On the second order holonomic equation for Sobolev-type orthogonal polynomials

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Abstract

It is presented a general approach to the study of orthogonal polynomials related to Sobolev inner products which are defined in terms of divided-difference operators having the fundamental property of leaving a polynomial of degree $n-1$ when applied to a polynomial of degree n. This paper gives analytic properties for the orthogonal polynomials, including the second order holonomic difference equation satisfied by them.

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1 Introduction

The origins of Sobolev-type orthogonal polynomials, also known as Sobolev discrete orthogonal polynomials (see [19]), can be traced back to the study of inner products such as [6, 7],

$$\langle f, g\rangle_S = \int_{\mathbb{R}} f(x)g(x)d\psi(x) + \lambda(\mathbb{D}f)(c)(\mathbb{D}g)(c)\,, \quad \lambda > 0\,, \tag{1}$$

where ψ is some distribution function with infinite support on $\mathbb{R}$, $\mathbb{D}$ is an operator involving differences, and c is some real constant fulfilling some conditions regarding the support of ψ. In [6, 7], it was considered $\mathbb{D} = \Delta$, being Δ the forward difference operator, $\Delta f(c) = \frac{f(c+h)-f(c)}{h}$, $h \in \mathbb{R}$, with c such that ψ has no points of increase in the interval $]c, c+1[$. More recently, in [12], it was considered (1) $\mathbb{D}$ replaced by the q-difference operator, $D_q(x) = \frac{f(qx)-f(x)}{(q-1)x}$; see also, [16], on new properties of (1) with with ψ the Poisson distribution and $\mathbb{D} = \Delta$. Sequences of orthogonal polynomials with respect to inner products such as (1), are nowadays commonly called Sobolev-type orthogonal polynomials or discrete Sobolev orthogonal polynomials. Note that the presence of derivatives produces significant changes, most of the nice properties of the standard orthogonal polynomials (e.g, the three-term recurrence relation, Christoffel-Darboux formula, etc) no longer hold for Sobolev

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orthogonal polynomials. This is due to the fact that the Sobolev inner products, $\langle\cdot,\cdot\rangle_S$, are non-standard, that is, $\langle xf,g\rangle_S \neq \langle f,xg\rangle_S$.

Sobolev orthogonal polynomials find many applications in several areas of Mathematics. They were primarily introduced in the framework of the least square approximation to a function and, simultaneously, to its derivatives in [1]. Since the 1980s, the research in this field has been widely intensive, falling into several directions [19, 20]. The analysis of algebraic, differential, and asymptotic properties of Sobolev orthogonal polynomials are nowadays broadly active topics. We refer the interested reader to the survey [20].

In the present paper we shall consider inner products such as (cf. Section 3)

$$\langle f,g\rangle_\lambda = \langle \mathbf{u}, fg\rangle + \lambda(\mathbb{D}f)(c)(\mathbb{D}g)(c)\,, \quad \lambda \geq 0\,, \tag{2}$$

where $\mathbf{u}$ is a linear functional defined in the linear space of polynomials, c is some real or complex number, and $\mathbb{D}$ is a general divided-difference operator (given in [17, Eq.(1.1)]), having the fundamental property of leaving a polynomial of degree $n-1$ when applied to a polynomial of degree n. Such operators $\mathbb{D}$ are related to the so-called special non-uniform lattices (snul) [17, 18]. Essentially, there are four primary classes of divided difference operators [17, Eq.(1.1)] (cf. (3)): the forward difference operator ([25, 24]), the q-difference operator [15]; the Wilson operator [4], the Askey-Wilson operator [3, 4]. Further details on the hierarchy of operators and the main properties of the divided-difference calculus to be used in the sequel are given in Sub-Section 2.1.

The purpose of the present paper is twofold: on the one hand, to provide un updated analysis of the families of orthogonal polynomials with respect to linear functionals $\mathbf{u}$ members of the semi-classical class (cf. Section 2), specifically, on the second-order holonomic equation - a second-order divided-difference equation with polynomial coefficients, for the corresponding orthogonal polynomials; on the other hand, to deduce (general) identities for the Sobolev polynomials related to the inner product (2). Several key formulas, commonly used in the literature on Sobolev orthogonal polynomials, will be presented: connection formulae, expressing the orthogonal polynomials related to (2), say S_n, $n=0,1,\dots$, in terms of the orthogonal polynomials related to $\mathbf{u}$, say P_n, $n=0,1,\dots$, and a three-term recurrence relation (with rational coefficients) for S_n, $n=0,1,\dots$. Indeed, the sequence $\{S_n\}_{n\geq 0}$ inherit properties from $\{P_n\}_{n\geq 0}$: when $\{P_n\}_{n\geq 0}$ is a member of the so-called semi-classical class on (snul), then we also derive a second order holonomic difference equation for $\{S_n\}_{n\geq 0}$. To the best of the author's knowledge, the derivation of the second order holonomic difference equation for $\{S_n\}_{n\geq 0}$ is novel. For the semi-classical case on snul, we can find the second order holonomic difference equation in [17, Sec. 6] and [21, Sec. 4.2] using different aproaches/techniques than the ones in the present paper (here, we take advantage of some of difference systems deduced in [8]). Also, let us note that standard techniques, such as the ladder-operator approach (see, for for instance, [2] or [16, Sec. 4]), do not hold for the snul case.

The remainder of the paper is organized as follows. In Section 2 we give the background on the divided-difference calculus on special non-uniform lattices (snul), as well as on the corresponding sequences of orthogonal polynomials; the second order holonomic equation for semi-classical orthogonal polynomials on snul is deduced in 2.3. In Section 3 we introduce the Sobolev-type orthogonal polynomials related to (2): the connection formulae are deduced in Sub-Section 3.1; the three-term recurrence relation is deduced in 3.2; the second order holonomic equation for Sobolev-type orthogonal polynomials is deduce in sub-Section 3.3. Final remarks are presented in Section 4.

2 The second order holonomic difference equation for semi-classical orthogonal polynomials

2.1 Special non-uniform lattices (snul), and the divided-difference calculus

We consider the divided difference operator $\mathbb{D}$ given in [17, Eq.(1.1)], with the property that $\mathbb{D}$ leaves a polynomial of degree $n-1$ when applied to a polynomial of degree n. The operator $\mathbb{D}$, defined on the space of arbitrary functions, is given by

$$(\mathbb{D}f)(x) = \frac{f(y_2(x)) - f(y_1(x))}{y_2(x) - y_1(x)}\,, \tag{3}$$

where y_1 and y_2 are functions that satisfy

$$y_1(x) + y_2(x) = \text{polynomial of degree } 1\,, \tag{4}$$
$$(y_1(x))^2 + y_1(x)y_2(x) + (y_2(x))^2 = \text{polynomial of degree } 2\,, \tag{5}$$

the later condition being equivalent to $y_1(x)y_2(x) =$ polynomial of degree less or equal than 2. Conditions (4)–(5) define y_1 and y_2 as the two y-roots of a quadratic equation

$$\hat{a}y^2 + 2\hat{b}xy + \hat{c}x^2 + 2\hat{d}y + 2\hat{e}x + \hat{f} = 0\,, \quad \hat{a} \neq 0\,. \tag{6}$$

Identities involving y_1 and y_2, following from the fact that y_1, y_2 are the y-roots of (6):

$$y_1(x) + y_2(x) = -2(\hat{b}x + \hat{d})/\hat{a}\,, \tag{7}$$
$$y_1(x)y_2(x) = (\hat{c}x^2 + 2\hat{e}x + \hat{f})/\hat{a}\,, \tag{8}$$
$$(y_2(x) - y_1(x))^2 = 4\left((\hat{b}^2 - \hat{a}\hat{c})x^2 + 2(\hat{b}\hat{d} - \hat{a}\hat{e})x + \hat{d}^2 - \hat{a}\hat{f}\right)/\hat{a}^2\,, \tag{9}$$
$$y_1(x) = p(x) - \sqrt{r(x)}\,, \quad y_2(x) = p(x) + \sqrt{r(x)}\,, \tag{10}$$

with p, r polynomials given by

$$p(x) = -\frac{\hat{b}x + \hat{d}}{\hat{a}}\,, \quad r(x) = \frac{(\hat{b}^2 - \hat{a}\hat{c})}{\hat{a}^2}x^2 + \frac{2(\hat{b}\hat{d} - \hat{a}\hat{e})}{\hat{a}^2}x + \frac{\hat{d}^2 - \hat{a}\hat{f}}{\hat{a}^2}\,. \tag{11}$$

When $\hat{b}^2 - \hat{a}\hat{c} \neq 0$, then we have

$$r(x) = \frac{\hat{\lambda}}{\hat{a}^2}\left(x + \frac{\hat{b}\hat{d} - \hat{a}\hat{e}}{\hat{\lambda}}\right)^2 + \frac{\tau}{\hat{a}\hat{\lambda}}\,,$$

where $\hat{\lambda} = \hat{b}^2 - \hat{a}\hat{c}$, $\tau = \left((\hat{b}^2 - \hat{a}\hat{c})(\hat{d}^2 - \hat{a}\hat{f}) - (\hat{b}\hat{d} - \hat{a}\hat{e})^2\right)/\hat{a}$.

There are four primary classes of divided difference operators (3) and related lattices. Such a classification is done according to the two parameters λ and τ defined above, assuming $\hat{a}\hat{c} \neq 0$:
(i) $\hat{\lambda} = \tau = 0$ - the linear lattice, related to the forward difference operator [25, Chapter 2, Section 12];
(ii) $\hat{\lambda} \neq 0$, $\tau = 0$ - the q-linear lattice, related to the q-difference operator [15];

(iii) $\hat{\lambda} = 0,\ \tau \neq 0$ - the quadratic lattice, related to the Wilson operator [4];
(iv) $\hat{\lambda}\tau \neq 0$ - the q-quadratic lattice, related to the Askey-Wilson operator [4].

Each of the operators in (i)—(iv) is an extension of the preceding one, which is recovered as a particular case or as a limit case, up to a linear transformation of the variable.

For the quadratic and $q-$quadratic lattices, there is the parametrization of the conic (6), say $x = x(s), y = y(s)$, such that

$$y_1(x) = x(s - 1/2)\,, \quad y_2(x) = x(s + 1/2)\,,$$

given as [5, 17, 18]

$$x(s) = \begin{cases} \kappa_1 q^s + \kappa_2 q^{-s} + \kappa_3\,, & \text{if} \quad -\dfrac{\hat{b}}{\sqrt{\hat{a}\hat{c}}} \neq \pm 1\,, \\ \kappa_4 s^2 + \kappa_5 s + \kappa_6\,, & \text{if} \quad -\dfrac{\hat{b}}{\sqrt{\hat{a}\hat{c}}} = 1 \end{cases}$$

for some appropriate constants κ's, and q defined through

$$\frac{q^{1/2} + q^{-1/2}}{2} = -\frac{\hat{b}}{\sqrt{\hat{a}\hat{c}}}\,.$$

Note that, in this case, we have the divided-difference operator (3) given as

$$\mathbb{D} f(x(s)) = \frac{f(x(s+1/2)) - f(x(s-1/2))}{x(s+1/2) - x(s-1/2)}\,,$$

and the polynomials p, r are recovered via

$$x(s+1/2) + x(s-1/2) = 2p(x(s))\,, \quad (x(s+1/2) - x(s-1/2))^2 = 4r(x(s))\,.$$

In the present paper we will consider the general case, $\hat{\lambda}\tau \neq 0$, and we shall operate with the divided difference operator $\mathbb{D}$ given in its general form (3), with y_1, y_2 given in (10), defined in terms of the polynomials p and r in (11). Throughout the paper we shall use the notation $\Delta_y = y_2 - y_1$. From (10), it follows that

$$\Delta_y = 2\sqrt{r}\,. \tag{12}$$

By defining the operators $\mathbb{E}_1$ and $\mathbb{E}_2$ (see [17]), acting on arbitrary functions f as

$$(\mathbb{E}_1 f)(x) = f(y_1(x))\,, \quad (\mathbb{E}_2 f)(x) = f(y_2(x))\,,$$

then the formula (3) is given by

$$(\mathbb{D} f)(x) = \frac{(\mathbb{E}_2 f)(x) - (\mathbb{E}_1 f)(x)}{(\mathbb{E}_2 x)(x) - (\mathbb{E}_1 x)(x)}\,.$$

The companion operator of $\mathbb{D}$ is defined as (see [17])

$$(\mathbb{M} f)(x) = \frac{(\mathbb{E}_1 f)(x) + (\mathbb{E}_2 f)(x)}{2}\,.$$

The operators $\mathbb{D}$ and $\mathbb{M}$ satisfy the product and quotient rules listed below (see [17]):

$$\mathbb{D}(gf) = \mathbb{D}g\,\mathbb{M}f + \mathbb{M}g\,\mathbb{D}f\,, \tag{13}$$

$$\mathbb{D}(g/f) = \frac{\mathbb{D}g\,\mathbb{M}f - \mathbb{D}f\,\mathbb{M}g}{\mathbb{E}_1 f\,\mathbb{E}_2 f}\,, \tag{14}$$

$$\mathbb{M}(gf) = \mathbb{M}g\,\mathbb{M}f + \frac{\Delta_y^2}{4}\,\mathbb{D}g\,\mathbb{D}f\,, \tag{15}$$

$$\mathbb{M}(g/f) = \frac{\mathbb{E}_1 g\,\mathbb{E}_2 f + \mathbb{E}_2 g\,\mathbb{E}_1 f}{2\mathbb{E}_1 f\,\mathbb{E}_2 f}\,. \tag{16}$$

The operators $\mathbb{D}$ and $\mathbb{M}$ also satisfy the product rules II (see [14, Eq. 16])

$$\mathbb{D}\,\mathbb{M} = \alpha\,\mathbb{M}\,\mathbb{D} + U_1\,\mathbb{D}^2\,, \quad \mathbb{M}^2 = U_1\,\mathbb{M}\,\mathbb{D} + \alpha\,U_2\,\mathbb{D}^2 + \mathbb{I}\,, \tag{17}$$

where $\mathbb{I}$ is the identity operator, α is the constant

$$\alpha = -\hat{b}/\sqrt{\hat{a}\hat{c}}\,, \tag{18}$$

where $\hat{a}, \hat{b}, \hat{c}$ are the coefficients in (6), and

$$U_2(x) = \frac{\Delta_y^2}{4}\,, \quad U_1(x) = r_2 x + \frac{r_1}{2}\,, \tag{19}$$

with the notation $r(x) = r_2x^2 + r_1x + r_0$ for the polynomial r in (11) (cf. also (12) and (15)).

Note that $\mathbb{M}f$ is a polynomial whenever f is a polynomial. Furthermore, if $\deg(f) = n$, then $\deg(\mathbb{M}f) = n$ [9, Lemma 1]. Let us emphasize that, throughout the text, unless stated in contrary, by a polynomial we mean a polynomial in the variable x, that is, an element in $\mathbb{C}[x]$.

2.2 Orthogonal polynomials on snul

We shall consider formal orthogonal polynomials related to a (formal) Stieltjes function defined by

$$S(x) = \sum_{n=0}^{+\infty} \frac{u_n}{x^{n+1}} \tag{20}$$

where $(u_n)_{n\geq 0}$, the sequence of moments, satisfies the regularity condition

$$\det\left[u_{i+j}\right]_{i,j=0}^{n} \neq 0\,, \quad n = 0, 1, 2, \ldots\,, \tag{21}$$

and, without loss of generality, $u_0 = 1$. It is well-known [11, Th. 4.4] that (21) is a necessary and sufficient condition for the existence of a sequence of polynomials, say $\{P_n\}_{n\geq 0}$, orthogonal with respect to the linear functional $\mathbf{u} : \mathbb{C}[x] \longrightarrow \mathbb{C}$, defined by $\langle \mathbf{u}, x^n \rangle = u_n,\ n = 0, 1, 2, \ldots$, that is, the system $\{P_n\}_{n\geq 0}$ satisfies

$$\langle \mathbf{u}, P_n P_m \rangle = h_n \delta_{n,m}\,, \quad n, m = 0, 1, \ldots\,,$$

where $h_n \neq 0$ and $\delta_{n,m}$ is the Kronecker's delta.

If the moments satisfy the condition $\det((u_{i+j})_{i,j=0}^n) > 0,\ n \geq 0$ (that is, u is positive-definite), then u has an integral representation in terms of a positive Borel measure, μ, supported on an infinite point set, I, such that

$$\langle \mathbf{u}, x^n \rangle = \int_I x^n \, d\mu(x)\,, \quad n \geq 0\,.$$

In this case, S is the so-called Stieltjes transform of the measure,

$$S(x) = \int_I \frac{d\mu(y)}{x-y}\,, \quad x \in \mathbb{C} \setminus I\,. \tag{22}$$

Given the above context, throughout the text we shall also refer to $\{P_n\}_{n\geq 0}$ as the sequence of orthogonal polynomials related to S.

Throughout the paper we consider each P_n monic, and we will denote the sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ by SMOP. Monic orthogonal polynomials satisfy a three-term recurrence relation [26]

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots\,, \tag{23}$$

with $P_{-1}(x) = 0,\ P_0(x) = 1$, and $\gamma_n \neq 0,\ n \geq 1,\ \gamma_0 = 1$. The parameters β_n, γ_n are commonly called the recurrence coefficients of $\{P_n\}_{n\geq 0}$.

A central object in our study is the reproducing Kernel,

$$K_n(x,y) = \sum_{k=0}^{n-1} \frac{P_k(x)P_k(y)}{\langle \mathbf{u}, P_k^2 \rangle}\,, \quad n = 1, 2, \ldots\,. \tag{24}$$

Proposition 1. *(Christoffel-Darboux formula)[10] Let $\{P_n\}_{n\geq 0}$ be a SMOP with respect to the linear functional $\mathbf{u}$. For all $n \geq 1$,*

$$K_n(x,y) = \frac{1}{\langle \mathbf{u}, P_{n-1}^2 \rangle} \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{x-y}\,. \tag{25}$$

We shall make use of the following notation:

$$K_n^{(i,j)}(x,y) = \sum_{k=0}^{n-1} \frac{(\mathbb{D}^i P_k)(x)(\mathbb{D}^j P_k)(y)}{\langle \mathbf{u}, P_k^2 \rangle}\,, \quad n = 1, 2, \ldots\,, \tag{26}$$

where, naturally, $\mathbb{D}^k$ refers to the kth derivative. The reproducing property of the Kernel gives us the following identity, for an arbitrary polynomial f of degree less of equal than $n-1$,

$$\langle \mathbf{u}, K_n^{(0,j)}(x,y) f(x) \rangle = (\mathbb{D}^j f)(y)\,.$$

Thus, if $j = 0$ in the formula above, then

$$\langle \mathbf{u}, K_n(x,y) f(x) \rangle = f(y)\,.$$

2.2.1 Semi-classical orthogonal polynomials on snul

Definition 1 ([17]). *A SMOP $\{P_n\}_{n\geq 0}$ related to a Stieltjes function, S, is said to be semi-classical (on snul) if S satisfies a difference equation*

$$A(x)\mathbb{D}S(x) = C(x)\mathbb{M}S(x) + D(x)\,, \tag{27}$$

where $A(x), C(x), D(x)$ are irreducible polynomials in x, $A \neq 0$.

Note that [17]

$$\deg(A) \leq m+2\,,\ \deg(C) \leq m+1\,,\ \deg(D) \leq m\,, \tag{28}$$

where m is some nonnegative integer. When $m = 0$ we get the so-called classical polynomials on snul [14, 23, 24].

Semi-classical orthogonal polynomials on snul are characterized through the following equivalent properties:
(i) a distributional equation for the linear functional [14, 22],

$$\mathbb{D}(\phi\mathbf{u}) = \mathbb{M}(\psi\mathbf{u})\,, \tag{29}$$

with polynomials ϕ, ψ such that $\phi \neq 0\,,\ \deg(\psi) \geq 1$;
(ii) a difference equation (27) for the Stieltjes function [17, 27],

$$A\mathbb{D}S = C\mathbb{M}S + D\,,$$

with A, C, D irreducible polynomials (in x).

Furthermore, whenever S is defined through (22), with μ defined in terms of a weight w as $d\mu(x) = w(x)dx$, then we also have the equivalence between (i) and (ii) and the Pearson equation for the weight [8, 27],

$$A\mathbb{D}w = C\mathbb{M}w\,, \tag{30}$$

where A and C are the same polynomials as in (27).

The polynomials in (29)–(30) are related via

$$A = \mathbb{M}\phi - \alpha\frac{\Delta_y^2}{4}\mathbb{D}\psi - U_1\mathbb{M}\psi\,, \quad C = -\mathbb{D}\phi + \alpha\mathbb{M}\psi + U_1\mathbb{D}\psi\,, \tag{31}$$

with α, U_1 given in (18) and (19), respectively.

Remark 1. *The polynomial D in (27) depends on A, C. It relates to ϕ, ψ as follows:*

$$D = \mathbb{D}(\mathbf{u}\,\theta_0\phi) - \alpha\mathbb{M}(\mathbf{u}\,\theta_0\psi) - U_1\mathbb{D}(\mathbf{u}\,\theta_0\psi)\,,$$

with $\theta_0 f$ defined by $\theta_0 f(x) = \frac{f(x)-f(0)}{x}$, and the right product of the functional $\mathbf{u}$ by a polynomial $g(x) = \sum_{k=0}^{n} g_k x^k$ defined as the following polynomial,

$$(\mathbf{u}\,g)(x) = \sum_{k=0}^{n}\left(\sum_{j=k}^{n} g_j u_{j-k}\right)x^k\,.$$

In the sequel we will use the following matrices:

$$\mathcal{P}_n = \begin{bmatrix} P_{n+1} \\ P_n \end{bmatrix}, \quad n \geq 0 . \tag{32}$$

In the account of (23), $\mathcal{P}_n$ satisfies the difference equation

$$\mathcal{P}_n = \mathcal{A}_n \mathcal{P}_{n-1} , \quad \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1 , \tag{33}$$

with initial condition $\mathcal{P}_0 = \begin{bmatrix} x-\beta_0 & 1 \\ 1 & 0 \end{bmatrix}$. The matrix $\mathcal{A}_n$ is usually known as transfer matrix.

According to [8, Cor. 1], if $\{P_n\}_{n\geq 0}$ is a SMOP related to a semi-classical weight, w, such that $A\mathbb{D}w = C\mathbb{M}w$, then $\{\mathcal{P}_n\}_{n\geq 1}$ satisfies the following equation:

$$A_{n+1}\, \mathbb{D}\mathcal{P}_n = (\mathcal{B}_n - C/2\, I)\mathbb{M}\mathcal{P}_n , \quad n \geq 1 , \tag{34}$$

where A_{n+1} is the polynomial

$$A_{n+1} = A + \frac{\Delta_y^2}{2}\pi_n , \tag{35}$$

I is the identity matrix, and $\mathcal{B}_n$ is a matrix with polynomial entries of uniformly bounded degrees, given by (in the account of [8, Eq. (57)])

$$\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & -l_n \end{bmatrix} . \tag{36}$$

The polynomials Θ_n, l_n and π_n satisfy the fundamental relations, for all $n \geq 0$ [9, 13]:

$$\pi_{n+1} = -\frac{1}{2}\sum_{k=0}^{n+1} \frac{\Theta_{k-1}}{\gamma_k} , \tag{37}$$

$$l_{n+1} + l_n + \mathbb{M}(x-\beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} = 0 , \tag{38}$$

$$-A + \mathbb{M}(x-\beta_{n+1})(l_{n+1}-l_n) - \frac{\Delta_y^2}{2}(\pi_{n+1}+\pi_n) + \Theta_{n+1} = \frac{\gamma_{n+1}}{\gamma_n}\Theta_{n-1} , \tag{39}$$

$$l_{n+1} + l_n = 2\mathbb{M}(x-\beta_{n+1})(\pi_{n+1}-\pi_n) , \tag{40}$$

together with the initial conditions

$$\pi_{-1} = 0, \ \pi_0 = -D/2, \tag{41}$$

$$\Theta_{-1} = D, \ \Theta_0 = A - \frac{\Delta_y^2}{4}D - (l_0 - C/2)\mathbb{M}(x-\beta_0) + B, \tag{42}$$

$$l_{-1} = C/2, \ l_0 = -\mathbb{M}(x-\beta_0)D - C/2 . \tag{43}$$

2.3 The second order holonomic divided-difference equation for semi-classical orthogonal polynomials on snul

Let us take a SMOP $\{P_n\}_{n\geq 0}$ related to a semi-classical weight, w, such that (30) holds, i.e., $A\mathbb{D}w = C\mathbb{M}w$, or, in more general terms, let us take the difference equation (27),

$A\mathbb{D}S = C\mathbb{M}S + D$, for the corresponding Stieltjes function. Recall the equations enclosed by (34),

$$A_{n+1}\mathbb{D}P_{n+1} = (l_n - C/2)\,\mathbb{M}P_{n+1} + \Theta_n\,\mathbb{M}P_n\,, \tag{44}$$

$$A_{n+1}\mathbb{D}P_n = -\frac{\Theta_{n-1}}{\gamma_n}\,\mathbb{M}P_{n+1} - (l_n + C/2)\,\mathbb{M}P_n\,. \tag{45}$$

The above equations will play a fundamental role in the sequel, as they fully determine the coefficients of the second order divided-difference equation - the holonomic equation, to be deduced later on.

For matters of simplification in the sequel, let us write (44) and (45) as

$$\mu_n = \Theta_n\,\mathbb{M}P_n\,, \tag{46}$$

$$A_{n+1}\mathbb{D}P_n = -\eta_n + \tilde{l}_n\,\mathbb{M}P_n\,, \tag{47}$$

that is, we use the notation

$$\mu_n = A_{n+1}\mathbb{D}P_{n+1} - (l_n - C/2)\,\mathbb{M}P_{n+1}\,,\ \ \eta_n = \frac{\Theta_{n-1}}{\gamma_n}\,\mathbb{M}P_{n+1}\,,\ \ \tilde{l}_n = -l_n - C/2\,.$$

Proposition 2. *Let the previous notations hold. The following difference equations hold,*

$$\begin{aligned}
\mathbb{D}\mu_n &= \mathbb{D}\Theta_n\,\mathbb{M}^2P_n + \mathbb{M}\Theta_n\,\mathbb{D}\mathbb{M}P_n\,, &(48)\\
\mathbb{M}\mu_n &= \mathbb{M}\Theta_n\,\mathbb{M}^2P_n + \frac{\Delta_y^2}{4}\mathbb{D}\Theta_n\,\mathbb{D}\mathbb{M}P_n\,, &(49)\\
-U_1\mathbb{D}\eta_n &= -U_1\mathbb{D}\tilde{l}_n\,\mathbb{M}^2P_n + E_n\,\mathbb{M}\mathbb{D}P_n + F_n\,\mathbb{D}\mathbb{M}P_n\,, &(50)\\
-U_1\mathbb{M}\eta_n &= -U_1\mathbb{M}\tilde{l}_n\,\mathbb{M}^2P_n + G_n\mathbb{M}\mathbb{D}P_n + H_n\,\mathbb{D}\mathbb{M}P_n\,, &(51)
\end{aligned}$$

with

$$E_n = U_1\mathbb{D}A_{n+1} - \alpha\mathbb{M}A_{n+1}\,,\ \ F_n = \mathbb{M}A_{n+1} - U_1\mathbb{M}\tilde{l}_n\,, \tag{52}$$

$$G_n = U_1\mathbb{M}A_{n+1} - \alpha\frac{\Delta_y^2}{4}\mathbb{D}A_{n+1}\,,\ \ H_n = \frac{\Delta_y^2}{4}\left(\mathbb{D}A_{n+1} - U_1\mathbb{D}\tilde{l}_n\right)\,. \tag{53}$$

Proof. By applying $\mathbb{D}$ to (46) and using the product rule (13), we get (48).

By applying $\mathbb{M}$ to (46) and using the product rule (15), we get (49).

Equation (50) is obtained by applying $U_1\mathbb{D}$ to (47), with U_1 the polynomial defined in (19), and using the product rule $U_1\mathbb{D}^2 = \mathbb{D}\mathbb{M} - \alpha\mathbb{M}\mathbb{D}$ from (17). Equation (51) is obtained by applying $U_1\mathbb{M}$ to (47), with U_1 the polynomial defined in (19), and using the product rule $U_1\mathbb{D}^2 = \mathbb{D}\mathbb{M} - \alpha\mathbb{M}\mathbb{D}$ from (17). □

As a consequence of the previous proposition, we get the result that follows.

Proposition 3. *Under the previous notations, the following equations hold,*

$$D_n\,\mathbb{M}^2P_n = (E_nH_n - F_nG_n)\mathbb{M}\mu_n + \frac{\Delta_y^2}{4}U_1\mathbb{D}\Theta_n(E_n\mathbb{M}\eta_n - G_n\mathbb{D}\eta_n)\,, \tag{54}$$

$$D_n\,\mathbb{D}\mathbb{M}P_n = U_1\mathbb{M}\Theta_n(G_n\mathbb{D}\eta_n - E_n\mathbb{M}\eta_n) + U_1(E_n\mathbb{M}\tilde{l}_n - G_n\mathbb{D}\tilde{l}_n)\mathbb{M}\mu_n\,, \tag{55}$$

with the polynomial D_n defined by

$$D_n = (E_nH_n - F_nG_n)\mathbb{M}\Theta_n + \frac{\Delta_y^2}{4}U_1(E_n\mathbb{M}\tilde{l}_n - G_n\mathbb{D}\tilde{l}_n)\mathbb{D}\Theta_n\,. \tag{56}$$

Proof. Take the system (49), (50), (51), in the unknowns $\mathbb{M}^2 P_n, \mathbb{M}\mathbb{D}P_n$ and $\mathbb{D}\mathbb{M}P_n$. The polynomial D_n given by (56) is the determinant of that system. Solving for $\mathbb{M}^2 P_n$ and $\mathbb{D}\mathbb{M}P_n$, then equations (54) and (55) follow. □

It is well to emphasize that $\mathbb{D}\mu_n, \mathbb{M}\mu_n$ and $\mathbb{D}\eta_n, \mathbb{M}\eta_n$ in Proposition 2 are constituted by polynomials times factors of $\mathbb{D}^2 P_{n+1}, \mathbb{M}\mathbb{D}P_{n+1}$, and P_{n+1}. The precise formulae are given in the next proposition.

Proposition 4. *Under the previous notations, the following equations hold,*

$$\mathbb{D}\mu_n = R_{n,1}\,\mathbb{D}^2 P_{n+1} + R_{n,2}\,\mathbb{M}\mathbb{D}P_{n+1} - \mathbb{D}\left(l_n - C/2\right) P_{n+1}\,, \tag{57}$$

$$\mathbb{M}\mu_n = \frac{\Delta_y^2}{4} R_{n,2}\,\mathbb{D}^2 P_{n+1} + R_{n,1}\,\mathbb{M}\mathbb{D}P_{n+1} - \mathbb{M}\left(l_n - C/2\right) P_{n+1}\,, \tag{58}$$

$$\mathbb{D}\eta_n = T_{n,1}\,\mathbb{D}^2 P_{n+1} + T_{n,2}\,\mathbb{M}\mathbb{D}P_{n+1} + \mathbb{D}\frac{\Theta_{n-1}}{\gamma_n} P_{n+1}\,, \tag{59}$$

$$\mathbb{M}\eta_n = \frac{\Delta_y^2}{4} T_{n,2}\,\mathbb{D}^2 P_{n+1} + T_{n,1}\,\mathbb{M}\mathbb{D}P_{n+1} + \mathbb{M}\frac{\Theta_{n-1}}{\gamma_n} P_{n+1}\,, \tag{60}$$

with

$$R_{n,1} = \mathbb{M}A_{n+1} - \alpha\frac{\Delta_y^2}{4}\mathbb{D}\left(l_n - C/2\right) - U_1\mathbb{M}\left(l_n - C/2\right)\,,$$
$$R_{n,2} = \mathbb{D}A_{n+1} - \alpha\mathbb{M}\left(l_n - C/2\right) - U_1\mathbb{D}\left(l_n - C/2\right)\,,$$
$$T_{n,1} = \alpha\frac{\Delta_y^2}{4}\mathbb{D}\frac{\Theta_{n-1}}{\gamma_n} + U_1\mathbb{M}\frac{\Theta_{n-1}}{\gamma_n}\,,\quad T_{n,2} = U_1\mathbb{D}\frac{\Theta_{n-1}}{\gamma_n} + \alpha\mathbb{M}\frac{\Theta_{n-1}}{\gamma_n}\,.$$

Proof. Use the product rules (13), (15), as well as (17), to obtain only factors of $\mathbb{D}^2 P_{n+1}$, $\mathbb{M}\mathbb{D}P_{n+1}$, and P_{n+1}. □

We now deduce the second-order divided-difference equation with polynomial coefficients - the holonomic difference equation.

Theorem 1. *Let $\{P_n\}_{n\geq 0}$ be a semi-classical SMOP on snul. Assuming that $\{P_n\}_{n\geq 0}$ is related to a Stieltjes function satisfying the difference equation (27), $A\mathbb{D}S = C\mathbb{M}S + D$, then, for all $n \geq 1$, the following second-order difference equation holds,*

$$\widetilde{A_n}\,\mathbb{D}^2 P_{n+1} + \widetilde{B_n}\,\mathbb{M}\mathbb{D}P_{n+1} + \widetilde{C_n} P_{n+1} = 0\,, \tag{61}$$

with the polynomial coefficients given by

$$\widetilde{A_n} = \left(\mathbb{M}A_{n+1} - \alpha\frac{\Delta_y^2}{4}\mathbb{D}\left(l_n - C/2\right) - U_1\mathbb{M}\left(l_n - C/2\right)\right) D_n$$
$$-\frac{\Delta_y^2}{4}\left(\mathbb{D}A_{n+1} - \alpha\mathbb{M}\left(l_n - C/2\right) - U_1\mathbb{D}\left(l_n - C/2\right)\right) J_n$$
$$+U_1\left(\left(\alpha\frac{\Delta_y^2}{4}\mathbb{D}\frac{\Theta_{n-1}}{\gamma_n} + U_1\mathbb{M}\frac{\Theta_{n-1}}{\gamma_n}\right) G_n - \frac{\Delta_y^2}{4}\left(\alpha\mathbb{M}\frac{\Theta_{n-1}}{\gamma_n} + U_1\mathbb{D}\frac{\Theta_{n-1}}{\gamma_n}\right) E_n\right) V_n\,, \quad (62)$$
$$\widetilde{B_n} = \left(\mathbb{D}A_{n+1} - \alpha\mathbb{M}\left(l_n - C/2\right) - U_1\mathbb{D}\left(l_n - C/2\right)\right) D_n$$
$$-\left(\mathbb{M}A_{n+1} - \alpha\frac{\Delta_y^2}{4}\mathbb{D}\left(l_n - C/2\right) - U_1\mathbb{M}\left(l_n - C/2\right)\right) J_n$$
$$+U_1\left(\left(U_1\mathbb{D}\frac{\Theta_{n-1}}{\gamma_n} + \alpha\mathbb{M}\frac{\Theta_{n-1}}{\gamma_n}\right) G_n - \left(U_1\mathbb{M}\frac{\Theta_{n-1}}{\gamma_n} + \alpha\frac{\Delta_y^2}{4}\mathbb{D}\frac{\Theta_{n-1}}{\gamma_n}\right) E_n\,,\right) V_n\,, \quad (63)$$
$$\widetilde{C_n} = -D_n\mathbb{D}\left(l_n - C/2\right) + J_n\mathbb{M}\left(l_n - C/2\right) + U_1\left(G_n\mathbb{D}\frac{\Theta_{n-1}}{\gamma_n} - E_n\mathbb{M}\frac{\Theta_{n-1}}{\gamma_n}\right) V_n\,, \quad (64)$$

where D_n is given by (56), and

$$J_n = (E_nH_n - F_nG_n)\mathbb{D}\Theta_n + U_1\mathbb{M}\Theta_n(E_n\mathbb{M}\tilde{l}_n - G_n\mathbb{D}\tilde{l}_n)\,, \quad (65)$$
$$V_n = \frac{\Delta_y^2}{4}\left(\mathbb{D}\Theta_n\right)^2 - \left(\mathbb{M}\Theta_n\right)^2\,. \quad (66)$$

with E_n, F_n, G_n, H_n given in (52)–(53).

Proof. The difference equation (61) is obtained starting with (48). Indeed, by multiplying (48) by D_n and using $\mathbb{M}^2P_n$ and $\mathbb{D}\mathbb{M}P_n$ from (54) and (55), we get

$$D_n\mathbb{D}\mu_n - J_n\mathbb{M}\mu_n + U_1G_nV_n\mathbb{D}\eta_n - U_1E_nV_n\mathbb{M}\eta_n = 0\,, \quad (67)$$

with J_n, V_n given in (65)–(66). The use of equations (57)–(60) in (67) yields the second order divided-difference equation (61) with coefficients (62)–(64). □

Remark 2. *Alternatively, the difference equation (61) can be deduced starting with (49) instead of (48), and following the analogue procedure as the one described in the proof of the previous theorem.*

3 Sobolev-type orthogonal polynomials on snul

Let $\mathbf{u}$ be a linear functional defined in the linear space of polynomials with real or complex coefficients. We consider the Sobolev-type inner product (2), that is,

$$\langle f, g\rangle_\lambda = \langle \mathbf{u}, fg\rangle + \lambda(\mathbb{D}f)(c)(\mathbb{D}g)(c)\,, \quad \lambda \geq 0\,,$$

where $\mathbb{D}$ is the general divided-difference operator given by (3). Here, c will be taken as a general real/complex number. Usually, c should be taken outside the support of $\mathbf{u}$.

We denote by $\{S_n\}_{n\geq 0}$ the sequence of orthogonal polynomials with respect to the inner product (2). Without loss of generality, S_n will be taken monic, for $n = 0, 1, 2 \ldots$.

Let $\{P_n\}_{n\geq 0}$ be the SMOP related to $\mathbf{u}$. The goal of the next section is to deduce relations expressing S_n in terms of (quantities related to) $\{P_n\}_{n\geq 0}$.

3.1 Connection formulae

Proposition 5. *Under the previous notations, the following connection formulae take place:*

$$S_n(x) = P_n(x) - \lambda(\mathbb{D}S_n)(c)K_n^{(0,1)}(x,c)\,, \tag{68}$$

$$S_n(x) = P_n(x) - \lambda_n K_n^{(0,1)}(x,c)\,, \tag{69}$$

$$\begin{aligned} S_n(x) = P_n(x) - \frac{\lambda_n}{\|P_{n-1}\|^2} \Bigg\{ & \frac{P_n(x)(\mathbb{M}P_{n-1})(c) - P_{n-1}(x)(\mathbb{M}P_n)(c)}{(x-y_1(c))(x-y_2(c))} \\ & + \frac{(P_n(x)(\mathbb{D}P_{n-1})(c) - P_{n-1}(x)(\mathbb{D}P_n)(c))}{(x-y_1(c))(x-y_2(c))}(x-p(c)) \Bigg\}, \end{aligned} \tag{70}$$

where

$$\lambda_n = \lambda \frac{(\mathbb{D}P_n)(c)}{1+\lambda K_n^{(1,1)}(c,c)}\,, \tag{71}$$

$K_n^{(0,1)}(x,y)$ *denotes the derivatives of the reproducing Kernel defined in (26), and*

$$p(c) = (y_1(c)+y_2(c))/2\,, \tag{72}$$

with y_1, y_2 *the functions defined in (11).*

Proof. Recalling that P_n and S_n are both monic, let us write

$$S_n(x) = P_n(x) + \sum_{k=0}^{n-1} a_{n,k}P_k(x)\,, \quad n \geq 1\,. \tag{73}$$

The Fourier coefficients $a_{n,k}$ are given by

$$a_{n,k} = \frac{\langle \mathbf{u}, S_nP_k\rangle}{\langle \mathbf{u}, P_k^2\rangle}\,, \quad k = 0,1,\ldots,n-1\,. \tag{74}$$

On the other hand, by writing

$$\langle \mathbf{u}, S_nP_k\rangle = \langle S_n, P_k\rangle_\lambda - \lambda(\mathbb{D}S_n)(c)(\mathbb{D}P_k)(c)\,,$$

and using the orthogonality of $\{S_n\}$ with respect to $\langle\cdot,\cdot\rangle_\lambda$, then $\langle S_n, P_k\rangle_\lambda = 0$, thus, from (74), we get

$$a_{n,k} = -\frac{\lambda(\mathbb{D}S_n)(c)(\mathbb{D}P_k)(c)}{\langle \mathbf{u}, P_k^2\rangle}\,, \quad k = 0,1,\ldots,n-1\,.$$

Thus, the Fourier expansion (73) is given by

$$S_n(x) = P_n(x) - \lambda(\mathbb{D}S_n)(c)\sum_{k=0}^{n-1}\frac{(\mathbb{D}P_k)(c)}{\langle \mathbf{u}, P_k^2\rangle}P_k(x)\,. \tag{75}$$

Noting that (cf. (26))

$$\sum_{k=0}^{n-1}\frac{(\mathbb{D}P_k)(c)P_k(x)}{\langle \mathbf{u}, P_k^2\rangle} = K_n^{(0,1)}(x,c)\,,$$

then (75) can be also written as (68).

Furthermore, let us apply $\mathbb{D}$ to (68) and compute its value at $x = c$. We get

$$(\mathbb{D}S_n)(c) = (\mathbb{D}P_n)(c) - \lambda(\mathbb{D}S_n)(c)K_n^{(1,1)}(c,c)\,,$$

thus,

$$(\mathbb{D}S_n)(c) = \frac{(\mathbb{D}P_n)(c)}{1+\lambda K_n^{(1,1)}(c,c)}\,.$$

Therefore, (68) can also be written as

$$S_n(x) = P_n(x) - \lambda\frac{(\mathbb{D}P_n)(c)}{1+\lambda K_n^{(1,1)}(c,c)}K_n^{(0,1)}(x,c)\,,$$

or, using the notation (71), we write the previous equation as (69).

To deduce (70) we consider the following representation for the Kernel:

$$K_n^{(0,1)}(x,c) = \partial_y K_n(x,y)|_{y=c}\,,$$

where ∂_y here is used to denote the derivative of $K_n(x,y)$ in the argument y evaluated at $y = c$. In the account of (25), that is, $K_n(x,y) = \dfrac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{(x-y)\|P_{n-1}\|^2}$, we have, by using the the quotient rule (14),

$$K_n^{(0,1)}(x,c) = \frac{P_n(x)}{\|P_{n-1}\|^2}\left\{\frac{(\mathbb{D}P_{n-1})(c)(x-p(c)) + (\mathbb{M}P_{n-1})(c)}{(x-y_1(c))(x-y_2(c))}\right\} - \frac{P_{n-1}(x)}{\|P_{n-1}\|^2}\left\{\frac{(\mathbb{D}P_n)(c)(x-p(c)) + (\mathbb{M}P_n)(c)}{(x-y_1(c))(x-y_2(c))}\right\}, \quad (76)$$

where $p(c) = (y_1(c)+y_2(c))/2$. Plugging (76) into (69) we get, after rearranging, (70). □

In what follows it is shown a property to be used later on.

Proposition 6. *Considering the inner product $\langle\cdot,\cdot\rangle_\lambda$ defined in (2), the following property holds, for arbitrary polynomials f, g:*

$$\langle(x-y_1(c))(x-y_2(c))f, g\rangle_\lambda = \langle f, (x-y_1(c))(x-y_2(c))g\rangle_\lambda\,. \quad (77)$$

Proof. We have

$$\langle(x-y_1(c))(x-y_2(c))f, g\rangle_\lambda = \langle\mathbf{u}, (x-y_1(c))(x-y_2(c))fg\rangle + \lambda\left(\mathbb{D}((x-y_1(c))(x-y_2(c))f)\right)(c)\,(\mathbb{D}g)\,(c)\,.$$

and

$$\langle f, (x-y_1(c))(x-y_2(c))g\rangle_\lambda = \langle\mathbf{u}, (x-y_1(c))(x-y_2(c))fg\rangle + \lambda\,(\mathbb{D}f)\,(c)\left(\mathbb{D}((x-y_1(c))(x-y_2(c))g)\right)(c)\,.$$

Taking into account that (cf. (3))

$$\mathbb{D}((x-y_1(c))(x-y_2(c))h) = 0\,, \quad h \in \mathbb{C}[x]\,, \quad (78)$$

the result follows. □

Further expansions follow.

Proposition 7. *The following expansion holds, for all $n \geq 2$:*

$$(x - y_1(c))(x - y_2(c))S_n(x) = \sum_{k=n-2}^{n+2} a_{n,k} P_k(x)\,, \tag{79}$$

with $a_{n,n+2} = 1$, and

$$\begin{aligned}
a_{n,n+1} &= b_{n+1} - \frac{\lambda_n}{\|P_{n-1}\|^2} (\mathbb{D}P_{n-1})(c)\,,\\
a_{n,n} &= b_n - \frac{\lambda_n}{\|P_{n-1}\|^2} \left\{ (\mathbb{M}P_{n-1})(c) + (\mathbb{D}P_{n-1})(c)(\beta_n - p(c)) - (\mathbb{D}P_n)(c)(\beta_{n-1} - p(c)) \right\}\,,\\
a_{n,n-1} &= b_{n-1} + \frac{\lambda_n}{\|P_{n-1}\|^2} \left\{ (\mathbb{M}P_n)(c) - \gamma_n (\mathbb{D}P_{n-1})(c) + (\mathbb{D}P_n)(c)(\beta_{n-1} - p(c)) \right\}\,,\\
a_{n,n-2} &= b_{n-2} + \frac{\lambda_n}{\|P_{n-1}\|^2} \gamma_{n-1} (\mathbb{D}P_n)(c)\,,
\end{aligned}$$

where the b_k's are defined in terms of the recurrence coefficients of $\{P_n\}_{n\geq 0}$ as follows,

$$b_{n+1} = \beta_{n+1} + \beta_n - 2p(c)\,, \quad b_n = \gamma_{n+1} + \gamma_n + \beta_n^2 - 2\beta_n p(c) + y_1(c) y_2(c)\,,$$
$$b_{n-1} = \gamma_n(\beta_n + \beta_{n-1} - 2p(c))\,, \quad b_{n-2} = \gamma_n \gamma_{n-1}\,.$$

Here, $p(c) = (y_1(c) + y_2(c))/2$, with y_1, y_2 the functions defined in (11).

Proof. Let us write

$$(x - y_1(c))(x - y_2(c))S_n(x) = \sum_{k=0}^{n+2} a_{n,k} P_k(x)\,. \tag{80}$$

The Fourier coefficients $a_{n,k}$ are given by

$$a_{n,k} = \frac{\langle \mathbf{u}, (x - y_1(c))(x - y_2(c)) S_n P_k \rangle}{\langle \mathbf{u}, P_k^2 \rangle}\,. \tag{81}$$

By virtue of (78), we can write

$$\begin{aligned}
\langle \mathbf{u}, (x - y_1(c))(x - y_2(c)) S_n P_k \rangle &= \langle \mathbf{u}, (x - y_1(c))(x - y_2(c)) S_n P_k \rangle\\
&\quad + \lambda (\mathbb{D}((x - y_1(c))(x - y_2(c)) S_n))(c) (\mathbb{D}P_k)(c)\,,
\end{aligned}$$

thus, we have

$$a_{n,k} = \frac{\langle (x - y_1(c))(x - y_2(c)) S_n, P_k \rangle_\lambda}{\langle \mathbf{u}, P_k^2 \rangle}\,.$$

Recalling (77), there holds

$$\langle (x - y_1(c))(x - y_2(c)) S_n, P_k \rangle_\lambda = \langle S_n, (x - y_1(c))(x - y_2(c)) P_k \rangle_\lambda\,.$$

Thus,

$$a_{n,k} = \frac{\langle S_n, (x - y_1(c))(x - y_2(c)) P_k \rangle_\lambda}{\langle \mathbf{u}, P_k^2 \rangle}\,.$$

From the orthogonality of $\{S_n\}_{n\geq 0}$ there follows $a_{n,k}=0,\quad k=0,1,\ldots,n-3$. Hence, the expansion (80) is reduced to (79).

Let us now deduce the coefficients $a_{n,k},\ k=n-2,n-1,\ldots,n+2$.

As S_n and P_n are monic, then $a_{n,n+2}=1$. In order to compute the remaining $a_{n,k}$'s recall (81),

$$a_{n,k}\langle \mathbf{u}, P_k^2\rangle = \langle \mathbf{u}, (x-y_1(c))(x-y_2(c))S_nP_k\rangle\,.$$

On the other hand, from (70), we have

$$\begin{aligned}(x-y_1(c))(x-y_2(c))S_n(x) &= (x-y_1(c))(x-y_2(c))P_n(x)\\ &- \frac{\lambda_n}{\|P_{n-1}\|^2}\{P_n(x)(\mathbb{M}P_{n-1})(c) - P_{n-1}(x)(\mathbb{M}P_n)(c)\\ &+ (P_n(x)(\mathbb{D}P_{n-1})(c) - P_{n-1}(x)(\mathbb{D}P_n)(c))\,(x-p(c))\}\,. \quad (82)\end{aligned}$$

Using the three-term recurrence relation (23) for $\{P_n\}_{n\geq 0}$ in (82), we get the required coefficients. □

3.2 Three-term recurrence relations for Sobolev-type Orthogonal polynomials

In order to deduce a recurrence relation for $\{S_n\}_{n\geq 0}$, it is useful to deduce the expansion given in the following proposition.

Proposition 8. *The following expansion holds, for all $n\geq 1$:*

$$(x-y_1(c))(x-y_2(c))S_n(x) = \sum_{j=n-2}^{n+2} c_{n,j}S_j(x)\,,\quad n\geq 1\,, \tag{83}$$

with the coefficients $c_{n,j}$ given by

$$c_{n,j} = \frac{a_{n,j}\|P_j\|^2 - \lambda_j \sum_{k=n-2}^{j-1} a_{n,k}(\mathbb{D}P_k)(c)}{\|S_j\|_\lambda^2}\,,\quad j=n-2,\ldots,n+2\,, \tag{84}$$

with the convention $\sum_{k=a}^{b}\cdot = 0$ whenever $a>b$.

Here, the $a_{n,j}$'s are the coefficients given in Proposition 7.

Proof. Let us write

$$(x-y_1(c))(x-y_2(c))S_n(x) = \sum_{j=0}^{n+2} c_{n,j}S_j(x)\,.$$

The Fourier coefficients $c_{n,k}$ are given by

$$c_{n,j}\langle S_j, S_j\rangle_\lambda = \langle (x-y_1(c))(x-y_2(c))S_n, S_j\rangle_\lambda\,. \tag{85}$$

By virtue of property (77), we have

$$\langle (x-y_1(c))(x-y_2(c))S_n, S_j\rangle_\lambda = \langle S_n, (x-y_1(c))(x-y_2(c))S_j\rangle_\lambda\,,$$

thus, due to the orthogonality of $\{S_n\}_{n\geq 0}$, there follows that $c_{n,j}=0,\ j=0,1,\ldots,n-3$. Hence, we get the expansion (83).

In order to compute the coefficients $c_{n,j},\ j=n-2,\ldots,n+2$, we start by recalling property (78), thus (85) yields

$$c_{n,j}\langle S_j,S_j\rangle_\lambda=\langle \mathbf{u},(x-y_1(c))(x-y_2(c))S_nS_j\rangle\,. \tag{86}$$

Using the two representations (69) and (79), respectively,

$$S_j(x)=P_j(x)-\lambda_jK_j^{(0,1)}(x,c)\,,\quad (x-y_1(c))(x-y_2(c))S_n(x)=\sum_{k=n-2}^{n+2}a_{n,k}P_k(x)\,,$$

into (86), and using the orthogonality of $\{P_n\}_{n\geq 0}$, we obtain the required coefficients given in (84). □

Remark 3. *As S_n is monic, for all $n=0,1,\ldots,$ then $c_{n,n+2}=1$. Therefore, taking $j=n+2$ in (84) we obtain the norm of S_{n+2} in terms of the norm of P_{n+2}, as follows:*

$$\|S_{n+2}\|_\lambda^2=\|P_{n+2}\|^2-\lambda_{n+2}\sum_{k=n-2}^{n+1}a_{n,k}(\mathbb{D}P_k)(c)\,.$$

Proposition 9. *Let Φ be the polynomial defined by $\Phi(x)=(x-y_1(c))(x-y_2(c))$. The following equations take place:*

$$\Phi(x)S_n(x) = \hat{A}_n(x)P_n(x)+\hat{B}_n(x)P_{n-1}(x)\,,\quad n\geq 2\,, \tag{87}$$
$$\Phi(x)S_{n-1}(x) = \hat{C}_n(x)P_n(x)+\hat{D}_n(x)P_{n-1}(x)\,,\quad n\geq 2\,, \tag{88}$$

where $\hat{A}_n,\hat{B}_n,\hat{C}_n,\hat{D}_n$, are polynomials given by

$$\hat{A}_n(x) = x^2-(\beta_{n+1}+\beta_n+a_{n,n+1})\,x-\frac{a_{n,n-2}}{\gamma_{n-1}}+a_{n,n}-\beta_na_{n,n+1}+\beta_{n+1}\beta_n-\gamma_{n+1} \tag{89}$$
$$\hat{B}_n(x) = (a_{n,n-2}/\gamma_{n-1}-\gamma_n)\,x-\frac{a_{n,n-2}}{\gamma_{n-1}}\beta_{n-1}+a_{n,n-1}-\gamma_na_{n,n+1}+\beta_{n+1}\gamma_n\,. \tag{90}$$
$$\hat{C}_n(x) = -\frac{\hat{B}_{n-1}(x)}{\gamma_{n-1}}\,, \tag{91}$$
$$\hat{D}_n(x) = \hat{A}_{n-1}(x)+\frac{(x-\beta_{n-1})}{\gamma_{n-1}}\hat{B}_{n-1}(x)\,, \tag{92}$$

where the $a_{n,j}$'s are the coefficients given in Proposition 7.

Proof. Let us start with equation (79). Using the three-term recurrence relation (23) together with the (consequent) following equations

$$P_{n+2}(x) = ((x-\beta_{n+1})(x-\beta_n)-\gamma_{n+1})\,P_n(x)-(x-\beta_{n+1})\gamma_nP_{n-1}(x)\,, \tag{93}$$
$$P_{n-2}(x) = \frac{1}{\gamma_{n-1}}\left((x-\beta_{n-1})P_{n-1}(x)-P_n(x)\right)\,, \tag{94}$$

into (79), we get (87) with polynomials $\hat{A}_n,\hat{B}_n$ given by (89)–(90).

To obtain (88) we take (87) for $n-1$ and use (94), thus getting the required equation with polynomials $\hat{C}_n,\hat{D}_n$ given by (91)–(92). □

Theorem 2. *Let the previous notations hold. The sequence $\{S_n\}_{n\geq 0}$ of Sobolev-type orthogonal polynomials related to the inner product (2) satisfies a three-term recurrence relation with rational coefficients,*

$$S_{n+1}(x) = \hat{\beta}_n(x)S_n(x) + \hat{\gamma}_n(x)S_{n-1}(x)\,, \quad n \geq 1\,, \tag{95}$$

with initial conditions $S_0(x) = 1, S_1(x) = P_1(x)$, and

$$\hat{\beta}_n(x) = \frac{\left(\gamma_n \hat{C}_n(x) + (x-\beta_n)\hat{D}_n(x)\right)\hat{A}_{n+1}(x) + \hat{B}_{n+1}(x)\hat{D}_n(x)}{\hat{A}_n(x)\hat{D}_n(x) - \hat{B}_n(x)\hat{C}_n(x)}\,, \tag{96}$$

$$\hat{\gamma}_n(x) = \frac{-\left(\gamma_n \hat{A}_n(x) + (x-\beta_n)\hat{B}_n(x)\right)\hat{A}_{n+1}(x) - \hat{B}_{n+1}(x)\hat{B}_n(x)}{\hat{A}_n(x)\hat{D}_n(x) - \hat{B}_n(x)\hat{C}_n(x)}\,. \tag{97}$$

Proof. For simplicity matter, let us write the system (87)–(88) in the matrix form

$$\Phi \mathcal{S}_{n-1} = \mathcal{E}_n \mathcal{P}_{n-1}\,, \tag{98}$$

where

$$\mathcal{S}_{n-1} = \begin{bmatrix} S_n \\ S_{n-1} \end{bmatrix}, \quad \mathcal{E}_n = \begin{bmatrix} \hat{A}_n & \hat{B}_n \\ \hat{C}_n & \hat{D}_n \end{bmatrix}, \quad \mathcal{P}_{n-1} = \begin{bmatrix} P_n \\ P_{n-1} \end{bmatrix}. \tag{99}$$

Standard computations give us $\det(\mathcal{E}_n) \neq 0$.

By taking $n+1$ in (98), we get

$$\Phi \mathcal{S}_n = \mathcal{E}_{n+1}\mathcal{P}_n\,.$$

Using $\mathcal{P}_n = \mathcal{A}_n \mathcal{P}_{n-1}$ (cf. (33)) in the equation above and taking into account the regularity of the matrix $\mathcal{E}_n$, we obtain

$$\mathcal{S}_n = \mathcal{E}_{n+1}\mathcal{A}_n\mathcal{E}_n^{-1}\mathcal{S}_{n-1}\,.$$

Hence, we have the difference equation

$$\mathcal{S}_n = \hat{\mathcal{A}}_n \mathcal{S}_{n-1}\,, \tag{100}$$

with the transfer-type matrix

$$\hat{\mathcal{A}}_n = \mathcal{E}_{n+1}\mathcal{A}_n\mathcal{E}_n^{-1}\,.$$

Standard computations give us

$$\hat{\mathcal{A}}_n = \begin{bmatrix} \hat{\beta}_n(x) & \hat{\gamma}_n(x) \\ 1 & 0 \end{bmatrix}$$

with $\hat{\beta}_n(x), \hat{\gamma}_n(x)$ given by (96) and (97), respectively. □

3.3 The second order holonomic divided-difference equation for Sobolev-type orthogonal polynomials

Let us now assume the semi-classical property of the linear functional $\mathbf{u}$ in (2). Let us take $\mathbf{u}$ satisfying a Pearson equation such as (29), $\mathbb{D}(\phi\mathbf{u}) = \mathbb{M}(\psi\mathbf{u})$, or, equivalently, such that the corresponding Stieltjes function satisfies (27), $A\mathbb{D}S = C\mathbb{M}S + D$.

Recalling the process from Sub-Section 2.3, the divided-difference equations of type (44)–(45) are the starting point to deduce the holonomic equation satisfied by a sequence of orthogonal polynomials.

In what follows we will make use of the connection relations between $\{S_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ given by (87)–(88), written in the matrix form (98), that is,

$$\Phi \mathcal{S}_n = \mathcal{E}_{n+1} \mathcal{P}_n \,.$$

Recall that $\{\mathcal{P}_n\}_{n\geq 0}$ satisfies the system (34), thus we have

$$A_{n+1}\, \mathbb{D}\mathcal{P}_n = (\mathcal{B}_n - C/2\, I)\mathbb{M}\mathcal{P}_n \,, \quad n \geq 1 \,.$$

In what follows we shall deduce similar systems similar to (34) for $\{S_n\}_{n\geq 0}$. These systems are fundamental to the deduction of the the second order holonomic difference equation.

Proposition 10. *Let $\{S_n\}_{n\geq 0}$ be the sequence of Sobolev-type orthogonal polynomials related to the inner product (2), with $\mathbf{u}$ a semi-classical linear functional. Under the previous notations, the sequence $\{S_n\}_{n\geq 0}$ satisfies the divided-difference equations, for all $n \geq 1$,*

$$\widehat{A}_{n+1}\mathbb{D}S_{n+1} = L_{n,1}\mathbb{M}S_{n+1} + \Theta_{n,1}\mathbb{M}S_n \,, \tag{101}$$

$$\widehat{A}_{n+1}\mathbb{D}S_n = L_{n,2}\mathbb{M}S_{n+1} + \Theta_{n,2}\mathbb{M}S_n \,, \tag{102}$$

where $\widehat{A}_{n+1}$ is the polynomial defined by $\widehat{A}_{n+1} = \det \mathcal{G}_{n+1}$, with

$$\mathcal{G}_{n+1} = A_{n+1}\mathbb{M}\Phi\, I - \frac{\Delta_y^2}{4}\mathcal{F}_{n+1}\mathbb{D}\mathcal{E}_{n+1}^{-1} \,, \;\; \mathcal{F}_{n+1} = A_{n+1}\mathbb{D}\mathcal{E}_{n+1} + \mathbb{M}\mathcal{E}_{n+1}\left(\mathcal{B}_n - C/2\, I\right) \,, \tag{103}$$

where $L_{n,1}, \Theta_{n,1}, L_{n,2}, \Theta_{n,2}$ are, respectively, the entries $(1,1), (1,2), (2,1), (2,2)$ in the matrix $(\operatorname{adj} \mathcal{G}_{n+1})\left(\mathcal{F}_{n+1}\mathbb{M}\mathcal{E}_{n+1}^{-1} - A_{n+1}\mathbb{D}\Phi\, I\right)$.

Proof. By applying $\mathbb{D}$ to $\Phi\mathcal{S}_n = \mathcal{E}_{n+1}\mathcal{P}_n$ we get, using the product rule (13),

$$\mathbb{D}\Phi\, \mathbb{M}\mathcal{S}_n + \mathbb{M}\Phi\, \mathbb{D}\mathcal{S}_n = \mathbb{D}\mathcal{E}_{n+1}\, \mathbb{M}\mathcal{P}_n + \mathbb{M}\mathcal{E}_{n+1}\, \mathbb{D}\mathcal{P}_n \,. \tag{104}$$

Multiplying (104) by the polynomial A_{n+1} defined in (34) we get, using the equation from (34),

$$A_{n+1}\mathbb{D}\Phi\, \mathbb{M}\mathcal{S}_n + A_{n+1}\mathbb{M}\Phi\, \mathbb{D}\mathcal{S}_n = \left(A_{n+1}\mathbb{D}\mathcal{E}_{n+1} + \mathbb{M}\mathcal{E}_{n+1}\left(\mathcal{B}_n - C/2\, I\right)\right)\, \mathbb{M}\mathcal{P}_n \,. \tag{105}$$

Also, from $\mathcal{P}_n = \mathcal{E}_{n+1}^{-1}\mathcal{S}_n$, using the product rule (15), we get

$$\mathbb{M}\mathcal{P}_n = \mathbb{M}\mathcal{E}_{n+1}^{-1}\, \mathbb{M}\mathcal{S}_n + \frac{\Delta_y^2}{4}\mathbb{D}\mathcal{E}_{n+1}^{-1}\, \mathbb{D}\mathcal{S}_n \,. \tag{106}$$

Now, using (106) into (105) we get

$$\mathcal{G}_{n+1}\, \mathbb{D}\mathcal{S}_n = \left(\mathcal{F}_{n+1}\mathbb{M}\mathcal{E}_{n+1}^{-1} - A_{n+1}\mathbb{D}\Phi\, I\right)\mathbb{M}\mathcal{S}_n \,, \tag{107}$$

where I denotes the identity matrix of order two, and $\mathcal{F}_n, \mathcal{G}_n$ are the matrices given by (103). By multiplying (107) by the adjoint matrix $\operatorname{adj} \mathcal{G}_{n+1}$, and noting that $\det \mathcal{G}_{n+1} \neq 0$, we obtain the system

$$\det \mathcal{G}_{n+1}\, \mathbb{D}\mathcal{S}_n = \operatorname{adj} \mathcal{G}_{n+1}\left(\mathcal{F}_{n+1}\mathbb{M}\mathcal{E}_{n+1}^{-1} - A_{n+1}\mathbb{D}\Phi\, I\right)\mathbb{M}\mathcal{S}_n \,,$$

thus we get the required equations. □

Theorem 3. *Let $\{S_n\}_{n\geq 0}$ be the sequence of Sobolev-type orthogonal polynomials related to the inner product (2), with* $\mathbf{u}$ *a semi-classical linear functional. Let the previous notations hold. The sequence $\{S_n\}_{n\geq 0}$ satisfies the second order divided-difference equation*

$$\widetilde{\widetilde{A}}_n\, \mathbb{D}^2 S_{n+1} + \widetilde{\widetilde{B}}_n\, \mathbb{M}\mathbb{D} S_{n+1} + \widetilde{\widetilde{C}}_n S_{n+1} = 0\,, \tag{108}$$

with rational coefficients given by

$$\begin{aligned}
\widetilde{\widetilde{A}}_n &= \left(\mathbb{M}\widehat{A}_{n+1} - \alpha\frac{\Delta_y^2}{4}\mathbb{D}L_{n,1} - U_1\mathbb{M}L_{n,1}\right) D_n - \frac{\Delta_y^2}{4}\left(\mathbb{D}\widehat{A}_{n+1} - \alpha\mathbb{M}L_{n,1} - U_1\mathbb{D}L_{n,1}\right)\widehat{J}_n \\
&\quad - U_1\left(\left(\alpha\frac{\Delta_y^2}{4}\mathbb{D}L_{n,2} + U_1\mathbb{M}L_{n,2}\right)\widehat{G}_n + \frac{\Delta_y^2}{4}\left(\alpha\mathbb{M}L_{n,2} + U_1\mathbb{D}L_{n,2}\right)E_n\right)\widehat{V}_n\,, \\
\widetilde{\widetilde{B}}_n &= \left(\mathbb{D}\widehat{A}_{n+1} - \alpha\mathbb{M}L_{n,1} - U_1\mathbb{D}L_{n,1}\right)\widehat{D}_n - \left(\mathbb{M}\widehat{A}_{n+1} - \alpha\frac{\Delta_y^2}{4}\mathbb{D}L_{n,1} - U_1\mathbb{M}L_{n,1}\right)\widehat{J}_n \\
&\quad - U_1\left(\left(U_1\mathbb{D}L_{n,2} + \alpha\mathbb{M}L_{n,2}\right)\widehat{G}_n + \left(U_1\mathbb{M}L_{n,2} + \alpha\frac{\Delta_y^2}{4}\mathbb{D}L_{n,2}\right)\widehat{E}_n\,,\right)\widehat{V}_n\,, \\
\widetilde{\widetilde{C}}_n &= -\widehat{D}_n\mathbb{D}L_{n,1} + \widehat{J}_n\mathbb{M}L_{n,1} - U_1\left(\widehat{G}_n\mathbb{D}L_{n,2} - \widehat{E}_n\mathbb{M}L_{n,2}\right)\widehat{V}_n\,,
\end{aligned}$$

where

$$\begin{aligned}
\widehat{D}_n &= (\widehat{E}_n\widehat{H}_n - \widehat{F}_n\widehat{G}_n)\mathbb{M}\Theta_{n,1} + \frac{\Delta_y^2}{4}U_1(\widehat{E}_n\mathbb{M}\Theta_{n,2} - \widehat{G}_n\mathbb{D}\Theta_{n,2})\mathbb{D}\Theta_{n,1}\,, \\
\widehat{J}_n &= (\widehat{E}_n\widehat{H}_n - \widehat{F}_n\widehat{G}_n)\mathbb{D}\Theta_{n,1} + U_1\mathbb{M}\Theta_{n,1}(\widehat{E}_n\mathbb{M}\Theta_{n,2} - \widehat{G}_n\mathbb{D}\Theta_{n,2})\,, \\
\widehat{V}_n &= \frac{\Delta_y^2}{4}\left(\mathbb{D}\Theta_{n,1}\right)^2 - \left(\mathbb{M}\Theta_{n,1}\right)^2\,,
\end{aligned}$$

with

$$\begin{aligned}
\widehat{E}_n &= U_1\mathbb{D}\widehat{A}_{n+1} - \alpha\mathbb{M}\widehat{A}_{n+1}\,, \quad \widehat{F}_n = \mathbb{M}\widehat{A}_{n+1} - U_1\mathbb{M}\Theta_{n,2}\,, \\
\widehat{G}_n &= U_1\mathbb{M}\widehat{A}_{n+1} - \alpha\frac{\Delta_y^2}{4}\mathbb{D}\widehat{A}_{n+1}\,, \quad \widehat{H}_n = \frac{\Delta_y^2}{4}\left(\mathbb{D}\widehat{A}_{n+1} - U_1\mathbb{D}\Theta_{n,2}\right)\,.
\end{aligned}$$

Here, Δ_y, α and U_1 are given in (12), (18) and (19), respectively.

Proof. The proof follows the same technique of the proof of Theorem 1. □

4 Final remarks

In this paper it is given an unified treatment to the study of Sobolev-type orthogonal polynomials, related to inner products (2),

$$\langle f, g\rangle_\lambda = \langle \mathbf{u}, fg\rangle + \lambda(\mathbb{D}f)(c)(\mathbb{D}g)(c)\,, \quad \lambda \geq 0\,,$$

where $\mathbb{D}$ is a general divided-difference operator (given in [17, Eq.(1.1)]) having the fundamental property of leaving a polynomial of degree $n-1$ when applied to a polynomial of degree n. Here, it was deduced several key properties of the sequences of orthogonal polynomials with respect to (2), namely, connection formulae; a three-term recurrence relation; difference systems such as (101)–(102), inherited when the semi-classical character of $\mathbf{u}$ holds, and a second order divided-difference equation with polynomial coefficients. There are several works on Sobolev orthogonal polynomials related to inner products such as (2) with specific operators $\mathbb{D}$, for instance, [6, 7, 12, 16] (see also their references lists). These works provide examples where the results of the present paper apply.

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