.3. *Assume $\beta^+=\max_{t\in[t_0,\,t_0+\omega]}\beta(t)$ and*

$$1<\min_{t\in[t_0,\,t_0+\omega]}\frac{\beta(t)}{\delta(t)}\leq\max_{t\in[t_0,\,t_0+\omega]}\frac{\beta(t)}{\delta(t)}\leq\max_{t\in[t_0,\,t_0+\omega]}\frac{\beta(t)}{\delta(t)-\tau(t)\beta(t)\beta^+}\leq e^2. \tag{3.3}$$

Then, equation (1.3) *possesses a globally asymptotically stable positive periodic solution.*

Proof. As is seen from Lemma 2.3 and Lemma 2.4, we only need to verify that the positive ω-periodic solution $x^*(t)$ of (1.3) satisfies $\max_{t\in[t_0,\ t_0+\omega]} x^*(t) \le 2$. In order to do this, denote $x(t) = x(t; t_0, \varphi)$ for arbitrary $t \in [t_0 - \bar{\tau},\ +\infty)$, and

$$L = \limsup_{t\to+\infty} x(t), \qquad l = \liminf_{t\to+\infty} x(t). \tag{3.4}$$

Apparently, Lemma 2.1 yields $l > 0$. Now, we verify $L \le 2$. Again from the fluctuation lemma [16, Lemma A.1.], one can pick $\{t_k\}_{k=1}^{+\infty}$ such that

$$\lim_{k\to+\infty} t_k = +\infty, \qquad \lim_{k\to+\infty} x'(t_k) = 0, \qquad \lim_{k\to+\infty} x(t_k) = L. \tag{3.5}$$

Also, we suppose that $\lim_{k\to+\infty} \delta(t_k), \lim_{k\to+\infty} \beta(t_k)$ and $\lim_{k\to+\infty} \tau(t_k)$ exist.

For any $\varepsilon > 0$, it is easy to find $N > 0$ satisfying that $x(t) < L + \varepsilon$ for arbitrary $t > N$, and hence for arbitrary $t \in (N + \bar{\tau},\ +\infty)$,

$$-\delta(t)(L+\varepsilon) < -\delta(t)x(t) + \beta(t)x(t-\tau(t))e^{-x(t-\tau(t))} < \beta(t)(L+\varepsilon).$$

Furthermore,

$$|x'(t)| < \beta(t)(L+\varepsilon), \qquad t \in (N+\bar{\tau},\ +\infty),$$

and

$$\begin{aligned} x'(t_k) &= -\delta(t_k)x(t_k) + \beta(t_k)x(t_k)e^{-x(t_k)} + \beta(t_k)[x(t_k-\tau(t_k))e^{-x(t_k-\tau(t_k))} - x(t_k)e^{-x(t_k)}] \\ &\le -\delta(t_k)x(t_k) + \beta(t_k)x(t_k)e^{-x(t_k)} + \beta(t_k)|(1-\theta)e^{-\theta}||x(t_k) - x(t_k-\tau(t_k))| \\ &\le -\delta(t_k)x(t_k) + \beta(t_k)x(t_k)e^{-x(t_k)} + \beta(t_k)\int_{t_k-\tau(t_k)}^{t_k} |x'(s)|ds \\ &\le -\delta(t_k)x(t_k) + \beta(t_k)x(t_k)e^{-x(t_k)} + \beta^+\beta(t_k)\tau(t_k)(L+\varepsilon), \qquad t_k > N+\bar{\tau}, \end{aligned} \tag{3.6}$$

where θ is the mean value in the Differential Mean Value Theorem. From (3.5), taking the limits on both sides of (3.6) leads to

$$e^L \le \frac{\lim\limits_{k\to+\infty} \beta(t_k)}{\lim\limits_{k\to+\infty} \delta(t_k) - \lim\limits_{k\to+\infty} \beta^+\beta(t_k)\tau(t_k)\frac{L+\varepsilon}{L}}.$$

Let $\varepsilon \to 0$, from (3.3), we derive

$$e^L \le \lim_{k\to+\infty} \frac{\beta(t_k)}{\delta(t_k) - \beta^+\beta(t_k)\tau(t_k)} \le \max_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t) - \tau(t)\beta(t)\beta^+}, \quad \text{and} \quad L \le 2.$$

Thus, the positive ω-periodic solution $x^*(t)$ of system (1.3) obeys $\max_{t\in[t_0,\ t_0+\omega]} x^*(t) \le 2$. The verification of Theorem 3.3 is completed. □

Remark 3.4. Theorem 3.3 is established without the assumption of $\tau(t) \equiv m\omega$, and it is easy to verify the feasibility of the conditions (3.3) when the delay is small. Meanwhile, the condition (3.3) is equivalent to the sharp condition (1.6) when the delay vanishes to zero.

4 Numerical simulations

Regard the following scalar delay Nicholson's blowflies equation:

$$x'(t) = -(1+|\sin t|)x(t) + (1+|\sin t|)(1.01 + (e^2 - 1.01)|\cos t|)x(t-2\pi)e^{-x(t-2\pi)}, \quad (4.1)$$

and

$$x'(t) = -(1+|\sin t|)x(t) + (1+|\sin t|)(1.05 + (e^2 - 1.1)|\cos t|)x\left(t - \frac{1}{50e^4}|\cos t|\right)e^{-x(t-\frac{1}{50e^4}|\cos t|)}, \quad (4.2)$$

where $t \geq t_0 = 0$. It is easy to verify that (4.1) and (4.2) satisfy

$$\tau(t) \equiv 2\pi, \qquad 1.01 = \min_{t\in[t_0,\, t_0+\pi]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\, t_0+\pi]} \frac{\beta(t)}{\delta(t)} = e^2, \quad (4.3)$$

and

$$1.05 = \min_{t\in[t_0,\, t_0+\pi]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\, t_0+\pi]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\, t_0+\pi]} \frac{\beta(t)}{\delta(t) - \tau(t)\beta(t)\beta^+} \approx e^2 - 0.02, \quad (4.4)$$

respectively. Therefore, from Theorems 3.1 and 3.3, we know that the above two scalar Nicholson's blowflies models possess global asymptotic stable positive π-periodic solutions. The numerical simulation results of the two examples are shown in Figures 4.1–4.2, and the trajectories of the solutions strongly confirm the correctness and validity of the results in this paper.

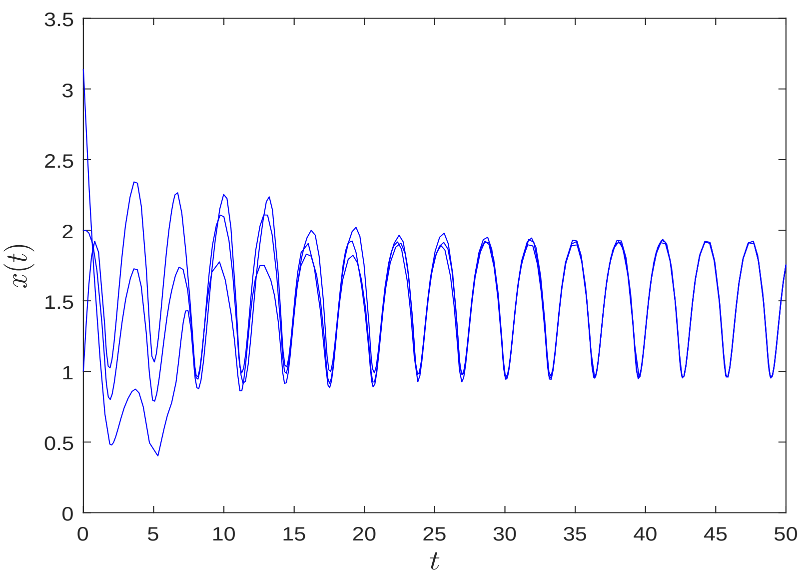

Figure 4.1: Numerical state trajectories of model (4.1) involving the initial values: $1, 2, \pi$, respectively.

Remark 4.1. Nicholson's blowflies equation (4.1) does not satisfy the condition (1.4), equation (4.2) does not obey the conditions (1.4) and (1.5), which have been adopted as fundamental assumptions for the considered periodicity of (1.3) in [6, 12]. Consequently, the conclusions in [6, 12] can not be directly employed to illustrate the globally asymptotic stability of the

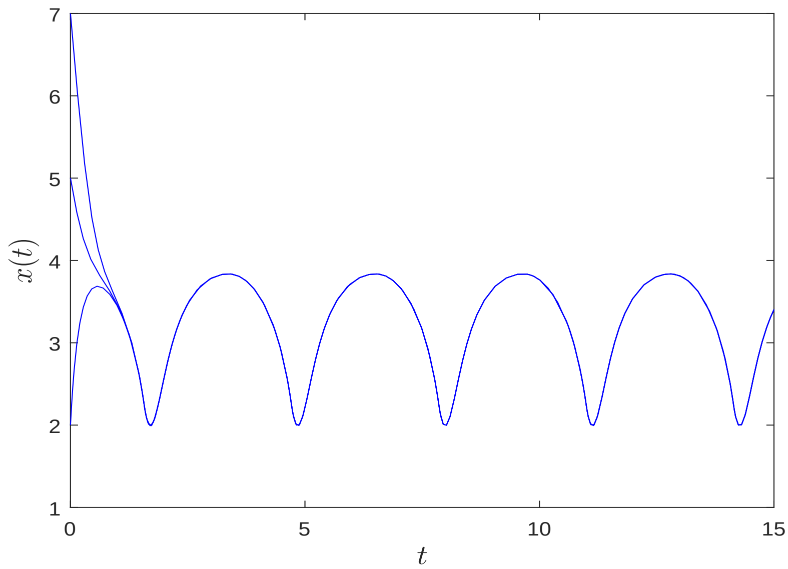

Figure 4.2: Numerical state trajectories of model (4.2) involving the initial values: $2, 5, 7$, respectively.

positive periodic solutions for (4.1) and (4.2), which indicates that the results of this paper improve and extend the corresponding ones of [6,9,12,13,17] and the references cited therein. It is noteworthy that, the method presented in this article can be used to explore the sharp condition of the existence and global asymptotic stability of positive periodic solutions to the scalar Nicholson's blowflies models involving multiple time-varying delays in [12] and the delay Nicholson's blowflies systems accompanying patch structure in [6].

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