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Matching with Quorums

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Matching with Quorums

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Abstract

We study the problem of allocating workers to different projects in which each project requires having a minimum number of workers assigned to it or else it does not open. We show that the well-known serial dictatorship mechanism is neither strategy-proof nor Pareto efficient. Thus, we propose an algorithm, denoted as the serial dictatorship with project closures, which is strategy-proof and also Pareto efficient over the set of all feasible allocations.

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KEYWORDS: matching, stability, efficiency, serial dictatorship

1 Introduction

We consider the problem of allocating agents to different projects that have a minimum quorum and a maximum capacity. Firms with multiple projects routinely face this problem: they must decide how to best allocate the workforce into different projects, and each worker must be allocated to one and only one project. In addition, projects typically require a minimum number of workers in order to be successfully completed; hence, firms do not initiate a given project if the minimum quorum is not satisfied. This could be the case, for example, of projects that have a large fixed cost, or that present economies of scale. In addition, allocating too many workers to a project is inefficient and the firms may require a maximal capacity for each project. Some educational institutions face a similar problem when assigning students to classes. Students must choose which classes to take in a given

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semester, where there are many potential course offerings. Once all students ask for their classes– some of which are not mandatory– the courses will only be offered if a minimum quorum is satisfied.[1]

First we show that in our setting, the well known serial dictatorship mechanism may not satisfy both efficiency and strategy-proofness, regardless of how the agents are ordered. This is because agents who make the initial choices must consider the possibility that the projects chosen will be closed due to a lack of enrollment. Consequently, agents might want to choose a less preferred project with a lower quorum. Motivated by this, we propose a strategy-proof and efficient mechanism which we call the serial dictatorship with project closures. Our mechanism is a stronger form of the serial dictatorship in that the set of available projects evolves so that already-chosen projects are not closed.

The serial dictatorship mechanism in problems without the minimum quorum restriction satisfies many positive properties. In the house allocation setting,[2] Svensson (1999) shows that it is the only deterministic algorithm that is strategy-proof, non-bossy and neutral. Abdulkadiroglu and Sonmez (1998) show that the serial dictatorship spans the whole set of efficient allocations through different orders and that the core from random endowments is equivalent to random serial dictatorship,[3] providing a further justification for the use of the random serial dictatorships in practice.

The study of matching problems with a minimum quorum is recent: K. Hamada and Miyazaki (2008) and Biro *et al.* (2010) study two-sided matching problems with a quorum in which both sides have well defined preferences. They concentrate on stability and show that stable matchings do not necessarily exist. The question of how to find a stable matching when it exists is still under study. Meanwhile, the current paper studies efficiency and strategy-proofness.

Manea (2007) considers environments in which agents want to consume more than one object and he studies a weak form of serial dictatorship: agents choose one object at a time according to an order in which any given agent could appear more than once. Manea (2007) shows that, in such environments, this weak version of the serial dictatorship mechanism may fail strategy-proofness and efficiency. In fact, Papai (2001), Ehlers and Klaus (2003) and Hatfield (2009) establish a general result in such environments. The only strategy-proof, Pareto optimal and nonbossy mechanisms are the strong form of the sequential dictatorship. Each agent chooses his or her favorite set of available objects according to a predefined

[1] A Director of Graduate Studies might find this problem to be a familiar one. In fact, as anecdotal evidence, some Ph.D. programs in the US regularly face this situation, in which if there is only one student enrolled in a course, the course ends up not being offered.

[2] See Hylland and Zeckhauser (1979) and Abdulkadiroglu and Sonmez (1998).

[3] A serial dictatorship in which the order is the outcome of a lottery.

order. In contrast, in our setting, each agent is entitled to only one object, yet the serial dictatorship mechanism fails (constrained) efficiency and strategy-proofness.

2 Model

There is a finite number of workers and projects. The set of workers is denoted $I = \{1, \cdots, n\}$ whereas $P = \{p_1, \cdots, p_m\}$ is the set of projects. Each project $p \in P$ has a maximum capacity $k_p \leq \infty$, but also a minimum quorum $q_p \geq 1$. This means that a project may not be assigned more than k_p workers and a project will not open if there are less than q_p workers. There may be projects with no restrictions whatsoever ($k_p = \infty$ and $q_p = 1$) or with only one restriction: an upper (lower) bound of workers.

We assume that each agent $i \in I$ has a strict preference ordering $\succ_i$ over the projects. We write $p \succeq_i p'$ if either $p \succ_i p'$ or $p = p'$. A preference profile $\succ$ is $(\succ_i)_{i \in I}$ and let $\mathcal{P}$ be the set of all possible strict preference profiles.

Each worker can be assigned to at most one project and we assume that for each player, being assigned to any project is better than not being assigned at all, i.e., individual rationality constraints are always satisfied. Formally, a matching μ is a correspondence $\mu : I \cup P \to I \cup P$ such that (i) $\mu(i) \in P$ (ii) $\mu(p) \subseteq I \cup \emptyset$ and (iii) $\mu(i) = s$ if and only if $i \in \mu(s)$.

Definition 1 (Feasible Matching)**.** *A matching μ is feasible if for all $p \in P$, either $q_p \leq |\mu(p)| \leq k_p$ or $|\mu(p)| = 0$.*

For simplicity, we will restrict attention to problems in which there exists at least one feasible allocation.

The definition of Pareto efficiency in our setting coincides with the standard definition.

Definition 2 (Pareto Dominance)**.** *A matching $\bar{\mu}$ Pareto dominates μ if, for some $i \in I$, $\bar{\mu}(i) \succ_i \mu(i)$ and for $\forall j \in I$, $\bar{\mu}(j) \succeq_j \mu(j)$.*

Definition 3 (Constrained Pareto Efficiency)**.** *A matching μ is constrained Pareto efficient if it is feasible and in addition, there does not exist a* feasible *matching $\bar{\mu}$ that Pareto dominates μ.*

A mechanism φ maps the reported preferences into a feasible matching. A mechanism is said to be strategy-proof if reporting the true preferences is a weakly dominant strategy for all the workers.

Definition 4 (Strategy-Proofness)**.** *A mechanism φ is* strategy-proof *if for any preference profile $\succ$, and any player $i \in I$:*

$$\varphi_i(\succ) \succeq_i \varphi_i(\succ_i', \succ_{-i}), \text{ for } \forall \succ_i' \neq \succ_i,$$

where $\succ_{-i} = (\succ_j)_{j \neq i}$.

3 Serial Dictatorship

The algorithm known as serial dictatorship (*SD*) has been widely used in matching problems, both in theory and practice. At an environment without the minimum quorum restriction, the algorithm runs as follows. All workers are placed in an exogenously given order and select a project according to this order. A worker can select any project available, as long as its maximum capacity has not yet been reached. The algorithm terminates once all workers have chosen their projects. The resulting matching is unique given the choosing order and Pareto efficient. Moreover, the algorithm is clearly strategy-proof.

When there is a minimum quorum restriction, it may be the case that once all agents have pointed to their preferred project among the available ones, some projects do not meet the quorum. For simplicity, in our setting, we consider *SD* to be such that whenever some projects do not satisfy the minimum quorum restriction, the resulting matching leaves the agents who chose these projects unassigned.[4] All other agents are allocated to the projects of their choice. This mechanism induces a dynamic game of complete and perfect information and we look for the subgame perfect equilibria. We show below that in our setting the *SD* fails strategy-proofness and efficiency.

Proposition 1 (Failure of (Constrained) Efficiency and Strategy-Proofness)**.** *If the minimum quorum for some project p (strictly) exceeds 1 ($q_p > 1$), then the SD is not necessarily strategy-proof and (constrained) Pareto efficient.*

Proof. The proof consists in showing a preference profile for which the *SD* fails strategy-proofness and constrained Pareto efficiency. Consider an example in which there are 3 workers, $\{i_1, i_2, i_3\}$ and 5 projects $\{A, B, C, D, E\}$. Projects A, B, and C have a minimum quorum of 2, and no capacity restriction, while projects D and E have a minimum quorum of 1, but a capacity restriction of 1. Formally, $k_j = \infty$ and $q_j = 2$; for $j = A, B, C$ and

[4] Alternatively, one could have assumed an algorithm in which after all agents have made one choice, some of the projects with less agents than the minimum quorum restriction could be closed and the unassigned agents would be allowed to choose again. Given that we have a complete information environment, the two specifications yield the same results.

$k_j = q_j = 1$, for $j = D, E$. The workers' preferences are given by the table below.

$$\begin{array}{llllllllll} i_1: & A & \succ_{i_1} & D & \succ_{i_1} & E & \succ_{i_1} & B & \succ_{i_1} & C \\ i_2: & B & \succ_{i_2} & D & \succ_{i_2} & E & \succ_{i_2} & A & \succ_{i_2} & C \\ i_3: & C & \succ_{i_3} & D & \succ_{i_3} & E & \succ_{i_3} & A & \succ_{i_3} & B \end{array}$$

Recall that the *SD* induces a sequential game of complete and perfect information. Using backwards induction, we conclude that the outcome is such that the first agent chooses D, the second agent chooses E (thus, the mechanism is not strategy-proof), and the third agent is left unassigned. This outcome is Pareto dominated by the feasible matching in which the third agent and the first are allocated to the first agent's most preferred project. Note that this is true for any order of the *SD*. □

In some special cases, the *SD* might be strategy-proof and efficient and we characterize the class of problems in which this happens. First note that the total number of potential spots is the sum of the maximum capacities of each project: $\sum_{i=1}^{m} k_{p_i}$, where, recall $P = \{p_1, p_2, ..., p_m\}$.

Proposition 2. *The SD mechanism is strategy-proof and Pareto efficient (for all preference profiles) if the total number of potential spots is the same or less than the number of agents:* $\sum_{i=1}^{m} k_{p_i} \leq n$.

Proof. Suppose that the total number of potential spots is the same as, or less than, the number of agents in the environment. In this case, all projects will be opened once all agents have made their choices (recall that individual rationality is assumed throughout the paper). Given that all projects will be opened, the first agent chooses her most preferred project, knowing that it will open. The second agent chooses her most preferred project among all the projects with remaining open slots, again knowing that the project will be opened. Thus, for exactly the same reasons as in problems without project closure, the algorithm is strategy-proof and efficient. □

4 Serial Dictatorship with Project Closures

We propose a mechanism for the class of problems considered in this paper. This mechanism is denoted serial dictatorship with project closures (*SDPC*), and may be considered as a strong form of the *SD*. As we will show, this mechanism has many desirable properties: it yields a feasible matching (assuming one exists), it is strategy-proof and also constrained Pareto efficient.

The mechanism runs in successive phases and can be described as follows. Each agent is exogenously assigned an order. The agents will make their choices according to the assigned order and without loss of generality we label the agents according to their order. i.e., agent i_1 is the first to choose, followed by agent i_2, and so on. The last agent to choose is agent i_n. Let us denote $\theta_p(k)$ to be the number of agents that have chosen project p at the start of phase k of the algorithm. Moreover, for simplicity, we denote p_j as the project that was first chosen by agent i_j.

Phase 1: The first agent chooses her preferred project p_1.

Phase 2: If $q_{p_1} - 1 > n - 2$, the second agent must choose project p_1. Otherwise, agent i_2 may choose her project among all projects that still have remaining spots and such that $(q_{p_1} - 1) + (q_{p_2} - 1) \leq n - 2$

In general, at phase k:

Phase k: If the project chosen by player i_1 is such that $q_{p_1} - \theta_{p_1}(k) > n - k$, agent i_k must choose p_1. If not, then if the project chosen by player i_2 is such that $q_{p_2} - \theta_{p_2}(k) > n - k$, agent i_k must choose p_2. Repeat the procedure for the projects chosen by agents $i_3, ..., i_{k-1}$. If $q_{p_l} - \theta_{p_l}(k) \leq n - k$, for all agents $l \in \{1, 2, ..., k-1\}$, agent i_k may choose a project among all projects that still have remaining spots and such that

$$\sum_{j=1}^{k-1} \max\left\{0, \left(q_{p_j} - \theta_{p_j}\right)\right\} \leq n - k.$$

This procedure terminates when the last agent makes her choice.

Our algorithm provides an easy and straightforward way of implementing the following mechanism. The first worker chooses the set of feasible allocations under which she obtains her most preferred project, call it S_1. Then, the second worker chooses a subset $S_2 \subseteq S_1$ under which she obtains her most preferred project that is feasible under S_1. The process goes on until the last agent makes her choice.

Theorem 1 (SDPC: Strategy-Proofness and Efficiency)**.** *The SDPC mechanism is strategy-proof and always yields a constrained Pareto efficient matching (provided that a feasible matching exists).*

Proof. By construction, the algorithm always yields a feasible matching. Moreover, it is clearly strategy-proof. The way the mechanism is constructed implies that if the agent has an option between projects, the chosen project will necessarily be opened (otherwise, she would not be given the option) and she will be allocated to it. The agent will choose whatever is best for her given the available options. It is also constrained Pareto efficient. The first agent receives her first option. The second agent either receives her first option, or receives a less valuable option. In this last case, it is due to the fact that agent 1's project demands a minimum quorum which made player 2's option not possible, or because player 2's first option is also player 1's and this first option had a maximum capacity of 1. Thus, player 2 could only be made better off, by closing the project that player 1 chose. Player 3 will choose the best option available to her, given that player 2 has chosen her best offer and so did player 1. To improve player 3, either player 2 or player 1 must be made worse off. The argument extends to the remaining agents. □

We remark that the serial dictatorship with project closures does not span the entire set of Pareto optimal matchings. This is in contrast to the serial dictatorship in matching problems without the minimum quorum restriction (Abdulkadiroglu and Sonmez (1998)).

Proposition 3 (Pareto Frontier)**.** *The serial dictatorship with project closures does not necessarily yield all possible constrained Pareto efficient allocations.*[5]

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Proof of Proposition 3. (Proof upon Request) Consider the following example. There are n workers: $\{i_1, i_2, ..., i_n\}$ and $n+1$ projects $\{\bar{p}, p_1, p_2, ..., p_n\}$. Further, assume that for any project the minimum quorum is the entire set of agents, i.e. $q_p = n$, $\forall p$ and there is no restriction regarding the maximum capacity: $k_p = \infty, \forall p$.

The preference ordering of worker k, for $\forall k = 1, ..., n$, is given by:

$$i_k : p_1 \succ p_2 \succ ... \succ p_{k-1} \succ p_{k+1} \succ ... \succ \bar{p} \succ p_k.$$

That is, p_k is agent k's least preferred allocation. Moreover, p_1 is the most preferred project of all workers, except for the first worker, who view it as her least preferred option.

The allocation $\mu(i) = \bar{p}$ for $\forall i \in \{i_1, i_2, ..., i_n\}$ is constrained Pareto efficient. Any other allocation would make at least one agent worse off.

As a remark, in the serial dictatorship with project closures, the resulting allocation is $\mu(j) = p_1$, if $j \neq 1$ and $\mu(j) = p_2$, if $j = 1$, where i_j is the first worker to choose. □

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