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Oscillation of half-linear differential equations with mixed type of argument

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Abstract. This paper is devoted to the study of the oscillatory behavior of half-linear functional differential equations with deviating argument of the form

$$\big(r(t)(y'(t))^{\alpha}\big)' = p(t)y^{\alpha}(\tau(t)). \tag{E}$$

We introduce new technique based on monotonic properties of nonoscillatory solutions to offer new oscillatory criteria for (E). We will show that presented results essentially improve existing ones even for linear differential equations.

Keywords: second order differential equations, delay, advanced, monotonic properties, oscillation.

2020 Mathematics Subject Classification: 34K11, 34C10.

1 Introduction

We consider half-linear second order differential equations with deviating argument

$$\big(r(t)(y'(t))^{\alpha}\big)' = p(t)y^{\alpha}(\tau(t)). \tag{E}$$

Throughout the paper it is assumed that

(H_1) $p, r \in C([t_0, \infty))$, $p(t) > 0$, $r(t) > 0$, α is the ratio of two positive odd integers,

(H_2) $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) \geq 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a solution of Eq. (E) we mean a function $y(t) \in C^1([T_y, \infty))$, $T_y \geq t_0$, such that $r(t)(y'(t))^{\alpha} \in C^1([T_y, \infty))$ and $y(t)$ satisfies Eq. (E) on $[T_y, \infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_y, \infty)$ and otherwise it is called to be nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

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Throughout the paper we consider (E) in canonical form, that is,

$$R(t)=\int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}\,\mathrm{d}s\to\infty \quad \text{as } t\to\infty.$$

The problem of establishing oscillatory criteria for various types of differential equations has been a very active research area over the past decades (see [1–11]).

The half-linear ordinary differential equation

$$\left((y'(t))^{\alpha}\right)'=p(t)y^{\alpha}(t)$$

to which (E) reduces when $\tau(t)\equiv t$ and $r(t)\equiv 1$ is nonoscillatory in the sense that all of its solutions are nonoscillatory; see Elbert [6]. However, the presence of deviating argument $\tau(t)\not\equiv t$ may generate oscillation of some or all of its solutions.

It is known that (E) may possess only two types of nonoscillatory solutions. So, if $y(t)$ is a nonoscillatory solution of (E) it is easy to see that $y'(t)$ is eventually of constant sign, so that either

$$y(t)y'(t)<0 \tag{1.1}$$

or

$$y(t)y'(t)>0, \tag{1.2}$$

eventually. Moreover, if $y(t)$ is an eventually positive solution satisfying inequality (1.2), then $r(t)(y'(t))^{\alpha}>k>0$, and an integration of $y'(t)>\frac{k^{1/\alpha}}{r^{1/\alpha}(t)}$ yields

$$y(t)\ge k^{1/\alpha}\int_{t_1}^{t}\frac{1}{r^{1/\alpha}(s)}\,\mathrm{d}s\to\infty \quad \text{as } t\to\infty.$$

Consequently, if (E) is in canonical form, then $y(t)$ is bounded or unbounded according to whether (1.1) or (1.2) holds. Effort of mathematicians was aimed to show that (E) admits no bounded or unbounded nonoscillatory solutions in the case where $\tau(t)$ is retarded ($\tau(t)\le t$) or advanced argument ($\tau(t)\ge t$), respectively. To illustrate this we recall classical result of Kusano and Lalli [10].

Theorem A. Suppose that

(i) $\tau(t)<t$ and

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}\left(\frac{1}{r(u)}\int_{s}^{t}p(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u>1.$$

Then (E) has no bounded nonoscillatory solutions.

(ii) If $\tau(t)>t$ and

$$\limsup_{t\to\infty}\int_{t}^{\tau(t)}\left(\frac{1}{r(u)}\int_{t}^{s}p(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u>1.$$

Then (E) has no unbounded nonoscillatory solutions.

In this paper we are interested in the situation when $\tau(t)$ is of mixed type which means that its retarded part

$$\mathcal{R}_{\tau}=\{t\in(t_0,\infty):\tau(t)<t\}$$

and its advanced part

$$\mathcal{A}_{\tau}=\{t\in(t_0,\infty):\tau(t)>t\}$$

are both unbounded subset of (t_0, ∞). The presence of mixed argument may cause that (E) has neither bounded nor unbounded nonoscillatory solutions which means oscillation of (E). This fact has been observed by Kusano [9], who showed that the second order differential equation

$$y''(t) = p_0 y(t + \sin t) \tag{E_x}$$

is oscillatory provided that

$$p_0 \geq \frac{1}{\sin 1 - 0.5} \approx 2.9285. \tag{1.3}$$

In this paper we present new technique for investigation of (E) with mixed argument and the progress achieved will be demonstrated via equation (E_x) and its oscillatory criterion (1.3).

2 Main results

We are about to establish new criteria for (E) to do not possess neither bounded nor unbounded solutions. We start with some useful lemma concerning monotonic properties of nonoscillatory solutions for studied equations.

Lemma 2.1. *Let that there exist a sequence $\{t_k\}$ such that $t_k \in \mathcal{R}_\tau$, $t_k \to \infty$ as $k \to \infty$. Assume that $y(t)$ is a positive bounded solution of (E). If there exists some positive constant β such that for all $k \in \{1, 2, \dots\}$*

$$\left[\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s\right]^{1/\alpha} \geq \beta \quad \text{on } [\tau(t_k), t_k], \tag{2.1}$$

then $y(\tau(t))\mathrm{e}^{\beta R(\tau(t))}$ is decreasing on all $[\tau(t_k), t_k]$.

Proof. Assume that $y(t)$ is a positive decreasing solution of (E) and $t \in [\tau(t_k), t_k]$. An integration of (E) from $\tau(t)$ to t yields

$$r(t)\left(y'(t)\right)^{\alpha} - r(\tau(t))\left(y'(\tau(t))\right)^{\alpha} \geq y^{\alpha}(\tau(t)) \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s \geq \beta^{\alpha} y^{\alpha}(\tau(t)).$$

That is

$$-r^{1/\alpha}(\tau(t))y'(\tau(t)) \geq \beta\, y(\tau(t)).$$

Therefore

$$\left[y(\tau(t))\mathrm{e}^{\beta R(\tau(t))}\right]' = \frac{\mathrm{e}^{\beta R(\tau(t))}\tau'(t)}{r^{1/\alpha}(\tau(t))}\left[\beta y(\tau(t)) + r^{1/\alpha}(\tau(t))y'(\tau(t))\right] \leq 0$$

and we conclude that function $y(\tau(t))\mathrm{e}^{\beta R(\tau(t))}$ is decreasing. The proof is complete. □

Now we apply the above monotonicity to establish criterion for absence of decreasing solutions.

Theorem 2.2. *Let that there exist a sequence $\{t_k\}$ such that $t_k \in \mathcal{R}_\tau$, $t_k \to \infty$ as $k \to \infty$ and (2.1) hold. If*

$$\limsup_{k\to\infty} \mathrm{e}^{\beta R(\tau(t_k))} \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left[\int_{u}^{t_k} p(s)\,\mathrm{e}^{-\alpha\beta R(\tau(s))}\,\mathrm{d}s\right]^{1/\alpha} \mathrm{d}u > 1, \tag{2.2}$$

then (E) has no bounded nonoscillatory solutions.

Proof. Assume on the contrary, that (E) possesses an eventually positive decreasing solution $y(t)$. We assume that $u \in [\tau(t_k), t_k]$ Integrating (E) from u to t_k and using monotonic property of $\mathrm{e}^{\beta R(\tau(t))} y(\tau(t))$, we obtain

$$\begin{aligned} -r(u)(y'(u))^\alpha &\geq \int_u^{t_k} p(s) y^\alpha(\tau(s)) \mathrm{e}^{\alpha\beta R(\tau(s))} \mathrm{e}^{-\alpha\beta R(\tau(s))}\, \mathrm{d}s \\ &\geq y^\alpha(\tau(t_k)) \mathrm{e}^{\alpha\beta R(\tau(t_k))} \int_u^{t_k} p(s) \mathrm{e}^{-\alpha\beta R(\tau(s))}\, \mathrm{d}s. \end{aligned}$$

Extracting the α root and integrating once more from $\tau(t_k)$ to t_k, we get

$$y(\tau(t_k)) \geq \mathrm{e}^{\beta R(\tau(t_k))} y(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(v)} \left[\int_v^{t_k} p(s) \mathrm{e}^{-\alpha\beta R(\tau(s))}\, \mathrm{d}s \right]^{1/\alpha} \mathrm{d}v$$

which contradicts to condition (2.2) and we conclude, that (E) does not possess decreasing solutions. □

Now we turn our attention to monotonic properties for possible unbounded solutions of (E).

Lemma 2.3. *Let that there exist a sequence $\{s_k\}$ such that $s_k \in \mathcal{A}_\tau$, $s_k \to \infty$ as $k \to \infty$. Assume that $y(t)$ is a positive unbounded solution of (E). If there exists some positive constant γ such that for all $k \in \{1, 2, \dots\}$*

$$\left[\int_t^{\tau(t)} p(s)\, \mathrm{d}s \right]^{1/\alpha} \geq \gamma \quad \text{on } [s_k, \tau(s_k)], \tag{2.3}$$

then $y(\tau(t))\mathrm{e}^{-\gamma R(\tau(t))}$ is increasing on all $[s_k, \tau(s_k)]$.

Proof. Assume that $y(t)$ is a positive increasing solution of (E) and $t \in [s_k, \tau(s_k)]$. An integration of (E) from t to $\tau(t)$ yields

$$r(\tau(t)) \left(y'(\tau(t)) \right)^\alpha \geq y^\alpha(\tau(t)) \int_t^{\tau(t)} p(s)\, \mathrm{d}s \geq \gamma^\alpha y^\alpha(\tau(t)).$$

This means

$$r^{1/\alpha}(\tau(t)) y'(\tau(t)) \geq \gamma\, y(\tau(t)).$$

it is easy to see that

$$\left[y(\tau(t)) \mathrm{e}^{-\gamma R(\tau(t))} \right]' = \frac{\mathrm{e}^{-\gamma R(\tau(t))} \tau'(t)}{r^{1/\alpha}(\tau(t))} \left[-\gamma y(\tau(t)) + r^{1/\alpha}(\tau(t)) y'(\tau(t)) \right] \geq 0$$

and we verified that function $y(\tau(t))\mathrm{e}^{-\gamma R(\tau(t))}$ is increasing. The proof is complete. □

Theorem 2.4. *Let that there exist a sequence $\{s_k\}$ such that $s_k \in \mathcal{A}_\tau$, $s_k \to \infty$ as $k \to \infty$ and (2.3) hold. If*

$$\limsup_{k \to \infty} \mathrm{e}^{-\gamma R(\tau(s_k))} \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(u)} \left[\int_{s_k}^u p(s)\, \mathrm{e}^{\alpha\gamma R(\tau(s))}\, \mathrm{d}s \right]^{1/\alpha} \mathrm{d}u > 1, \tag{2.4}$$

then (E) has no unbounded nonoscillatory solutions.

Proof. Assume on the contrary, that (E) has an eventually positive increasing solution $y(t)$. We consider $u \in [s_k, \tau(s_k)]$. Integrating (E) from s_k to u and employing monotonic property of $\mathrm{e}^{-\gamma R(\tau(t))} y(\tau(t))$, one gets

$$\begin{aligned} r(u)(y'(u))^\alpha &\geq \int_{s_k}^{u} p(s) y^\alpha(\tau(s)) \mathrm{e}^{-\alpha\gamma R(\tau(s))} \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s \\ &\geq y^\alpha(\tau(s_k)) \mathrm{e}^{-\alpha\gamma R(\tau(s_k))} \int_{s_k}^{u} p(s) \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s. \end{aligned}$$

Simplifying and then integrating once more from s_k to u we obtain

$$y(u) \geq \mathrm{e}^{-\gamma R(\tau(s_k))} y(\tau(s_k)) \int_{s_k}^{u} \frac{1}{r^{1/\alpha}(v)} \left[\int_{s_k}^{v} p(s) \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}v.$$

Putting $u = \tau(s_k)$, we have

$$y(\tau(s_k)) \geq \mathrm{e}^{-\gamma R(\tau(s_k))} y(\tau(s_k)) \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(v)} \left[\int_{s_k}^{v} p(s) \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}v$$

which contradicts to condition (2.4) and we conclude, that (E) does not possess decreasing solutions. □

Picking up the previous results we can formulate the following oscillatory criterion.

Theorem 2.5. *Assume that there exist two sequences $\{t_k\}$ and $\{s_k\}$ such that $t_k \in \mathcal{R}_\tau$, $s_k \in \mathcal{A}_\tau$ $t_k, s_k \to \infty$ as $k \to \infty$. Let β and γ be defined by* (2.1) *and* (2.3)*, respectively. If* (2.2) *and* (2.4) *are satisfied, then* (E) *is oscillatory.*

3 Examples

Example 3.1. We consider the differential equation

$$y''(t) = py(t + \sin t), \qquad p > 0. \tag{E_x}$$

We shall show that (E_x) is oscillatory provided that $p \geq p_0 = 1.5955$.

To verify that (E_x) has no bounded nonoscillatory solutions we set $t_k = \frac{3}{2}\pi + 2k\pi$. Then it is easy to see that $\tau(t_k) = \frac{3}{2}\pi - 1 + 2k\pi$. So condition (2.1) reduces to

$$-p\sin(t) \geq -p_0 \sin(t) \geq \beta \quad \text{on } [\tau(t_k), t_k], \quad k = 1, 2, \dots$$

Since $-p_0 \sin(t)$ is increasing function on $[\tau(t_k), t_k]$, we can choose

$$\beta = -p_0 \sin(\tau(t_k)) = p_0 \cos 1,$$

On the other hand, condition (2.2) for (E_x) takes the form

$$\limsup_{k\to\infty} \mathrm{e}^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} \int_{u}^{t_k} p_0 \,\mathrm{e}^{-\beta\tau(s)} \,\mathrm{d}s \,\mathrm{d}u > 1. \tag{3.1}$$

Changing order of integration in (3.1) we get simpler form

$$\limsup_{k\to\infty} p_0 \mathrm{e}^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} \mathrm{e}^{-\beta\tau(s)} (s - \tau(t_k)) \,\mathrm{d}s > 1. \tag{3.2}$$

Setting the corresponding values into (3.2) one gets

$$p_0 \mathrm{e}^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} \mathrm{e}^{-\beta\tau(s)}(s-\tau(t_k))\,\mathrm{d}s$$
$$= p_0 \mathrm{e}^{\beta\left(\frac{3}{2}\pi-1+2k\pi\right)} \int_{\tau(t_k)}^{t_k} e^{-\beta(s+\sin s)} \left(s-\left(\frac{3}{2}\pi-1+2k\pi\right)\right) \mathrm{d}s.$$

Substitution $s-\left(\frac{3}{2}\pi-1+2k\pi\right)=t$ yields

$$p_0 \mathrm{e}^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} \mathrm{e}^{-\beta\tau(s)}(s-\tau(t_k))\,\mathrm{d}s = p_0 \int_0^1 t e^{-\beta(t-\cos(1-t))}\,\mathrm{d}t$$
$$= 0.6268 p_0 = 1.000004 > 1,$$

where for evaluation of the above integral we employed Matlab. We have verified that (2.2) holds true and by Theorem 2.2 (E_x) has no bounded solutions.

On the other hand, to ensure that (E_x) has no unbounded nonoscillatory solutions we chose $s_k = \frac{\pi}{2} + 2k\pi$. Then $\tau(s_k) = \frac{\pi}{2} + 1 + 2k\pi$. Now,condition (2.4) takes the form

$$p\sin(t) \ge p_0 \sin(t) \ge \gamma \quad \text{on } [s_k, \tau(s_k)], \quad k = 1, 2, \ldots$$

Using the fact that $p_0 \sin(t)$ is decreasing function on $[s_k, \tau(s_k)]$, we set

$$\gamma = p_0 \sin(\tau(s_k)) = p_0 \cos 1 = \beta,$$

Condition (2.4) reduces to

$$\limsup_{k\to\infty} \mathrm{e}^{-\gamma\tau(t_k)} \int_{s_k}^{\tau(s_k)} \int_{s_k}^{u} p_0\, \mathrm{e}^{\gamma\tau(s)}\,\mathrm{d}s\,\mathrm{d}u > 1$$

which is equivalent to

$$\limsup_{k\to\infty} p_0 \mathrm{e}^{-\gamma\tau(s_k)} \int_{s_k}^{\tau(s_k)} \mathrm{e}^{\gamma\tau(s)}(\tau(s_k)-s)\,\mathrm{d}s > 1,$$

which for parameters of (E_x) means

$$p_0 \mathrm{e}^{-\gamma\tau(s_k)} \int_{s_k}^{\tau(s_k)} \mathrm{e}^{\gamma\tau(s)}(\tau(s_k)-s)\,\mathrm{d}s$$
$$= p_0 \mathrm{e}^{-\gamma\left(\frac{\pi}{2}+1+2k\pi\right)} \int_{s_k}^{\tau(s_k)} e^{\gamma(s+\sin s)} \left(\frac{\pi}{2}+1+2k\pi-s\right) \mathrm{d}s.$$

Substitution $\frac{\pi}{2}+1+2k\pi-s=t$ provides

$$p_0 \mathrm{e}^{-\gamma\tau(s_k)} \int_{s_k}^{\tau(s_k)} \mathrm{e}^{\gamma\tau(s)}(\tau(s_k)-s)\,\mathrm{d}s = p_0 \int_0^1 t e^{-\beta(t-\cos(1-t))}\,\mathrm{d}t$$
$$= 0.6268 p_0 = 1.000004 > 1,$$

Consequently, condition (2.4) is satisfied and by Theorem 2.4 Eq. (E_x) has no unbounded nonoscillatory solutions. By comparing with Kusano's result mentioned in the motivation part, our oscillatory constant is significantly better.

4 Summary

In this paper we improved Kusano's technique for investigation of differential equations with mixed arguments. The progress achieved has been presented via Kusano's differential equation.

As a matter of fact the results presented in this paper can be rewritten also for differential equation of the form

$$\big(r(t)|y'(t)|^{\alpha}\operatorname{sgn} y'(t)\big)' = p(t)|y(\tau(t))|^{\alpha}\operatorname{sgn} y(\tau(t)).$$

The details are left to the reader.

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References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluver Academic Publishers, Dordrecht 2002. https://doi.org/10.1007/978-94-017-2515-6; MR2091751; Zbl 1073.34002

[2] B. Baculikova, Oscillation of second-order nonlinear noncanonical differential equations with deviating argument, *Appl. Math. Lett.* **91**(2019), 68–75. https://doi.org/10.1016/j.aml.2018.11.021; MR3892107; Zbl 1408.34046

[3] B. BaculÃ■kovÃą, Oscillation of second-order half-linear differential equations with deviating arguments of mixed type, *Appl. Math. Lett.* **119**(2021), Paper No. 107228, 8 pp. https://doi.org/10.1016/j.aml.2021.107228; MR4236317; Zbl 07410066

[4] B. BaculÃ■kovÃą, Oscillatory behavior of the second order noncanonical differential equation, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 89, 1–11. https://doi.org/10.14232/ejqtde.2019.1.89; MR4040408; Zbl 1449.34220

[5] O. Došlý, P. Řehák, *Half-linear differential equations*, North-Holland Mathematics Studies, Vol. 202, Elsevier Science B.V., Amsterdam, 2005. MR2158903 Zbl 1090.34001

[6] ÃĄ. Elbert, A half-linear second order differential equation, in: *Qualitative theory of differential equations, Vol. I, II (Szeged, 1979)*, Colloq. Math. Soc. János Bolyai, Vol. 30, Szeged, 1979, pp. 153–180. MR680591; Zbl 0511.34006

[7] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Acad. Publ., Dordrecht 1993. https://doi.org/10.1007/978-94-011-1808-8; MR1220223 Zbl 0782.34002

[8] R. Koplatadze, G. Kvinikadze, I. P. Stavroulakis, Properties A and B of nth order linear differential equations with deviating argument, *Georgian Math. J.* **6**(1999), No. 6, 553–566. https://doi.org/10.1023/A:1022962129926; MR1693798; Zbl 0939.34062

[9] T. Kusano, Oscillation of even order linear functional differential equations with deviating arguments of mixed type, *J. Math. Anal. Appl.* **98**(1984), 341–347. https://doi.org/10.1016/0022-247X(84)90253-1; MR730511; Zbl 0546.34056

[10] T. Kusano, B. S. Lalli, On oscillation of half-linear functional differential equations with deviating arguments, *Hiroshima Math. J.* **24**(1994), 549–563. MR1309139; Zbl 0836.34081

[11] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Marcel Dekker, New York, 1987. MR1017244; Zbl 0832.34071