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On a solvable class of nonlinear difference equations of fourth order

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Abstract. We consider a class of nonlinear difference equations of the fourth order, which extends some equations in the literature. It is shown that the class of equations is solvable in closed form explaining theoretically, among other things, solvability of some previously considered very special cases. We also present some applications of the main theorem through two examples, which show that some results in the literature are not correct.

Keywords: nonlinear difference equation, solvable difference equation, general solution, closed-form formula for solutions.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ stand for the sets of natural, integer and real numbers respectively, and let $\mathbb{N}_k = \{n \in \mathbb{Z} : n \geq k\}$ where $k \in \mathbb{Z}$. If $l \in \mathbb{Z}$, then, as usual, we regard that $\prod_{j=l}^{l-1} c_j = 1$.

To obtain some information on solutions of difference equations and systems of difference equations scientists first tried to find some closed form formulas for their solutions. The first important results can be found, for example, in [7,10,11,17,18], as well as in the books [15,16] where many results up to the end of the eighteenth century can be found.

The linear homogeneous second order difference equation with constant coefficients

$$x_{n+2} + ax_{n+1} + bx_n = 0, \quad n \in \mathbb{N}_0, \tag{1.1}$$

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where $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$, was solved by de Moivre [10].

If the coefficients a and b satisfy the condition $a^2 \neq 4b$, then the general solution to equation (1.1) is given by the formula

$$x_n = \frac{(x_1 - \lambda_2 x_0)\lambda_1^n - (x_1 - \lambda_1 x_0)\lambda_2^n}{\lambda_1 - \lambda_2}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \quad \text{and} \quad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

are the roots of the polynomial equation $p_2(\lambda) := \lambda^2 + a\lambda + b = 0$ (see [10, p.84]).

If $a^2 = 4b$, then the polynomial has two equal roots

$$\lambda_1 = \lambda_2 = -\frac{a}{2},$$

and the general solution to equation (1.1) in this case is given by the closed-form formula

$$x_n = ((x_1 - \lambda_1 x_0)n + \lambda_1 x_0)\lambda_1^{n-1}, \quad n \in \mathbb{N}_0. \tag{1.3}$$

See [7], where, among other things, the method for finding solutions to linear homogeneous difference equations with constant coefficients of arbitrary order in the form

$$x_n = \lambda^n, \quad n \in \mathbb{N}_0,$$

was described.

Closed-form formulas for solutions to linear homogeneous difference equations with constant coefficients of the third order were presented by Euler in [11]. For some later presentations of results in the topic see, for example, the books [8,13,19,20,22].

Beside solvability of difference equations and systems of difference equations, some recent investigations in the topic include also finding invariants of the equations and systems. For some recent results in the topics, as well as their applications see, for example, [4–6,12,23–37], as well as many related references cited therein.

One of the difference equations which, by using some changes of variables, reduces to equation (1.1) is

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

the bilinear/fractional linear difference equation. Equation (1.4) and some of related systems of difference equations have been investigated since the time of Laplace and frequently appear in the literature (see, for example, [1,2,6,8,9,14–16,19,21,22,31,32,34,35,37]).

Many other classes of difference equations can be reduced to linear difference equations with constant coefficients. It is of some interest to find such classes, as well as some which reduces to equation (1.4). By using some changes of variables it is easy to form many such classes.

There have been some investigations on solvability and behaviour of solutions to the difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where $a, b, c, d \in \mathbb{R}$.

Here we show that a more general class of difference equations can be solved in closed form, extending some of the results on equation (1.5) in the literature. We use some methods and ideas related to the ones, e.g., in [12,31,32,34,37]. By using obtained closed-form formulas for solutions to equation (1.5), we present some applications of our main theorem by giving two examples which show that some results in [3] are not correct.

2 Main results

This section presents the main result in the paper. It shows the solvability of a generalization of equation (1.5), by finding closed form formulas for their solutions.

Theorem 2.1. *Assume $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0 \neq \gamma^2 + \delta^2$, g is a strictly monotone and continuous function, $g(\mathbb{R}) = \mathbb{R}$ and $g(0) = 0$. Then, the equation*

$$x_{n+1} = g^{-1}\left(g(x_n)\frac{\alpha g(x_{n-2}) + \beta g(x_{n-3})}{\gamma g(x_{n-2}) + \delta g(x_{n-3})}\right), \quad n \in \mathbb{N}_0, \tag{2.1}$$

is solvable in closed form.

Proof. By a well known theorem in real analysis we see that the conditions posed on function g imply the existence of the inverse function g^{-1} which satisfies the same conditions as the function g ([38]).

Assume that $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. Then from (2.1) we have $x_{n_0+1} = 0$. These facts along with (2.1) imply that x_{n_0+4} is not defined. Thus, of interest are the solutions of equation (2.1) such that $x_n \neq 0$, $n \in \mathbb{N}_0$. We may also assume that $x_{-j} \neq 0$, $j = \overline{1,3}$, otherwise the equation can be considered only on the domain $\mathbb{N}_0$. Hence, we suppose

$$x_n \neq 0, \quad \text{for } n \in \mathbb{N}_{-3}. \tag{2.2}$$

From (2.2) and the conditions of the theorem we have

$$g(x_n) \neq 0, \quad \text{for } n \in \mathbb{N}_{-3}. \tag{2.3}$$

First, assume $\alpha\delta \neq \beta\gamma$ and $\gamma \neq 0$. Let

$$y_n = \frac{g(x_n)}{g(x_{n-1})}, \quad n \in \mathbb{N}_{-2}. \tag{2.4}$$

From (2.1) and monotonicity of g, we have

$$g(x_{n+1}) = g(x_n)\frac{\alpha g(x_{n-2}) + \beta g(x_{n-3})}{\gamma g(x_{n-2}) + \delta g(x_{n-3})}, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Employing the change of variables (2.4) in (2.5) we have

$$y_{n+1} = \frac{\alpha y_{n-2} + \beta}{\gamma y_{n-2} + \delta}, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Let

$$z_m^{(j)} = y_{3m-j}, \quad m \in \mathbb{N}_0, \ j = \overline{0,2}. \tag{2.7}$$

Then, from (2.6) and (2.7) we have

$$z^{(j)}_{m+1} = \frac{\alpha z^{(j)}_m + \beta}{\gamma z^{(j)}_m + \delta}, \tag{2.8}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, which is a bilinear difference equation.

Let

$$z^{(j)}_m = \frac{u^{(j)}_{m+1}}{u^{(j)}_m} + f_j, \quad m \in \mathbb{N}_0,\ j = \overline{0,2}, \tag{2.9}$$

for some $f_j \in \mathbb{R}$, $j = \overline{0,2}$.

Then from (2.8) and (2.9) we have

$$\left(\frac{u^{(j)}_{m+2}}{u^{(j)}_{m+1}} + f_j\right)\left(\gamma \frac{u^{(j)}_{m+1}}{u^{(j)}_m} + \gamma f_j + \delta\right) - \left(\alpha \frac{u^{(j)}_{m+1}}{u^{(j)}_m} + \alpha f_j + \beta\right) = 0,$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

Let

$$f_j = -\frac{\delta}{\gamma}, \quad j = \overline{0,2}.$$

Then we have

$$\gamma^2 u^{(j)}_{m+2} - \gamma(\alpha+\delta)u^{(j)}_{m+1} + (\alpha\delta - \beta\gamma)u^{(j)}_m = 0, \tag{2.10}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

Suppose $\Delta := (\alpha+\delta)^2 - 4(\alpha\delta - \beta\gamma) \neq 0$. Then by using formula (1.2) we have that

$$u^{(j)}_m = \frac{(u^{(j)}_1 - \lambda_2 u^{(j)}_0)\lambda_1^m - (u^{(j)}_1 - \lambda_1 u^{(j)}_0)\lambda_2^m}{\lambda_1 - \lambda_2}, \tag{2.11}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, where

$$\lambda_1 = \frac{\alpha+\delta+\sqrt{\Delta}}{2\gamma} \quad \text{and} \quad \lambda_2 = \frac{\alpha+\delta-\sqrt{\Delta}}{2\gamma},$$

is the general solution to (2.10).

Formulas (2.9) and (2.11) imply

$$\begin{aligned} z^{(j)}_m &= \frac{(u^{(j)}_1 - \lambda_2 u^{(j)}_0)\lambda_1^{m+1} - (u^{(j)}_1 - \lambda_1 u^{(j)}_0)\lambda_2^{m+1}}{(u^{(j)}_1 - \lambda_2 u^{(j)}_0)\lambda_1^{m} - (u^{(j)}_1 - \lambda_1 u^{(j)}_0)\lambda_2^{m}} - \frac{\delta}{\gamma} \\ &= \frac{(z^{(j)}_0 - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (z^{(j)}_0 - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(z^{(j)}_0 - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (z^{(j)}_0 - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma}, \end{aligned}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, from which along with (2.7) it follows that

$$y_{3m-j} = \frac{(y_{-j} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (y_{-j} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(y_{-j} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (y_{-j} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma}, \tag{2.12}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.4) and (2.12) it follows that

$$g(x_{3m-j}) = \left(\frac{\left(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma} \right) g(x_{3m-j-1}),$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.4) we easily get

$$g(x_{3m-j}) = y_{3m-j}y_{3m-j-1}y_{3m-j-2}g(x_{3m-j-3}), \tag{2.13}$$

for $m \in \mathbb{N}$, $j = \overline{1,3}$.

Hence

$$\begin{aligned} g(x_{3m}) &= g(x_{-3}) \prod_{i=0}^{m} y_{3i}y_{3i-1}y_{3i-2}, \\ g(x_{3m+1}) &= g(x_{-2}) \prod_{i=0}^{m} y_{3i+1}y_{3i}y_{3i-1}, \\ g(x_{3m+2}) &= g(x_{-1}) \prod_{i=0}^{m} y_{3i+2}y_{3i+1}y_{3i}, \end{aligned}$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{3m} = g^{-1}\left(g(x_{-3}) \prod_{i=0}^{m} y_{3i}y_{3i-1}y_{3i-2} \right), \tag{2.14}$$

$$x_{3m+1} = g^{-1}\left(g(x_{-2}) \prod_{i=0}^{m} y_{3i+1}y_{3i}y_{3i-1} \right), \tag{2.15}$$

$$x_{3m+2} = g^{-1}\left(g(x_{-1}) \prod_{i=0}^{m} y_{3i+2}y_{3i+1}y_{3i} \right), \tag{2.16}$$

for $m \in \mathbb{N}_0$, where

$$\begin{aligned} y_{3m}y_{3m-1}y_{3m-2} = &\left(\frac{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\ &\times \left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\ &\times \left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma} \right), \end{aligned} \tag{2.17}$$

$$y_{3m+1}y_{3m}y_{3m-1} = \left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+2} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}} - \frac{\delta}{\gamma}\right)$$
$$\times\left(\frac{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right)$$
$$\times\left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right), \tag{2.18}$$

$$y_{3m+2}y_{3m+1}y_{3m} = \left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+2} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}} - \frac{\delta}{\gamma}\right)$$
$$\times\left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+2} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}} - \frac{\delta}{\gamma}\right)$$
$$\times\left(\frac{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right), \tag{2.19}$$

for $m \in \mathbb{N}_0$. Formulas (2.14)–(2.19) present general solution to equation (2.1) in this case.

Assume $\Delta = 0$. Then, by using formula (1.3) we see that the general solution to equation (2.10) in this case is given by

$$u_m^{(j)} = ((u_1^{(j)} - \lambda_1 u_0^{(j)})m + \lambda_1 u_0^{(j)})\lambda_1^{m-1}, \tag{2.20}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, where

$$\lambda_1 = \frac{\alpha+\delta}{2\gamma} \neq 0.$$

From (2.9) and (2.20) we have

$$z_m^{(j)} = \frac{((u_1^{(j)} - \lambda_1 u_0^{(j)})(m+1) + \lambda_1 u_0^{(j)})\lambda_1}{(u_1^{(j)} - \lambda_1 u_0^{(j)})m + \lambda_1 u_0^{(j)}} - \frac{\delta}{\gamma}$$
$$= \frac{((z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, from which along with (2.7) it follows that

$$y_{3m-j} = \frac{((y_{-j} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(y_{-j} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}, \tag{2.21}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.4) and (2.21) we have

$$g(x_{3m-j}) = \left(\frac{(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right) g(x_{3m-j-1}), \tag{2.22}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

We also have

$$y_{3m}y_{3m-1}y_{3m-2} = \left(\frac{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right), \tag{2.23}$$

$$y_{3m+1}y_{3m}y_{3m-1} = \left(\frac{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+2) + \lambda_1)\lambda_1}{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right), \tag{2.24}$$

$$y_{3m+2}y_{3m+1}y_{3m} = \left(\frac{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})(m+2) + \lambda_1)\lambda_1}{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+2) + \lambda_1)\lambda_1}{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right), \tag{2.25}$$

for $m \in \mathbb{N}_0$.

The above consideration, shows that the general solution to equation (2.1) in this case is given by formulas (2.14)–(2.16), (2.23)–(2.25).

Now assume $\gamma = 0$. Then $\delta \neq 0$ and equation (2.6) becomes

$$y_{n+1} = \frac{\alpha}{\delta}y_{n-2} + \frac{\beta}{\delta}, \quad n \in \mathbb{N}_0. \tag{2.26}$$

Hence,

$$z_{m+1}^{(j)} = \frac{\alpha}{\delta}z_m^{(j)} + \frac{\beta}{\delta}, \quad m \in \mathbb{N}_0, \ j = \overline{0,2}. \tag{2.27}$$

If $\alpha = \delta$, then from (2.27) we obtain

$$z_m^{(j)} = \frac{\beta}{\delta}m + z_0^{(j)}, \quad m \in \mathbb{N}_0, \ j = \overline{0,2}.$$

that is

$$y_{3m-j} = \frac{\beta}{\delta} m + y_{-j}, \quad m \in \mathbb{N}_0,\ j = \overline{0,2},$$

from which along with (2.4) and (2.13) it follows that

$$\begin{aligned} g(x_{3m}) &= \left(\frac{\beta}{\delta} m + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta} m + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta} m + \frac{g(x_{-2})}{g(x_{-3})}\right) g(x_{3m-3}), \\ g(x_{3m+1}) &= \left(\frac{\beta}{\delta}(m+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta} m + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta} m + \frac{g(x_{-1})}{g(x_{-2})}\right) g(x_{3m-2}), \\ g(x_{3m+2}) &= \left(\frac{\beta}{\delta}(m+1) + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}(m+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta} m + \frac{g(x_0)}{g(x_{-1})}\right) g(x_{3m-1}), \end{aligned}$$

for $m \in \mathbb{N}_0$, from which it follows that

$$\begin{aligned} g(x_{3m}) &= g(x_{-3}) \prod_{j=0}^{m} \left(\frac{\beta}{\delta} j + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_{-2})}{g(x_{-3})}\right), \\ g(x_{3m+1}) &= g(x_{-2}) \prod_{j=0}^{m} \left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_{-1})}{g(x_{-2})}\right), \\ g(x_{3m+2}) &= g(x_{-1}) \prod_{j=0}^{m} \left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_0)}{g(x_{-1})}\right), \end{aligned}$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{3m} = g^{-1}\left(g(x_{-3}) \prod_{j=0}^{m} \left(\frac{\beta}{\delta} j + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_{-2})}{g(x_{-3})}\right)\right), \tag{2.28}$$

$$x_{3m+1} = g^{-1}\left(g(x_{-2}) \prod_{j=0}^{m} \left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_{-1})}{g(x_{-2})}\right)\right), \tag{2.29}$$

$$x_{3m+2} = g^{-1}\left(g(x_{-1}) \prod_{j=0}^{m} \left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta} j + \frac{g(x_0)}{g(x_{-1})}\right)\right), \tag{2.30}$$

for $m \in \mathbb{N}_0$. Hence, the general solution to equation (2.1) in this case is given by formulas (2.28)–(2.30).

If $\alpha \neq \delta$, then from (2.27) we have

$$z_m^{(j)} = \frac{\beta}{\alpha - \delta}\left(\left(\frac{\alpha}{\delta}\right)^m - 1\right) + \left(\frac{\alpha}{\delta}\right)^m z_0^{(j)},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, that is,

$$y_{3m-j} = \beta \frac{(\alpha/\delta)^m - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^m y_{-j}, \tag{2.31}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.4), (2.13) and (2.31) we have

$$\begin{aligned}
g(x_{3m}) = &\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_0)}{g(x_{-1})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_{-1})}{g(x_{-2})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_{-2})}{g(x_{-3})}\right)g(x_{3m-3}),\\
g(x_{3m+1}) = &\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{g(x_{-2})}{g(x_{-3})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_0)}{g(x_{-1})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_{-1})}{g(x_{-2})}\right)g(x_{3m-2}),\\
g(x_{3m+2}) = &\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{g(x_{-1})}{g(x_{-2})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{g(x_{-2})}{g(x_{-3})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_0)}{g(x_{-1})}\right)g(x_{3m-1}),
\end{aligned}$$

for $m \in \mathbb{N}_0$.

Hence

$$\begin{aligned}
g(x_{3m}) = g(x_{-3})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-1})}{g(x_{-2})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-2})}{g(x_{-3})}\right),\\
g(x_{3m+1}) = g(x_{-2})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-2})}{g(x_{-3})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-1})}{g(x_{-2})}\right),\\
g(x_{3m+2}) = g(x_{-1})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-1})}{g(x_{-2})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-2})}{g(x_{-3})}\right)\\
&\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right),
\end{aligned}$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{3m} = g^{-1}\Bigg(g(x_{-3})\prod_{j=0}^{m}\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right)$$
$$\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-1})}{g(x_{-2})}\right)$$
$$\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-2})}{g(x_{-3})}\right)\Bigg), \tag{2.32}$$

$$x_{3m+1} = g^{-1}\Bigg(g(x_{-2})\prod_{j=0}^{m}\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-2})}{g(x_{-3})}\right)$$
$$\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right)$$
$$\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-1})}{g(x_{-2})}\right)\Bigg), \tag{2.33}$$

$$x_{3m+2} = g^{-1}\Bigg(g(x_{-1})\prod_{j=0}^{m}\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-1})}{g(x_{-2})}\right)$$
$$\times\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-2})}{g(x_{-3})}\right)$$
$$\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right)\Bigg), \tag{2.34}$$

for $m \in \mathbb{N}_0$. Hence, the general solution to equation (2.1) in this case is given by formulas (2.32)–(2.34).

Assume $\alpha\delta = \beta\gamma$. If $\alpha = 0$, then $\beta \neq 0$. This implies $\gamma = 0$ and $\delta \neq 0$. Hence

$$x_{n+1} = g^{-1}\left(\frac{\beta}{\delta}g(x_n)\right), \quad n \in \mathbb{N}_0. \tag{2.35}$$

From (2.35) we easily get

$$x_n = g^{-1}\left(\left(\frac{\beta}{\delta}\right)^n g(x_0)\right), \tag{2.36}$$

for $n \in \mathbb{N}_0$.

If $\alpha \neq 0$ and $\beta = 0$, then $\delta = 0$, from which it follows that $\gamma \neq 0$. Hence

$$x_{n+1} = g^{-1}\left(\frac{\alpha}{\gamma}g(x_n)\right), \quad n \in \mathbb{N}_0. \tag{2.37}$$

From (2.37) we obtain

$$x_n = g^{-1}\left(\left(\frac{\alpha}{\gamma}\right)^n g(x_0)\right), \quad n \in \mathbb{N}_0. \tag{2.38}$$

If $\delta = 0$, then $\gamma \neq 0$. This implies $\beta = 0$, and consequently $\alpha \neq 0$, so we get equation (2.37) whose solutions are given by formula (2.38). If $\gamma = 0$, then $\delta \neq 0$. Hence $\alpha = 0$ which implies $\beta \neq 0$, so we get equation (2.35) whose solutions are given by formula (2.36).

If $\alpha\beta\gamma\delta \neq 0$, then $\alpha = \beta\gamma/\delta$, so we again get equation (2.35), which in this case coincides with equation (2.37).

From above obtained closed-form formulas for solutions to equation (2.1) the theorem follows. □

3 Some applications and discussions

A part of recent literature on difference equations contains many claims which are not established and/or explained. In some of our papers we discussed some aspects of the phenomena (see, e.g., [32–34, 36]). Here we discuss some incorrect claims on long-term behaviour of solutions to equation (1.5) given in [3].

Note that equation (1.5) can be written in the form

$$x_{n+1} = x_n \frac{acx_{n-2} + (ad+b)x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

In [3] was first tried to find the equilibria of the equation. After some simple algebraic manipulations it was concluded that $\bar{x} = 0$ is a unique equilibrium point of equation (1.5), when

$$(1-a)(c+d) \neq b.$$

Assume that $\bar{x}$ is an equilibrium of equation (1.5). Then it must satisfy the algebraic equation

$$\bar{x} = a\bar{x} + \frac{b\bar{x}^2}{(c+d)\bar{x}}. \tag{3.2}$$

From (3.2) we see that it must be

$$\bar{x} \neq 0 \quad \text{and} \quad c+d \neq 0.$$

This eliminates the possibility $\bar{x} = 0$.

If $\bar{x} \neq 0$, then (3.2) implies

$$\bar{x}\left(1 - a - \frac{b}{c+d}\right) = 0,$$

and consequently

$$1 - a - \frac{b}{c+d} = 0.$$

Therefore, under the last condition any $\bar{x} \neq 0$ is an equilibrium of the difference equation.

This means that the claim in [3, Theorem 1] that, under a condition, the zero equilibrium point of equation (1.5) is locally asymptotically stable is not correct, since it is not an equilibrium at all.

Further, Theorem 2 in [3] claims the following:

Theorem 3.1. *The equilibrium point $\bar{x}$ of equation (1.5) is global attractor if $d(1-a) \neq b$.*

Note that equation (3.1) is a special case of equation (2.1) with

$$g(x) = x, \quad \alpha = ac, \quad \beta = ad + b, \quad \gamma = c \quad \text{and} \quad \delta = d.$$

Example 3.2. Consider the equation (1.5) with

$$a = 3, \quad b = -5, \quad c = 1 \quad \text{and} \quad d = 2, \tag{3.3}$$

that is, the equation

$$x_{n+1} = x_n \frac{3x_{n-2} + x_{n-3}}{x_{n-2} + 2x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

This is the equation (2.1) with $g(x) = x$, $x \in \mathbb{R}$,

$$\alpha = 3, \quad \beta = \gamma = 1, \quad \delta = 2. \tag{3.5}$$

The associated characteristic polynomial to the corresponding linear equation in (2.10) is

$$p_2(\lambda) = \lambda^2 - 5\lambda + 5,$$

and its roots are

$$\lambda_1 = \frac{5+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{5-\sqrt{5}}{2}.$$

Since in this case we have

$$d(1-a) - b = 1 \neq 0,$$

the condition $d(1-a) \neq b$ in Theorem 3.1 is satisfied.

Employing the formulas in (2.14)–(2.19), where $g(x) = x$, $x \in \mathbb{R}$, and the coefficients $\alpha, \beta, \gamma, \delta$ are as in (3.5), we have

$$x_{3m} = x_{-3} \prod_{i=0}^{m} y_{3i} y_{3i-1} y_{3i-2}, \tag{3.6}$$

$$x_{3m+1} = x_{-2} \prod_{i=0}^{m} y_{3i+1} y_{3i} y_{3i-1}, \tag{3.7}$$

$$x_{3m+2} = x_{-1} \prod_{i=0}^{m} y_{3i+2} y_{3i+1} y_{3i}, \tag{3.8}$$

for $m \in \mathbb{N}_0$, where

$$\begin{aligned} y_{3m}y_{3m-1}y_{3m-2} = {} & \left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \right) \\ & \times \left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \right) \\ & \times \left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} y_{3m+1}y_{3m}y_{3m-1} = {} & \left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+2} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+2}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+1}} - 2 \right) \\ & \times \left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \right) \\ & \times \left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} y_{3m+2}y_{3m+1}y_{3m} = {} & \left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+2} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+2}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+1}} - 2 \right) \\ & \times \left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+2} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+2}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+1}} - 2 \right) \\ & \times \left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \right), \end{aligned} \tag{3.11}$$

for $m \in \mathbb{N}_0$.

Now note that

$$\begin{aligned}
\lim_{m\to+\infty} & \frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \\
&= \lim_{m\to+\infty} \frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \\
&= \lim_{m\to+\infty} \frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \\
&= \lambda_1 - 2 = \frac{1+\sqrt{5}}{2} > 1,
\end{aligned}$$

when

$$\frac{x_{-i}}{x_{-(i+1)}} \neq \lambda_2 - 2 = \frac{1-\sqrt{5}}{2}, \quad i = \overline{0,2}. \tag{3.12}$$

By choosing positive initial values satisfying (3.12) and using formulas (3.6)–(3.11) we have

$$\lim_{n\to+\infty} x_n = +\infty.$$

This means that the solution is not convergent, which is a counterexample to the claim in Theorem 3.1.

Bearing in mind that in [3] is stated that it considers equation (1.5) for the case when all the coefficients a, b, c and d are positive, and that one of the coefficients in Example 3.2 is negative (see (3.3)), in the following example we also give a counterexample to the statement in Theorem 3.1 for the case of positive coefficients.

Example 3.3. Consider the equation (1.5) with

$$a = b = c = d = 1, \tag{3.13}$$

that is, the equation

$$x_{n+1} = x_n \frac{x_{n-2} + 2x_{n-3}}{x_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{3.14}$$

This is the equation (2.1) with $g(x) = x$, $x \in \mathbb{R}$,

$$\alpha = \gamma = \delta = 1, \quad \beta = 2. \tag{3.15}$$

The associated characteristic polynomial to the corresponding linear equation in (2.10) is

$$p_2(\lambda) = \lambda^2 - 2\lambda - 1,$$

and its roots are

$$\lambda_1 = 1 + \sqrt{2} \quad \text{and} \quad \lambda_2 = 1 - \sqrt{2}.$$

Since in this case we have

$$d(1-a) - b = -1 \neq 0,$$

the condition $d(1-a) \neq b$ in Theorem 3.1 is satisfied.

Employing (2.14)–(2.19), where $g(x) = x$, $x \in \mathbb{R}$, and the coefficients $\alpha, \beta, \gamma, \delta$ are as in (3.15) we have that the relations in (3.6)–(3.8) hold for $m \in \mathbb{N}_0$, where

$$\begin{aligned} y_{3m}y_{3m-1}y_{3m-2} = &\left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 1\right)\lambda_2^{m}} - 1\right) \\ &\times \left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 1\right)\lambda_2^{m}} - 1\right) \\ &\times \left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 1\right)\lambda_2^{m}} - 1\right), \end{aligned} \tag{3.16}$$

$$\begin{aligned} y_{3m+1}y_{3m}y_{3m-1} = &\left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 1\right)\lambda_1^{m+2} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 1\right)\lambda_2^{m+2}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 1\right)\lambda_2^{m+1}} - 1\right) \\ &\times \left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_2^{m}} - 1\right) \\ &\times \left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 1\right)\lambda_2^{m}} - 1\right), \end{aligned} \tag{3.17}$$

$$\begin{aligned} y_{3m+2}y_{3m+1}y_{3m} = &\left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 1\right)\lambda_1^{m+2} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 1\right)\lambda_2^{m+2}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 1\right)\lambda_2^{m+1}} - 1\right) \\ &\times \left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 1\right)\lambda_1^{m+2} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 1\right)\lambda_2^{m+2}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 1\right)\lambda_2^{m+1}} - 1\right) \\ &\times \left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 1\right)\lambda_2^{m}} - 1\right), \end{aligned} \tag{3.18}$$

for $m \in \mathbb{N}_0$.

Now note that

$$\begin{aligned} &\lim_{m\to+\infty} \frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 1\right)\lambda_2^{m}} - 1 \\ &\quad = \lim_{m\to+\infty} \frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 1\right)\lambda_2^{m}} - 1 \\ &\quad = \lim_{m\to+\infty} \frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 1\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 1\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 1\right)\lambda_1^{m} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 1\right)\lambda_2^{m}} - 1 \\ &\quad = \lambda_1 - 1 = \sqrt{2} > 1, \end{aligned}$$

when

$$\frac{x_{-i}}{x_{-(i+1)}} \neq \lambda_2 - 1 = -\sqrt{2}, \quad i = \overline{0,2}. \tag{3.19}$$

By choosing positive initial values satisfying (3.19), and using formulas (3.6)–(3.8), (3.16)–(3.18) we have

$$\lim_{n\to+\infty} x_n = +\infty.$$

Hence, the solutions are not convergent, which is a counterexample to the claim in Theorem 3.1 in the case $\min\{a, b, c, d\} > 0$.

Remark 3.4. The closed-form formulas for some special cases of equation (1.5) presented in [3] easily follow from the ones in Theorem 2.1. We leave the verification of the fact to the interested reader as some simple exercises. Hence, our Theorem 2.1 gives a theoretical explanation for the closed-form formulas therein.

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References

[1] D. Adamović, Problem 194, *Mat. Vesnik* **22**(1970), No. 2, 270.

[2] D. Adamović, Solution to problem 194, *Mat. Vesnik* **23**(1971), 236–242.

[3] R. P. Agarwal, E. M. Elsayed, On the solutions of fourth-order rational recursive sequence, *Adv. Stud. Contemp. Math. (Kyungshang)* **20**(2010), No. 4, 525–545. MR2984760; Zbl 1248.39006

[4] A. Andruch-Sobilo, M. Migda, Further properties of the rational recursive sequence $x_{n+1} = ax_{n-1}/(b + cx_n x_{n-1})$, *Opuscula Math.* **26**(2006), No. 3, 387–394. MR2280266; Zbl 1131.39003

[5] A. Andruch-Sobilo, M. Migda, On the rational recursive sequence $x_{n+1} = ax_{n-1}/(b + cx_n x_{n-1})$, *Tatra Mt. Math. Publ.* **43**(2009), 1–9. MR2588871; Zbl 1212.39008

[6] L. Berg, S. Stević, On some systems of difference equations, *Appl. Math. Comput.* **218**(2011), 1713–1718. https://doi.org/10.1016/j.amc.2011.06.050; MR2831394; Zbl 1243.39009

[7] D. Bernoulli, Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur (in Latin), *Commentarii Acad. Petropol. III*, 1728 (1732), 85–100.

[8] G. Boole, *A treatise on the calculus of finite differences*, Third Edition, Macmillan and Co., London, 1880.

[9] L. Brand, A sequence defined by a difference equation, *Amer. Math. Monthly* **62**(1955), No. 7, 489–492. https://doi.org/10.2307/2307362; MR1529078

[10] A. de Moivre, *Miscellanea analytica de seriebus et quadraturis* (in Latin), J. Tonson & J. Watts, Londini, 1730.

[11] L. Euler, *Introductio in analysin infinitorum, tomus primus* (in Latin), Lausannae, 1748.

[12] B. Iričanin, S. Stević, On some rational difference equations, *Ars Combin.* **92**(2009), 67–72. MR2532566; Zbl 1224.39014

[13] C. Jordan, *Calculus of finite differences*, 3rd edition, Chelsea Publishing Company, New York, 1965.

[14] V. A. Krechmar, *A problem book in algebra*, Mir Publishers, Moscow, 1974.

[15] S. F. Lacroix, *Traité des differénces et des séries* (in French) J. B. M. Duprat, Paris, 1800.

[16] S. F. Lacroix, *An elementary treatise on the differential and integral calculus, with an appendix and notes by J. Herschel*, J. Smith, Cambridge, 1816.

[17] J.-L. Lagrange, Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes (in French), *Miscellanea Taurinensia,* t. I, (1759), 33–42 (Lagrange OEuvres, I, 23–36, 1867).

[18] P. S. Laplace, Recherches sur l'intégration des équations différentielles aux différences finies et sur leur usage dans la théorie des hasards (in French), *Mémoires de l'Académie Royale des Sciences de Paris* 1773, t. VII, (1776) (Laplace OEuvres, VIII, 69–197, 1891).

[19] H. Levy, F. Lessman, *Finite difference equations*, The Macmillan Company, New York, NY, USA, 1961. MR0118984

[20] A. A. Markoff, *Differenzenrechnung* (in German), Teubner, Leipzig, 1896.

[21] D. S. Mitrinović, D. D. Adamović, *Nizovi i redovi [Sequences and series] (in Serbian)*, Naučna Knjiga, Beograd, Serbia, 1980.

[22] D. S. Mitrinović, J. D. Kečkić, *Metodi izračunavanja konačnih zbirova [Methods for calculating finite sums]* (in Serbian), Naučna Knjiga, Beograd, 1984.

[23] G. Papaschinopoulos, C. J. Schinas, Invariants for systems of two nonlinear difference equations, *Differential Equations Dynam. Systems* **7**(1999), 181–196. MR1860787; Zbl 0978.39014

[24] G. Papaschinopoulos, C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations, *Nonlinear Anal.* **46**(2001), 967–978. `https://doi.org/10.1016/S0362-546X(00)00146-2`; MR1866733; Zbl 1003.39007

[25] G. Papaschinopoulos, C. J. Schinas, G. Stefanidou, On a k-order system of Lyness-type difference equations, *Adv. Difference Equ.* **2007**, Art. ID 31272, 13 pp. `https://doi.org/10.1155/2007/31272`; MR2322487; Zbl 1154.39013

[26] G. Papaschinopoulos, G. Stefanidou, Asymptotic behavior of the solutions of a class of rational difference equations, *Int. J. Difference Equ.* **5**(2010), No. 2, 233–249. MR2771327

[27] M. H. Rhouma, The Fibonacci sequence modulo π, chaos and some rational recursive equations, *J. Math. Anal. Appl.* **310**(2005), 506–517. https://doi.org/10.1016/j.jmaa.2005.02.038; MR2022941; Zbl 1173.11306

[28] C. Schinas, Invariants for some difference equations, *J. Math. Anal. Appl.* **212**(1997), 281–291. https://doi.org/10.1006/jmaa.1997.5499; MR1460198; Zbl 0879.39001

[29] C. Schinas, Invariants for difference equations and systems of difference equations of rational form, *J. Math. Anal. Appl.* **216**(1997), 164–179. https://doi.org/10.1006/jmaa.1997.5667; MR1487258; Zbl 0889.39006

[30] S. Stević, On the recursive sequence $x_{n+1} = A / \prod_{i=0}^{k} x_{n-i} + 1 / \prod_{j=k+2}^{2(k+1)} x_{n-j}$, *Taiwanese J. Math.* **7**(2003), No. 2, 249–259. MR1978014; Zbl 1054.39008

[31] S. Stević, Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 67, 1–15. https://doi.org/10.14232/ejqtde.2014.1.67; MR3304193; Zbl 1324.39004

[32] S. Stević, Representations of solutions to linear and bilinear difference equations and systems of bilinear difference equations, *Adv. Difference Equ.* **2018**, Art. No. 474, 21 pp. https://doi.org/10.1186/s13662-018-1930-2; MR3894606; Zbl 1448.39001

[33] S. Stević, A. E. Ahmed, W. Kosmala, Z. Šmarda, Note on a difference equation and some of its relatives, *Math. Methods Appl. Sci.* **44**(2021), 10053–10061. https://doi.org/doi.org/10.1002/mma.7389; MR4284829; Zbl 1473.39005

[34] S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, Note on the bilinear difference equation with a delay, *Math. Methods Appl. Sci.* **41**(2018), 9349–9360. https://doi.org/10.1002/mma.5293; MR3897790; Zbl 1404.39001

[35] S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, Note on a solution form to the cyclic bilinear system of difference equations, *Appl. Math. Lett.* **111**(2021), Article No. 106690, 8 pp. https://doi.org/10.1016/j.aml.2020.106690; MR4142097; Zbl 1448.39008

[36] S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, Note on difference equations with the right-hand side function nonincreasing in each variable, *J. Inequal. Appl.* **2022**, Article No. 25, 7 pp. https://doi.org/10.1186/s13660-022-02761-9; MR4386416; Zbl 07512180

[37] S. Stević, B. Iričanin, Z. Šmarda, On a symmetric bilinear system of difference equations, *Appl. Math. Lett.* **89**(2019), 15–21. https://doi.org/10.1016/j.aml.2018.09.006; MR3886971; Zbl 1409.39002

[38] V. A. Zorich, *Mathematical analysis I*, Springer, Berlin, Heidelberg, 2015. https://doi.org/10.1007/978-3-662-48792-1; MR3495809; Zbl 1332.00009