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Superconvergent methods based on quasi-interpolating operators for Fredholm integral equations of the second kind[*]

C. Allouch, S. Remogna, D. Sbibih, M. Tahrichi[†]

Abstract

In this paper, we apply spline quasi-interpolating operators on a bounded interval to solve numerically linear Fredholm integral equations of second kind by using superconvergent Nyström and degenerate kernel methods introduced in [4]. We give convergence orders associated with approximate solutions and their iterated versions in terms of spline quasi-interpolating order. Moreover, asymptotic expansions at the node/partition points for second kind Fredholm integral equations with Green's type kernel are obtained in the Nyström method based on quadratic and cubic quasi-interpolants. Therefore, the Richardson extrapolation technique is used to improve the convergence orders. Finally, numerical examples and comparison with existing methods are given to illustrate the theoretical results and to show that the proposed methods improve the convergence orders.
Keywords: Fredholm integral equation, Quasi-interpolating spline, Nyström method, Degenerate kernel method, Superconvergence.

1 Introduction

Let us consider the linear Fredholm integral equation of the second kind

$$x - \mathcal{K}x = f, \tag{1.1}$$

where $\mathcal{K}$ is the compact linear operator defined on the space $\mathcal{C}[0,1]$ by

$$\mathcal{K}x(s) = \int_0^1 k(s,t)x(t)dt, \qquad s \in [0,1], \tag{1.2}$$

with $k(.,.) \in \mathcal{C}([0,1] \times [0,1])$ and $f \in \mathcal{C}[0,1]$. The operator $\mathcal{I} - \mathcal{K}$ is assumed to be invertible, so that the equation (1.1) has a unique solution $x^* \in \mathcal{C}[0,1]$.

The Galerkin, collocation, Nyström and degenerate kernel methods are commonly used methods to solve numerically the equation (1.1). They have been extensively studied in the literature (see [5, 10, 16]) and references therein. Sloan in [21] improved Galerkin and collocation methods by an iteration technique. Many authors have used Sloan iteration technique to solve different types of integral equations, see for example [7]. In [14], Kulkarni et al. introduced an efficient method to solve (1.1) based on a sequence of projectors onto a space of piecewise polynomials of order $\leq r$. Inspired by Kulkarni's scheme, authors in [4], introduced superconvergent Nyström and degenerate

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†The Faculty of Nador, Team of Modeling and Scientific Computing, Nador, Morocco (c.allouch@ump.ma); University of Torino, Department of Mathematics, Torino, Italy (sara.remogna@unito.it); University Mohammed I, FSO, LANO Laboratory, Oujda, Morocc (sbibih@yahoo.fr); University Mohammed I, Team ANAA, EST, LANO Laboratory, Oujda, Morocco (mtahrichi@gmail.com)

kernel methods based on the same sequence of projectors. Asymptotic expansions of approximate solutions of (1.1) in the cases of Nyström, iterated Galerkin and iterated collocation methods have been obtained in [8, 17]. In [15], the authors obtained asymptotic expansions of approximate solutions of (1.1) when the function kernel $k(.,.)$ is of Green's type. The case of nonlinear Fredholm integral equations with Green's type kernel is studied in [13]. Furthermore, superconvergence is a practically useful topic in finite element analysis and several techniques have been proposed by many authors to obtain the superconvergence of finite elements methods (see [23, 22]).

The previously mentioned methods are usually based on a sequence of linear projectors onto finite dimensional approximating subspaces and in the most cases these projectors are chosen to be interpolatory or orthogonal projectors on piecewise polynomial spaces. But, piecewise polynomials being only continuous in the best configurations of collocation parameters, do not preserve the smoothness of the exact solution. Recently, many authors have been interested in using spline methods to solve (1.1), obtaining thereby approximate solutions that are smooth over all the domain. For instance, the authors in [18] constructed a global approximate solution of (1.1) by means of cubic splines. In [1, 3], degenerate kernel method, collocation method and a modified Kulkarni's scheme based on spline quasi-interpolating operators (abbr. QIs) are studied. The authors in [11] applied spline QIs projectors for numerical solution of (1.1) by Galerkin, Kantorovich, Sloan and Kulkarni schemes. The same operators are used in [12] for the solution of nonlinear integral equations.

In the present paper, we consider the use of spline QI operators that are not necessarily projectors to solve equation (1.1) by the superconvergent Nyström and degenerate kernel methods introduced in [4]. Moreover, in the case of (1.1) with Green's function type kernel, we investigate in depth the error in the Nyström method based on quadratic and cubic QIs to get asymptotic expansions and consequently use Richardson extrapolation technique. On the one hand, the methods presented in this paper are useful when we want to preserve the smoothness of the exact solution. On the other hand, these methods are more accurate than the classical Nyström and degenerate kernel methods and more economical than Kulkarni's scheme based on spline QI projectors.

Here is an outline of the paper. In Section 2, we briefly introduce univariate spline QI operators on a bounded interval. In Section 3, we describe superconvergent Nyström and degenerate kernel methods based on QI operators and we give details on the linear system which need to be solved to obtain the corresponding approximate solutions. Section 4 contains a general framework for the convergence analysis of the approximate and the iterated solutions. Section 5 is devoted to considering the case when the kernel $k(.,.)$ is of Green's function type. Finally, we give in Section 6 numerical examples to illustrate theoretical results.

2 Spline quasi-interpolating operator

Let $\Delta_n := \{s_k, \quad 0 \le k \le n\}$ be a uniform partition of the interval $I := [0,1]$ into n subintervals of length $h = \frac{1}{n}$, i.e. $s_k = kh, \quad 0 \le k \le n$. For a fixed $r \ge 1$, we denote by $\mathcal{S}_{n,r} := \mathcal{S}_r^{r-2}(I, \Delta_n)$ the space of splines of order r (degree $r-1$) and of class $\mathcal{C}^{r-2}$ on the partition Δ_n. A basis of this space is formed by $n+r-1$ normalized B-splines $\{\mathcal{B}_j, j \in \mathcal{J}\}$, with $\mathcal{J} := \{1, 2, \ldots, n+r-1\}$. According to these notations, $\text{supp}(\mathcal{B}_j) = [s_{j-r}, s_j]$ and $\mathcal{N}_j = \{s_{j-r+1}, \ldots, s_{j-1}\}$ is the set of the $r-1$ interior knots in the support of $\mathcal{B}_j$. As usual, multiple knots are added at the endpoints. We put $t_0 := s_0$, $t_{n+1} := s_n$, $t_j := \frac{s_{j-1}+s_j}{2}$, $1 \le j \le n$ and we define the set of quasi-interpolation nodes by $\mathcal{E}_n := \{\xi_i\}_{i=0}^N$, with

$$\begin{cases} \xi_i = s_i, \ N = n, & \text{for } r \text{ even}, \\ \xi_i = t_i, \ N = n+1, & \text{for } r \text{ odd}. \end{cases}$$

We consider the discrete spline QI operator defined from $\mathcal{C}([0,1])$ to $\mathcal{S}_{n,r}$ by

$$\mathcal{Q}_n^r x := \sum_{j \in \mathcal{J}} \lambda_j(x) \mathcal{B}_j, \tag{2.3}$$

where $\lambda_j(x)$ are linear combinations of values of x at some points of $\mathcal{E}_n$ in the neighbourhood of supp$(\mathcal{B}_j)$. i.e.,

$$\lambda_j(x) := \sum_i \alpha_{ij} x(\xi_i).$$

The coefficients α_{ij} are chosen such that

$$\mathcal{Q}_n^r x = x, \quad \text{for all } x \in \mathbb{P}_r,$$

where $\mathbb{P}_r$ denotes the polynomial space of order r. The QI $\mathcal{Q}_n^r$ can be written under the following quasi-Lagrange form

$$\mathcal{Q}_n^r x = \sum_{j=0}^{N} x(\xi_j) L_j, \tag{2.4}$$

where the quasi-Lagrange functions L_j are linear combinations of a finite number of B-splines.

It is well known, see e.g. [9], that the operators $\mathcal{Q}_n^r$ are uniformly bounded and for smooth functions $x \in \mathcal{C}^r[0,1]$, there exists a positive constant C_1 independent of h such that

$$\|\mathcal{Q}_n^r x - x\|_\infty \leq C_1 \|x^{(r)}\| h^r. \tag{2.5}$$

In the case of odd order, QI $\mathcal{Q}_n^r$ present an interesting property regarding the associated quadrature rule. More precisely, we have the following theorem (see [2] for the proof).

Theorem 2.1. Let r be an odd number. Let $\mathcal{Q}_n^r$ be the discrete QI operator of order r defined on $\mathcal{S}_{n,r}$ by (2.3). Then, there exists a positive constant C_2 independent of h such that

$$\left| \int_0^1 g(t)(\mathcal{Q}_n^r x(t) - x(t)) dt \right| \leq C_2 h^{r+1} \|x^{(r+1)}\|, \tag{2.6}$$

for any function $x \in \mathcal{C}^{r+1}([0,1])$ and any weight function g with $\|g'\|_1$ bounded.

Even if the results given in this paper are valid for general QI operators, we report in what follows two examples of spline QI operators denoted by $\mathcal{Q}_n^3$ and $\mathcal{Q}_n^4$ that we use as particular cases.

- $\mathcal{Q}_n^3$ is defined on the space $\mathcal{S}_3^1(I, \Delta_n)$ of $\mathcal{C}^1$ quadratic splines by (see [9], [20])

$$\mathcal{Q}_n^3 x := \sum_{j=0}^{n+1} x(t_j) L_j, \tag{2.7}$$

with

$$
\begin{aligned}
L_0(t) &= \mathcal{B}_0(t) - \frac{1}{3}\mathcal{B}_1(t), \qquad L_1(t) = \frac{3}{2}\mathcal{B}_1(t) - \frac{1}{8}\mathcal{B}_2(t),\\
L_2(t) &= -\frac{1}{6}\mathcal{B}_1(t) + \frac{5}{4}\mathcal{B}_2(t) - \frac{1}{8}\mathcal{B}_3(t),\\
L_{n+1}(t) &= \mathcal{B}_{n+1}(t) - \frac{1}{3}\mathcal{B}_n(t), \qquad L_n(t) = \frac{3}{2}\mathcal{B}_n(t) - \frac{1}{8}\mathcal{B}_{n-1}(t),\\
L_{n-1}(t) &= -\frac{1}{6}\mathcal{B}_n(t) + \frac{5}{4}\mathcal{B}_{n-1}(t) - \frac{1}{8}\mathcal{B}_{n-2}(t),\\
L_j(t) &= -\frac{1}{8}\mathcal{B}_{j-1}(t) + \frac{5}{4}\mathcal{B}_j(t) - \frac{1}{8}\mathcal{B}_{j+1}(t), \quad 3 \le j \le n-2.
\end{aligned}
$$

- $\mathcal{Q}_n^4$ is defined on the space $\mathcal{S}_4^2(I, \Delta_n)$ of $\mathcal{C}^2$ cubic splines by (see [9], [20])

$$
\mathcal{Q}_n^4 x := \sum_{j=0}^{n} x(s_j) L_j, \tag{2.8}
$$

with

$$
\begin{aligned}
L_0(t) &= \mathcal{B}_0(t) + \frac{7}{18}\mathcal{B}_1(t) - \frac{1}{6}\mathcal{B}_2(t),\ L_1(t) = \mathcal{B}_1(t) + \frac{4}{3}\mathcal{B}_2(t) - \frac{1}{6}\mathcal{B}_3(t),\\
L_2(t) &= -\frac{1}{2}\mathcal{B}_1(t) - \frac{1}{6}\mathcal{B}_2(t) + \frac{4}{3}\mathcal{B}_3(t) - \frac{1}{6}\mathcal{B}_4(t),\\
L_3(t) &= \frac{1}{9}\mathcal{B}_1(t) - \frac{1}{6}\mathcal{B}_3(t) + \frac{4}{3}\mathcal{B}_4(t) - \frac{1}{6}\mathcal{B}_5(t),\\
L_n(t) &= \mathcal{B}_{n+2}(t) + \frac{7}{18}\mathcal{B}_{n+1}(t) - \frac{1}{6}\mathcal{B}_n(t),\ L_{n-1}(t) = \mathcal{B}_{n+1}(t) + \frac{4}{3}\mathcal{B}_n(t) - \frac{1}{6}\mathcal{B}_{n-1}(t),\\
L_{n-2}(t) &= -\frac{1}{2}\mathcal{B}_{n+1}(t) - \frac{1}{6}\mathcal{B}_n(t) + \frac{4}{3}\mathcal{B}_{n-1}(t) - \frac{1}{6}\mathcal{B}_{n-2}(t),\\
L_{n-3}(t) &= \frac{1}{9}\mathcal{B}_{n+1}(t) - \frac{1}{6}\mathcal{B}_{n-1}(t) + \frac{4}{3}\mathcal{B}_{n-2}(t) - \frac{1}{6}\mathcal{B}_{n-3}(t),\\
L_j(t) &= -\frac{1}{6}\mathcal{B}_j(t) + \frac{4}{3}\mathcal{B}_{j+1}(t) - \frac{1}{6}\mathcal{B}_{j+2}(t), \quad 4 \le j \le n-4.
\end{aligned}
$$

In general, the QI operator $\mathcal{Q}_n^r$ is superconvergent at some points on the interval [0,1], which means that the error at this points are more accurate than the optimal one given by (2.5). In the following theorem, we give the set of superconvergent points associated with the quadratic QI $\mathcal{Q}_n^3$ given by (2.7).

Theorem 2.2. If $x \in \mathcal{C}^4([0,1])$, then there exists a positive constant C_3 independent of h such that it holds

$$
\max_{\xi_i \in \mathfrak{S}_n} |\mathcal{Q}_n^3 x(\xi_i) - x(\xi_i)| \le C_3 h^4 \|x^{(4)}\|, \tag{2.9}
$$

where the set $\mathfrak{S}_n$ is defined by

$$
\mathfrak{S}_n := \{t_i,\ 0 \le i \le n,\ i \ne 1, n-1\} \cup \{s_i,\ 0 \le i \le n+1,\ i \ne 1, 2, n-1, n\} \tag{2.10}
$$

Proof. The proof can be carried out by using the same logical scheme as in the proof of Lemma 4.1 in [11]. □

3 Methods based on $\mathcal{Q}_n^r$

We propose to solve the integral equation (1.1) by the following two methods based on $\mathcal{Q}_n^r$.

3.1 Superconvergent degenerate kernel method

By approximating the kernel $k(s,t)$ as function on t with the QI $\mathcal{Q}_n^r$, we obtain the following degenerate kernel

$$k_n(s,t) := \mathcal{Q}_n^r k(s,.)(t) = \sum_{j=0}^{N} k(s,\xi_j)L_j(t) = \sum_{j=0}^{N} \tilde{k}_j(s)L_j(t),$$

where $\tilde{k}_j(s) := k(s,\xi_j)$. The degenerate kernel integral operator is defined by

$$\mathcal{K}_n^D x := \int_0^1 k_n(.,t)x(t)dt.$$

We propose to approximate the operator $\mathcal{K}$ by the following finite rank operator

$$\mathcal{K}_n^{SD} := \mathcal{Q}_n^r\mathcal{K} + \mathcal{K}_n^D - \mathcal{Q}_n^r\mathcal{K}_n^D. \tag{3.11}$$

The corresponding approximate equation of (1.1) becomes

$$x_n^{SD} - \mathcal{K}_n^{SD}x_n^{SD} = f. \tag{3.12}$$

It is also interesting to consider the iterated solution defined by

$$\tilde{x}_n^{SD} = \mathcal{K}x_n^{SD} + f. \tag{3.13}$$

In order to give more information about the implementation of x_n^{SD}, it is easy to show from (3.11) and (3.12), that x_n^{SD} has the following form

$$x_n^{SD} = f + \sum_{i=0}^{N} \mathcal{X}_i L_i + \sum_{i=0}^{N} \mathcal{Y}_i \tilde{k}_i,$$

where the coefficients $\{\mathcal{X}_i, \mathcal{Y}_i,\ i = 0\ldots,N\}$ are obtained by substituting x_n^{SD} in equation (3.12). Then, we successively have

$$\begin{aligned}
\mathcal{Q}_n^r\mathcal{K}x_n^{SD} &= \sum_{i=0}^{N}\left(\mathcal{K}f(\xi_i) + \sum_{j=0}^{N}\mathcal{X}_j\mathcal{K}L_j(\xi_i) + \sum_{j=0}^{N}\mathcal{Y}_j\mathcal{K}\tilde{k}_j(\xi_i)\right)L_i,\\
\mathcal{K}_n^D x_n^{SD} &= \sum_{i=0}^{N}\left(\int_0^1 L_i(t)f(t)dt + \sum_{j=0}^{N}\mathcal{X}_j\int_0^1 L_i(t)L_j(t)dt + \sum_{j=0}^{N}\mathcal{Y}_j\int_0^1 L_i(t)\tilde{k}_j(t)dt\right)\tilde{k}_i,\\
\mathcal{Q}_n^r\mathcal{K}_n^D x_n^{SD} &= \sum_{i=0}^{N}\sum_{j=0}^{N}\left(\int_0^1 L_j(t)f(t)dt + \sum_{\ell=0}^{N}\mathcal{X}_\ell\int_0^1 L_j(t)L_\ell(t)dt + \sum_{\ell=0}^{N}\mathcal{Y}_\ell\int_0^1 L_j(t)\tilde{k}_\ell(t)dt\right)\tilde{k}_j(\xi_i)L_i.
\end{aligned}$$

Except for very specific situations, the family of functions $\{L_i, \tilde{k}_i\}$ is linearly independent, therefore, we can identify the coefficients of L_i and $\tilde{k}_i$, respectively and we obtain

$$\begin{aligned}
\mathcal{X}_i &= \mathcal{K}f(\xi_i) + \sum_{j=0}^{N}\mathcal{X}_j\mathcal{K}L_j(\xi_i) + \sum_{j=0}^{N}\mathcal{Y}_j\mathcal{K}\tilde{k}_j(\xi_i)\\
&\quad - \sum_{j=0}^{N}\left(\int_0^1 L_j(t)f(t)dt + \sum_{\ell=0}^{N}\mathcal{X}_\ell\int_0^1 L_j(t)L_\ell(t)dt + \sum_{\ell=0}^{N}\mathcal{Y}_\ell\int_0^1 L_j(t)\tilde{k}_\ell(t)dt\right)\tilde{k}_j(\xi_i)\\
\mathcal{Y}_i &= \int_0^1 L_i(t)f(t)dt + \sum_{j=0}^{N}\mathcal{X}_j\int_0^1 L_i(t)L_j(t)dt + \sum_{j=0}^{N}\mathcal{Y}_j\int_0^1 L_i(t)\tilde{k}_j(t)dt.
\end{aligned}$$

In matrix form, we write

$$\mathcal{X} = a + \mathcal{A}\mathcal{X} + \mathcal{D}\mathcal{Y} - \mathcal{B}(b + \mathcal{C}\mathcal{X} + \mathcal{E}\mathcal{Y})$$
$$\mathcal{Y} = b + \mathcal{C}\mathcal{X} + \mathcal{E}\mathcal{Y}$$

where a and b are vectors with components

$$a_i = \mathcal{K}f(\xi_i), \quad b_i = \int_0^1 f(t)L_i(t)dt,$$

and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ are matrices with entries

$$\mathcal{A}_{i,j} = \mathcal{K}L_j(\xi_i), \qquad \mathcal{B}_{i,j} = \tilde{k}_j(\xi_i),$$
$$\mathcal{C}_{i,j} = \int_0^1 L_i(t)L_j(t)dt, \qquad \mathcal{D}_{i,j} = \mathcal{K}\tilde{k}_j(\xi_i), \qquad \mathcal{E}_{i,j} = \int_0^1 \tilde{k}_j(t)L_i(t)dt.$$

Replacing $\mathcal{Y}$ by its value in the first equation, we get the following linear system

$$(\mathcal{I} - \mathscr{F})\mathcal{Z} = c, \tag{3.14}$$

with

$$\mathscr{F} = \begin{bmatrix} \mathcal{A} & \mathcal{D} - \mathcal{B} \\ \mathcal{C} & \mathcal{E} \end{bmatrix}, \quad c = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \mathcal{Z} = \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}.$$

3.2 Superconvergent Nyström method

Let's define the numerical scheme based on $\mathcal{Q}_n^r$ by

$$\int_0^1 x(t)dt \simeq \mathcal{I}_{\mathcal{Q}_n^r}(f) := \int_0^1 \mathcal{Q}_n^r x(t)dt = \sum_{i=0}^{N} \omega_i x(\xi_i) \tag{3.15}$$

where $\omega_i = \int_0^1 L_i(t)dt$. The associated Nyström integral operator is given by

$$\mathcal{K}_n^N x := \sum_{j=0}^{N} \omega_j x(\xi_j)\tilde{k}_j.$$

In the superconvergent Nyström method, the approximate operator of $\mathcal{K}$ is given by

$$\mathcal{K}_n^{SN} := \mathcal{Q}_n^r\mathcal{K} + \mathcal{K}_n^N - \mathcal{Q}_n^r\mathcal{K}_n^N, \tag{3.16}$$

and the approximate solution satisfies

$$x_n^{SN} - \mathcal{K}_n^{SN}x_n^{SN} = f, \tag{3.17}$$

while the iterated solution is defined by

$$\tilde{x}_n^{SN} = \mathcal{K}x_n^{SN} + f. \tag{3.18}$$

From (3.16) and (3.17), we show that the approximate solution x_n^{SN} can be written as

$$x_n^{SN} = f + \sum_{i=0}^{N} \mathcal{X}_i L_i + \sum_{i=0}^{N} \omega_i \mathcal{Y}_i \tilde{k}_i.$$

where the coefficients $\{\mathcal{X}_i, \mathcal{Y}_i,\ i = 0\ldots, N\}$ are obtained by substituting x_n^{SN} in equation (3.17). It holds

$$\mathcal{Q}_n^r\mathcal{K}x_n^{SN} = \sum_{i=0}^{N}\left(\mathcal{K}f(\xi_i) + \sum_{j=0}^{N}\mathcal{X}_j\mathcal{K}L_j(\xi_i) + \sum_{j=0}^{N}\omega_j\mathcal{Y}_j\mathcal{K}\tilde{k}_j(\xi_i)\right)L_i,$$
$$\mathcal{K}_n^N x_n^{SN} = \sum_{i=0}^{N}\omega_i\left(f(\xi_i) + \sum_{j=0}^{N}\mathcal{X}_jL_j(\xi_i) + \sum_{j=0}^{N}\omega_j\mathcal{Y}_j\tilde{k}_j(\xi_i)\right)\tilde{k}_i,$$
$$\mathcal{Q}_n^r\mathcal{K}_n^N x_n^{SN} = \sum_{i=0}^{N}\sum_{j=0}^{N}\omega_j\left(f(\xi_j) + \sum_{\ell=0}^{N}\mathcal{X}_\ell L_\ell(\xi_j) + \sum_{\ell=0}^{N}\omega_\ell\mathcal{Y}_\ell\tilde{k}_\ell(\xi_j)\right)\tilde{k}_j(\xi_i)L_i.$$

Now, by identifying respectively the coefficients of L_i and $\tilde{k}_i$, we get

$$\begin{aligned}\mathcal{X}_i &= \mathcal{K}f(\xi_i) + \sum_{j=0}^{N} \mathcal{X}_j \mathcal{K} L_j(\xi_i) + \sum_{j=0}^{N} \omega_j \mathcal{Y}_j \mathcal{K}\tilde{k}_j(\xi_i) \\ &\quad - \sum_{j=0}^{N} \omega_j \left(f(\xi_j) + \sum_{\ell=0}^{N} \mathcal{X}_\ell L_\ell(\xi_j) + \sum_{\ell=0}^{N} \omega_\ell \mathcal{Y}_\ell \tilde{k}_\ell(\xi_j) \right) \tilde{k}_j(\xi_i) \\ \mathcal{Y}_i &= f(\xi_i) + \sum_{j=0}^{N} \mathcal{X}_j L_j(\xi_i) + \sum_{j=0}^{N} \omega_j \mathcal{Y}_j \tilde{k}_j(\xi_i).\end{aligned}$$

In matrix form, we write

$$\begin{aligned}\mathcal{X} &= a + \mathcal{A}\mathcal{X} + \tilde{\mathcal{D}}\mathcal{Y} - \tilde{\mathcal{B}}(\tilde{b} + \tilde{\mathcal{C}}\mathcal{X} + \tilde{\mathcal{B}}\mathcal{Y}) \\ \mathcal{Y} &= \tilde{b} + \tilde{\mathcal{C}}\mathcal{X} + \tilde{\mathcal{B}}\mathcal{Y},\end{aligned}$$

where $\tilde{b}$ is the vector with components

$$\tilde{b}_i = f(\xi_i),$$

and $\tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{\mathcal{B}}$ are matrices with entries

$$\tilde{\mathcal{C}}_{i,j} = L_j(\xi_i), \qquad \tilde{\mathcal{D}}_{i,j} = \omega_j \mathcal{K}\tilde{k}_j(\xi_i), \qquad \tilde{\mathcal{B}}_{i,j} = \omega_j \tilde{k}_j(\xi_i),$$

Replacing $\mathcal{Y}$ by its value in the first equation, we get the following linear system

$$(\mathcal{I} - \tilde{\mathcal{F}})\mathcal{Z} = c, \tag{3.19}$$

with

$$\tilde{\mathcal{F}} = \begin{bmatrix} \mathcal{A} & \tilde{\mathcal{D}} - \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{B}} \end{bmatrix}, \quad c = \begin{bmatrix} a \\ \tilde{b} \end{bmatrix} \quad \text{and} \quad \mathcal{Z} = \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}.$$

Remark 3.1. There are integrals in sitting up the linear systems (3.14), (3.19) and in evaluating the iterated solutions $\tilde{x}_n^{SD}$ and $\tilde{x}_n^{SN}$. These integrals are calculated numerically to high accuracy, to imitate exact integration.

4 Convergence orders

At the beginning of this section, we prove the existence and uniqueness of approximate solutions presented in the previous section. Afterwards, we give the corresponding convergence orders.

Lemma 4.1. Assume that equation (1.1) has a unique solution x^*. Then, for n sufficiently large, the approximate equations (3.12) and (3.17) have unique solutions and the following error estimates hold

$$\begin{aligned}\|x^* - x_n^{SD}\| &\leq \Gamma_1 \|(\mathcal{I} - \mathcal{Q}_n^r)(\mathcal{K} - \mathcal{K}_n^D)x^*\|, \\ \|x^* - x_n^{SN}\| &\leq \Gamma_2 \|(\mathcal{I} - \mathcal{Q}_n^r)(\mathcal{K} - \mathcal{K}_n^N)x^*\|,\end{aligned} \tag{4.20}$$

where Γ_1 and Γ_2 are positive constants independent of n.

Proof. Let us consider the case of superconvergent degenerate kernel method. Using (2.5) and the fact that $(\mathcal{I} - \mathcal{K})$ is invertible, we deduce that $(\mathcal{I} - \mathcal{K}_n^{SD})$ is invertible for n large enough, and we have

$$\|(\mathcal{I} - \mathcal{K}_n^{SD})^{-1}\| \leq \Gamma_1.$$

Furthermore

$$\|\mathcal{K} - \mathcal{K}_n^{SD}\| \leq \|\mathcal{I} - \mathcal{Q}_n^r\| \max_{0 \leq s \leq 1} \int_0^1 |(\mathcal{I} - \mathcal{Q}_n^r)k(s,.)(t)|dt.$$

Consequently, we get

$$\begin{aligned}\|x^* - x_n^{SD}\| &= \|(\mathcal{K} - \mathcal{K}_n^{SD})^{-1}(\mathcal{K} - \mathcal{K}_n^{SD})x^*\| \\ &\leq \Gamma_3 \|(\mathcal{I} - \mathcal{Q}_n^r)(\mathcal{K} - \mathcal{K}_n^D)x^*\|,\end{aligned}$$

which gives the first estimation in (4.20). The proof is similar for the case of superconvergent Nyström method. □

The following proposition can be proved by similar way as in ([4], Proposition 1).

Proposition 4.1. Let r be an odd number. For $y \in \mathcal{C}^{r+1}([0,1])$ and for $0 \leq j \leq r$, we have

$$\begin{aligned}\| \left[(\mathcal{K} - \mathcal{K}_n^D)y \right]^{(j)} \| &\leq C h^{r+1}, \\ \| \left[(\mathcal{K} - \mathcal{K}_n^N)y \right]^{(j)} \| &\leq C' h^{r+1},\end{aligned}$$

where C and C' are two positive constants independent of h.

Using the above lemma and proposition, we get the convergence orders associated with each approximate solution. More precisely, the following theorem holds.

Theorem 4.1. Assume that $k \in \mathcal{C}^{r+1,r+1}([0,1] \times [0,1])$ and $f \in \mathcal{C}^{r+1}([0,1])$. Then, if x_n is either x_n^{SD} or x_n^{SN}, we have

$$\|x^* - x_n\| = \begin{cases} \mathcal{O}(h^{2r+1}), & \text{if } r \text{ is odd}, \\ \mathcal{O}(h^{2r}), & \text{if } r \text{ is even}. \end{cases}$$

Remark 4.1. Denote by x_n^D and x_n^N the approximate solutions of (1.1) using degenerate and Nyström methods respectively. From Proposition 4.1 and Theorem 4.1, we show that the superconvergent versions are more accurate. In other respects, the authors in [11] have used Kulkarni's method based on QI projectors to solve Fredholm integral equations. Only quadratic and cubic QI projectors have been used and similar convergence orders to those given in Theorem 4.1 for $r = 3$ and $r = 4$ are obtained. We note that the approximate Kulkarni's solution is given as

$$x_n^{Ku} = f + \sum_{i=1}^{\mathcal{N}} \mathcal{X}_i \mathcal{B}_j + \sum_{i=1}^{\mathcal{N}} \mathcal{Y}_i \tilde{\mathcal{B}}_j,$$

where $\tilde{\mathcal{B}}_j = \mathcal{K}\mathcal{B}_j$ and $\mathcal{N} = n + r - 1$. The vector $\mathcal{Y}$ is obtained by solving the following system of size $\mathcal{N}$

$$((\mathcal{I} - \mathscr{B})^2 + \mathscr{B} - \mathcal{C})\mathcal{Y} = c + (\mathcal{I} - \mathscr{B})b. \tag{4.21}$$

Afterwards $\mathcal{X}$ is calculated by

$$\mathcal{X} = (\mathcal{I} - \mathscr{B})\mathcal{Y} - b, \tag{4.22}$$

where

$$b_i = \lambda_i(f), \; c_i = \lambda_i(\mathcal{K}f), \; \mathscr{B}_{ij} = \lambda_i(\tilde{\mathcal{B}}_j), \quad \text{and} \quad \mathcal{C}_{ij} = \lambda_i(\mathcal{K}\tilde{\mathcal{B}}_j).$$

λ_i is the linear coefficient functional of the used QI projector. We point out that this λ_i uses more evaluation points than the one associated with the QI $\mathcal{Q}_n^r$ given in (2.3).
Also, even if the system (4.21) is of size $\mathcal{N}$, its matrix is more complicated to evaluate than $\mathscr{F}$ and $\tilde{\mathscr{F}}$ because it requires evaluation of multiple integrals. Moreover, the approximate solution x_n^{Ku} is more complicated than x_n^{SD} and x_n^{SN} since one have to evaluate $\mathcal{K}B_j$ instead of $\tilde{k}_j$. Consequently, the methods proposed here are faster and easier to implement. For a detailed comparison between superconvergent Nyström and degenerate kernel methods and Kulkarni's method based on interpolatory projectors, the reader is referred to [4].
In the rest of this section, we show that the convergence order is improved in the iterated versions of x_n^{SD} and x_n^{SN}.

Theorem 4.2. Assume that $k \in \mathcal{C}^{r+1,r+1}([0,1]\times[0,1])$ and $f \in \mathcal{C}^{r+1}([0,1])$. Then, it holds

$$\|x^* - \tilde{x}_n\| = \begin{cases} \mathcal{O}(h^{2r+2}) & \text{if } r \text{ is odd,} \\ \mathcal{O}(h^{2r}) & \text{if } r \text{ is even.} \end{cases}$$

where $\tilde{x}_n$ is either $\tilde{x}_n^{SD}$ or $\tilde{x}_n^{SN}$.

Proof. We give the proof for the case of the iterated superconvergent degenerate solution $\tilde{x}_n^{SD}$. For $\tilde{x}_n^{SN}$, the proof is similar. Moreover, the results are obvious when r is even. So, we will only process the case when r is odd. From (1.1) and (3.13), we have

$$x^* - \tilde{x}_n^{SD} = \mathcal{K}(x^* - x_n^{SD}) = \mathcal{K}(\mathcal{I}-\mathcal{K})^{-1}(\mathcal{I}-\mathcal{Q}_n^r)(\mathcal{K}-\mathcal{K}_n^D)(x^* + x_n^{SD} - x^*).$$

Then, we deduce that

$$\begin{aligned} \|x^* - \tilde{x}_n^{SD}\| \le\ & \|(\mathcal{I}-\mathcal{K})^{-1}\|\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n^r)(\mathcal{K}-\mathcal{K}_n^D)x^*\| \\ & + \|(\mathcal{I}-\mathcal{K})^{-1}\|\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n^r)(\mathcal{K}-\mathcal{K}_n^D)\|\|(x^* - x_n^{SD})\|. \end{aligned} \tag{4.23}$$

Using the error estimate (2.6), we obtain

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n^r)(\mathcal{K}-\mathcal{K}_n^D)x^*\| \le C_2\|\big[(\mathcal{K}-\mathcal{K}_n^D)x^*\big]^{(r+1)}\|h^{r+1}. \tag{4.24}$$

Next,

$$\big[(\mathcal{K}-\mathcal{K}_n^D)x^*\big]^{(r+1)}(s) = \int_0^1 (\mathcal{I}-\mathcal{Q}_n^r)\frac{\partial^{r+1}}{\partial s^{r+1}}k(s,.)(t)x(t)dt$$

Using again the error estimate (2.6), it follows

$$\|\big[(\mathcal{K}-\mathcal{K}_n^D)x^*\big]^{(r+1)}\| \le C_2 \max_{0\le s,t\le 1} \frac{\partial^{2r+2}}{\partial t^{r+1}\partial s^{r+1}}k(s,t)h^{r+1}. \tag{4.25}$$

By combining (4.24) and (4.25), we deduce

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n^r)(\mathcal{K}-\mathcal{K}_n^D)x^*\| = \mathcal{O}(h^{2r+2}). \tag{4.26}$$

On the other hand, it is easy to see that

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n^r)(\mathcal{K}-\mathcal{K}_n^D)\| = \mathcal{O}(h^r). \tag{4.27}$$

Finally, the required result follows from (4.23), (4.26) , (4.27) and Theorem 4.1. $\square$

Recall that from Theorem 2.2, the quadratic QI operator $\mathcal{Q}_n^3$ is superconvergent at some points in the interval $[0,1]$. This superconvergence property is also satisfied by the approximate solutions x_n^{SD} and x_n^{SN} as we show in the following proposition.

Proposition 4.2. Assume that $k \in \mathcal{C}^{4,4}([0,1]\times[0,1])$ and $f \in \mathcal{C}^4([0,1])$. Let x_n be either x_n^{SD} or x_n^{SN} based on the quadratic spline QI $\mathcal{Q}_n^3$. Then, we have the following superconvergence property

$$\max_{\xi_i \in \mathfrak{S}_n} |x^*(\xi_i) - x_n(\xi_i)| = \mathcal{O}(h^8). \tag{4.28}$$

Proof. Let x_n be the solution x_n^{SD}. We have

$$\max_{\xi_i \in \mathfrak{S}_n} |x^*(\xi_i) - x_n^{SD}(\xi_i)| \le \max_{\xi_i \in \mathfrak{S}_n} |x^*(\xi_i) - \tilde{x}_n^{SD}(\xi_i)| + \max_{\xi_i \in \mathfrak{S}_n} |\tilde{x}_n^{SD}(\xi_i) - x_n^{SD}(\xi_i)| \tag{4.29}$$

By using Theorem 4.2 for $r = 3$, we deduce that

$$\max_{\xi_i \in \mathfrak{S}_n} |x^*(\xi_i) - \tilde{x}_n^{SD}(\xi_i)| \le \|x^* - \tilde{x}_n^{SD}\| = \mathcal{O}(h^8).$$

On the other hand, we have

$$\tilde{x}_n^{SD}(\xi_i) - x_n^{SD}(\xi_i) = (\mathcal{K} - \mathcal{K}_n^{SD})x_n^{SD}(\xi_i) = (\mathcal{I} - \mathcal{Q}_n^3)(\mathcal{K} - \mathcal{K}_n^D)x_n^{SD}(\xi_i).$$

Then

$$\max_{\xi_i \in \mathfrak{S}_n} |\tilde{x}_n(\xi_i) - x_n(\xi_i)| \leq \|(\mathcal{I} - \mathcal{Q}_n^3)(\mathcal{K} - \mathcal{K}_n^D)\| \|x^* - x_n\| + \max_{\xi_i \in \mathfrak{S}_n} |(\mathcal{I} - \mathcal{Q}_n^3)(\mathcal{K} - \mathcal{K}_n^D)x^*(\xi_i)| \tag{4.30}$$

By using Theorem 4.2 and the fact that

$$\|(\mathcal{I} - \mathcal{Q}_n^3)(\mathcal{K} - \mathcal{K}_n^D)\| = \mathcal{O}(h^3),$$

we deduce that the first term on the right hand side of (4.30) is on $\mathcal{O}(h^{10})$. While the second term of this inequality verifies, see Theorem 2.2,

$$\max_{\xi_i \in \mathfrak{S}_n} |(\mathcal{I} - \mathcal{Q}_n^3)(\mathcal{K} - \mathcal{K}_n^D)x^*(\xi_i)| \leq Ch^4 \|[(\mathcal{K} - \mathcal{K}_n^D)x^*]^{(4)}\|.$$

Using (2.6) for $r = 3$, it follows

$$\max_{\xi_i \in \mathfrak{S}_n} |(\mathcal{I} - \mathcal{Q}_n^3)(\mathcal{K} - \mathcal{K}_n^D)x^*(\xi_i)| \leq CC_2 h^8 \left\| \frac{\partial^8 k(s,t)}{\partial t^4 \partial s^4} \right\|$$

which completes the proof of (4.28) for $x_n = x_n^{SD}$. For the case of x_n^{SN} the proof is quite similar. □

5 Case of Green's function type kernel

In this section, we consider the Fredholm integral equation (1.1) with kernel function $k(.,.)$ of Green's function type. More precisely, we assume that $k(.,.) \in \mathcal{C}([0,1] \times [0,1])$ and has the following form

$$k(s,t) = \begin{cases} k_1(s,t), & 0 \leq s \leq t \leq 1 \\ k_2(s,t), & 0 \leq t \leq s \leq 1 \end{cases}$$

such that $k_1 \in \mathcal{C}^m(\{0 \leq s \leq t \leq 1\})$ and $k_2 \in \mathcal{C}^m(\{0 \leq t \leq s \leq 1\})$, where m is a positive integer.

The operator K is still compact and, if we assume that (1.1) has a unique solution $x^* \in \mathcal{C}^m([0,1])$, then for n sufficiently large, all the approximate solutions previously defined exist. However, from Lemma 4.1, we can show that, in this case, the superconvergent degenerate kernel and Nyström methods have the same convergence orders as degenerate and Nyström ones, respectively, since the factor $\mathcal{I} - \mathcal{Q}_n^r$ in the error bound of the superconvergent approximate solutions does not improve convergence orders.
Let $g : [0,1] \to \mathbb{R}$ be m times differentiable function on $[0,1]$. The Euler-MacLaurin summation formula (see [17]) is given for $0 \leq \tau \leq 1$ by

$$g(\tau) = \int_0^1 g(t)dt + \sum_{\nu=1}^{m} \frac{\mathfrak{B}_\nu(\tau)}{\nu!} \left[g^{(\nu-1)}(1-) - g^{(\nu-1)}(0+) \right] - \int_0^1 \frac{\bar{\mathfrak{B}}_m(\tau - t)}{m!} g^{(m)}(t)dt. \tag{5.31}$$

In [15], the authors have proved an extension of the Euler-MacLaurin summation formula for a function $g : [0,1] \to \mathbb{R}$, m times differentiable on $[0,1]$ except at one point $s \in (0,1)$. That is

$$\begin{aligned} g(\tau) = & \int_0^1 g(t)dt + \sum_{\nu=1}^{m} \frac{\mathfrak{B}_\nu(\tau)}{\nu!} \left[g^{(\nu-1)}(1-) - g^{(\nu-1)}(0+) \right] \\ & - \sum_{\nu=2}^{m} \frac{\bar{\mathfrak{B}}_\nu(\tau - s)}{\nu!} \left[g^{(\nu-1)}(s+) - g^{(\nu-1)}(s-) \right] - \int_0^1 \frac{\bar{\mathfrak{B}}_m(\tau - t)}{m!} g^{(m)}(t)dt. \end{aligned} \tag{5.32}$$

Here, $\mathfrak{B}_\nu$ is the Bernoulli polynomial of degree ν and $\bar{\mathfrak{B}}_\nu$ is the corresponding periodic Bernoulli function. In the rest of this section, we put $\bar{r} := r$ if r is even, $\bar{r} := r+1$ if r is odd, and we assume that m is an even positive integer such that $m \geq \bar{r}$.

Proposition 5.1. Let $s \in (0,1)$. Then

$$\begin{aligned}\mathcal{I}_{\mathcal{Q}_n^r}(\kappa_s) = \int_0^1 \kappa_s(t)dt + \sum_{\substack{\nu=\bar{r}\\ \nu \text{ even}}}^{m} \frac{\mathcal{I}_{\mathcal{Q}_n^r}(\mathfrak{B}_\nu)}{\nu!}[\kappa_s^{(\nu-1)}(1-) - \kappa_s^{(\nu-1)}(0+)] \\ - \sum_{\nu=2}^{m} \frac{\mathcal{I}_{\mathcal{Q}_n^r}(\bar{\mathfrak{B}}_\nu(.-s))}{\nu!}[\kappa_s^{(\nu-1)}(s+) - \kappa_s^{(\nu-1)}(s-)] + \mathcal{R}_m,\end{aligned} \tag{5.33}$$

where

$$\mathcal{R}_m := -\int_0^1 \frac{\mathcal{I}_{\mathcal{Q}_n^r}(\bar{\mathfrak{B}}_m(\cdot - t))}{m!}\kappa_s^{(m)}(t)dt.$$

Proof. Since $x^* \in \mathcal{C}^m([0,1])$, the function κ_s, $s \in (0,1)$, is m times differentiable on $[0,1]$ except at s. Then, from (5.32), we get

$$\begin{aligned}\kappa_s(\tau) = \int_0^1 \kappa_s(t)dt + \sum_{\nu=1}^{m} \frac{\mathfrak{B}_\nu(\tau)}{\nu!}\left[\kappa_s^{(\nu-1)}(1-) - \kappa_s^{(\nu-1)}(0+)\right] \\ - \sum_{\nu=2}^{m} \frac{\bar{\mathfrak{B}}_\nu(\tau - s)}{\nu!}\left[\kappa_s^{(\nu-1)}(s+) - \kappa_s^{(\nu-1)}(s-)\right] - \int_0^1 \frac{\bar{\mathfrak{B}}_m(\tau - t)}{m!}\kappa_s^{(m)}(t)dt.\end{aligned} \tag{5.34}$$

Choosing τ in (5.34) as the nodes of the quadrature rule $\mathcal{I}_{\mathcal{Q}_n^r}$ and multiplying by the corresponding weights, we obtain

$$\begin{aligned}\mathcal{I}_{\mathcal{Q}_n^r}(\kappa_s) = \int_0^1 \kappa_s(t)dt + \sum_{\nu=1}^{m} \frac{\mathcal{I}_{\mathcal{Q}_n^r}(\mathfrak{B}_\nu)}{\nu!}[\kappa_s^{(\nu-1)}(1-) - \kappa_s^{(\nu-1)}(0+)] \\ - \sum_{\nu=2}^{m} \frac{\mathcal{I}_{\mathcal{Q}_n^r}(\bar{\mathfrak{B}}_\nu(.-s))}{\nu!}[\kappa_s^{(\nu-1)}(s+) - \kappa_s^{(\nu-1)}(s-)] + \mathcal{R}_m.\end{aligned}$$

Since $\mathcal{I}_{\mathcal{Q}_n^r}$ is exact on $\mathbb{P}_{\bar{r}-1}$, we obtain

$$\mathcal{I}_{\mathcal{Q}_n^r}(\mathfrak{B}_\nu) = \int_0^1 \mathfrak{B}_\nu(t)dt = 0, \qquad 1 \leq \nu \leq \bar{r} - 1.$$

Moreover, we have $\mathfrak{B}_\nu(1-t) = -\mathfrak{B}_\nu(t)$ for all odd ν, then using the symmetry of nodes and weights we get

$$\mathcal{I}_{\mathcal{Q}_n^r}(\mathfrak{B}_\nu) = 0, \quad \text{for all odd } \nu.$$

Thus (5.33) holds. □

From the above proposition, we see that for all s in $[0,1]$, the error of the quadrature rule on k_s is reduced to $\mathcal{O}(h^2)$. In what follows, we show that for particular choice of s, we get asymptotic expansions for the quadrature rules $\mathcal{I}_{\mathcal{Q}_n^3}$ and $\mathcal{I}_{\mathcal{Q}_n^4}$ based respectively on quadratic and cubic spline QIs.

Theorem 5.1. Let $s \in (0,1)$. For $r = 3,4$, let $\mathcal{I}_{\mathcal{Q}_n^r}$ be the quadrature rules based respectively on quadratic and cubic spline QIs $\mathcal{Q}_n^r$. Suppose that $s = s_i$ for $i = 0,\dots,n$ or $s = t_i$ for $i = 1,\dots,n$. Then, there exist constants C_1, C_2, C_3 independent of h such that

$$\mathcal{I}_{\mathcal{Q}_n^3}(\kappa_s) = \int_0^1 \kappa_s(t)dt + C_1h^2 + C_2h^4 + \mathcal{O}(h^5), \tag{5.35}$$

and

$$\mathcal{I}_{\mathcal{Q}_n^4}(\kappa_s) = \int_0^1 \kappa_s(t)dt + C_3h^2 + \mathcal{O}(h^5). \tag{5.36}$$

Proof. We give the proof of (5.35). The proof of (5.36) can be done in a similar way. The quadrature rule $\mathcal{I}_{\mathcal{Q}_n^3}$ based on $\mathcal{Q}_n^3$ is given by

$$\mathcal{I}_{\mathcal{Q}_n^3}(\mathfrak{B}_\nu) = h\sum_{i=0}^{n+1} \omega_i \mathfrak{B}_\nu(t_i), \tag{5.37}$$

where the weights ω_i are given explicitly by (see [19])

$$\omega_0 = \omega_{n+1} = \frac{1}{9}, \quad \omega_1 = \omega_n = \frac{7}{8}, \quad \omega_2 = \omega_{n-1} = \frac{73}{72}, \quad \omega_i = 1, \quad i = 3, \dots, n-2.$$

Using the symmetry of $\mathfrak{B}_\nu$ (ν even) on $[0, 1]$, we get

$$\mathcal{I}_{\mathcal{Q}_n^3}(\mathfrak{B}_\nu) = h\sum_{i=1}^{n} \mathfrak{B}_\nu(t_i) + h\left(\frac{2}{9}\mathfrak{B}_\nu(0) - \frac{1}{4}\mathfrak{B}_\nu\left(\frac{h}{2}\right) + \frac{1}{36}\mathfrak{B}_\nu\left(\frac{3h}{2}\right)\right).$$

From Euler-MacLaurin summation formula (5.31), it follows that

$$h\sum_{i=1}^{n} \mathfrak{B}_\nu(t_i) = h^\nu \mathfrak{B}_\nu\left(\frac{1}{2}\right).$$

Let

$$p_\nu(h) = \frac{2}{9}\mathfrak{B}_\nu(0) - \frac{1}{4}\mathfrak{B}_\nu\left(\frac{h}{2}\right) + \frac{1}{36}\mathfrak{B}_\nu\left(\frac{3h}{2}\right).$$

For $\nu = 4$, we have

$$p_4(h) = -\frac{h^3}{8} + \frac{h^4}{8}.$$

For $\nu > 4$ (ν even), we calculate $p_\nu^{(j)}(0), \quad j = 0, \dots, 4$ and we get

$$\begin{aligned} p_\nu(0) &= p'_\nu(0) = p''_\nu(0) = p'''_\nu(0) = 0, \\ p_\nu^{(4)}(0) &= \frac{1}{8}\nu(\nu-1)(\nu-2)(\nu-3)\mathfrak{B}_{\nu-4}(0), \end{aligned}$$

which implies that

$$p_\nu(h) = \frac{1}{132}\nu(\nu-1)(\nu-2)(\nu-3)\mathfrak{B}_{\nu-4}(0)h^4 + \mathcal{O}(h^5).$$

Thus

$$\mathcal{I}_{\mathcal{Q}_n^3}(\mathfrak{B}_\nu) = \begin{cases} -\frac{23}{240}h^4 + \frac{1}{8}h^5, & \text{if } \nu = 4 \\ \mathcal{O}(h^5), & \text{if } \nu > 4. \end{cases} \tag{5.38}$$

On the other hand, we have

$$\begin{aligned} \mathcal{I}_{\mathcal{Q}_n^3}(\bar{\mathfrak{B}}_\nu(. - s)) = h\sum_{i=1}^{n} \bar{\mathfrak{B}}_\nu(t_i - s) &+ h\left[\frac{2}{9}\bar{\mathfrak{B}}_\nu(-s) - \frac{1}{8}\left(\bar{\mathfrak{B}}_\nu(\frac{h}{2} - s) + \bar{\mathfrak{B}}_\nu(1 - \frac{h}{2} - s)\right)\right. \\ &\left. + \frac{1}{72}\left(\bar{\mathfrak{B}}_\nu(\frac{3h}{2} - s) + \bar{\mathfrak{B}}_\nu(1 - \frac{3h}{2} - s)\right)\right]. \end{aligned}$$

Using the Euler-MacLaurin summation formula, we get

$$h\sum_{i=1}^{n} \bar{\mathfrak{B}}_\nu(t_i - s) = h^\nu \bar{\mathfrak{B}}_\nu\left(\frac{1}{2} - \frac{s}{h}\right) = \begin{cases} h^\nu \mathfrak{B}_\nu(\frac{1}{2}), & \text{if } s = s_i, \\ h^\nu \mathfrak{B}_\nu(0), & \text{if } s = t_i. \end{cases}$$

Let

$$q_\nu(h) = h\left[\frac{2}{9}\bar{\mathfrak{B}}_\nu(-s) - \frac{1}{8}\left(\bar{\mathfrak{B}}_\nu(\frac{h}{2}-s) + \bar{\mathfrak{B}}_\nu(1-\frac{h}{2}-s)\right) + \frac{1}{72}\left(\bar{\mathfrak{B}}_\nu(\frac{3h}{2}-s) + \bar{\mathfrak{B}}_\nu(1-\frac{3h}{2}-s)\right)\right],$$

using same arguments as for $p_v(h)$, we show that

$$q_2(h) = 0, \quad q_3(h) = 0, \quad q_\nu(h) = \mathcal{O}(h^4), \quad \text{for } \nu > 2.$$

Then, we deduce that

$$\begin{aligned}
&\mathcal{I}_{\mathcal{Q}_n^3}(\bar{\mathfrak{B}}_2(.-s)) = Ch^2,\\
&\mathcal{I}_{\mathcal{Q}_n^3}(\bar{\mathfrak{B}}_4(.-s)) = Ch^4 + \mathcal{O}(h^5),\\
&\mathcal{I}_{\mathcal{Q}_n^3}(\bar{\mathfrak{B}}_\nu(.-s)) = \mathcal{O}(h^5), \quad \text{for } \nu > 4,
\end{aligned} \tag{5.39}$$

where $C = \mathfrak{B}_\nu(\frac{1}{2})$ if $s = s_i$ and $C = \mathfrak{B}_\nu(0)$ if $s = t_i$.
The required expansion (5.35) follows from (5.38), (5.39) and (5.33) of Proposition 5.1. □

In the previous theorem we have obtained asymptotic expansions for $(\mathcal{K}_n^N - \mathcal{K})x^*(\eta_i)$ where $\mathcal{K}_n^N$ is the Nyström operator based on $\mathcal{Q}_n^3$ or $\mathcal{Q}_n^4$. The points η_i are given by the partition points s_i, $i = 0, \dots, n$ or by the midpoints t_i, $i = 1, \dots, n$. Using the technique from Ford et al. [13], we deduce the following asymptotic expansions for $x_n^N(\eta_i) - x^*(\eta_i)$:

1. **Nyström method based on $\mathcal{Q}_n^3$**

$$x_n^N(\eta_i) - x^*(\eta_i) = \rho_1(\eta_i)h^2 + \rho_2(\eta_i)h^4 + \mathcal{O}(h^5). \tag{5.40}$$

2. **Nyström method based on $\mathcal{Q}_n^4$**

$$x_n^N(\eta_i) - x^*(\eta_i) = \rho_3(\eta_i)h^2 + \mathcal{O}(h^5). \tag{5.41}$$

Note that the functions ρ_1, ρ_2 and ρ_3 are independent of h.

From (5.40), (5.41) and using the Richardson extrapolation technique, it is easy to see that an order of $\mathcal{O}(h^5)$ can be restored in Nyström method based on $\mathcal{Q}_n^3$ or $\mathcal{Q}_n^4$ even if the kernel is of Green's function type. In the following section we validate the above results by numerical examples

6 Numerical results

At the beginning of this section, we consider the Fredholm integral equation

$$u(s) - \int_0^1 k(s,t)u(t)dt = e^{-s} + \frac{1}{2}(e^{-s-1} - 1), \quad 0 \le s \le 1,$$

with the smooth kernel

$$k(s,t) = \frac{1}{2}(s+1)e^{-st}.$$

The exact solution is $u(s) = e^{-s}$.
This equation is solved by using methods given in Section 3 based on quadratic spline QI ($r = 3$) and on cubic spline QI ($r = 4$) defined on the interval $I = [0,1]$ endowed with a uniform partition into n sub-intervals of length $h = \frac{1}{n}$.
For different values of n, and for $\alpha = SD, SN$, we estimate the maximum absolute errors

$$\mathcal{E}_n^\alpha = \|x^* - x_n^\alpha\|, \qquad \tilde{\mathcal{E}}_n^\alpha = \|x^* - \tilde{x}_n^\alpha\|,$$

on a set of 300 equally spaced points in I, and the maximum errors at the superconvergent points given by

$$\mathcal{S}_n^\alpha = \max_{\xi_i \in \mathfrak{S}_n} |x^*(\xi_i) - x_n^\alpha(\xi_i)|.$$

For the sake of comparison, we compute the maximum errors on degenerate kernel and Nyström methods based on the same QIs and we denote them respectively by $\mathcal{E}_n^D$ and $\mathcal{E}_n^N$. We also give the maximum errors noted $\mathcal{E}_n^{Ku}$ related to the Kulkarni's method based on QI projector of the same order which was introduced in [11].
The obtained results are illustrated in Tables 6.1-6.6, where the quantity $\mathcal{NCO}$ represents the numerical convergence order calculated as the logarithm to base 2 of the ratio between two consecutive errors. Programs are performed by Mathematica 11.3 for all the numerical examples.

Table 1: Superconvergent degenerate kernel method based on $\mathcal{Q}_n^3$

n	$\mathcal{E}_n^{SD}$	$\mathcal{NCO}$	$\tilde{\mathcal{E}}_n^{SD}$	$\mathcal{NCO}$	$\mathcal{S}_n^{SD}$	$\mathcal{NCO}$
8	1.99×10^{-10}		6.87×10^{-12}		5.40×10^{-11}	
16	1.38×10^{-12}	7.16	3.99×10^{-14}	6.36	2.26×10^{-13}	7.90
32	1.10×10^{-14}	6.97	1.88×10^{-16}	7.40	8.99×10^{-16}	7.97
64	5.61×10^{-17}	7.62	7.09×10^{-19}	7.73	3.73×10^{-18}	7.91
128	4.14×10^{-19}	7.08	2.81×10^{-21}	7.98	1.22×10^{-20}	8.26

Table 2: Superconvergent Nyström method based on $\mathcal{Q}_n^3$

n	$\mathcal{E}_n^{SN}$	$\mathcal{NCO}$	$\tilde{\mathcal{E}}_n^{SN}$	$\mathcal{NCO}$	$\mathcal{S}_n^{SN}$	$\mathcal{NCO}$
8	4.74×10^{-10}		5.05×10^{-11}		7.85×10^{-11}	
16	3.47×10^{-12}	7.09	2.59×10^{-13}	7.16	4.11×10^{-13}	7.58
32	3.17×10^{-14}	6.77	1.15×10^{-15}	7.81	1.92×10^{-15}	7.75
64	1.66×10^{-16}	7.58	4.79×10^{-18}	7.91	8.17×10^{-18}	7.87
128	1.29×10^{-18}	7.00	1.92×10^{-20}	7.96	1.46×10^{-20}	7.99

Table 3: Comparison with other methods based on $\mathcal{Q}_n^3$

n	$\mathcal{E}_n^{D}$	$\mathcal{E}_n^{N}$	$\mathcal{E}_n^{Ku}$ [11]	$\mathcal{E}_n^{SD}$	$\mathcal{E}_n^{SN}$
8	1.23×10^{-6}	7.92×10^{-6}	1.1×10^{-10}	1.99×10^{-10}	4.74×10^{-10}
16	8.69×10^{-8}	5.55×10^{-7}	4.2×10^{-13}	1.38×10^{-12}	3.47×10^{-12}
32	5.75×10^{-9}	3.66×10^{-8}	1.7×10^{-15}	1.10×10^{-14}	3.17×10^{-14}
64	3.63×10^{-10}	2.35×10^{-9}	6.7×10^{-16}	5.61×10^{-17}	1.66×10^{-16}
128	1.49×10^{-11}	1.49×10^{-10}	-	4.14×10^{-19}	1.29×10^{-18}

Table 4: Superconvergent degenerate kernel method based on $\mathcal{Q}_n^4$

n	$\mathcal{E}_n^{SD}$	$\mathcal{NCO}$	$\tilde{\mathcal{E}}_n^{SD}$	$\mathcal{NCO}$
8	3.45×10^{-10}		1.31×10^{-10}	
16	3.10×10^{-12}	6.80	1.60×10^{-12}	6.36
32	1.73×10^{-14}	7.49	9.46×10^{-15}	7.40
64	7.95×10^{-17}	7.76	4.45×10^{-17}	7.73
128	2.96×10^{-19}	8.06	1.72×10^{-19}	8.01

Table 5: Superconvergent Nyström method based on $\mathcal{Q}_n^4$

n	$\mathcal{E}_n^N$	$\mathcal{E}_n^{SN}$	$\tilde{\mathcal{E}}_n^{SN}$	$\mathcal{NCO}$
8	1.99×10^{-10}		4.93×10^{-11}	
16	2.30×10^{-12}	6.44	9.75×10^{-13}	7.57
32	1.53×10^{-14}	7.22	8.14×10^{-15}	7.81
64	7.53×10^{-17}	7.67	4.39×10^{-17}	7.91
128	3.03×10^{-19}	7.96	1.87×10^{-20}	8.00

Table 6: Comparison with other methods based on $\mathcal{Q}_n^4$

n	$\mathcal{E}_n^D$	$\tilde{\mathcal{E}}_n^N$	$\mathcal{E}_n^{Ku}$ [11]	$\mathcal{E}_n^{SD}$	$\mathcal{E}_n^{SN}$
8	1.45×10^{-6}	2.92×10^{-5}	1.1×10^{-10}	3.45×10^{-10}	1.99×10^{-10}
16	1.59×10^{-7}	3.08×10^{-6}	4.2×10^{-13}	3.10×10^{-12}	2.30×10^{-12}
32	1.23×10^{-8}	2.35×10^{-7}	1.7×10^{-15}	1.73×10^{-14}	1.53×10^{-14}
64	8.42×10^{-10}	1.61×10^{-8}	6.7×10^{-17}	7.95×10^{-17}	7.53×10^{-17}
128	5.52×10^{-11}	1.05×10^{-9}	-	2.96×10^{-19}	3.03×10^{-19}

It can be seen from Tables 6.1-6.6, that the numerical convergence orders match well with the expected values given in Theorem 4.1, Theorem 4.2 and Proposition 4.2. On the other hand, the results obtained in this paper improve clearly those provided by the classical Nyström and degenerate kernel methods, and are comparable with those obtained by Kukarni's methods based on QI projectors.

In the rest of this section, we consider the following example of Green's function type kernel given in Atkinson and Shampine [6] and defined by

$$u(s) - \int_0^1 k(s,t)u(t)dt = \left(1 - \frac{1}{\pi^2}\right) sin(\pi s), \quad 0 \le s \le 1,$$

with

$$k(s,t) = \begin{cases} s(1-t), & \text{if } s \le t, \\ t(1-s), & \text{if } t \le s, \end{cases}$$

and the exact solution is $u(s) = \sin(\pi s)$.
The above equation is solved by Nyström method based on quadratic spline QI ($r = 3$) and on a cubic spline QI ($r = 4$) defined on the interval $I = [0, 1]$ endowed with a uniform partition into n subintervals of length $h = \frac{1}{n}$.
Let $s = \frac{1}{2}$. For different values of n, we compute the following errors

$$\mathcal{E}_{n,\ell}^N = |x_{n,\ell}^N(s) - x^*(s)|, \quad l = 0, 1, 2,$$

where $x_{n,l}(s)$, $\ell = 0, 1, 2$, are given by

$$x_{n,0}^N(s) = x_n^N(s),$$
$$x_{n,\ell}(s) = \frac{2^{2\ell} x_{2n,\ell-1}^N(s) - x_{n,\ell-1}^N(s)}{2^{2\ell} - 1}.$$

The obtained results are given in Tables 6.5-6.6.
Note that the computed values of convergence orders in Tables 6.5-6.6 are as expected in Section 5.

7 Conclusion

In this paper, we have used QI operators to solve numerically linear Fredholm integral equations of second kind by superconvergent degenerate kernel and Nyström methods.

Table 7: Extrapolation of Nyström method based on $\mathcal{Q}_n^3$.

n	$\mathcal{E}_{n,0}^N$	$\mathcal{NCO}$	$\mathcal{E}_{n,1}^N$	$\mathcal{NCO}$	$\mathcal{E}_{n,2}^N$	$\mathcal{NCO}$
8	5.25×10^{-4}					
16	1.27×10^{-4}	2.05	5.10×10^{-6}			
32	3.15×10^{-5}	2.01	2.42×10^{-7}	4.53	1.15×10^{-7}	
64	7.88×10^{-6}	2.00	1.17×10^{-8}	4.37	3.69×10^{-9}	4.96
128	1.97×10^{-7}	2.00	6.20×10^{-10}	4.23	1.16×10^{-10}	4.99

Table 8: Extrapolation of Nyström method based on $\mathcal{Q}_n^4$.

n	$\mathcal{E}_{n,0}^N$	$\mathcal{NCO}$	$\mathcal{E}_{n,1}^N$	$\mathcal{NCO}$	$\mathcal{E}_{n,2}^N$	$\mathcal{NCO}$
8	1.22×10^{-3}					
16	3.54×10^{-4}	1.79	6.51×10^{-5}			
32	9.03×10^{-5}	1.97	2.28×10^{-6}	4.83	1.91×10^{-6}	
64	2.26×10^{-5}	2.00	6.76×10^{-8}	5.07	7.98×10^{-8}	4.58
128	5.66×10^{-6}	2.00	1.73×10^{-9}	5.29	2.66×10^{-9}	4.91

We have provided convergence order for each method, and we have showed that, for the iterated versions, the convergence order is $2r+2$ if r is odd. The integer r denotes the order of the used QI. The obtained numerical results confirm the theoretical ones and illustrate the efficiency of the proposed methods with respect to classical degenerate kernel and Nyström ones. Moreover, we have showed that the obtained error values are well comparable with those obtained in [11] by Kulkarni's method based on QI projectors. However, superconvergent Nyström and degenerate kernel methods are faster and simpler to implement. Finally, we have considered the case where the kernel is of the Green's function type and we have improved the convergence order in the Nyström method based on QI operators.

References

[1] C. Allouch, P. Sablonnière, D. Sbibih, A modified Kulkarni's method based on a discrete spline quasi-interpolant, Math. Comput. Simul, 81, 1991-2000 (2011).

[2] C. Allouch, P. Sablonnière, D. Sbibih, M. Tahrichi, Product integration methods based on discrete spline quasi-interpolants and application to weakly singular integral equations, J. Comput. Appl. Math, 233, 2855-2866 (2010).

[3] C. Allouch, P. Sablonnière, D. Sbibih, Solving Fredholm integral equations by approximating kernels by spline quasi-interpolants, Numer. Algorithms, 56, 437-453 (2011).

[4] C. Allouch, P. Sablonnière, D. Sbibih, M. Tahrichi, Superconvergent *Nyström* and degenerate kernel methods for the numerical solution of integral equations of the second kind, Journal of Integral Equations and Applications, 24, 463-485 (2012).

[5] K. Atkinson, The numerical solution of integral equations of the second kind, Cambridge University Press (1997).

[6] K. Atkinson and L.F. Shampine, Solving Fredholm integral equations of the second kind in MATLAB, ACM Trans. Math Software, 34, Article 21 (2008).

[7] K.E. Atkinson, F . Potra, Projection and iterated projection methods for nonlinear integral equations, SIAM J. numer. Anal, 14, 1352–1373 (1987).

[8] C.T.H. Baker, The numerical treatment of integral equations, Oxford University Press, Oxford (1977).

[9] C. de Boor, A practical guide to splines (Revised edition), Springer Verlag, Berlin (2001).

[10] M. Chen, Z. Chen and G. Chen, Approximate solutions of operator equations, World Scientific Publ. Co., Singapore (1997).

[11] C. Dagnino, S. Remogna, P. Sablonnière, On the solution of Fredholm integral equations based on spline quasi-interpolating projectors, BIT Numerical Mathematics, 54 (4), 979-1008 (2014).

[12] C. Dagnino, A. Dallefrate, S. Remogna, Spline quasi-interpolating projectors for the solution of nonlinear integral equations, J. Comput. Appl. Math, 354, 360-372 (2019).

[13] W.F. Ford, J.A. Pennline, Y. Xu and Y. Zhao, Asymptotic error analysis of a quadrature method for integral esuations with Green's function kernel, J. Integral. Eqns. Appl, 12, 349-384 (2000).

[14] R.P. Kulkarni, A superconvergence result for solutions of compact operator equations, Bull. Austral. Math. Soc, 68, 517-528 (2003).

[15] R. P. Kulkarni, A.S. Rane, Asymptotic expansions for approximate solutions of Fredholm integral equations with Green's function type kernels, J. Integral Equations Applications, 24 (1), 39-79, (2012).

[16] R. Kress, Linear Integral Equations. Berlin etc, Springer-Verlag, (1989).

[17] W. Mclean, Asymptotic error expansions for numerical solution of integral equations, IMA J. Numer. Anal. 9, 373-384, (1989).

[18] A.N. Netravali, R. J. P. de Figueiredo, Spline approximation to the solution of the linear Fredholm integral equation of the second kind, SIAM Journal on Numerical Analysis, 11 (3), 538-549, (1974).

[19] P. Sablonnière, A quadrature formula associated with a univariate quadratic spline quasi-interpolant, BIT Numerical Mathematics, 47, 825-837 (2007).

[20] P. Sablonnière, Univariate spline quasi-interpolants and applications to numerical analysis, Rend. Sem. Mat. Univ. Pol. Totino 63 (2), 107-118 (2005).

[21] I.H. Sloan, Improvement by iteration for compact operator equations, Mathematics of Computation, 30 (136), 758–764 (1976).

[22] M. Sun, J. Li, P. Wang, Z. Zhang, Superconvergence analysis of high-order rectangular edge elements for time-harmonic Maxwell's equations. Journal of Scientific Computing, 75(1), 510-535 (2018).

[23] C. Wu, Y. Huang, N. Yi, J. Yuan, Superconvergent recovery of edge finite element approximation for Maxwell's equations, Computer Methods in Applied Mechanics and Engineering, 371, 113-302 (2020).