Divided-difference operators from the geometric point of view

Maria das Neves Rebocho *

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Abstract

It is presented a study of general divided-difference operators having the fundamental property of leaving a polynomial of degree $n-1$ when applied to a polynomial of degree n.

1 Introduction

In the present paper it is shown a study on divided-difference operators having the fundamental property of leaving a polynomial of degree $n-1$ when applied to a polynomial of degree n. Primarily, the focus is on the geometric interpretation, by analysing the connection between the divided-difference operators and their relation with a corresponding conic, which, in turn, gives rise to a corresponding lattice of points that well-defines the operator (see [11]). Essentially, there are four primary classes of lattices and related divided-difference operators having the above mentioned property:
(i) the linear lattice, related to the forward difference operator [15, Chapter 2, Section 12] ; (ii) the q-linear lattice, related to the q-difference operator [6] ; (iii) the quadratic lattice, related to the Wilson operator [2] ; (iv) the q-quadratic lattice, related to the Askey-Wilson operator [2]. This list gives a hierarchy of operators, as each of the operators in (i)-(iv) is an extension of the preceding one, which can be recovered as a special case and/or a limit case, up to a linear transformation of the variable.

The analysis of divided-difference operators (i)-(iv) is rather sparse in the literature. For instance, they are a fundamental machinery for the study of certain special functions appearing in problems from Mathematical-Physics, e.g., within the general theory of orthogonal polynomials (see [2, 7, 9, 15]). Very often, when dealing with applications, final and combined formulae are given, together with a notation that may lead to a heavy reading for readers unaware of basic relations in the theory of divided-difference operators. With this idea in mind, the main goal of the present paper is to give a concise but detailed study of some basic aspects of the divided-difference operators above referred, showing details on fundamental formulae that emerge from the geometric interpretation (given in the seminal paper [10]) and its connection with algebraic aspects of operator calculus. Here, the following topics are covered: the geometric interpretation - namely,

*Departamento de Matemática, Universidade da Beira Interior, 6201-001 Covilhã, Portugal; CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal. Email: mneves@ubi.pt

the connection between the operator and a conic/lattice (cf. Section 2); the classification of operators in terms of a set of parameters in the given conic (cf. Section 3); the analysis of coalescences between the operators (cf. Section 4); basic and fundamental formulae in the divided-difference calculus (cf. Section 5).

2 The conic and the related lattice

We start by following the approach from [10], where it is considered a divided-difference operator involving the values of a function at two points, with the property that it leaves a polynomial of degree $n-1$ when applied to a polynomial of degree n. Let us take the divided-difference operator $\mathbb{D}_x$ as given in [10, Eq.(1.1)], defined on the space of arbitrary functions, by

$$\mathbb{D}_x f(x) = \frac{f(y_+(x)) - f(y_-(x))}{y_+(x) - y_-(x)}\,, \tag{1}$$

where, at this stage, y_+ and y_- are unknown functions. To define them, one starts by using the property that $\mathbb{D}_x f$ is a polynomial of degree $n-1$ whenever f is a polynomial of degree n. Then, applying $\mathbb{D}_x$ to $f(x) = x^2$ and $f(x) = x^3$, we obtain, respectively,

$$y_-(x) + y_+(x) = \text{polynomial of degree } 1\,, \tag{2}$$

$$(y_-(x))^2 + y_-(x)y_+(x) + (y_+(x))^2 = \text{polynomial of degree } 2\,, \tag{3}$$

the later condition being equivalent to $y_-(x)y_+(x) =$ polynomial of degree less or equal than two. From standard polynomial properties, the conditions (2)-(3) define y_- and y_+ as the two y-roots of a quadratic equation, say,

$$ay^2 + 2bxy + cx^2 + 2dy + 2ex + f = 0\,. \quad a \neq 0\,. \tag{4}$$

The conic defined by the equation above plays an essential role in the sequel. The following identities, to be used later on, follow from the fact that y_-, y_+ are the y-roots of (4):

$$y_-(x) + y_+(x) = -2(bx + d)/a\,, \tag{5}$$

$$y_-(x)y_+(x) = (cx^2 + 2ex + f)/a\,, \tag{6}$$

$$y_-(x) = p(x) - \sqrt{r(x)}\,, \quad y_+(x) = p(x) + \sqrt{r(x)}\,, \tag{7}$$

with p, r polynomials given by

$$p(x) = -\frac{b}{a}x - \frac{d}{a}\,, \quad r(x) = \frac{(b^2 - ac)}{a^2}x^2 + 2\frac{(bd - ae)}{a^2}x + \frac{(d^2 - af)}{a^2}\,. \tag{8}$$

By virtue of (7), the operator $\mathbb{D}_x$ defined in (1) is given as

$$\mathbb{D}_x f(x) = \frac{f(p(x) + \sqrt{r(x)}) - f(p(x) - \sqrt{r(x)})}{2\sqrt{r(x)}}\,. \tag{9}$$

Remark 1. *The polynomials p, r will play a fundamental role in the sequel. Note that, from (7), it follows that*

$$y_-(x) + y_+(x) = 2p(x)\,, \quad (y_-(x) - y_+(x))^2 = 4r(x)\,. \tag{10}$$

Let us now look at the lattices.

Associated to each conic (4) two lattices are determined: the x-lattice and the y-lattice. The construction is based on the parametric representations of the conic, as follows (see [11]):
Let $\{x(s),\, y(s)\}$ be a parametric representation of the conic (4). For a given $x = x(s)$ value, the quadratic (4) defines two y-roots, say $y_s := y(s)$ and $y_{s+1} := y(s+1)$, which are the two ordinates associated to the abcissa $x(s)$. Then one starts from some point $\{x_1 = x(s_1),\, y_1 = y(s_1)\}$ on the conic, and one looks for the points $\{x_k = x(s_1+k),\, y_k = y(s_1+k)\},\ k = 1, 2, \ldots$. This determines the so-called y-lattice, also known as the dual lattice. Conversely, if $c \neq 0$ in (4), then, for a given y-value, the quadratic (4) defines two x-roots, say $x_s := x(s),\ x_{s+1} := x(s+1)$, which are consecutive points on the so-called x-lattice, also known as the direct lattice.

Remark 2. *With the above notation, in terms of the operator $\mathbb{D}_x$ defined in (1), we have*

$$y_s = y_-(x(s))\,, \quad y_{s+1} = y_+(x(s))\,.$$

2.1 The quadratic class of lattices - explicit parameterizations

The quadratic class of lattices appears when the conic (4) is such that $(b^2-ac)(d^2-af)-(bd-ae)^2 \neq 0$. Two sub-cases hold: the conic is a parabola - when $b^2-ac=0$ - this corresponds to the quadratic case; the conic is a hyperbola or an ellipse - when $b^2-ac>0$ or $b^2-ac<0$, respectively - this corresponds to the q-quadratic case.

For the quadratic class of lattices there is a parametric representation of the conic, say $\{x(s),\, y(s)\}$, such that the functions y_- and y_+ in (1) satisfy [11, 16, 14]

$$y_-(x(s)) = y(s) = x(s-1/2)\,, \quad y_+(x(s)) = y(s+1) = x(s+1/2)\,. \tag{11}$$

Hence, the divided-difference operator (1) is given as

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+1/2)) - f(x(s-1/2))}{x(s+1/2) - x(s-1/2)}\,. \tag{12}$$

The parametrization on s is explicit [13], given by

$$x(s) = \tilde{\kappa}_2 s^2 + \tilde{\kappa}_1 s + \tilde{\kappa}_0 \tag{13}$$

where $\tilde{\kappa}_2 \neq 0$ in the quadratic case, and

$$x(s) = \kappa_1 q^s + \kappa_2 q^{-s} + \kappa_3 \tag{14}$$

where $\kappa_1\kappa_2 \neq 0$ in the q-quadratic case. Here, the κ's and $\tilde{\kappa}$'s are appropriate constants.

The parameterizations of the form (13) and (14) cover the whole set of canonical forms for the lattices. A formal deduction of formulae (13) and (14), based on properties of adjoint operators, will be given in Sub-Section 5.1.

Remark 3. *Note that, in the account of (10) and (11), the polynomials p, r in (9) are then recovered under*

$$x(s+1/2) + x(s-1/2) = 2p(x(s))\,, \quad (x(s+1/2) - x(s-1/2))^2 = 4r(x(s))\,.$$

Indeed, by writing $p(x) = p_1x + p_0$, $r(x) = r_2x^2 + r_1x + r_0$, *we get*

$$p_1 = 1\,,\ p_0 = \tilde\kappa_2/4\,,\ r_2 = 0\,,\ r_1 = \tilde\kappa_2\,,\ r_0 = \tilde\kappa_1^2/4 - \tilde\kappa_2\tilde\kappa_0 \tag{15}$$

in the case (13), and

$$p_1 = \frac{q^{1/2}+q^{-1/2}}{2}\,,\ p_0 = \kappa_3\left(1 - \frac{(q^{1/2}+q^{-1/2})}{2}\right)\,, \tag{16}$$

$$r_2 = \frac{(q^{1/2}-q^{-1/2})^2}{4}\,,\ r_1 = -\kappa_3\frac{(q^{1/2}-q^{-1/2})^2}{2}\,, \tag{17}$$

$$r_0 = (-\kappa_1\kappa_2 + \frac{\kappa_3^2}{4})(q^{1/2}-q^{-1/2})^2 \tag{18}$$

in the case (14).

A.P. Magnus, in [11, p. 255], gives the following precise parameterizations.

Proposition 1. *Consider the conic (4),* $ay^2+2bxy+cx^2+2dy+2ex+f=0$, *with* $ac \neq 0$. *The following assertions hold.*
(a) *If the conic has a center* $\lambda := b^2 - ac \neq 0$, *then, with the center coordinates*

$$x_c = \frac{ae-bd}{\lambda}\,,\quad x_c = \frac{cd-be}{\lambda}\,,$$

one has (4) written in the form

$$a(y-y_c)^2 + 2b(x-x_c)(y-y_c) + c(x-x_c)^2 + \tilde f = 0\,,$$

with

$$\tilde f = f - ay_c^2 - 2bx_cy_c - cx_c^2 = f + dy_c + ex_c = f + \frac{cd^2 - 2bde + ae^2}{\lambda}\,.$$

(a.1) *If* $\tilde f \neq 0$, *then*

$$x(s) = x_c + \xi\sqrt{a}(q^s + q^{-s})\,,\quad y(s) = y_c + \xi\sqrt{c}(q^{s-1/2} + q^{-s+1/2})\,,$$

is a parametric representation of (4), where $\xi^2 = \tilde f/(4\lambda)$, *and*

$$q^{1/2} + q^{-1/2} = -\frac{2b}{\sqrt{ac}},\quad i.e.,\quad q + q^{-1} = \frac{4b^2}{ac} - 2\,.$$

(a.2) *If* $\tilde f = 0$, *then one finds the parametric representation*

$$x(s) = x_c + X\sqrt{a}q^s\,,\quad y(s) = y_c + X\sqrt{c}q^{s\pm 1/2}\,,$$

for arbitrary parameters X.
(b) *If the conic has a center* $\lambda := b^2 - ac = 0$, *then*

$$\begin{cases} x(s) = \sqrt{a}\left(\dfrac{d^2 - af}{2a(d\sqrt{c}+e\sqrt{a})} - 2\dfrac{(d\sqrt{c}+e\sqrt{a})}{ac}s^2\right) \\[2ex] y(s) = \sqrt{c}\left(\dfrac{e^2 - cf}{2c(d\sqrt{c}+e\sqrt{a})} - 2\dfrac{(d\sqrt{c}+e\sqrt{a})}{ac}(s-1/2)^2\right) \end{cases}$$

is a parametric representation of (4).

Remark 4. *In the generic case q-quadratic case* $|q| \neq 1$ *the conic gives a hyperbola. In such a case, the asymptotes are given by* $y = (c/a)^{1/2}q^{\pm 1/2}x$, *thus,* q *is precisely the ratio of the slopes of the asymptotes of the conic.*

3 Classification

There are four primary classes of lattices and related divided-difference operators:
(i) the linear lattice, related to the forward difference operator [15, Chapter 2, Section 12];
(ii) the q-linear lattice, related to the q-difference operator [6];
(iii) the quadratic lattice, related to the Wilson operator [2];
(iv) the q-quadratic lattice, related to the Askey-Wilson operator [2].

Such a classification can be done according to the two parameters λ, τ defined in terms of the conic (4), $ay^2 + 2bxy + cx^2 + 2dy + 2ex + f = 0$, as follows:

$$\lambda = b^2 - ac, \quad \tau = \left((b^2 - ac)(d^2 - af) - (bd - ae)^2\right)/a\,, \tag{19}$$

or, using the determinant notation,

$$\tau = \det \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

Note that $\lambda \neq 0$ allows us to write the polynomial r in (8) as

$$r(x) = \frac{\lambda}{a^2}\left(x + \frac{bd - ae}{\lambda}\right)^2 + \frac{\tau}{a\lambda}. \tag{20}$$

A detailed analysis of each case (i)-(iv), showing each of the operators in the form (9) with the corresponding polynomials p, r, is given in the following sub-sections.

3.1 The linear lattice: $\lambda = \tau = 0$ in (19)

If $\lambda = 0$ and $\tau = 0$, then, from (19), $bd - ae = 0$, thus, the the polynomial r defined in (8) is constant, $r(x) = \dfrac{d^2 - af}{a^2}$. Hence, we have the polynomials p, r defined in (8) given by

$$p(x) = -\frac{b}{a}x - \frac{d}{a}, \quad r(x) = \frac{d^2 - af}{a^2}.$$

Recalling (7), it follows that

$$y_{\pm}(x) = -\frac{b}{a}x - \frac{d}{a} \pm \frac{\sqrt{d^2 - af}}{a^2}, \tag{21}$$

that is, we have two parallel lines,

$$y_{\pm}(x) = mx \pm b_{\pm}\,,$$

with

$$m = -\frac{b}{a}, \quad b_{\pm} = -\frac{d}{a} \pm \frac{\sqrt{d^2 - af}}{a^2}.$$

Proposition 2. *The canonical divided-difference operator related to the linear lattices is the forward difference operator $\mathbb{D}_x = \Delta_w$ - the so-called Hahn's operator [6], where*

$$\Delta_w f(x) = \frac{f(x + w) - f(x)}{w}, \quad w \neq 0\,, \tag{22}$$

for arbitrary functions f. Hence, the operator Δ_w can be written in the form (9), with the polynomials p, r given by

$$p(x) = x + \frac{w}{2}\,, \quad r(x) = \frac{w^2}{4}\,.$$

Proof. Combining (1) with (21), the operator (22) is recovered through the specialization

$$b = -a \quad c = a\,, \quad d = -aw/2\,, \quad e = aw/2\,, \quad f = 0\,, \tag{23}$$

and it follows the assertion on the polynomials p, r. □

Also, by using the values of (23) into (4), we get the conic with equation

$$y^2 - 2xy + x^2 - wy + wx = 0\,,$$

which can be factorized as

$$(y - x)(y - x - w) = 0\,.$$

The linear lattice, obtained via two parallel lines, is illustrated through Fig. 2.d) in [11, pp. 256]).

3.2 The q-linear lattice: $\lambda \neq 0\,, \;\; \tau = 0$ in (19)

If $\lambda \neq 0$ and $\tau = 0$, the polynomials p, r defined in (8) are given by

$$p(x) = -\frac{b}{a}x - \frac{d}{a}\,, \quad r(x) = \frac{\lambda}{a^2}\left(x + \frac{bd - ae}{\lambda}\right)^2\,.$$

Recalling (7), it follows that

$$y_{\pm}(x) = -\frac{b}{a}x - \frac{d}{a} \pm \frac{\sqrt{\lambda}}{a}\left(x + \frac{bd - ae}{\lambda}\right)\,, \tag{24}$$

that is, we have two intersecting lines,

$$y_{+}(x) = m_{+}x + b_{+}\,, \quad y_{-}(x) = m_{-}x + b_{-}\,,$$

with

$$m_{+} = -\frac{b}{a} + \frac{\sqrt{\lambda}}{a}\,, \quad m_{-} = -\frac{b}{a} - \frac{\sqrt{\lambda}}{a}\,,$$
$$b_{+} = \frac{\sqrt{\lambda}}{a}\left(\frac{bd - ae}{\lambda}\right) - \frac{d}{a}\,, \quad b_{-} = -\frac{\sqrt{\lambda}}{a}\left(\frac{bd - ae}{\lambda}\right) - \frac{d}{a}\,.$$

Proposition 3. *The canonical divided-difference operator related to the q-linear lattices is the q-linear difference operator, $\mathbb{D}_x = \Delta_{q,w}$ [6], where*

$$\Delta_{q,w}f(x) = \frac{f(qx + w) - f(x)}{(q - 1)x + w}\,, \quad q \neq 1\,, \tag{25}$$

for arbitrary functions f. Hence, the operator $\Delta_{q,w}$ can be written in the form (9), with the polynomials p, r given by

$$p(x) = \frac{(q + 1)}{2}x + \frac{w}{2}\,, \quad r(x) = \frac{(q - 1)^2}{4}\left(x + \frac{w}{q - 1}\right)^2\,.$$

Proof. Combining (1) with (24), the operator (25) is recovered through the specialization

$$b=-\frac{(q+1)}{2}a\,,\quad c=qa\,,\quad d=-\frac{w}{2}a\,,\quad e=\frac{w}{2}a\,,\quad f=0\,, \tag{26}$$

and it follows the assertion on the polynomials p, r. □

Also, by using the values of (26) into (4), we get the conic with equation

$$y^2-(q+1)xy+qx^2-wy+wx=0\,,$$

which can be factorized as

$$(y-x)(y-qx-w)=0\,.$$

The q-linear lattice, obtained via two intersecting lines, is illustrated through Fig. 2.b) in [11, pp. 256]).

Remark 5. *In [6, pp. 6], it is shown that, whenever $q\neq 1$, the constant w in (25) can be eliminated through a linear transformation: by setting $x=\hat{a}z+\hat{b}$ and $f(x)=h(z)$, the operator $\Delta_{q,w}$ can be written as*

$$\Delta_{q,w}f(x)=\frac{h\left(qz+\dfrac{(q-1)\hat{b}+w}{\hat{a}}\right)-h(z)}{(q-1)z+\dfrac{(q-1)\hat{b}+w}{\hat{a}}}\,.$$

Now, choosing $\hat{a}=1$, $\hat{b}=\dfrac{w}{1-q}$, we get the operator

$$\mathcal{D}_qf(x)=\frac{f(qx)-f(x)}{(q-1)x}\,. \tag{27}$$

3.3 The quadratic lattice: $\lambda=0\,,\quad \tau\neq 0$ in (19)

If $\lambda=0$ and $\tau\neq 0$, the polynomials p, r defined in (8) are both of degree one, given by

$$p(x)=-\frac{b}{a}x-\frac{d}{a}\,,\quad r(x)=2\frac{(bd-ae)}{a^2}x+\frac{(d^2-af)}{a^2}\,.$$

Recalling (7), it follows that

$$y_{\pm}(x)=-\frac{b}{a}x-\frac{d}{a}\pm\frac{\sqrt{2(bd-ae)x+(d^2-af)}}{a}\,. \tag{28}$$

Proposition 4. *The canonical divided-difference operator related to the quadratic lattices is the Wilson operator [1, 2], $\mathbb{D}_x=\mathcal{W}$ where*

$$\mathcal{W}f(x)=\frac{f\left((\sqrt{x}+\frac{i}{2})^2\right)-f\left((\sqrt{x}-\frac{i}{2})^2\right)}{2i\sqrt{x}}\,, \tag{29}$$

for arbitrary functions f. Hence, the operator $\mathcal{W}$ can be written in the form (9), with the polynomials p, r given by

$$p(x)=x-\frac{1}{4}\,,\quad r(x)=-x\,.$$

Proof. Combining (1) with (28), the operator (29) is recovered through the specialization

$$b = -a\,, \quad c = a\,, \quad d = e = \frac{a}{4}\,, \quad f = \frac{a}{16}\,. \tag{30}$$

and it follows the assertion on the polynomials p, r. □

Also, by using the values of (30) into (4), we get the conic with equation

$$y^2 - 2xy + x^2 + \frac{y}{2} + \frac{x}{2} + \frac{1}{16} = 0\,,$$

which is a parabola (we have $\lambda = 0$ and $\tau < 0$). The corresponding lattice, obtained via a parabola, is illustrated through Fig. 2.c) in [11, pp. 256]).

3.4 The q-quadratic lattice: $\lambda \neq 0\,, \;\; \tau \neq 0$ in (19)

If $\lambda \neq 0$ and $\tau \neq 0$, the polynomials p, r defined in (8) are of degree one and two, respectively, given as

$$p(x) = -\frac{b}{a}x - \frac{d}{a}\,, \quad r(x) = r(x) = \frac{\lambda}{a^2}\left(x + \frac{bd - ae}{\lambda}\right)^2 + \frac{\tau}{a\lambda}\,.$$

Recalling (7), it follows that

$$y_{\pm}(x) = -\frac{b}{a}x - \frac{d}{a} \pm \sqrt{\frac{\lambda}{a^2}\left(x + \frac{bd - ae}{\lambda}\right)^2 + \frac{\tau}{a\lambda}}\,. \tag{31}$$

Under some specializations, by considering the centred and symmetrised forms of the lattice, one can recover the Askey-Wilson operator [1, 2] (see also [7, Eq. (12.1.12)]), given by

$$\mathbb{D}_x f(x) = \frac{f(\frac{1}{2}(q^{1/2}z + q^{-1/2}z^{-1})) - f(\frac{1}{2}(q^{-1/2}z + q^{1/2}z^{-1}))}{\frac{1}{2}(q^{1/2} - q^{-1/2})(z - z^{-1})}\,. \tag{32}$$

Indeed, let us begin by defining the base $q = e^{2i\eta}$ and consider the projection map from the unit circle $\{z = e^{i\theta},\ \theta \in [-\pi, \pi[\}$ onto $[-1, 1]$ by

$$x = \frac{1}{2}(z + z^{-1})\,.$$

Note that we have

$$y_-(x) = \frac{1}{2}(q^{-1/2}z + q^{1/2}z^{-1})\,, \;\; y_+(x) = \frac{1}{2}(q^{1/2}z + q^{-1/2}z^{-1})\,. \tag{33}$$

Proposition 5. *The canonical divided-difference operator related to the q-quadratic lattices, in the symmetrical form, is the Askey-Wilson operator (32) [1, 2]. The operator (32) can be written in the form (9), with the polynomials p, r given by*

$$p(x) = \frac{(q^{1/2} + q^{-1/2})}{2}x\,, \quad r(x) = \frac{(q^{1/2} - q^{-1/2})}{4}(x^2 - 1)\,.$$

Proof. Combining (1) with (33), we have, after basic computations,

$$y_-(x) + y_+(x) = 2\cos(\eta)x = (q^{1/2} + q^{-1/2})x\,, \tag{34}$$

$$(y_-(x) - y_+(x))^2 = (q^{1/2} - q^{-1/2})(x^2 - 1)\,. \tag{35}$$

In the account of (10), that is, $y_-(x)+y_+(x) = 2p(x)$ and $(y_-(x)-y_+(x))^2 = 4r(x)$, there follow the polynomials p, r as stated.

The operator (32) is recovered through the specialization

$$a = c\,, \text{ arbitrary and non-zero}, \quad b = -a\cos(\eta)\,, \quad d = e = 0\,, \quad f = -a\sin^2(\eta)\,.$$

□

In the q-quadratic case, the conic is an hyperbola (when $\lambda > 0$ and $\tau < 0$), or an ellipse (when $\lambda < 0$ and $\tau < 0$, respectively). The corresponding lattice, obtained via an hyperbola or an ellipse, is illustrated through Figs. 1 and 2.a) in [11, pp. 256]).

4 Coalescence

The set of lattices previously defined can be classified through specifications on the constants in the parametrization formulae (13) and (14), that is, in

$$x(s) = \tilde{\kappa}_2 s^2 + \tilde{\kappa}_1 s + \tilde{\kappa}_0$$

and

$$x(s) = \kappa_1 q^s + \kappa_2 q^{-s} + \kappa_3\,,$$

respectively. Indeed, depending on the constants κ's and $\tilde{\kappa}$'s, we recover the four primary classes for the lattices $x(s)$:
(i) Linear lattices : $\tilde{\kappa}_2 = 0$ and $\tilde{\kappa}_1 \neq 0$ in (13);
(ii) q-linear lattices : $\kappa_2 = 0$ and $\kappa_1 \neq 0$ in (14);
(iii) Quadratic lattices : $\tilde{\kappa}_2 \neq 0$ in (13);
(iv) q-Quadratic lattices : $\kappa_1\kappa_2 \neq 0$ in (14).

The q-quadratic lattice, in its general non-symmetrical form, is the most general case and the other lattices can be found from this by limiting processes.

It turns out that each of the operators listed in (i)-(iii) of the previous section, specified in Sub-Sections 3.1–3.3, can be recovered as a particular case or as a limit case, up to a linear transformation of the variable, from one of the operators in the list. Details are given as follows.

Recall the polynomials p, r in (8): by writing $p(x) = p_1x + p_0$, $r(x) = r_2x^2 + r_1x + r_0$, we have

$$p_1 = -\frac{b}{a}\,, \quad p_0 = -\frac{d}{a}\,, \tag{36}$$

$$r_2 = \frac{b^2 - ac}{a^2}\,, \quad r_1 = 2\frac{(bd - ae)}{a^2}\,, \quad r_0 = \frac{d^2 - af}{a^2}\,. \tag{37}$$

4.1 From q-quadratic to quadratic

Taking limits $q \to 1$ in (16) as well as in (17) we get $p_1 = 1$ and $r_2 = 0$. In the account of (37), $r_2 = 0$ yields $b^2 - ac = 0$. Furthermore, in the account of (37), note that $\tau \neq 0$ in (19) if, and only if, $r_0 r_2 - (r_1/2)^2 \neq 0$. As we have $r_2 = 0$, then $\tau \neq 0$ if, and only if, $r_1 \neq 0$, which must hold upon a suitable choice of κ_3. Thus, we get the quadratic case: $\lambda = 0$ and $\tau \neq 0$ (cf. Sub-Section 3.3).

4.2 From q-quadratic to q-linear

Recalling the remark 5, let us take the operator $\mathcal{D}_q$ defined by (27),

$$\mathcal{D}_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}\,.$$

We begin by fixing the parameter $q \neq 1$. Taking limits $\kappa_2 \to 0$, $\kappa_3 \to 0$, and fixing $q \neq 1$ in (14) we get $r_2 \neq 0$, $r_1 = 0$, $r_0 = 0$ in (17)-(18), that, in the account of (37), yields $b^2 - ac \neq 0$, $bd - ae = 0$, $d^2 - af = 0$. Thus, we get the q-linear case: $\lambda \neq 0$ and $\tau = 0$ (cf. Sub-Section 3.2).

Note that, in such a situation, the operator $\mathcal{D}_q$ obtained via the above limiting process is given by

$$\mathcal{D}_q f(x(s)) = \frac{f(\kappa_1 q^{s+1/2}) - f(\kappa_1 q^{s-1/2})}{\kappa_1 (q^{s+1/2} - q^{s-1/2})}\,,$$

which can be easily written as (27) trough the change of variable $x(s) = \kappa_1 q^{s-1/2}$.

4.3 From q-linear to linear

The linear case follows easily by taking limits $q \to 1$ in (25). Indeed, we get the coefficients of the polynomials p, r as given in Proposition 2, thus, in the account of (37), we have $\lambda = 0$ and $\tau = 0$ (cf. Sub-Section 3.1).

5 Divided-difference operator calculus

Recall the operator $\mathbb{D}_x$ in its general form given by (1), together with the corresponding conic (4) and the polynomials p, r defined in (8). In the sequel we shall take $\Delta_y = y_+ - y_-$. From (7), there follows

$$\Delta_y = 2\sqrt{r}\,. \tag{38}$$

In order to deduce further properties, let us now introduce the operators $\mathbb{E}_x^+$ and $\mathbb{E}_x^-$ (see [10]), acting on arbitrary functions f, as

$$\mathbb{E}^{\pm} f(x) = f(y_{\pm}(x))\,.$$

With this notation, (1) is also given by

$$\mathbb{D}_x f(x) = \frac{\mathbb{E}_x^+ f - \mathbb{E}_x^- f}{\mathbb{E}_x^+ x - \mathbb{E}_x^- x}\,.$$

The companion operator of $\mathbb{D}$ is then defined as (see [10])

$$\mathbb{M}_x f(x) = \frac{\mathbb{E}_x^+ f(x) + \mathbb{E}_x^- f(x)}{2} \,. \tag{39}$$

Note that $\mathbb{M}_x f$ is a polynomial whenever f is a polynomial. Furthermore, if $\deg(f) = n$, then $\deg(\mathbb{M}_x f) = n$.

The operators $\mathbb{D}_x$ and $\mathbb{M}_x$ satisfy the product and quotient rules listed below (see [10]):

$$\mathbb{D}_x(fg) = \mathbb{D}_x f\, \mathbb{M}_x g + \mathbb{M}_x f\, \mathbb{D}_x g \,, \tag{40}$$

$$\mathbb{D}_x(f/g) = \frac{\mathbb{D}_x f\, \mathbb{M}_x g - \mathbb{D}_x g\, \mathbb{M}_x f}{\mathbb{E}_x^- f\, \mathbb{E}_x^+ f} \,, \tag{41}$$

$$\mathbb{M}_x(fg) = \mathbb{M}_x f\, \mathbb{M}_x g + \frac{\Delta_y^2}{4}\, \mathbb{D}_x f\, \mathbb{D}_x g \,, \tag{42}$$

$$\mathbb{M}_x(f/g) = \frac{\mathbb{E}_x^- f\, \mathbb{E}_x^+ g + \mathbb{E}_x^+ f\, \mathbb{E}_x^- g}{2\mathbb{E}_x^- g\, \mathbb{E}_x^+ g} \,. \tag{43}$$

Eq. (40) has the equivalent forms:

$$\mathbb{D}_x(gf) = \mathbb{D}_x g\, \mathbb{E}_x^- f + \mathbb{D}_x f\, \mathbb{E}_x^+ g \,,$$
$$\mathbb{D}_x(gf) = \mathbb{D}_x g\, \mathbb{E}_x^+ f + \mathbb{D}_x f\, \mathbb{E}_x^- g \,.$$

Also, one has two equivalent forms for (41):

$$\mathbb{D}_x(g/f) = \frac{\mathbb{D}_x g\, \mathbb{E}_x^- f - \mathbb{D}_x f\, \mathbb{E}_x^- g}{\mathbb{E}_x^- f\, \mathbb{E}_x^+ f} \,,$$
$$\mathbb{D}_x(g/f) = \frac{\mathbb{D}_x g \mathbb{E}_x^+ f - \mathbb{D}_x f \mathbb{E}_x^+ g}{\mathbb{E}_x^- f\, \mathbb{E}_x^+ f} \,.$$

The operators $\mathbb{D}_x$ and $\mathbb{M}_x$ also satisfy the product rules II (see [5, Eq. 15] and [4])

$$\mathbb{D}_x\, \mathbb{M}_x = \alpha \mathbb{M}_x\, \mathbb{D}_x + U_1\, \mathbb{D}_x^2 \,, \quad \mathbb{M}_x^2 = U_1\, \mathbb{M}_x\, \mathbb{D}_x + \alpha \frac{\Delta_y^2}{4}\, \mathbb{D}_x^2 + \mathbb{I} \,, \tag{44}$$

where $\mathbb{I}$ is the identity operator, $\mathbb{I} f(x) = f(x)$, α is defined in terms of the conic (4) as $\alpha = -\frac{b}{\sqrt{ac}}$, and

$$U_1(x) = (p_1^2 - 1)x + \frac{r_1}{2} \,, \tag{45}$$

with p_1 and r_1 defined in (15) in the quadratic case, or in (16)-(18) in the q-quadratic case.

5.1 The explicit parameterizations revisited

Let us recall the conic (4), $ay^2 + 2bxy + cx^2 + 2dy + 2ex + f = 0\,, \quad a \neq 0$, as well as its two y-roots, satisfying (5) and (6). Assuming $c \neq 0$ in (4), then one defines the inverse functions of y_- and y_+, denoted by y_-^{-1} and y_+^{-1}, respectively, such that

$$y_-^{-1}(y_-(x)) = x \,, \quad y_+^{-1}(y_+(x)) = x \,,$$

together with the corresponding operators

$$\left(\mathbb{E}_x^-\right)^{-1} f(x) = f\left(y_-^{-1}(x)\right) \,, \quad \left(\mathbb{E}_x^+\right)^{-1} f(x) = f\left(y_+^{-1}(x)\right) \,. \tag{46}$$

Let us also define the operators $\mathbb{E} = \left(\mathbb{E}_x^-\right)^{-1} \mathbb{E}_x^+$, $\mathbb{E}^{-1} = \left(\mathbb{E}_x^+\right)^{-1} \mathbb{E}_x^-$ by (see [10])

$$\mathbb{E} f(x) = f\left(y_+(y_-^{-1}(x))\right) \,, \quad \mathbb{E}^{-1} f(x) = f\left(y_-(y_+^{-1}(x))\right) \,. \tag{47}$$

In order to deduce the parameterizations of the quadratic and q-quadratic cases, we first present the following lemma. The results are gathered in [10], but here we detail its proof.

Lemma 1. *Recalling the conic (4) and the operators previously defined, the following equalities hold:*

$$\mathbb{E}x + x = \frac{-2(by_-^{-1}(x) + d)}{a} \,, \tag{48}$$

$$\mathbb{E}^{-1}x + x = \frac{-2(by_+^{-1}(x) + d)}{a} \,, \tag{49}$$

$$y_-^{-1}(x) + y_+^{-1}(x) = \frac{-2(bx + e)}{c} \,, \tag{50}$$

$$\mathbb{E}x + \mathbb{E}^{-1}x = 2\left(\frac{2b^2}{ac} - 1\right) x + 4\left(\frac{be - cd}{ac}\right) \,. \tag{51}$$

Proof. Equations (48) and (49) follow by taking $x = y_-^{-1}(X)$ and $x = y_+^{-1}(X)$, respectively, in (5), $y_-(x) + y_+(x) = -2(bx + d)/a$.

To deduce (50) we start by evaluating (6) at $y_-^{-1}(x)$ as well as at $y_+^{-1}(x)$, thus getting

$$x\, y_+(y_-^{-1}(x)) = \frac{c(y_-^{-1}(x))^2 + 2ey_-^{-1}(x) + f}{a} \,, \tag{52}$$

$$x\, y_-(y_+^{-1}(x)) = \frac{c(y_+^{-1}(x))^2 + 2ey_+^{-1}(x) + f}{a} \,. \tag{53}$$

Subtracting (53) to (52) yields

$$x\left(y_+(y_-^{-1}(x)) - y_-(y_+^{-1}(x))\right) = \frac{c\left((y_-^{-1}(x))^2 - (y_+^{-1}(x))^2\right) + 2e\left(y_-^{-1}(x) - y_+^{-1}(x)\right)}{a} \,.$$

Thus, we have

$$\mathbb{E}x + x - (\mathbb{E}^{-1}x + x) = \frac{\left(y_-^{-1}(x) - y_+^{-1}(x)\right)}{xa}\left(c\left(y_-^{-1}(x) + y_+^{-1}(x)\right) + 2e\right) \,. \tag{54}$$

Using (48) and (49) in (54) gives us, after simplifications, equation (50).

Equation (51) follows from the sum of (48) with (49), and using (50). □

Applying $\mathbb{E}^n$ to (51) we obtain the difference equation

$$\mathbb{E}^{n+1}x + \mathbb{E}^{n-1}x = 2\left(\frac{2b^2}{ac} - 1\right)\mathbb{E}^n x + 4\left(\frac{be - cd}{ac}\right) \,. \tag{55}$$

The solution of the equation (55) leads us to the form of the parameterizations already discussed in Sub-Section 2.1(see [10, pp. 264] and [13]). Here, it is given the detailed proof in what follows.

Theorem 1. *Let q satisfy*

$$q + q^{-1} = 2\left(\frac{2b^2}{ac} - 1\right) . \tag{56}$$

The solution of the difference equation (55) is given by

$$\mathbb{E}^n x = k_1 q^n + k_2 q^{-n} + \frac{cd - be}{b^2 - ac} , \quad \text{if } q \neq 1 \tag{57}$$

or

$$\mathbb{E}^n x = k_1 + k_2 n + \frac{2(be - cd)}{ac} n^2 , \quad \text{if } q = 1 , \tag{58}$$

where k_1, k_2 are constants.

Proof. Recall that the solution of a difference equation such as (55), say,

$$X_{n+1} - \xi X_n + X_{n-1} = 4\left(\frac{be - cd}{ac}\right) , \quad \xi = 2\left(\frac{2b^2}{ac} - 1\right) , \tag{59}$$

can be written as $X_n = X_{h,n} + X_p$, with $X_{h,n}$ the solution of the homogeneous equation

$$X_{n+1} - \xi X_n + X_{n-1} = 0 \tag{60}$$

and X_p a particular solution of the complete equation (59). Also, denoting by ξ_1, ξ_2 the two roots of the so-called associated characteristic equation of (60),

$$x^2 - \xi x + 1 = 0 , \tag{61}$$

the solution of (60) is given by (see [12])

$$X_{h,n} = \begin{cases} k_1 \xi_1^n + k_2 \xi_2^n & \text{if } \xi_1 \neq \xi_2 , \\ k_1 \xi_1^n + k_2 n \xi_1^n & \text{if } \xi_1 = \xi_2 . \end{cases}$$

Note that the roots of $x^2 - \xi x + 1 = 0$ are $q_\pm := \dfrac{\xi \pm \sqrt{\xi^2 - 4}}{2}$. Hence, when $\xi^2 - 4 \neq 0$, we have two different roots of the quadratic equation, which satisfy indeed $q_- = (q_+)^{-1}$, and $q_- + q_+ = \xi$. Thus, we have the parameter q, say $q = q_+$, defined as in (56). If $\xi^2 - 4 = 0$, then $\xi = 2$, which implies the double root of the quadratic equation being $q := q_- = q_+ = 1$, thus, also defined as in (56).

Finally, we get (57) in the account that $\tilde{\lambda} := \dfrac{cd - be}{b^2 - ac}$ is a particular solution of the complete equation (59) in the case of two different roots of (61), and we get (58) in the account that $\tilde{\lambda} := \frac{2(be-cd)}{ac} n^2$ is a particular solution of the complete equation (59) in the case of a double root of (61). □

5.2 The divided-difference operators as exact lowering operators

We now give the analogues of the well-known formulae for the continuous case $\dfrac{d}{dx} x^n = n x^{n-1}$, as proposed by [16]. Further details are given in the more recent approach [18].

Let $\{l_n(x;a)\}_{n=0}^{+\infty}\}$ be a polynomial basis of $L^2(w(x)\mathbb{D}x, G)$, where l_n is a polynomial of exact degree n and the support is $G = \{\mathbb{E}^{+k}x :\ k \in 2\mathbb{Z}\}$ or, if finite, $G = \{x_0, \dots, x_{n_0}\}$, and a denotes the set of parameters characterising the lattice. The general requirements for the polynomial basis are:
(i) $l_n(x)$ is of precise degree n in x,
(ii) $\mathbb{D}_x$ is an exact lowering operator in this basis, that is, $\mathbb{D}_x l_n(x) = c_n l_{n-1}(x),\ n \geq 1$, where $c_n = c_n(\breve{a})$ is a constant with respect to x, depending on a set of parameters $\breve{a} := \{a_1, a_2, \dots, a_{m_0}\}$, characterizing the lattice.

A general solution of the above requirements is the polynomial defined by (see [18, Sec. 2])

$$l_n(x;\breve{a}) = g_n(\breve{a}) \prod_{j=0}^{n-1} \left(x - \left(\mathbb{E}_x^{+}\right)^{2j} x(\breve{a}) \right) ,$$

where $x(\breve{a})$ denotes the so-called basal point, parameterized by $\breve{a}$, and $g_n(\breve{a}) \neq 0$.

We have the following.

1. In the q-quadratic lattice $x(s) = \kappa_1 q^s + \kappa_2 q^{-s} + \kappa_3$, with $q \neq 1$ and $\kappa_1 > 0$, $\kappa_2 > 0$, the basis is

$$l_n(x(s)) = g_n \left(\frac{q^{-\frac{n}{2}+s+\frac{1}{4}}\sqrt{\kappa_1}}{\sqrt{\kappa_2}}; q \right)_n \left(\frac{q^{-\frac{n}{2}-s+\frac{1}{4}}\sqrt{\kappa_2}}{\sqrt{\kappa_1}}; q \right)_n , \quad n \geq 1 , \tag{62}$$

with

$$g_n = g_n(\kappa_1, \kappa_2, q) = \left(-\frac{\kappa_1^{3/2} q^{1/4}}{\sqrt{\kappa_2}} \right)^n .$$

The divided-difference operator satisfies $\mathbb{D}_x l_n(x(s)) = c_n l_{n-1}(x(s))$, $n \geq 1$, that is,

$$\mathbb{D}_x l_n(x(s)) = \frac{l_n(x(s+1/2)) - l_n(x(s-1/2))}{x(s+1/2) - x(s-1/2)} = c_n l_{n-1}(x(s))$$

with

$$c_n = c_n(\kappa_1, \kappa_2, q) = \frac{\kappa_1 q^{\frac{1-n}{2}} [n]_q}{\kappa_2} .$$

Here, it is used the Pochhammer symbol, given by

$$(a;q)_0 = 1 , \quad (a;q)_n = \prod_{j=0}^{n-1} (1 - a q^j) , \quad n = 1, 2, \dots ,$$

and the number $[z]_q$ defined by

$$[z]_q = \frac{q^z - 1}{q - 1} .$$

2. In the quadratic lattice $x(s) = \tilde{\kappa}_2 s^2 + \tilde{\kappa}_1 s + \tilde{\kappa}_0$, with $\tilde{\kappa}_2 \neq 0$, the basis is

$$l_n(x(s)) = 4^{-n} (-\tilde{\kappa}_2)^n \left(-\frac{\tilde{\kappa}_1}{\tilde{\kappa}_2} - 2s + \frac{1}{2} \right)_n \left(\frac{\tilde{\kappa}_1}{\tilde{\kappa}_2} + 2s + \frac{1}{2} \right)_n , \quad n \geq 1 . \tag{63}$$

The divided-difference operator satisfies $\mathbb{D}_x l_n(x(s)) = c_n l_{n-1}(x(s))$, $n \geq 1$, that is,

$$\mathbb{D}_x l_n(x(s)) = \frac{l_n(x(s+1/2)) - l_n(x(s-1/2))}{x(s+1/2) - x(s-1/2)} = c_n l_{n-1}(x(s))$$

with

$$c_n = n\,.$$

Here, it is used the Pochhammer symbol $(A)_n = A(A+1)\cdots(A+n-1)$.
3. In the q-linear lattice, the basis is

$$l_n(x) = (\breve{a}x;q)_n = \prod_{j=0}^{n-1}(1-\breve{a}q^j x)\,, \quad n \geq 1\,. \tag{64}$$

The divided-difference operator, taken in its canonical form as the $\mathcal{D}_q$ operator given in (27), satisfies $\mathcal{D}_q l_n(x) = c_n l_{n-1}(x)$, $n \geq 1$, that is,

$$\mathcal{D}_q l_n(x) = \frac{l_n(qx) - l_n(x)}{(q-1)x} = c_n l_{n-1}(x)$$

with

$$c_n = -\frac{1-\breve{a}q^n}{q-1}\,.$$

4. In the linear lattice, the basis is

$$l_n(x) = \prod_{j=0}^{n-1}(x-j) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}\,, \quad n \geq 1\,, \tag{65}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The divided-difference operator, taken in its canonical form as the forward difference operator $\Delta f(x) = f(x+1) - f(x)$, satisfies

$$\Delta l_n(x) = l_n(x+1) - l_n(x) = c_n l_{n-1}(x)$$

with

$$c_n = n\,.$$

5.3 Integrals

Let the lattice points be denoted by $G[x] = \{x(s) : s \in \mathbb{Z}\}$, with the point $x(0)$ as the basal point, and let us denote the dual lattice by $\tilde{G}[x] = \{x(s+1/2) : s \in \mathbb{Z}\}$. The $\mathbb{D}$-integral of a function defined on the x-lattice, $f : G[x] \to \mathbb{C}$ with basal point $x_0 = x(0)$, is defined by the Riemmann sum over the lattice points (see [18, Sec. 2])

$$I[f](x_0) = \int_G f(x(s))\mathbb{D}x(s) := \sum_{s\in\mathbb{Z}^*} f(x(s))(y_+(x(s)) - y_-(x(s)))\,. \tag{66}$$

Recalling that, in the quadratic case, $y_+(x(s)) = x(s+1/2)$, $y_-(x(s)) = x(s-1/2)$, and also recalling the notation $x_s := x(s)$), then we can write

$$I[f](x_0) = \sum_{s\in\mathbb{Z}^*} f(x(s))((x(s+1/2)) - (x(s-1/2))) = \sum_{s\in\mathbb{Z}^*} f(x_s)\Delta_y(x_s)\,.$$

Here, $\mathbb{Z}^*$ is a finite subset of $\mathbb{Z}$, namely $\{0,1,\ldots,n_0\}$, or $\mathbb{Z}_{\geq 0}$, or $\mathbb{Z}$.

Recalling that $\mathbb{E}_x^{\pm} f(x(s)) = f(x(s \pm 1/2))$, for $x(s) \in G[x]$, the following properties follow from (66) (see [18]):
1. an analog of the fundamental theorem of calculus:

$$\int_{x_0 \leq x_s \leq x_{n_0}} \mathbb{D}_x f(x(s)) \mathbb{D}x(s) = f(\mathbb{E}_x^+ x_{n_0}) - f(\mathbb{E}_x^- x_0) \,. \tag{67}$$

2. an analog of integration by parts for two functions $f(x), g(x)$:

$$\int_{x_0 \leq x_s \leq x_{n_0}} f(x(s)) \mathbb{D}_x g(x(s)) \mathbb{D}x(s) = f(\mathbb{E}_x^{+\,2} x_{n_0}) g(\mathbb{E}_x^+ x_{n_0}) - f(x_0) g(\mathbb{E}_x^- x_0)$$
$$- \int_{x_0 \leq x_s \leq x_{n_0}} \mathbb{D}_x f(\mathbb{E}_x^+ x(s)) g(\mathbb{E}_x^+ x(s)) \mathbb{D}\left(\mathbb{E}_x^+ x(s)\right) \,. \tag{68}$$

Remark 6. *The definition (66) reduces to the ususal definition of the difference integral and the Thomae-Jackson q-integrals in the canonical forms of the linear and q-linear lattices, respectively [8, 17].*

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