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On a viscoelastic heat equation with logarithmic nonlinearity

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Abstract. This work deals with the following viscoelastic heat equations with logarithmic nonlinearity

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u\ln|u|.$$

In this paper, we show the effects of the viscoelastic term and the logarithmic nonlinearity to the asymptotic behavior of weak solutions. Our results extend the results of Peng and Zhou [*Appl. Anal.* **100**(2021), 2804–2824] and Messaoudi [*Progr. Nonlinear Differential Equations Appl.* **64**(2005), 351–356.].

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1 Introduction

In this paper, we study the following heat equations with viscoelastic term and logarithmic nonlinearity

$$\begin{cases} u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u\ln|u|, & \text{in} \quad \Omega\times(0,\infty), \\ u = 0, & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x), & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

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where $u_0 \in H_0^1(\Omega)$ and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and the parameter p satisfy

$$2 < p < \begin{cases} \infty, & \text{if} \quad n \leq 2, \\ \frac{2(n-1)}{n-2}, & \text{if} \quad n > 2. \end{cases} \tag{1.2}$$

The equation of the form

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = f(u), \tag{1.3}$$

is used to model many natural phenomena in physical science and engineering. For example, in the study of heat conduction in materials with memory, from the heat balance equation the temperature $u(x,t)$ will satisfy Eq. (1.3) (see [3,5,12,13] for more detail).

In the last several decades, the initial-boundary valued problem to Eq. (1.3) has been studied extensively when the source $f(u)$ is the power functions $f(u) = |u|^{p-2}u$, or power like-functions satisfying:

(1) $f \in C^1$ and $f(0) = f'(0) = 0$.

(2) (a) f is monotone and is convex for $u > 0$, and concave for $u < 0$; or (b) f is convex.

(3) $(p+1)\int_0^u f(z)dz \leq uf(u)$, and $|uf(u)| \leq \kappa \int_0^u f(z)dz$, where

$$2 < p+1 \leq \kappa < 2^* =: \begin{cases} \infty, & \text{if} \quad n \leq 2, \\ \frac{2n}{n-2}, & \text{if} \quad n \geq 3. \end{cases}$$

For example, Messaoudi [12] studied Eq. (1.3) in the case $f(u) = |u|^{p-2}u$ associated with homogeneous Dirichlet boundary condition. By the convexity method, the author showed that if the relaxation function g is non-negative and non-increasing satisfying

$$\int_0^\infty g(s)ds < \frac{2(p-2)}{2p-3},$$

then weak solution to (1.3) blows up in finite time provided initial energy is positive. In [20], Truong and Y also studied the problem of the above type with $f(u)$ in the general polynomial type and they obtained the existence, blow up and asymptotic behavior for weak solution under suitable conditions. For further results on the existence, blow-up or asymptotic behavior of solutions, we refer the reader to [5,13,16,19] in case of power or power-like sources.

With regard to the logarithmic nonlinearity, there are a few results (see [1,2,7,9,15]). In case the relaxation function g vanishes, the problem (1.1) reduces to the following:

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u \ln|u|, & \text{in} \quad \Omega \times (0,\infty), \\ u = 0, & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & \text{in} \quad \Omega. \end{cases} \tag{1.4}$$

In case $p = 2$, self-similar solutions and their asymptotic stability for $(1.4)_1$ has been studied by Samarskii et al. [17]. With regard to weak solutions, by using the potential well method and the logarithmic Sobolev inequality in $H_0^1(\Omega)$ (see [6,11]), Chen et al. [1] prove that the weak solution blows up at infinite time and exists globally provided that the initial data start in the stable sets and unstable sets respectively. This result is so interesting because it showed the different effect of logarithmic nonlinearity compared to the power one. Inspired by this result the second and third authors [9] extended (1.4) to the evolution p-Laplacian equations and showed a different result compared to the case $p = 2$, confirming that weak solutions blow up in finite time. Afterward the PDEs with logarithmic nonlinearity have been attracted many researchers, see [2,7,15] for example. In particular, Peng and Zhou [15] have showed recently that in case $p > 2$ the solutions of (1.4) behave like the nonlinear case $f(u) = |u|^{p-2}u$. These results shows that $p = 2$ is the critical exponent for the blow-up at infinite time.

Motivated by all these works, our aim in this paper is to study the effect of the viscoelastic term $\int_0^t g(t-s)\Delta u(s)ds$ and the logarithmic nonlinearity $|u|^{p-2}u \ln|u|$ to the blow-up and global existence of weak solutions to (1.1). Firstly, the presence of logarithmic nonlinearity help us relax conditions on g compared to [12], that is,

$$\int_0^\infty g(s)ds < \frac{p(p-2)}{(p-1)^2},$$

where $\frac{p(p-2)}{(p-1)^2} > \frac{2(p-2)}{2p-3}$ since $p > 2$. Secondly, because of the presence of $\int_0^t g(t-s)\Delta u(s)ds$ we need more restriction on the range of p and for small energy levels $E(0) < d_\delta \le d$ (see (2.2) below) compared to [15].

Our result is twofold in the sense that it is not only study the blow-up in finite time but also global existence of weak solutions. In addition, we also give the lower and upper bound for blow-up time and decay estimate of global solutions. Also notice that our method differs from [12]. To obtain the main results, we employ the ideas from the potential well method due to Sattinger [18] (see also [14]). However, since the presence of the relaxation g we could not apply the stable and unstable sets as in [14]. To overcome this difficulty we construct a family of potential wells (see (2.3) and (2.4)) that is more suitable for the PDEs involving viscoelastic terms. Also notice that the asymptotic behavior of global solutions in [15] has not been studied and it can be done by using the method employed in this paper.

This paper is organized as follows. In the next section, we present some preliminaries and define the family of modified potential wells. Our main results are stated in the Section 3 and the rest of the paper is devoted to their proofs.

Notation. Throughout this paper, we denote $L^p(\Omega)$-norm by $\|\cdot\|_p$, especially $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. And let $\langle\cdot,\cdot\rangle$ denote L^2-inner product.

2 Preliminaries and Modified potential wells

2.1 Preliminary lemmas

The following lemmas will be needed in our proof of the main results.

Lemma 2.1 ([21, Lemma 3.1.7 and Remark 3.1.4]). *Let* $\mathbf{B}$ *be a reflexive Banach space and* $0 < T < \infty$. *Suppose* $1 < q < \infty$, $\varphi \in L^q(0,T;\mathbf{B})$, *and the sequence* $\{\varphi_m\}_{m=1}^\infty \subset L^q(0,T;\mathbf{B})$ *satisfy (as* $m \to \infty$*)*

$$\begin{cases} \varphi_m \to \varphi & \text{weakly in} \quad L^q(0,T;\mathbf{B}), \\ \varphi_{mt} \to \varphi_t & \text{weakly in} \quad L^q(0,T;\mathbf{B}). \end{cases}$$

Then $\varphi_m(0) \to \varphi(0)$ *weakly in* $\mathbf{B}$.

Lemma 2.2 ([21, Thereom 3.1.1])**.** *Let* (1.2) *hold and* $T \in (0, \infty)$ *be fixed. Then the embedding*

$$\left\{ \varphi \middle| \varphi \in L^2\left(0, T; H_0^1(\Omega)\right), \varphi_t \in L^2\left(0, T; L^2(\Omega)\right) \right\} \hookrightarrow L^2\left(0, T; L^p(\Omega)\right)$$

is compact.

Lemma 2.3 ([9])**.** *Let* ρ *be a positive number. Then we have the following elementary inequalities:*

$$\Psi^p \ln \Psi \le \frac{e^{-1}}{\rho} \Psi^{p+\rho},\ \forall \Psi \ge 1 \quad and \quad |\Psi^p \ln \Psi| \le (ep)^{-1},\ \forall 0 < \Psi < 1.$$

Lemma 2.4 ([8, 10])**.** *Suppose that* $\Phi(t) \in C^2[0, \infty)$ *is a positive function satisfying the following inequality*

$$\Phi(t)\Phi''(t) - (1+\gamma)\left(\Phi'(t)\right)^2 \ge 0,$$

where $\gamma > 0$ *is a constant. If* $\Phi(0) > 0, \Phi'(0) > 0$*, then* $\Phi(t) \to \infty$ *for* $t \to t_* \le t^* = \frac{\Phi(0)}{\gamma\Phi'(0)}$.

2.2 Modified potential wells

For $0 < \delta \le \ell$ with $\ell := 1 - \int_0^\infty g(s)ds$, we define potential energy functional

$$J_\delta(u) = \frac{\delta}{2}\|\nabla u\|^2 - \frac{1}{p}\int_\Omega |u|^p \ln|u| dx + \frac{1}{p^2}\|u\|_p^p,$$

and the associated Nehari functional

$$I_\delta(u) = \delta\|\nabla u\|^2 - \int_\Omega |u|^p \ln|u| dx.$$

then we have that

$$J_\delta(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\delta\|\nabla u\|^2 + \frac{1}{p}I_\delta(u) + \frac{1}{p^2}\|u\|_p^p.$$

We have the following lemma.

Lemma 2.5. *Let* $u \in H_0^1(\Omega)\backslash\{0\}$*. Then we have:*

(i) $\lim_{\lambda \to 0^+} J_\delta(\lambda u) = 0$ *and* $\lim_{\lambda \to \infty} J_\delta(\lambda u) = -\infty$.

(ii) there is a unique $\lambda_1 = \lambda_1(u) > 0$ *such that* $\frac{d}{d\lambda} J_\delta(\lambda u)\Big|_{\lambda=\lambda_1} = 0$.

(iii) $J_\delta(\lambda u)$ *is strictly increasing on* $(0, \lambda_1)$ *and strictly decreasing on* (λ_1, ∞)*, and attains its the maximum value at* $\lambda = \lambda_1$*. In addition, one has*

$$I_\delta(\lambda u) \begin{cases} > 0, & if \quad 0 \le \lambda < \lambda_1, \\ = 0, & if \quad \lambda = \lambda_1, \\ < 0, & if \quad \lambda_1 < \lambda < \infty. \end{cases}$$

Proof. (i) From the definition of J_δ, we have for $\lambda > 0$ that

$$J_\delta(\lambda u) = \frac{\delta\lambda^2}{2}\|\nabla u\|^2 - \frac{\lambda^p}{p}\int_\Omega |u|^p \ln|u|dx - \frac{\lambda^p}{p}\ln\lambda\,\|u\|_p^p + \frac{\lambda^p}{p^2}\|u\|_p^p,$$

which implies $\lim_{\lambda\to 0^+} J_\delta(\lambda u) = 0$ and $\lim\limits_{\lambda\to\infty} J_\delta(\lambda u) = -\infty$ thanks to $p > 2$.

For (ii). An easy calculation shows that

$$\frac{d}{d\lambda}J_\delta(\lambda u) = \lambda\left(\delta\|\nabla u\|^2 - \lambda^{p-2}\int_\Omega |u|^p\ln|u|dx - \lambda^{p-2}\ln\lambda\,\|u\|_p^p\right) := \lambda K_\delta(\lambda u),$$

where

$$K_\delta(\lambda u) = \delta\|\nabla u\|^2 - \lambda^{p-2}\int_\Omega |u|^p\ln|u|dx - \lambda^{p-2}\ln\lambda\,\|u\|_p^p. \tag{2.1}$$

A direct calculations yields

$$\frac{d}{d\lambda}K_\delta(\lambda u) = -\lambda^{p-3}\left((p-2)\int_\Omega |u|^p\ln|u|dx + (p-2)\ln\lambda\,\|u\|_p^p + \|u\|_p^p\right),$$

Hence if we choose

$$\lambda_* = \exp\left(\frac{(2-p)\int_\Omega |u|^p\ln|u|dx - \|u\|_p^p}{(p-2)\|u\|_p^p}\right),$$

then one has $\frac{d}{d\lambda}K_\delta(\lambda_* u) = 0$, $\frac{d}{d\lambda}K_\delta(\lambda u) > 0$ for $0 < \lambda < \lambda_*$ and $\frac{d}{d\lambda}K_\delta(\lambda u) < 0$ for $\lambda_* < \lambda < \infty$. On the other hand, from the definition of K, we have

$$\lim_{\lambda\to 0^+} K_\delta(\lambda u) = \delta\|\nabla u\|^2 > 0 \quad \text{and} \quad \lim_{\lambda\to\infty} K_\delta(\lambda u) = -\infty.$$

By these facts we obtain that there exists a unique $\lambda_1 > \lambda_*$ such that $K_\delta(\lambda_1 u) = 0$. Hence we obtain (ii).

The last statement (iii) follows from (i)–(ii) and the relation

$$I_\delta(\lambda u) = \lambda\frac{d}{d\lambda}J_\delta(\lambda u).$$

The proof is complete. □

Let us state here the Sobolev imbedding which can be found in [4].

Lemma 2.6. *Assume that p is a constant such that*

$$1 \le p \le \begin{cases} \frac{2n}{n-2}, & \text{if} \quad n > 2,\\ \widetilde{p}, & \text{if} \quad n = 2,\\ \infty, & \text{if} \quad n = 1,\end{cases}$$

where $\widetilde{p} \in [1,\infty)$ can be any constant. Then $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ continuously, and there exists a positive constant C_p depending on n, p and Ω such that

$$\|u\|_p \le C_p\|\nabla u\|$$

holds for all $u \in H_0^1(\Omega)$. We choose C_p be the optimal constant satisfying the above inequality, i.e.

$$C_p = \sup_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_p}{\|\nabla u\|}.$$

Since $p < \frac{2(n-1)}{n-2} < 2^*$, let

$$\sigma^* = \begin{cases} \frac{2n}{n-2} - p, & \text{if} \quad n > 2, \\ \infty, & \text{if} \quad n = 1, 2, \end{cases}$$

then $\sigma^* > 0$ and by Lemma 2.6, we have $H_0^1(\Omega) \hookrightarrow L^{p+\sigma}(\Omega)$ continuously for any $\sigma \in [0, \sigma^*)$. Denote $C_{p+\sigma}$ by C_*, then we have the following lemma.

Lemma 2.7. *Let* (1.2) *hold and* $u \in H_0^1(\Omega) \setminus \{0\}$. *Then we have*

(i) if $I_\delta(u) < 0$, *then* $\|\nabla u\| > r_\delta(\sigma)$,

(ii) if $\|\nabla u\| \le r_\delta(\sigma)$ *then* $I_\delta(u) \ge 0$,

where $r_\delta(\sigma) = \left(\frac{e\sigma\delta}{C_*^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}}$ *for* $0 < \sigma < \sigma^*$.

Proof. For $0 < \sigma < \sigma^*$, by Lemma 2.3 and the Sobolev inequality, we have

$$\begin{aligned} \int_\Omega |u|^p \ln |u| dx &= \int_{\{\Omega:|u|\le 1\}} |u|^p \ln |u| dx + \int_{\{\Omega:|u|\ge 1\}} |u|^p \ln |u| dx \\ &\le \frac{e^{-1}}{\sigma} \|u\|_{p+\sigma}^{p+\sigma} \le \frac{e^{-1}}{\sigma} C_*^{p+\sigma} \|\nabla u\|^{p+\sigma}. \end{aligned}$$

It follows that

$$\begin{aligned} I_\delta(u) &= \delta \|\nabla u\|^2 - \int_\Omega |u|^p \ln |u| dx \\ &\ge \delta \|\nabla u\|^2 - \frac{e^{-1}}{\sigma} C_*^{p+\sigma} \|\nabla u\|^{p+\sigma} = \|\nabla u\|^2 \left(\delta - \frac{e^{-1}}{\sigma} C_*^{p+\sigma} \|\nabla u\|^{p+\sigma-2} \right). \end{aligned}$$

The conclusions then follow from the above inequality. □

Let us define the so-called Nehari manifold associated to the energy functional J_δ by

$$\mathcal{N}_\delta = \left\{ u \in H_0^1(\Omega) \setminus \{0\} : I_\delta(u) = \langle J_\delta'(u), u \rangle = 0 \right\}.$$

By Lemma 2.5 we know that $\mathcal{N}_\delta$ is not empty set. It is clear that $J_\delta(u)$ is coercive on the Nehari manifold $\mathcal{N}_\delta$, hence we can define

$$d_\delta = \inf_{u \in \mathcal{N}_\delta} J_\delta(u). \tag{2.2}$$

The standard variational method shows that d_δ is a positive finite number and therefore it is well-defined.

We end this section by giving the definitions of the modified stable and unstable sets as in [14].

$$\mathcal{W}_\delta = \left\{ u \in H_0^1(\Omega) : J_\delta(u) < d_\delta,\ I_\delta(u) > 0 \right\} \cup \{0\}, \tag{2.3}$$

$$\mathcal{U}_\delta = \left\{ u \in H_0^1(\Omega) : J_\delta(u) < d_\delta,\ I_\delta(u) < 0 \right\}. \tag{2.4}$$

3 Main results

Throughout this paper, we make the following usual assumptions on the relaxation function g:

(G) $g : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to $C^1(\mathbb{R}^+)$ and satisfies the conditions

(i) $g(0) \geq 0, \quad \ell := 1 - \int_0^\infty g(s)ds > 0, \quad g'(t) \leq 0,$

(ii) $\int_0^\infty g(s)ds < \frac{p(p-2)}{(p-1)^2},$

(iii) There exists a positive differentiable function $\xi(t)$ such that

$$g'(t) \leq -\xi(t)g(t), \quad \xi'(t) \leq 0, \quad \int_0^\infty \xi(t)dt = \infty, \ \forall t > 0.$$

Let us now give the definition of weak solutions to (1.1).

Definition 3.1. Let $0 < T \leq \infty$, a function u is called a weak solution of problem (1.1) on $\Omega \times (0, T)$ if $u \in L^\infty(0, T; H_0^1(\Omega))$ with $u_t \in L^2(0, T; L^2(\Omega))$ satisfies $u(x, 0) = u_0(x) \in H_0^1(\Omega)$ and the equality

$$\langle u_t, \varphi\rangle + \langle \nabla u, \nabla \varphi\rangle - \int_0^t g(t-s) \langle \nabla u(\tau), \nabla u(s)\rangle \, ds = \left\langle |u|^{p-2} u \ln |u|, \varphi \right\rangle, \tag{3.1}$$

holds for a.e. $t \in (0, T)$ and any $\varphi \in H_0^1(\Omega)$.

Let u be a weak solution of problem (1.1), we define the total energy functional as follows

$$\begin{aligned} E(t) = {} & \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right) \|\nabla u(t)\|^2 + \frac{1}{2} (g \circ \nabla u)(t) \\ & - \frac{1}{p}\int_\Omega |u(t)|^p \ln |u(t)| dx + \frac{1}{p^2} \|u(t)\|_p^p, \end{aligned} \tag{3.2}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s) \|\nabla u(t) - \nabla u(s)\|^2 \, ds.$$

By the Definition 3.1, $u \in L^\infty(0, T; H_0^1(\Omega))$ and $u_t \in L^2(0, T; L^2(\Omega))$. So $E(t)$ is well-define for a.e. $t \in [0, T)$. In addition, the next lemma shows that $E(t)$ is a non-increasing functional.

Lemma 3.2. *Let (G, (i)) hold. The energy functional $E(t)$ defined in (3.2) is nonincreasing and*

$$\frac{d}{dt}E(t) = -\|u_t(t)\|^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t) \|\nabla u(t)\|^2 \leq 0. \tag{3.3}$$

Proof. By substituting $\varphi = u_t$ in (3.1), we get after some simple calculations that

$$\frac{d}{dt}E(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t) \|\nabla u(t)\|^2 - \|u_t(t)\|^2.$$

Then, using the assumption (G, (i)), it follows that $E(t)$ is an non-increasing functional and satisfies the energy inequality

$$E(t) + \int_0^t \|u_t(s)\|^2 ds \leq E(0). \tag{3.4}$$

The proof is complete. □

We are now in the position to state the main theorems of this paper.

Theorem 3.3 (Global existence). *Assume that* (1.2) *and (G, (i)) hold. Let* $u_0 \in H_0^1(\Omega)$ *and*

$$E(0) = \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{p}\int_\Omega |u_0|^p \ln |u_0| dx + \frac{1}{p^2}\|u_0\|_p^p < d_\delta, \quad I_\delta(u_0) > 0.$$

Then problem (1.1) *has a global weak solution* u *such that* $u \in L^\infty(0,\infty; H_0^1(\Omega))$ *with* $u_t \in L^2(0,\infty; L^2(\Omega))$.

Theorem 3.4 (Blow-up). *Assume that* (1.2) *hold and* g *satisfies (G, (i), (ii)). Assume further that* $u_0 \in H_0^1(\Omega)$ *and*

$$E(0) = \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{p}\int_\Omega |u_0|^p \ln |u_0| dx + \frac{1}{p^2}\|u_0\|_p^p < d_\kappa, \quad I_\kappa(u_0) < 0,$$

where

$$0 < \kappa = \ell - \frac{1}{p(p-2)}\int_0^\infty g(s)ds. \tag{3.5}$$

Then the weak solution $u(t)$ *to* (1.1) *blows up in finite time and the lifespan time* T *satisfies*

$$T \leq \frac{8\|u_0\|^2}{(p-2)^2(d_\kappa - E(0))}.$$

Furthermore, T *is bounded below by*

$$T \geq \int_{R(0)}^\infty \frac{1}{K_1 z^{p-1+\sigma} + K_2} dz, \tag{3.6}$$

for some $0 < \sigma < \frac{2(n-1)}{n-2} - p$*, where* $R(0) = \frac{1}{2}\|\nabla u_0\|^2$ *and*

$$K_1 = \frac{1}{2}(e\sigma)^{-2} S_{2(p-1+\sigma)}^{2(p-1+\sigma)} \left(2(p-1)^2\right)^{p-1+\sigma}, \quad K_2 = \frac{1}{2}(e(p-1))^{-2}|\Omega|.$$

Here $S_{2(p-1+\sigma)}$ *is the optimal embedding constants of* $H_0^1(\Omega) \hookrightarrow L^{2(p-1+\sigma)}(\Omega)$.

Theorem 3.5 (Decay estimate). *Assume that* (1.2) *holds and* g *satisfies (G,(i), (iii)). Assume further that* $u_0 \in H_0^1(\Omega)$ *with* $u_0 \in \mathcal{W}_\delta$ $(0 < \delta \leq \ell)$ *and*

$$E(0) < \left(\frac{\ell}{2\delta}\right)^{\frac{p}{p-2}} d_\delta.$$

Then solution $u(t)$ *to* (1.1) *decays exponentially.*

4 Proof of Theorem 3.3

Based on the Faedo–Galerkin method, this proof consists of three steps.

Step 1. Finite-dimensional approximations. Let $\{w_j\}$ be the orthogonal complete system of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, which is orthonormal in $L^2(\Omega)$. We find the approximate solution of the problem (1.1) in the forms

$$u_m(t) = \sum_{j=1}^m c_{mj}(t) w_j, \tag{4.1}$$

where the coefficients functions c_{mj}, $1 \le j \le m$, satisfy the system of integro-differential equations

$$\langle u_{mt}, w_j \rangle + \langle \nabla u_m, \nabla w_j \rangle - \int_0^t g(t-s) \langle \nabla u_m(s), \nabla w_j \rangle ds = \left\langle |u_m|^{p-2} u_m \ln |u_m|, w_j \right\rangle, \quad (4.2)$$

and

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \longrightarrow u_0 \quad \text{strongly in} \quad H_0^1(\Omega). \quad (4.3)$$

It is obvious that for each m, there exists a solution u_m of the form (4.1) which satisfies (4.2) and (4.3) almost everywhere on $t \in [0, T_m]$, for some sufficiently small $T_m > 0$. In what follows, we present a brief proof that a solution of (4.2)–(4.3) of the form (4.1) exists. It is obvious that the system (4.2)–(4.3) can be rewritten in the vectorial form

$$c_m'(t) + A_m c_m(t) = A_m \int_0^t g(t-s) c_m(s) ds + \mathcal{F}(c_m(t)),$$

with the initial condition

$$c_m(0) = \alpha_m,$$

where

$$\begin{cases} c_m(t) = (c_{m1}(t), c_{m1}(t), \ldots, c_{m1}(t))^T, \ \alpha = (\alpha_{m1}, \alpha_{m2}, \ldots, \alpha_{mm})^T, \\ A_m = \left[\langle \nabla w_i, \nabla w_j \rangle \right]_{i,j=1}^m, \ \mathcal{F}(c_m(t)) = (\mathcal{F}_1(c_m(t)), \mathcal{F}_2(c_m(t)), \ldots, \mathcal{F}_m(c_m(t)))^T, \\ \mathcal{F}_j(c_m(t)) = \left\langle |u_m|^{p-2} u_m \ln |u_m|, w_j \right\rangle, \quad \forall j = \overline{1, m}, \end{cases}$$

which is also equivalent to the integral equation

$$c_m(t) = \alpha_m - \int_0^t A_m c_m(s) ds + \int_0^t A_m \int_0^s g(s-\tau) c_m(\tau) d\tau ds + \int_0^t \mathcal{F}(c_m(s)) ds. \quad (4.4)$$

By the Schauder theorem, the integral equation (4.4) has a solution $c_m(t)$ in a certain closed ball of the Banach space $C([0, T_m]; \mathbb{R}^m)$ with $T_m \in (0, T]$. Therefore, there exists $u_m(t)$ of the form (4.1) which satisfies (4.2)–(4.3) on $0 \le t \le T_m$.

Step 2. A priori estimate. Multiplying (4.2) by $c_{mj}'(t)$ and summing for j from 1 to m, we get

$$\langle u_{mt}, u_{mt} \rangle + \langle \nabla u_m, \nabla u_{mt} \rangle - \int_0^t g(t-s) \langle \nabla u_m(s), \nabla u_{mt} \rangle ds = \left\langle |u_m|^{p-2} u_m \ln |u_m|, u_{mt} \right\rangle. \quad (4.5)$$

Integrating (4.5) with respect to time variable on $[0, t]$, we have

$$E_m(t) + \int_0^t \|u_{mt}(s)\|^2 ds = E_m(0) - \frac{1}{2} \int_0^t g(s) \|\nabla u_m(s)\|^2 ds + \frac{1}{2} \int_0^t (g' \circ \nabla u_m)(s) ds, \quad (4.6)$$

where we have for $0 < \delta \le \ell$

$$\begin{aligned} E_m(t) &= \frac{1}{2}\left(1 - \int_0^t g(s) ds\right) \|\nabla u_m(t)\|^2 + \frac{1}{2} (g \circ \nabla u_m)(t) \\ &\quad - \frac{1}{p} \int_\Omega |u_m|^p \ln |u_m| dx + \frac{1}{p^2} \|u_m(t)\|_p^p \\ &\ge J_\delta(u_m(t)) + \frac{1}{2} (g \circ \nabla u_m)(t). \end{aligned} \quad (4.7)$$

From $E(0) < d_\delta$ and (4.3), we deduce that $E_m(0) < d_\delta$ for sufficiently large m. And then, we deduce from (4.6) and (4.7) that

$$\frac{1}{2}\left(g \circ \nabla u_m\right)(t) + J_\delta(u_m(t)) + \int_0^t \|u_{mt}(s)\|^2 ds < d_\delta, \quad 0 \le t \le T_m, \tag{4.8}$$

holds for sufficiently large m. Take note of $I_\delta(u_0) > 0$, we can conclude that $u_0 \in \mathcal{W}_\delta$. It implies from (4.3) that $u_m(0) \in \mathcal{W}_\delta$ for sufficiently large m. Now, we will show that $u_m(t) \in \mathcal{W}_\delta$ for any $t \in [0, T_m]$ and sufficiently large m. In fact, if not, there exists a $t_0 \in (0, T_m]$ and a sufficient large m such that $I_\delta(u_m(t_0)) = 0$ and $u_m(t_0) \neq 0$, then we get that $u_m(t_0) \in \mathcal{N}_\delta$. So we deduce from the definition of d_δ that $J_\delta\left(u_m(t_0)\right) \ge d_\delta$, which contradicts (4.8). Thus, $u_m(t) \in \mathcal{W}_\delta$ for any $t \in [0, T_m]$ and sufficient large m, which implies $I_\delta(u_m(t)) \ge 0$ for any $t \in [0, T_m]$ and sufficient large m.

Thanks to the definition of J_δ and $I_\delta(u_m(t)) \ge 0$, we deduce from (4.8) that

$$\begin{aligned} \frac{p-2}{2p}\delta \|\nabla u_m(t)\|^2 + \frac{1}{p^2}\|u_m(t)\|_p^p + \frac{1}{2}\left(g \circ \nabla u_m\right)(t) + \int_0^t \|u_{mt}(s)\|^2 ds < d_\delta, \\ 0 \le t \le T_m, \end{aligned} \tag{4.9}$$

From (4.9) we obtain

$$\begin{cases} \|\nabla u_m(t)\|^2 < \dfrac{2p}{(p-2)\delta} d_\delta, \ \|u_m(t)\|_p^p < p^2 d_\delta, \\ \displaystyle\int_0^t \|u_{mt}(t)\|^2 < d_\delta, \ \left(g \circ \nabla u_m\right)(t) < 2d_\delta. \end{cases} \tag{4.10}$$

So $T_m = \infty$. And hence $u_m(t) \in \mathcal{W}_\delta$ for $t \in [0, \infty)$ and (4.10) holds for $t \in [0, \infty)$.

On the other hand, by (4.10), we get

$$\begin{aligned} \int_\Omega |\rho_m(x,t)|^{p'} dx &= \int_{\Omega_1} |\rho_m(x,t)|^{p'} dx + \int_{\Omega_2} |\rho_m(x,t)|^{p'} dx \\ &\le \left(e(p-1)\right)^{-p'} |\Omega_1| + (e\sigma)^{-p'} \|u_m\|_{p+p'\sigma}^{p+p'\sigma} \\ &\le \left(e(p-1)\right)^{-p'} |\Omega_1| + (e\sigma)^{-p'} S_{p+p'\sigma}^{p+p'\sigma} \|\nabla u_m\|^{p+p'\sigma} \\ &\le \left(e(p-1)\right)^{-p'} |\Omega_1| + (e\sigma)^{-p'} S_{p+p'\sigma}^{p+p'\sigma} \left(\frac{2pd_\delta}{(p-2)\delta}\right)^{\frac{p+p'\sigma}{2}} \equiv C_\delta, \end{aligned} \tag{4.11}$$

where $p' = \frac{p}{p-1}$, $0 < \sigma < \frac{1}{p'}\left(\frac{2n}{n-2} - p\right)$, $\rho_m(x,t) = |u_m(x,t)|^{p-1} \ln |u_m(x,t)|$,

$$\Omega_1 = \{x \in \Omega : |u_m(x,t)| \le 1\}, \quad \Omega_2 = \{x \in \Omega : |u_m(x,t)| \ge 1\},$$

and S_q is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$.

Step 3. Passage to the limit. From (4.10) and (4.11), we deduce that for each $T > 0$, there exists a function $u(t)$ and the subsequences of $\{u_m\}$, still denoted by $\{u_m\}$ such that

$$\begin{cases} u_m \to u & \text{in } L^\infty\left(0, T; H_0^1(\Omega)\right) \text{ weakly*}, \\ u_m \to u & \text{in } L^2\left(0, T; H_0^1(\Omega)\right) \text{ weakly}, \\ u_m \to u & \text{in } L^\infty\left(0, T; L^p(\Omega)\right) \text{ weakly*}, \\ u_m \to u & \text{in } L^2\left(0, T; L^p(\Omega)\right) \text{ weakly}, \\ u_{mt} \to u_t & \text{in } L^2\left(0, T; L^2(\Omega)\right) \text{ weakly*}. \end{cases} \tag{4.12}$$

By Lemma 2.1, it follows from $(4.12)_{2,5}$ that there exists the existence of a subsequence still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{strongly in } L^2(0,T;L^p(\Omega)) \quad \text{and} \quad u_m \to u \quad \text{a.e } (x,t) \in \Omega \times (0,T),$$

which yields

$$|u_m|^{p-2}u_m \ln|u_m| \to |u|^{p-2}u\ln|u| \quad \text{a.e } (x,t) \in \Omega \times (0,T). \tag{4.13}$$

From (4.11) and (4.13) by the Aubin–Lions Lemma, we deduce that

$$|u_m|^{p-2}u_m \ln|u_m| \to |u|^{p-2}u\ln|u| \quad \text{weakly* in } L^\infty\left(0,T;L^{p'}(\Omega)\right).$$

By using Lemma 2.1, it follows from $(4.12)_{2,5}$ that

$$u_m(0) \to u(0) \quad \text{weakly in } L^2(\Omega). \tag{4.14}$$

Passing to the limit in (4.2), by (4.3), (4.12), (4.13)–(4.14), we have u satisfying equation

$$\begin{cases} \langle u_t, \varphi\rangle + \langle \nabla u, \nabla\varphi\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla\varphi\rangle\, ds = \left\langle |u|^{p-2}u\ln|u|, \varphi\right\rangle, \\ u(0) = u_0. \end{cases}$$

The proof is complete.

5 Proof of Theorem 3.4

We begin this section by the following useful lemma which is useful later on.

Lemma 5.1. *Under the assumptions of the Theorem 3.4 and let $u(t)$ be any weak solution of the problem (1.1) on $[0,T)$ where T is the maximum existence time. Then we possess*

$$d_\kappa \le \frac{p-2}{2p}\kappa\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p, \tag{5.1}$$

where κ is defined by (3.5).

Proof. Firstly, we show that $u(t) \in \mathcal{U}_\kappa$ for all $t \in [0,T)$. Indeed, if it is false, then there exists a $t_0 > 0$ such that $I_\kappa(u(t)) < 0$ for $t \in [0,t_0)$ and $I_\kappa(u(t_0)) = 0$. By Lemma 2.7, we have $\|\nabla u(t)\| > r_\kappa(\sigma) > 0$, for $t \in [0,t_0)$ and $\|\nabla u(t_0)\| \ge r_\kappa(\sigma) > 0$, which yields $u(t_0) \in \mathcal{N}_\kappa$. So by the definition of d_κ we get $J_\kappa(u(t_0)) \ge d_\kappa$, which contradicts to $J_\kappa(u(t_0)) \le E(t_0)) \le E(0) < d_\kappa$. Hence, we obtain $u(t) \in \mathcal{U}_\kappa$ for $t \in [0,T)$.

By Lemma 2.5 we imply that there is a unique $\lambda_1 < 1$ such that $I_\kappa(\lambda_1 u(t)) = 0$. We next define $j(\lambda) = J_\kappa(\lambda u) - \frac{1}{p}I_\kappa(\lambda u)$, for $\lambda > 0$. By direct calculation, we have that

$$j(\lambda) = \frac{\kappa(p-2)}{2p}\lambda^2\|\nabla u(t)\|^2 + \frac{\lambda^p}{p^2}\|u(t)\|_p^p.$$

Since $u(t) \in \mathcal{U}_\kappa$, by Lemma 2.7 we have

$$j'(\lambda) = \frac{\kappa(p-2)}{p}\lambda\|\nabla u(t)\|^2 + \frac{\lambda^{p-1}}{p}\|u(t)\|_p^p > \kappa(p-2)\lambda r_\kappa^2(\sigma) > 0.$$

Hence, $j(\lambda)$ is strictly increasing on $(0,\infty)$ which implies $j(1) > j(\lambda_1)$, that is

$$J_\kappa(u(t)) - \frac{1}{p}I_\kappa(u(t)) > J_\kappa(\lambda_1 u(t)) - \frac{1}{p}I_\kappa(\lambda_1 u(t)) = J_\kappa(\lambda_1 u) \ge d_\kappa.$$

The proof of lemma is complete. □

We now divide the proof of the Theorem 3.4 into two following steps:

Step 1: Blow-up in finite time and upper bound estimate of the blow-up time.

By contradiction, we assume that $u(t)$ exists globally and define the function

$$\theta(t) = \int_0^t \|u(s)\|^2 \, ds + (T-t)\|u_0\|^2 + b(t+T_0)^2, \quad t \in [0,T], \tag{5.2}$$

where b and T_0 are positive constants to be determined later. Then we have

$$\begin{aligned} \theta'(t) &= \|u(t)\|^2 - \|u_0\|^2 + 2b(t+T_0) = \int_0^t \frac{d}{dt}\|u(s)\|^2 \, ds + 2b(t+T_0) \\ &= 2\int_0^t \langle u_t(s), u(s)\rangle \, ds + 2b(t+T_0), \end{aligned} \tag{5.3}$$

and

$$\theta''(t) = 2\int_\Omega u(t)u_t(t)dx + 2b. \tag{5.4}$$

By using (1.1), we deduce from (5.4) that

$$\theta''(t) = -2\|\nabla u(t)\|^2 + 2\int_0^t g(t-s)\langle \nabla u(s), \nabla u(t)\rangle \, ds + 2\int_\Omega |u(t)|^p \ln|u(t)|dx + 2b. \tag{5.5}$$

On the other hand, by the Hölder inequality and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \frac{1}{4}\left(\theta'(t)\right)^2 &\le \left(\int_0^t \|u(s)\|^2 \, ds + b(t+T_0)^2\right)\left(\int_0^t \|u_t(s)\|^2 \, ds + b\right) \\ &\le \theta(t)\left(\int_0^t \|u_t(s)\|^2 \, ds + b\right), \end{aligned} \tag{5.6}$$

and by the Young inequality, one has

$$\begin{aligned} &2\int_0^t g(t-s)\langle \nabla u(s), \nabla u(t)\rangle \, ds \\ &\quad = 2\int_0^t g(s)ds\,\|\nabla u(t)\|^2 + 2\int_0^t g(t-s)\langle \nabla u(s) - \nabla u(t), \nabla u(t)\rangle \, ds \\ &\quad \ge \left(2 - \frac{1}{p}\right)\int_0^t g(s)ds\,\|\nabla u(t)\|^2 - p\,(g \circ \nabla u)(t). \end{aligned} \tag{5.7}$$

It follows from (5.2)–(5.7) that

$$\theta''(t)\theta(t) - \frac{p+2}{4}\left(\theta'(t)\right)^2 \ge \theta(t)\zeta(t), \tag{5.8}$$

where $\zeta : [0,T] \to \mathbb{R}$ is the function defined by

$$\begin{aligned} \zeta(t) = &-2\|\nabla u(t)\|^2 + \left(2 - \frac{1}{p}\right)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 - p\,(g \circ \nabla u)(t) \\ &+ 2\int_\Omega |u(t)|^p \ln|u(t)|dx - (p+2)\int_0^t \|u_t(s)\|^2 \, ds - pb. \end{aligned} \tag{5.9}$$

On the other hand, from (3.2) we have that

$$\begin{aligned} \int_\Omega |u(t)|^p \ln|u(t)|dx = &-pE(t) + \frac{p}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|^2 \\ &+ \frac{p}{2}(g \circ \nabla u)(t) + \frac{1}{p}\|u(t)\|_p^p. \end{aligned} \tag{5.10}$$

And hence, (5.9) and (5.10) yield

$$\zeta(t) = -2pE(t) + \left[p - 2 - \left(p - 2 + \frac{1}{p}\right)\int_0^t g(s)ds\right]\|\nabla u(t)\|^2 + \frac{2}{p}\|u(t)\|_p^p - (p+2)\int_0^t \|u_t(s)\|^2 ds - pb. \tag{5.11}$$

By virtue of the energy inequality (3.4), we deduce from (5.11) that

$$\begin{aligned}\zeta(t) \geq &-2pE(0) + \left[p - 2 - \left(p - 2 + \frac{1}{p}\right)\int_0^t g(s)ds\right]\|\nabla u(t)\|^2 \\ &+ \frac{2}{p}\|u(t)\|_p^p + (p-2)\int_0^t \|u_t(s)\|^2 ds - pb \\ \geq &\, 2p\left[\frac{p-2}{2p}\left(1 - \int_0^t g(s)ds - \frac{1}{p(p-2)}\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p - E(0) - \frac{b}{2}\right] \\ \geq &\, 2p\left[\frac{p-2}{2p}\kappa\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p - E(0) - \frac{b}{2}\right],\end{aligned} \tag{5.12}$$

where κ is a constant given by

$$0 < \kappa = \ell - \frac{1}{p(p-2)}\int_0^\infty g(s)ds \leq \ell$$

thanks to $p > 2$ and $\ell = 1 - \int_0^\infty g(s)ds$.

By virtue of Lemma 5.1, it follows from (5.12) that

$$\zeta(t) \geq 2p\left(d_k - E(0) - \frac{b}{2}\right).$$

Since $E(0) < d_\kappa$, choosing b small enough such that

$$0 < b \leq 2\left(d_\kappa - E(0)\right), \tag{5.13}$$

we get

$$\zeta(t) > \rho > 0. \tag{5.14}$$

Combining (5.8) and (5.14), we arrive at

$$\theta''(t)\theta(t) - \frac{p+2}{4}\left(\theta'(t)\right)^2 \geq \rho\theta(t) \geq 0.$$

Applying Lemma 2.4 with $\gamma = \frac{p-2}{4}$ we have that $\theta(t) \to \infty$ for $t \to t^* < \infty$, which contradicts $T = \infty$. And hence $u(t)$ blows up at finite time T. Moreover, we have also

$$T \leq \frac{4\theta(0)}{(p-2)\theta'(0)} = \frac{4\left(T\|u_0\|^2 + bT_0^2\right)}{2(p-2)bT_0} = \frac{2\|u_0\|^2}{(p-2)bT_0}T + \frac{2T_0}{p-2}.$$

By choosing $T_0 \in \left(\frac{2\|u_0\|^2}{(p-2)b}, \infty\right)$, we get

$$T \leq \frac{2bT_0^2}{(p-2)bT_0 - 2\|u_0\|^2}.$$

Since b satisfies (5.13), by minimizing the above inequality for $T_0 > \frac{2\|u_0\|^2}{(p-2)b}$, we arrive at

$$T \leq \frac{8\|u_0\|^2}{(p-2)^2(d_\kappa - E(0))}.$$

Step 2: Lower bound estimate of the blow up time.

By Step 1 we know that $\lim_{t\to T^-} \|u(t)\|^2 = \infty$ which implies

$$\lim_{t\to T^-} \|\nabla u(t)\|^2 = \infty, \tag{5.15}$$

thanks to the continuous embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

Let us now define an auxiliary function

$$\begin{aligned} R(t) &= \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g\circ\nabla u)(t) \\ &= E(t) + \frac{1}{p}\int_\Omega |u(t)|^p \ln|u(t)|dx - \frac{1}{p^2}\|u(t)\|_p^p. \end{aligned}$$

Then by assumption (G, (ii)), we have

$$\frac{1}{2(p-1)^2}\|\nabla u(t)\|^2 \leq \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g\circ\nabla u)(t) = R(t).$$

which implies $\lim_{t\to T^-} R(t) = \infty$ thanks to (5.15).

Recalling the Lemma 3.2, we have

$$\begin{aligned} R'(t) &= E'(t) + \int_\Omega |u(t)|^{p-2}u(t)u_t(t)\ln|u(t)|dx \\ &\leq -\|u_t(t)\|^2 + \int_\Omega |u(t)|^{p-2}u(t)u_t(t)\ln|u(t)|dx. \end{aligned}$$

Let us divide Ω into two parts as follows:

$$\Omega_1 = \{x\in\Omega : |u(x,t)| \leq 1\} \quad \text{and} \quad \Omega_2 = \{x\in\Omega : |u(x,t)| \geq 1\}.$$

Applying Lemma 2.3, Hölder's inequality, Young's inequality, we reach

$$\begin{aligned} R'(t) &\leq -\|u_t\|^2 + \int_\Omega |u|^{p-2}uu_t\ln|u|dx \\ &= -\|u_t\|^2 + \int_{\Omega_1}|u|^{p-2}uu_t\ln|u|dx + \int_{\Omega_2}|u|^{p-2}uu_t\ln|u|dx \\ &\leq -\|u_t\|^2 + (e(p-1))^{-1}\int_{\Omega_1}|u_t|dx + (e\sigma)^{-1}\int_{\Omega_2}|u|^{p-1+\sigma}|u_t|dx \\ &\leq -\|u_t\|^2 + (e(p-1))^{-1}|\Omega_1|^{\frac{1}{2}}\|u_t\| + (e\sigma)^{-1}\|u\|_{2(p-1+\sigma)}^{p-1+\sigma}\|u_t\| \\ &\leq -\|u_t\|^2 + \frac{1}{2}(e(p-1))^{-2}|\Omega_1| + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(e\sigma)^{-2}\|u\|_{2(p-1+\sigma)}^{2(p-1+\sigma)} + \frac{1}{2}\|u_t\|^2 \\ &\leq \frac{1}{2}(e(p-1))^{-2}|\Omega| + \frac{1}{2}(e\sigma)^{-2}\|u\|_{2(p-1+\sigma)}^{2(p-1+\sigma)}. \end{aligned} \tag{5.16}$$

Here, for simplicity, we write u instead of $u(t)$.

By $2 < 2p-2 < \frac{2n}{n-2}$, there exists $\sigma > 0$ such that $2 < 2(p-1+\sigma) < \frac{2n}{n-2}$. Using the embedding $H_0^1(\Omega) \hookrightarrow L^{2(p-1+\sigma)}(\Omega)$, we deduce from (5.16) that

$$\begin{aligned} R'(t) &\leq \frac{1}{2}\left(e(p-1)\right)^{-2}|\Omega| + \frac{1}{2}(e\sigma)^{-2} S_{2(p-1+\sigma)}^{2(p-1+\sigma)} \|\nabla u(t)\|^{2(p-1+\sigma)} \\ &\leq K_1 R^{p-1+\sigma}(t) + K_2, \end{aligned} \tag{5.17}$$

where S_q is the optimal constant of embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, and

$$K_1 = \frac{1}{2}(e\sigma)^{-2} S_{2(p-1+\sigma)}^{2(p-1+\sigma)} \left(2(p-1)^2\right)^{p-1+\sigma}, \quad K_2 = \frac{1}{2}\left(e(p-1)\right)^{-2}|\Omega|.$$

Integrating (5.17) from 0 to t, we get

$$\int_{R(0)}^{R(t)} \frac{1}{K_1 z^{p-1+\sigma} + K_2} dz \leq t,$$

combining with the fact $\lim_{t\to T^-} R(t) = \infty$ we obtain (3.6). Thus the proof is complete.

6 Proof of Theorem 3.5

We begin with the following lemma which is helpful to the proof of Theorem 3.5.

Lemma 6.1. *Under the assumptions of the Theorem 3.3. For any $0 < \delta \leq \ell$, we have that*

$$I_\delta(u(t)) \geq \left[1 - \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\right] \delta \|\nabla u(t)\|^2.$$

Proof. It is first noticed that $u_0 \in \mathcal{W}_\delta$ thanks to $E(0) < d_\delta$ and $I(u_0) > 0$. By using the similar method as in the proof of Lemma 5.1, we can show that $u(t) \in \mathcal{W}_\delta$ for $t \geq 0$. Taking this into account and using the Lemma 2.5 (iii), we imply that there is a constant $\lambda_1 > 1$ such that $I_\delta(\lambda_1 u(t)) = 0$.

On the other hand, from the definition of I_δ, we have

$$\begin{aligned} I_\delta(\lambda_1 u(t)) &= \delta(\lambda_1)^2 \|\nabla u(t)\|^2 - (\lambda_1)^p \int_\Omega |u(t)|^p \ln|u(t)| dx - (\lambda_1)^p \ln\lambda_1 \|u(t)\|_p^p \\ &= \left((\lambda_1)^2 - (\lambda_1)^p\right)\delta\|\nabla u(t)\|^2 + (\lambda_1)^p I_\delta(u(t)) - (\lambda_1)^p \ln\lambda_1 \|u(t)\|_p^p, \end{aligned}$$

which implies, thanks to $I_\delta(\lambda_1 u(t)) = 0$ and $\lambda_1 > 1$, that

$$I_\delta(u(t)) \geq \left[1 - (\lambda_1)^{2-p}\right]\delta\|\nabla u(t)\|^2 + \ln\lambda_1 \|u(t)\|_p^p \geq \left[1 - (\lambda_1)^{2-p}\right]\delta\|\nabla u(t)\|^2. \tag{6.1}$$

To end the proof it remains to estimate λ_1. By variational characterization of d_δ, we have

$$\begin{aligned} d_\delta \leq J_\delta(\lambda_1 u(t)) &= \frac{1}{p} I_\delta(\lambda_1 u(t)) + \delta\left(\frac{1}{2} - \frac{1}{p}\right)(\lambda_1)^2 \|\nabla u(t)\|^2 + \frac{(\lambda_1)^p}{p^2}\|u(t)\|_p^p \\ &\leq \left[\delta\left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p\right](\lambda_1)^p. \end{aligned} \tag{6.2}$$

On the other hand, by the non-increasing property of functional energy $E(t)$, we have that

$$E(0) \geq E(t) \geq J_\delta(u(t)) = \frac{1}{p} I_\delta(u(t)) + \delta\left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u(t)\|^2 + \frac{1}{p^2} \|u(t)\|_p^p$$
$$> \delta\left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u(t)\|^2 + \frac{1}{p^2} \|u(t)\|_p^p. \tag{6.3}$$

From (6.2)–(6.3), we deduce that

$$\lambda_1 \geq \left(\frac{d_\delta}{E(0)}\right)^{1/p} > 1. \tag{6.4}$$

The proof follows from (6.1) and (6.4). □

As a consequence of this lemma, we get the following estimates.

Lemma 6.2. *Under the assumptions of the Theorem 3.3. For any $0 < \delta \leq \ell$, we possess*

$$\int_\Omega |u(t)|^p \ln |u(t)| dx \leq \delta \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}} \|\nabla u(t)\|^2 \quad \text{and} \quad \|u(t)\|_p^p \leq C(p, d_\delta) \|\nabla u(t)\|^2, \tag{6.5}$$

where $C(p, d_\delta)$ is the constant given by

$$C(p, d_\delta) = S_p^p \left[\frac{p\delta^{-1} d_\delta}{1 - \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}}\right]^{\frac{p-2}{2}}.$$

Here S_p is the best constant in the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

Proof. The first estimate in (6.5) follows from the Lemma 6.1 and the identity

$$\int_\Omega |u(t)|^p \ln |u(t)| dx = \delta \|\nabla u(t)\|^2 - I_\delta(u(t))$$
$$\leq \delta \|\nabla u(t)\|^2 - \left[1 - \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\right] \delta \|\nabla u(t)\|^2 = \delta \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}} \|\nabla u(t)\|^2,$$

and since $2 < p < \frac{2(n-1)}{n-2}$, the second one follows from the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ and the Lemma 6.1

$$\|u(t)\|_p^p \leq S_p^p \|\nabla u(t)\|^p \leq S_p^p \left[\frac{p\delta^{-1} d_\delta}{1 - \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}}\right]^{\frac{p-2}{2}} \|\nabla u(t)\|^2 \equiv C(p, d_\delta) \|\nabla u(t)\|^2,$$

where S_p is the best constant in the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. □

For the proof of Theorem 3.5, we define the following auxiliary functional

$$L(t) = E(t) + \varepsilon\rho(t),$$

where ρ is given by

$$\rho(t) = \frac{1}{2}\xi(t) \|u(t)\|^2.$$

The next lemma tells us that $E(t)$ and $L(t)$ are equivalent functions.

Lemma 6.3. *For ε_1 and ε_2 small enough, we have*

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t)$$

holds for two positive constants α_1 and α_2.

Proof. By virtue of Lemma 6.1 and the definition of $E(t)$, we have that

$$E(t) \geq \frac{\delta}{p}\left[1-\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\right]\|\nabla u(t)\|^2 .$$

Taking this into account, we deduce from the definition of $\rho(t)$ that

$$|\rho(t)| \leq \frac{S_2^2}{2}\xi(t)\|\nabla u(t)\|^2 \leq \frac{pS_2^2}{2\delta}\left[1-\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\right]^{-1}\xi(t)E(t),$$

where S_2 is the optimal constant in the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

From (G, iii) we have $\xi(t) \leq \xi(0) \leq M$ for some constant $M > 0$. Combining with the above estimate to obtain

$$|L(t) - E(t)| \leq \varepsilon |\rho(t)| \leq \varepsilon C(M)E(t),$$

that is

$$(1-\varepsilon C(M))\, E(t) \leq L(t) \leq (1+\varepsilon C(M))\, E(t).$$

By choosing ε small such that $0 < \varepsilon < 1/C(M)$ we claim the lemma. □

The next lemma allow us to estimate $\rho'(t)$.

Lemma 6.4. *Let (G, (i, iii)) hold. Then we have that*

$$\rho'(t) \leq -\frac{\ell}{2}\xi(t)\|\nabla u(t)\|^2 + \xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx + \frac{1-\ell}{2\ell}\xi(t)\,(g\circ\nabla u)\,(t).$$

Proof. By using the differential equation in (1.1), we easily see that

$$\int_\Omega u_t(t)u(t)dx = -\|\nabla u(t)\|^2 + \int_\Omega |u(t)|^p \ln|u(t)|dx + \int_0^t g\,(t-s)\,\langle \nabla u(s), \nabla u(t)\rangle\, ds.$$

By using the Hölder and Young inequalities, we obtain for any $\eta > 0$

$$\begin{aligned}
&\int_0^t g\,(t-s)\,\langle \nabla u(s), \nabla u(t)\rangle\, ds \\
&\quad = \int_0^t g\,(t-s)\,\langle \nabla u(s) - \nabla u(t), \nabla u(t)\rangle\, ds + \left(\int_0^t g\,(s)\,ds\right)\|\nabla u(t)\|^2 \\
&\quad \leq \frac{1}{2\eta}(g\circ\nabla u)(t) + \left(1+\frac{\eta}{2}\right)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 .
\end{aligned}$$

And hence, we arrive at

$$\int_\Omega u_t(t)u(t)dx \leq -\left[1-\frac{(1-\ell)(2+\eta)}{2}\right]\|\nabla u(t)\|^2 + \int_\Omega |u(t)|^p \ln|u(t)|dx + \frac{1}{2\eta}\,(g\circ\nabla u)\,(t).$$

By assumption (G,iii) and definition of $\rho(t)$, we deduce that

$$\begin{aligned}\rho'(t) &= \frac{1}{2}\xi'(t)\left\|u(t)\right\|^2 + \xi(t)\int_\Omega u_t(t)u(t)dx \\ &\le -\left[1-\frac{(1-\ell)(2+\eta)}{2}\right]\xi(t)\left\|\nabla u(t)\right\|^2 \\ &\quad + \xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx + \frac{1}{2\eta}\xi(t)\left(g\circ\nabla u\right)(t).\end{aligned}$$

Choosing $\eta = \frac{\ell}{1-\ell}$, we obtain

$$\rho'(t) \le -\frac{\ell}{2}\xi(t)\left\|\nabla u(t)\right\|^2 + \xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx + \frac{1-\ell}{2\ell}\xi(t)\left(g\circ\nabla u\right)(t).$$

The proof is complete. □

We are now ready to give the proof of Theorem 3.5.

Proof of Theorem 3.5. Taking into account (3.3), we deduce from Lemma 6.4 that

$$\begin{aligned}L'(t) &= E'(t) + \varepsilon\rho'(t) \\ &\le -\left\|u_t(t)\right\|^2 - \varepsilon\frac{\ell}{2}\xi(t)\left\|\nabla u(t)\right\|^2 + \varepsilon\xi(t)\int_\Omega |u(t)|^p\ln|u(t)|dx \\ &\quad + \frac{1}{2}\left(g'\circ\nabla u\right)(t) + \varepsilon\frac{1-\ell}{2\ell}\xi(t)\left(g\circ\nabla u\right)(t). \end{aligned} \tag{6.6}$$

By (G,iii) we have $\left(g'\circ\nabla u\right)(t) \le -\xi(t)\left(g\circ\nabla u\right)(t)$. Using (3.2), (6.6) and Lemma 6.2, we have

$$\begin{aligned}L'(t) &\le -\varepsilon\Lambda\xi(t)E(t) + \frac{\varepsilon\Lambda}{2}\xi(t)\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|^2 + \frac{\varepsilon\Lambda}{2}\xi(t)\left(g\circ\nabla u\right)(t) \\ &\quad - \frac{\varepsilon\Lambda}{p}\xi(t)\int_\Omega |u(t)|^p\ln|u(t)|dx + \frac{\varepsilon\Lambda}{p^2}\xi(t)\left\|u(t)\right\|_p^p - \left\|u_t(t)\right\|^2 \\ &\quad - \frac{\varepsilon\ell}{2}\xi(t)\left\|\nabla u(t)\right\|^2 + \varepsilon\xi(t)\int_\Omega |u(t)|^p\ln|u(t)|dx - \frac{1}{2}\left(1-\varepsilon\frac{1-\ell}{\ell}\right)\xi(t)\left(g\circ\nabla u\right)(t) \\ &\le -\varepsilon\Lambda\xi(t)E(t) - \varepsilon\left(\frac{\ell}{2}-\frac{\Lambda\ell}{2}-\frac{\Lambda C(p,d_\delta)}{p^2}\right)\xi(t)\left\|\nabla u(t)\right\|^2 \\ &\quad + \varepsilon\left(1-\frac{\Lambda}{p}\right)\xi(t)\int_\Omega |u(t)|^p\ln|u(t)|dx - \frac{1}{2}\left(1-\varepsilon\frac{1-\ell}{\ell}-\varepsilon\Lambda\right)\xi(t)\left(g\circ\nabla u\right)(t),\end{aligned}$$

for any $\Lambda > 0$. By choosing $\varepsilon > 0$ and $\Lambda < p$ small enough such that

$$1-\varepsilon\frac{1-\ell}{\ell} > 0 \quad \text{and} \quad 1-\varepsilon\frac{1-\ell}{\ell}-\varepsilon\Lambda > 0$$

we obtain

$$\begin{aligned}L'(t) &\le -\varepsilon\Lambda\xi(t)E(t) - \varepsilon\left(\frac{\ell}{2}-\frac{\Lambda\ell}{2}-\frac{\Lambda C(p,d_\delta)}{p^2}\right)\xi(t)\left\|\nabla u(t)\right\|^2 \\ &\quad + \varepsilon\delta\left(1-\frac{\Lambda}{p}\right)\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\xi(t)\left\|\nabla u(t)\right\|^2 \\ &\le -\varepsilon\Lambda\xi(t)E(t) - \varepsilon\left(\frac{\ell}{2}-\delta\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}-\frac{\Lambda\ell}{2}-\frac{\Lambda C(p,d_\delta)}{p^2}\right)\xi(t)\left\|\nabla u(t)\right\|^2.\end{aligned}$$

Since $E(0) < \left(\frac{\ell}{2\delta}\right)^{\frac{p}{p-2}} d_\delta$, we can pick $0 < \Lambda < p$ such that

$$\frac{\ell}{2} - \delta\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}} - \frac{\Lambda\ell}{2} - \frac{\Lambda C(p, d_\delta)}{p^2} > 0.$$

Therefore, we get

$$L'(t) \leq -\varepsilon\Lambda\xi(t)E(t) \leq -\frac{\varepsilon\Lambda}{\alpha_2}\xi(t)L(t), \quad \forall t \geq t_0,$$

which implies

$$L(t) \leq L(0)e^{-\frac{\varepsilon\Lambda}{\alpha_2}\int_0^t \xi(s)ds}, \quad \forall t \geq t_0.$$

This completes the proof. □

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