The Royal Norwegian Society of Sciences and Letters

ON STEADY SOLUTIONS OF A GENERALIZED WHITHAM EQUATION

Skrifter nr. 3-2021

Obed Opoku Afram

THE ROYAL NORWEGIAN SOCIETY OF SCIENCES AND LETTERS

Comments regarding accepted manuscripts, order of reprints, subscriptions etc.
should be sent to the Publisher.

ON STEADY SOLUTIONS OF A GENERALIZED WHITHAM EQUATION

Obed Opoku Afram

ISBN 978-82-93175-70-4 Trykk
ISBN 978-82-93175-71-1 Web
ISSN 0368-6310

Layout and print: Skipnes Kommunikasjon AS

Skipnes.no Svanemerket trykksak, 2041 0731

Transactions of The Royal Norwegian Society of Sciences and Letters
(Trans. R. Norw. Soc. Sci. Lett. 2021(3), 5-29)

ON STEADY SOLUTIONS OF A GENERALIZED WHITHAM EQUATION

Obed Opoku Afram

ABSTRACT. In this paper, we study the steady solutions of a generalized Whitham equation $\eta_t + \frac{3c_0}{2}\eta\eta_x + L_s\eta_x = 0$, where L_s is the nonlocal Fourier multiplier operator given by the symbol $m_s(\xi) = (\tanh\xi\,/\xi)^s$ with $s \in (0,1)$, for which we investigate whether a similar local and global theory is available as for the Whitham equation, which is the case $s = \frac{1}{2}$. We prove that there is a local analytic curve bifurcating at wave speed $\mu_1 = (\tanh 1)^s$, and these waves may be extended to large ones by global bifurcation. In our quest to understand the regularity of a possible highest wave for this generalized equation, we study the regularity of waves along the global bifurcation curve. We find that any highest wave of the generalized equation is Hölder continuous and has the regularity $C^s(\mathbb{S})$ at the point when the maximal value is $\varphi(0) = \frac{\mu}{2}$. In finding the regularity at the global level, we use the techniques from [1] and [2] combined in a new way which is necessary for the case $\frac{1}{2} < s < 1$. In addition, we study the properties of the symbol $m_s(\xi)$, and the corresponding integral kernel.

Keywords: Steady solutions; Whitham equation; local bifurcation; global bifurcation.

1. INTRODUCTION

The water wave equations pose severe challenges for rigorous analysis, modeling, and numerical simulation, from a mathematical viewpoint. Although water waves have intrigued mankind for thousands of years, it was not until the middle of the nineteenth century that the modern theory appeared, principally in the work of Stokes. The nineteenth century also produced useful models for tidal waves, solitary waves, the Korteweg–de Vries (KdV) equation, the Boussinesq models for shallow water waves, the Kelvin–Helmholtz instability, Cauchy–Poisson circular waves, Gerstner's rotational waves, Stokes' model for the highest wave, and Kelvin's model for ship wakes [3].

The Korteweg-de Vries equation (KdV) was introduced in 1895 to model the behavior of long waves on shallow water in close agreement with the observations of J. S. Russell [4]. The KdV model admits solitary waves which present soliton interaction: two solitary waves keep their shape and size after interaction although the ultimate position of each wave has been affected by the nonlinear interaction [5]. KdV has a bi-Hamiltonian structure which permits to obtain very precise information about the structure of the equation by the inverse scattering method, the equation being integrable [6]. The main challenge of the KdV equation was that it could not describe the breaking of the wave.

In 1967, a British-born American mathematician, G.B. Whitham proposed in [7] a non-local shallow water wave model for capturing the balance between linear dispersion and nonlinear effects, so that one would have smooth periodic and solitary waves, but also the features of wave breaking and surface singularities. Whitham [5] emphasized that the breaking phenomena is one of the most intriguing long-standing problems of water wave theory, and since the KdV equation can not describe breaking, he suggested the model

$$\eta_t + \frac{3c_0}{2h_0}\eta\eta_x + K_{h_0} * \eta_x = 0 \tag{1.1}$$

known as the Whitham equation. This equation combines a generic non-linear quadratic term with the exact linear dispersion relation for surface water waves on finite depth. Here, the kernel

$$K_{h_0} = \mathcal{F}^{-1}(c_{h_0}) \tag{1.2}$$

is the inverse Fourier transform of the phase speed

$$c_{h_0}(\xi) = \sqrt{\frac{g\tanh(h_0\xi)}{\xi}} \tag{1.3}$$

for the linearized water-wave problem; the constants g, h_o and $c_o = \sqrt{gh_o}$ denote, respectively, the gravitational constant of acceleration, the undisturbed water depth, and the limiting long wave speed. The function $\eta(t,x)$ describes the deflection of the fluid surface from the rest position at a point x at time t [5].

The Whitham equation (1.1) with the kernel (1.2) has some very interesting mathematical features. That is, it is generically non-local, making pointwise estimates difficult. Moreover, $c_{h_0}(\xi)$ has slow decay, and the kernel K_{h_o} is singular (it blows up at $x = 0$). This makes the Whitham equation in some important respects different from many other equations of the form (1.1) [8]. Whitham's actual motivation was to find a model that could feature the breaking of waves (wave breaking in this context describes a situation in which the spatial derivative of the function η becomes unbounded in finite time, while η itself remains bounded). Another interest was wave peaking which means that a wave forms a sharp crest or peak, such as a stagnation point in the full water wave problem [9, 10]. The Whitham equation captures the peaking phenomenon of the Stokes waves for the full water-wave problem. Interest in breaking, peaking and other phenomena connected with (1.1) has spawned a large amount of mathematical work. The monograph by Naumkin and Shishmarev [11] is devoted entirely to equations like (1.1).

In recent years, Hur [12] also dealt with the issue of wave breaking of bounded solutions with unbounded derivatives. One another hand, Hur and Tao [10] showed the wave breaking for the Whitham equation in a range of fractional dispersion. Hur and Johnson [13] also show that periodic traveling waves with sufficiently small amplitudes of the Whitham equation are spectrally unstable to long-wavelengths perturbations if the wave number is greater than a critical value,bearing out the Benjamin-Feir instability of Stokes waves.

Borluk et al. [14] investigated the simulation properties of the Whitham equation as a model for waves at the surface of a body of fluid. They found out that the periodic traveling-waves solutions of the Whitham equation are good approximations to solutions of the full free-surface water wave problem. Their results were due to the comparison of numerical solutions of the Whitham equation to numerical approximations of solutions of the full Euler free-surface water-wave problem.

Ehrnström and Kalisch [9] in 2009 proved that there exist small-amplitude periodic traveling waves with sub-critical speeds and as the period of these traveling waves tends to infinity, their velocities approach the limiting long-wave speed c_o. They further showed that there can be no solitary waves with velocities much greater than c_o. Again after performing some numerical analysis, it was proven that there is a periodic wave of greatest height ~ 0.642 h_o. In 2013, Ehrnström and Kalisch [8] again proved the existence of a global bifurcation branch of 2π-periodic, smooth, traveling-wave solutions of the Whitham equation. Furthermore, [8] showed that the solutions converge uniformly to a solution of Hölder regularity $\alpha \in (0,1)$, except possibly at the highest crest point (where $\alpha \leq$ ½).

The kernel K_{h_0} of the Whitham equation has not thoroughly been understood. In 2009, [9] features the integrability of this kernel in certain L^p spaces and smoothness away from the origin. However, very recently Ehrnström and Wahlén [1] provided an explicit representation formula for it and again showed that the integral kernel is completely monotone on the interval $(0, \infty)$ and also analytic with exponential decay away from the origin. They further proved the existence of a highest, cusped periodic traveling wave using the global bifurcation theory. Again, they found that the solution is P-periodic, even and strictly increasing on the interval $\left(\frac{-p}{2}, 0\right)$, satisfying $\varphi(0) = \frac{\mu}{2}$. The solution is furthermore smooth away from any crest, and obtains its optimal Hölder regularity $C^{\frac{1}{2}}(\mathrm{R})$ exactly at the crest, there by resolving Whitham's conjecture (G. B. Whitham conjectured that for (1.1), there would be a highest, cusped, travelling-wave solution). Truong et al. [15] on the other hand used an approach based on a nonlocal version of the center manifold theorem and found the highest wave as a limit point of the global bifurcation curve.

The paper [16] identified a scaling regime in which the Whitham equation can be derived from the Hamiltonian theory of surface water waves. After integrating the Whitham equation numerically, they showed that the equation gives a close approximation of inviscid free surface dynamics as described by the Euler equations. They then concluded that in a wide parameter range of amplitudes and wavelengths, the Whitham equation performs on par with or better than the Korteweg-de Vries (KdV) equation, the Benjamin Bona Mahony (BBM) equation and thePadé model.

Sanford et al. [17] focused on the stability of solutions in view of [9]. The numerical results presented in [17] suggest that all large-amplitude solutions are unstable, while small-amplitude solutions with large enough wavelength L are stable. Additionally, [17] proved that the periodic solutions with wavelengths smaller than a certain cut-off period always exhibit modulational instability. However, the cut-off wavelength is characterized by $kh_o = 1.145$, where $k = \frac{2\pi}{L}$ is the wave number and h_o is the mean fluid depth. The periodic traveling waves to the KdV do not exhibit this property but are spectrally stable [18]. Bronski and Johnson [19] also investigated the spectral stability of a family of periodic standing wave solutions to the generalized KdV equation.

In this present work, we consider a general version of the Whitham equation defined in (1.1), (1.2) and (1.3). That is taking $g, h_o \sim 1$, we have the generalized Whitham equation

$$\eta_t + \frac{3c_0}{2}\eta\eta_x + K_s * \eta_x = 0. \tag{1.4}$$

We then define the generalized Whitham symbol as

$$m_s(\xi) = \widehat{K_s}(\xi) = \left(\frac{\tanh \xi}{\xi}\right)^s, \quad 0 < s < 1, \tag{1.5}$$

whilst we have the generalized Whitham kernel defined by

$$K_s(x) = \mathcal{F}^{-1}\{m_s(\xi)\} = \frac{1}{2\pi}\int_{\mathbb{R}} m_s(\xi)e^{ix\xi}\, d\xi. \tag{1.6}$$

The aim of this paper is to study the generalized Whitham equation (1.4) and to see if a similar local and global theory is available as for the Whitham equation with $s = \frac{1}{2}$ (see [1, 2]). As one goal, we wanted to understand the highest order regularity of the solution for (1.4). We use the techniques from [1] and [2], combine in a new way which is necessary for the case $\frac{1}{2} < s < 1$, to find the regularity at the global level. This research requires the study of Banach algebras, Hölder spaces, Fréchet differentiability, implicit function theorem in Banach spaces, Stieltjes and completely monotone functions and the bifurcation theory.

The outline for our investigation is as follows. In Section 2 we lay out the analytic preliminaries. Most importantly, we perform some studies on the generalized Whitham kernel. In Section 3 we report on the local bifurcation of the generalized Whitham equation, which we will then extend to the global continuous curves of solutions in the next section. In Section 4 we finally investigate the global bifurcation for the generalized Whitham equation, where we prove the highest order regularity of the solution for (1.4).

References for borrowed materials and proofs are provided throughout the text. The proofs in Sections 2, 3 and 4 are adaptations of the ones in [1, 2, 8, 9], where the generalized Whitham equation, kernel and symbol have been taken into consideration.

2. PRELIMINARIES

In this section, we discuss the generalized Whitham kernel and its properties. More precise details about the Whitham kernel (1.2) are presented in [1, 9]. We shall consider the definitions of both the generalized Whitham kernel and symbol in (1.6) and (1.5) respectively in our arguments. We also note that $m_s(\xi)$ is clearly an even function.

Monotonicity property of the generalized Whitham kernel. Our aim is to show that the generalized Whitham symbol (1.5) belongs to the class of completely monotone functions. A more general theory can be found in the monograph [20], however our discussion is centered around the generalized Whitham symbol (1.5).

The generalized Whitham symbol can be represented as $m_s(\xi) = f(\xi^2)$, where

$$f(\lambda) = \left(\frac{\tanh\sqrt{\lambda}}{\sqrt{\lambda}}\right)^s, \lambda \geq 0 \; and \; 0 < s < 1.$$

We have that $|\xi| = \sqrt{\lambda}$ and it is also clearly seen that $f(\lambda)$ is positive on the interval $(0, \infty)$ and also has a nite limit as $\lambda \to 0$. That is

$$\lim_{\lambda\to 0} f(\lambda) = \lim_{\lambda\to 0}\left(\frac{\tanh\sqrt{\lambda}}{\sqrt{\lambda}}\right)^s = \left(1 - \lim_{\lambda\to 0}\left(\tanh\sqrt{\lambda}\right)^2\right)^s = 1^s < \infty,$$

It is clearly seen that the function $m_s(\xi)$ in (1.5) is real analytic, even and strictly decreasing on $(0,1)$. Now as $\xi \to 0$, we have

$$\lim_{\xi\to\infty} f(\xi) = \lim_{\xi\to\infty}\left(\frac{\tanh\xi}{\xi}\right)^s = \left(\lim_{\xi\to\infty}\frac{\sinh\xi}{\xi}.\lim_{\xi\to\infty}\frac{1}{\cosh\xi}\right)^s = 0,$$

since $\lim_{\xi\to\infty}\frac{1}{\cosh\xi}$ rapidly turns to 0. We know that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$ by the Riemann-Lebesgue Lemma. Consequently,

$$\int_{-\infty}^{\infty} K_s(x)\, dx = 1. \tag{2.1}$$

Proof of (2.1). If $f \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} f(x)dx = \hat{f}(0) = \int_{\mathbb{R}} f(x)e^{ix\xi}|_{\xi=0}\, dx.$$

$$\Rightarrow \int_{\mathbb{R}} K_s(x)dx = \widehat{K}(0) = \left(\frac{\tanh\xi}{\xi}\right)^s|_{\xi=0} = 1.$$

□

We can therefore deduce from the proof of (2.1) that

$$\|K_s\|_{L^1(\mathbb{R})} = \left\|\mathcal{F}^{-1}\left\{\left(\frac{\tanh\xi}{\xi}\right)^s\right\}\right\|_{L^1(\mathbb{R})} = 1,$$

this implies that $K_s \in L^1(\mathbb{R})$, since the function $m_s(\xi)$ is analytic and the inverse Fourier transform has rapid decay.

Alternatively, to show that $K_s \in L^1(\mathbb{R})$ we split the integral according to

$$\|K_s\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |K_s(x)|dx = \int_{|x|\le 1} |K_s(x)|dx + \int_{|x|\ge 1} |K_s(x)|dx < \infty,$$

since the limit of $\widehat{K_s}(\xi)$ as ξ approaches both 0 and ∞ is finite, hence it is plain that K_s has finite $L^1(\mathbb{R})$-norm. We have that the smooth and even function $m_s(\xi)$ is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$, reaching its global maximum of unit size at $\xi = 0$. As $|\xi| \to \infty$, it vanishes with the rate $|\xi|^{-s}$. The function $m_s(\xi)$ is even and integrable and we must now show that it is a completely monotone function.

When you look at Corollary 7.4 in [20] we have that: *Let g be a positive function on* $(0, \infty)$. *Then g is a Stieltjes function if, and only if, g(0+) exists in* $[0, \infty]$ *and g extends analytically to* $\mathbb{C}\backslash(-\infty, 0]$ *such that* $\Im m\, z . \Im m\, g(z) \le 0$, *i.e. g maps* $\mathbb{H}^{\uparrow}$ *to* $\mathbb{H}^{\downarrow}$ *and vice versa*
Ehrnström and Wahlén [1, Remark 2.10], made a remark on Corollary 7.4 in [20] that positive constant functions are examples of Stieltjes functions. And it follows easily by basic properties of analytic functions that a non-constant Stieltjes function maps $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \Im m\, z > 0\}$ to $\mathbb{C}_- = \{z \in \mathbb{C} \colon \Im m\, z < 0\}$. Again they remarked that if g is not identically 0, then $1/g(z)$ is a Nevanlinna function (also known as Herglotz or Pick function) and the corresponding function $1/g(x)$ is then a complete Bernstein function.

We shall make use of [1, Proposition 2.20], which we state in the form suitable for our purpose.

Theorem 2.1. *Let g and f be two functions satisfying* $g(\xi) = f(\xi^2)$. *Then g is the Fourier transform of an even, integrable and completely monotone function if f is Stieltjes with* $\lim_{\lambda\to 0} f(\lambda) < \infty$ *and* $\lim_{\lambda\to\infty} f(\lambda) = 0$.
Proof of Theorem 2.1. It is clearly seen that $\lim_{\lambda\to 0} f(\lambda) < \infty$ and $\lim_{\lambda\to\infty} f(\lambda) = 0$. We now show that $f(\lambda) = (h(\lambda))^s$ is a Stieltjes function for any $s \in (0,1)$. Let

$$h(\lambda) = \left(\frac{\tanh\sqrt{\lambda}}{\sqrt{\lambda}}\right), \qquad \lambda \ge 0.$$

It is noted that the reciprocal of $h(\lambda)$

$$\lambda \mapsto \frac{\sqrt{\lambda}}{\tanh\sqrt{\lambda}}$$

is positive on $(0, \infty)$ with the finite limit 1 as $\lambda \to 0$, and extends to an analytic function on $\mathbb{C}\backslash(-\infty, 0]$ if we let $\sqrt{\lambda}$ denote the principal branch of the square root. It also maps $\mathbb{C}_+$ to $\mathbb{C}_+$. We note that $\sinh(z) = -i\sin(iz)$ and $\sinh z \ge z$ for $z \ge 0$ then by a straightforward calculation

$$\begin{aligned}\Im m\left(\frac{z}{\tanh z}\right) &= \Im m\left(\frac{z(e^z + e^{-z})}{(e^z - e^{-z})} \cdot \frac{\overline{(e^z - e^{-z})}}{\overline{(e^z - e^{-z})}}\right)\\ &= \frac{\Im m\, z\,(2\sinh(2\Re e\, z) - 2i\ \sin(2\Im m\, z))}{|e^z - e^{-z}|^2}\\ &> \frac{4}{|e^z - e^{-z}|^2}(\Im m\, z\, \Re e\, z - \Re e\, z\, \Im m\, z) = 0\end{aligned}$$

when $\Re e\, z, \Im m\, z > 0$ from which it follows that $\Im m\,(\sqrt{\lambda}/\tanh\sqrt{\lambda}) > 0$ when $\Im m\,\lambda > 0$. This implies that $\lambda \mapsto \tanh\sqrt{\lambda}/\sqrt{\lambda}$ satisfies the conditions of Corollary 7.4 in [20], hence the function h is a Stieltjes function. In agreement with Lemma 2.12 in [1], which states that: *If g is a Stieltjes function, then so is* g^{α} *for any* $\alpha \in (0,1]$, we can then say that $(h(\lambda))^{s} = f(\lambda)$ is a Stieltjes function. In conclusion, the function $g(\xi) = m_s(\xi)$ is the Fourier transform of an even, integrable and completely monotone function since *f* is Stieltjes with $\lim_{\lambda\to 0} f(\lambda) < \infty$ and $\lim_{\lambda\to\infty} f(\lambda) = 0$.

□

The generalized Whitham kernel $K_s(x)$ in (1.6) is completely monotone on $(0,\infty)$. In particular, it is positive, strictly decreasing and strictly convex for $x > 0$ as proved by [1].

We now briey discuss some properties of the convolution operator and also examine how it acts on periodic functions.

The convolution operator L_s. The convolution operator from the Whitham map is much needed in our bifurcation analysis and it is necessary that we know its properties. We refer the reader to [9, 21, 22] for more details. We define convolution operator by

$$L_s := K_s * \tag{2.2}$$

Theorem 2.2 (Bounded linear operator). *L_s is a bounded linear operator on $L^2(\mathbb{R})$; that is, if $f \in L^2(\mathbb{R})$ then $\|L_s f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}$.*

Proof of Theorem 2.2. By applying both the convolution and Plancherel's theorem, we have that

$$\begin{aligned}\|L_s f\|_{L^2(\mathbb{R})} &= \|\mathcal{F}(L_s f)\|_{L^2(\mathbb{R})}\\ &\leq \sqrt{\int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2 d\xi}, \quad \left|\frac{\tanh\xi}{\xi}\right| \leq 1\\ &= \left\|\hat{f}\right\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.\end{aligned}$$

□

Theorem 2.3 (Symmetric bounded linear operator). *The operator L_s is symmetric on $L^2(\mathbb{R})$; that is if $f, g \in L^2(\mathbb{R})$ then $(L_s f, g)_{L^2(\mathbb{R})} = (f, L_s g)_{L^2(\mathbb{R})}$.*

Proof of Theorem 2.3. Suppose that $f, g \in L^2(\mathbb{R})$, then applying Plancherel's theorem, we have that

$$\begin{aligned}(L_s f, g)_{L^2(\mathbb{R})} &= (\mathcal{F}(L_s f), \mathcal{F}(g))_{L^2(\mathbb{R})}\\ &= \int_{\mathbb{R}} \mathcal{F}(L_s f)\overline{\mathcal{F}(g)}\, d\xi\\ &= \int_{\mathbb{R}} \left(\frac{\tanh\xi}{\xi}\right)^{s} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi = (f, L_s g)_{L^2(\mathbb{R})}.\end{aligned}$$

It follows that L_s is a symmetric bounded linear operator on the space $L^2(\mathbb{R})$.

□

Next we discuss how the convolution operator acts on periodic functions. If $f \in L^{\infty}(\mathbb{R})$ is periodic and even (since K_s is in $L^1(\mathbb{R})$) then by the properties of periodic convolutions, we can write the integral

$$\int_{-\infty}^{\infty} K_s(x-y)f(y)dy = \sum_{n=-\infty}^{\infty} \int_{-p}^{p} K_s(x-y+2np)f(y)dy$$
$$= \int_{-p}^{p} T(x-y)f(y)dy,$$

where

$$T(x) = \sum_{n=-\infty}^{\infty} K_s(x+2np).$$

The definition of $T(x)$ shows that it is $2p$-periodic, even and continuous on $[-p,p]\backslash\{0\}$. It is proved by Ehrnström and Kalisch in [9] that $T(x)$ belongs to $L^p(-p,p)$, for $1 \leq p < 2$ using Minkowski's inequality. Therefore, according to the Carleson-Hunt theorem [23], $T(x)$ can be approximated pointwise by its Fourier series. Thus

$$T(x) = \frac{1}{2p}\hat{T}_0 + \frac{1}{p}\sum_{n=1}^{\infty} \hat{T}_n \cos\left(\frac{n\pi x}{p}\right) = \frac{1}{p}\sum_{n=1}^{\infty}{}' \hat{T}_n \cos\left(\frac{n\pi x}{p}\right),$$

where the prime indicates that the first term of the sum is multiplied by $\frac{1}{2}$. Now the Fourier coefficients of T are given by

$$\hat{T}_n = \int_{-p}^{p} \sum_{k=-\infty}^{\infty} K_s(x+2kp)e^{-\frac{ixn\pi}{p}}dx$$
$$= \sum_{k=-\infty}^{\infty} \int_{-(2k+1)p}^{-(2k-1)p} K_s(x+4kp)e^{-\frac{i(x+2kp)n\pi}{p}}dx$$
$$= \int_{-\infty}^{\infty} K_s(x)e^{-\frac{ixn\pi}{p}}dx$$
$$= \hat{K}_s\left(\frac{n\pi}{p}\right).$$

One can observe that the periodic problem is given by the same multiplier as the problem at hand, hence we have the representation

$$K_s * \ f(x) = \int_{-p}^{p} T(x-y)f(y)dy$$
$$= \int_{-p}^{p} \frac{1}{p}\sum_{n=0}^{\infty}{}' \hat{K}_n\left(\frac{n\pi}{p}\right)\cos\left(\frac{n\pi(x-y)}{p}\right)f(y)dy$$
$$= \frac{1}{p}\sum_{n=0}^{\infty}{}' \hat{K}_n\left(\frac{n\pi}{p}\right)\left\{\frac{e^{\frac{in\pi x}{p}}\hat{f}_n + e^{-\frac{-in\pi x}{p}}\hat{f}_{-n}}{2}\right\}$$
$$= \frac{1}{p}\sum_{n=0}^{\infty}{}' \hat{f}_n\hat{K}_s\left(\frac{n\pi}{p}\right)\cos\left(\frac{n\pi x}{p}\right),$$

Since $\hat{f}_n = \hat{f}_{-n}$ for even f.

Fourier multipliers on Hölder spaces. We present a brief summary of certain properties of the Fourier multiplier operators, given by classical symbols for the purpose of our analysis. We refer the reader to [24, 25] for a more detailed argument.

A smooth, real-valued function g on ℝ is said to be in the symbol class S^m if for some constant $c_k > 0$ and any non-negative integer k, the estimate

$$\left|\partial_\xi^k g(\xi)\right| \leq c_k(1+|\xi|)^{m-k}$$

holds. If $\alpha \geq 0$ is real, we may consider those functions in L^2 such that

$$\int (1+|\xi|^2)^\alpha |\hat{g}(\xi)|^2 d\xi < \infty \tag{2.3}$$

to define the Sobolev space $H^\alpha = W^{\alpha,2}$. We note that since $1 \leq (1+|\xi|^2)^\alpha$ the finiteness of this integral implies $\int \left|\hat{f}(\xi)\right|^2 d\xi < \infty$ which implies $f \in L^2$ by the Plancherel theorem.

We will now in the next section discuss the local bifurcation for the Whitham equation which will later be extended to the global continuous curves of solutions in Section 4.

3. LOCAL BIFURCATION FOR THE WHITHAM EQUATION

The solution of the Whitham equation is dene on the space C^α_{even}, $\alpha \in (0,1)$, that is, the space of even and α-Hölder continuous real-valued functions on the unit circle $\mathbb{S}$. We also take into consideration that the convolution operator (2.2) is a bounded linear operator on $C^\alpha_{\text{even}}(\mathbb{S})$ and there is nothing particular about the choice of α in this section and that all small enough solutions are smooth.

In considering steady solutions with the propagation speed $c > 0$ of a right-going traveling wave, we make the usual ansatz[1] $\eta(x,t) = \varphi(x - ct)$. Using this form, the equation (1.4) transforms into

$$-c\varphi' + \frac{3c_o}{2}\varphi\varphi' + K_s * \varphi' = 0$$

Which may be integrated to

$$-c\varphi + \frac{3c_o}{4}\varphi^2 + K_s * \varphi = \beta$$

for some real constant β. For solution $\varphi \in L^2(\mathbb{R})$, it appears that

$$\begin{aligned} \|K_s * \varphi\|_{L^2(\mathbb{R})} &= \left\|\widehat{K}_s\hat{\varphi}\right\|_{L^2(\mathbb{R})} \\ &\leq \|K_s\|_{L^1(\mathbb{R})}\|\hat{\varphi}\|_{L^2(\mathbb{R})} \\ &\leq \|\hat{\varphi}\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}, \end{aligned}$$

the convolution $K_s * \varphi$ is in $L^2(\mathbb{R})$ since K_s is in $L^1(\mathbb{R})$. Therefore, the left-hand side must vanish as $|x| \to \infty$, and we shall consider the case for which $\beta = 0$ [9]. The scalings $\frac{3}{4}\varphi \mapsto \varphi$ and $\frac{1}{c_o}K_s \mapsto K_s$ then yield the normalised equation

$$-\mu\varphi + \varphi^2 + K_s * \varphi = 0 \tag{3.1}$$

where $\mu \coloneqq c \backslash c_o$ is the non-dimensional wave speed.

[1] An assumption about the form of an unknown function which is made in order to facilitate solution of an equation or other problem.

Local bifurcation theory. Local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change [26]. The Whitham symbol in (1.5) is considered as a generic non-local smoothing operator in the form of a Fourier multiplier, that is, for all ξ

$$m_s(\xi) = \left(\frac{\tanh \xi}{\xi}\right)^s \approx \frac{1}{(1+|\xi|)^s}. \tag{3.2}$$

Indeed,

i. $\frac{\tanh \xi}{\xi} = \frac{\sinh \xi}{\xi \cosh \xi} = \frac{e^{\xi}-e^{-\xi}}{\xi(e^{\xi}+e^{-\xi})} = \frac{e^{2\xi}-1}{\xi(e^{2\xi}+1)} \approx \frac{1}{1+|\xi|}$, if $|\xi| \geq 1$.
   However, $\frac{\tanh \xi}{\xi} \approx \frac{1}{|\xi|} \approx \frac{1}{1+|\xi|}$, for $|\xi| \gg 1$.

ii. If $|\xi| \ll 1$, then

$$\frac{e^{2\xi}-1}{\xi(e^{2\xi}+1)} \approx \frac{e^{2\xi}-1}{2\xi} \approx \frac{\int_0^{\xi} e^{2\eta}\,d\eta}{\xi} \leq \frac{2\int_0^{\xi} e^{2\eta}\,d\eta}{\xi} \approx \frac{3}{2}.$$

   This implies $\frac{\tanh \xi}{\xi} \approx \frac{1}{1+|\xi|}$ for $|\xi| \ll 1$.

We can then say that m_s belongs to the symbol class $\mathcal{S}^{-s}(\mathbb{R})$ and therefore its estimate is given by

$$\left|\partial_{\xi}^{k} m_s(\xi)\right| \leq \frac{1}{|\xi|^{s+k}} \approx \frac{1}{(1+|\xi|)^{s+k}}.$$

We must also note that

$$\frac{\tanh \xi}{\xi} \approx \frac{1}{1+|\xi|} \approx \frac{1}{(1+|\xi|^2)^{\frac{1}{2}}}. \tag{3.3}$$

To illustrate how the analysis used for the Whitham equation can be applied to a larger class of equation, a local bifurcation is performed for the Whitham equation.

We shall make use of [8, Theorem 3.1], which we state in a form suitable for our purposes as we consider the generalized Whitham equation (1.4). The proof of Theorem 3.1 and Proposition 3.2 are the author's own adaption of the one in [8].

Theorem 3.1 (Functional-analytic formulation). *For fixed α and $\mu > 0$, the solutions in $C^{\alpha}_{\text{even}}(\mathbb{S})$ of the Whitham equation (3.1) coincide with the kernel of the analytic operator F : $C^{\alpha}_{\text{even}}(\mathbb{S}) \times \mathbb{R}_{>0} \to C^{\alpha}_{\text{even}}(\mathbb{S})$ given by*

$$F(\varphi,\mu) = \mu\varphi - L_s\varphi + N(\varphi)$$

where L_s is bounded linear and compact and the non-linear operator $N(\varphi)$ has zero linear part, meaning that $D_{\varphi}N[0,\mu] = 0$. Thus $D_{\varphi}N[0,\mu]$ is Fredholm of index 0.

We must note that the operators L_s and N are independent of μ.

Proof of Theorem 3.1. We first consider the Whitham equation (3.1) and define L_s as in (2.2). $C^{\alpha}_{\text{even}}(\mathbb{S})$ is a subalgebra of the Wiener algebra of 2π-periodic functions with absolutely convergent Fourier series [27]. Hence, for $f \in C^{\alpha}_{\text{even}}(\mathbb{S})$ and by the Fourier series expansion, we have that

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(kx) \quad \text{and} \quad \sum_{k=0}^{\infty} |a_k| < \infty.$$

Now, from (2.2) we have

$$L_s f(x) = K_s * f(x)$$
$$= \sum_{k=0}^{\infty} a_k \left(\frac{\tanh k}{k}\right)^s \cos(kx) \tag{3.4}$$

The Fourier multiplier symbol in the above expression belongs to the symbol class $\mathcal{S}^{-s}(\mathbb{R})$ as shown in (3.2) and $L_s f$ is a bounded linear operator on $C^{\alpha}_{\text{even}}(\mathbb{S}) \to C^{\alpha+s}_{\text{even}}(\mathbb{S})$ for $\alpha + s \notin \mathbb{Z}$. From (2.3), (3.2) and (3.3) we have that

$$\|L_s f\|^2_{H^{\alpha}(\mathbb{R})} = \int \left|\widehat{L_s f}(\xi)\right|^2 (1+|\xi|^2)^{\alpha} d\xi$$
$$\approx \int \left|\frac{\hat{f}(\xi)}{(1+|\xi|^2)^{\frac{s}{2}}}\right|^2 (1+|\xi|^2)^{\alpha} d\xi$$
$$= \int \left|\hat{f}(\xi)\right|^2 (1+|\xi|^2)^{\alpha-s} d\xi = \|f\|^2_{H^{\alpha-s}(\mathbb{R})}.$$

Conversely,

$$\|L_s f\|^2_{H^{\alpha+s}(\mathbb{R})} = \int \left|\widehat{L_s f}(\xi)\right|^2 (1+|\xi|^2)^{\alpha+s} d\xi$$
$$\approx \int \left|\frac{\hat{f}(\xi)}{(1+|\xi|^2)^{\frac{s}{2}}}\right|^2 (1+|\xi|^2)^{\alpha+s} d\xi$$
$$= \int \left|\hat{f}(\xi)\right|^2 (1+|\xi|^2)^{\alpha} d\xi = \|f\|^2_{H^{\alpha}(\mathbb{R})}.$$

Since $L_s: H^{\alpha-s} \to H^{\alpha}$ is continuous implies that $L_s: H^{\alpha} \to H^{\alpha+s}$ is also continuous, hence L_s Is invertible with bounded linear inverse $L_s^{-1}: C^{\alpha+s}_{\text{even}}(\mathbb{S}) \to C^{\alpha}_{\text{even}}(\mathbb{S})$. Due to the compactness of the embedding $C^{\beta}_{\text{even}}(\mathbb{S}) \hookrightarrow C^{\alpha}_{\text{even}}(\mathbb{S})$, $\beta > \alpha$, the operator is compact on $C^{\alpha}_{\text{even}}(\mathbb{S})$. We then define the mapping $L_s: C^{\alpha}_{\text{even}}(\mathbb{S}) \times \mathbb{R}_{>0} \to C^{\alpha}_{\text{even}}(\mathbb{S})$ by

$$F_s(\varphi,\mu) := \mu\varphi - L_s\varphi - \varphi^2, \tag{3.5}$$

where F_s is analytic. We also have $F_s(0,\mu) = 0$, and the linearization $D_{\varphi}F_s[0,\mu] = \mu - L_s$ is Fredholm of index 0.

□

We restate [8, Corollary 3.2], as we consider the general $s \in (0,1)$.

Proposition 3.2. *For each integer* $k \geq 1$, *there exist* $\mu_k := (\tanh(k)/k)^s$ *and a local, analytic curve*

$$\varepsilon \mapsto (\varphi(\varepsilon), \mu(\varepsilon)) \in C^{\alpha}_{\text{even}}(\mathbb{S}) \times (0,1)$$

of nontrivial $2\pi/k$*-periodic Whitham solutions with* $D_{\varepsilon}\varphi(0) = cos(kx)$ *that bifurcates from the trivial solution curve* $\mu \mapsto (0,1)$ *at* $\big(\varphi(0), \mu(0)\big) = (0, \mu_k)$. *In a neighborhood of the bifurcation point* $(0,\mu_k)$ *these are all nontrivial solutions of* $F_s(\varphi,\mu) = 0$ *in* $C^{\alpha}_{even}(\mathbb{S}) \times (0,1)$, *and there are no other bifurcation points* $\mu > 0$, $\mu \neq 1$ *for solutions in* $C^{\alpha}_{\text{even}}(\mathbb{S})$. *At* $\mu = 1$ *the trivial solution curve* $\mu \mapsto (0,\mu)$ *intersects the curve* $\mu \mapsto (\mu - 1,\ \mu)$ *of constant solutions* $\varphi = \mu - 1$*; together these constitute all solutions in* $C^{\alpha}_{\text{even}}(\mathbb{S})$ *in a neighborhood of* $(\varphi,\mu) = (0,1)$.

Proof of Proposition 3.2. We first consider the expansions[2];

$$\varphi(\varepsilon) = \varepsilon\cos(x) + \varepsilon^2\left(\frac{1}{2}C_1 + C_2\cos(2x)\right) + \mathcal{O}(\varepsilon^3),$$
$$\mu(\varepsilon) = \mu^* + \varepsilon^2(C_1 + C_2) + \mathcal{O}(\varepsilon^3),$$
$$\mu^* = \mu_1 = (\tanh(1))^s,$$
$$C_1 = \frac{1}{\mu^* - 1} \quad \text{and}$$
$$C_2 = \frac{1}{2\left(\mu^* - \left(\frac{\tanh(2)}{2}\right)^s\right)}.$$

At the limit where $\varepsilon \to 0$, we have that $\varphi(0) = (0)$, $\mu(0) = \mu^*$ and also $D_\varepsilon\varphi(0) = \cos(x)$. Now the fact that $D_\varphi F_s[0,\mu] = \mu\,\mathrm{id} - L_s$ is Fredholm of index 0 and the formula (3.4) shows that μ_k are all simple eigenvalues of L_s, and that no other eigenvalues $\mu > 0$ exist. The assertion then follows from the analytic version of the Crandall-Rabinowitz theorem for bifurcation from a simple eigenvalue [28, Theorem 8.4.1]. From the equation

$$-\mu\varphi + \varphi^2 + L_s\varphi = 0$$

we have that,

$$|\varphi|^2 = |\mu\varphi - L_s\varphi|$$
$$\leq \mu|\varphi| - |L_s\varphi|$$
$$\|\varphi\|_\infty^2 = |\mu|\|\varphi\|_\infty - \|L_s\|_{L^\infty(\mathbb{R})}\|\varphi\|_\infty$$
$$\|\varphi\|_\infty = (|\mu| - 1)\|\varphi\|_\infty$$

where $\varphi = \mu - 1$ is a solution in $\varphi \in C^\alpha_{\text{even}}(\mathbb{S})$. For the case $k = 0$ we have the limit as $k \mapsto 0$ of $\mu_k := (\tanh(k)/k)^s$ to be 1, that is $\mu_0 = 1$. Moreover, the fact that the solutions are $2\pi/k$-periodic can be seen by restricting attention to the subspaces $\{\varphi \in C^\alpha_{\text{even}}(\mathbb{S}) : \varphi \text{ is } 2\pi/k - \text{periodic}\}$, and corresponding to the case $k = 0$, it is instantly verified that $\varphi = \mu - 1$ is a solution. By uniqueness, this family must therefore constitute the local bifurcation curve at $\mu = 1$. Since for all other $\mu > 0$ the linearization $D_\varphi F_s[0,\mu]$ is Fredholm of index zero with trivial kernel. It is a consequence of the implicit function theorem that the vanishing solution is locally the unique solution in $C^\alpha_{\text{even}}(\mathbb{S})$.

□

In the next section we discuss the global bifurcation for the Whitham equation.

[2] These equations will be justified in the next section under Bifurcation formulas.

4. GLOBAL BIFURCATION FOR THE WHITHAM EQUATION

We now study the global bifurcation for the Whitham equation (1.1) and some properties along the bifurcation branch (Uniform convergence and the characterization of blow-up). The discussions in this section follow a similar pattern as presented by Ehrnström and Kalisch [8] and Ehrnström et al [2]. In this present discussion we consider the generalized Whitham symbol described in (1.5) instead of

$$\widehat{K}(\xi) = \sqrt{\frac{\tanh \xi}{\xi}}$$

as defined in [8]. The theorems in this section are true for the convolution operator L_s, which maps C^α into $C^{\alpha+s}$ for $\alpha + s \notin \mathbb{Z}$. The choice of α has some implications for the proof about the global argument. In particular $\alpha > s$ makes it easier to rule out alternative three in Theorem 4.4 along the curve of solutions and that alternative one occurs if $\alpha > s$.

Boundedness and smoothness of the Whitham solution. Let F_s be the Whitham operator from Theorem 3.1, defined by (3.4) and (3.5). With

$$U := \left\{(\varphi, \mu) \in C^\alpha_{\text{even}}(\mathbb{S}) \times (0,1) : \varphi < \frac{\mu}{2}\right\},$$

we let

$$S := \{(\varphi, \mu) \in U : F_s(\varphi, \mu) = 0\} \tag{4.1}$$

be our set of solutions (we refer readers to [1] for a detailed justification of the choice of *U* and *S*).

We restate Lemmas 4.1, 4.2 and 4.3 in [8].

Lemma 4.1 (L^∞-bound). *Let $\mu > 0$ and bounded. Any bounded Whitham solution satisfies*

$$\|\varphi\|_\infty \leq \mu + \|L_s\|_{\mathcal{L}(L^\infty(\mathbb{S}))} \tag{4.2}$$

where $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators on a Banach space X.

Lemma 4.2 (Fredholm). *The Fréchet derivative $D_\varphi F_s[\varphi, \mu]$ is a Fredholm operator of index 0 for all $(\varphi, \mu) \in U$.*

Lemma 4.3 *Suppose $(\varphi, \mu) \in S$, then the function φ is smooth and bounded, and the closed sets of S are compact in $C^\alpha_{\text{even}}(\mathbb{S}) \times (0,1)$.*

See [8, Section 4] for the proof of Lemmas 4.1, 4.2 and 4.3.

We next introduce the concept of global bifurcation in relation to the Whitham equation.

Global bifurcation theory. We shall make use of the global one-dimensional branches theorem [28, Theorem 9.1.1], which we state in the form suitable for our purposes.

Theorem 4.4 (Global bifurcation). *Suppose* $(0,\mu) \in U$ *and* $F_s(0,\mu) = 0$ *for all* $\mu \in \mathbb{R}$, *then the local bifurcation curves* $\varepsilon \mapsto (\varphi(\varepsilon), \mu(\varepsilon))$ *of solutions to the Whitham equation from Proposition 3.2 extend to global continuous curves of solutions* $\mathbb{R}_{\geq 0} \to S$, *with S as in (4.1). Moreover, at least one of the following alternatives holds:*

(i) $\|\varphi(\varepsilon)\|_{C^\alpha(\mathbb{S})} \to \infty$ *as* $\varepsilon \to \infty$.

(ii) $(\varphi(\varepsilon), \mu(\varepsilon))$ *approaches the boundary of S as* ε *tend to* ∞.

(iii) *The function* $\varepsilon \mapsto (\varphi(\varepsilon), \mu(\varepsilon))$ *is T-periodic, for some* $T \in (0, \infty)$.

We can rely on Lemma 4.2 and 4.3 and show that, for $\varepsilon > 0$ taken to be sufficiently small, $\mu(\varepsilon)$ is not identically equal to a constant; that is if any of the derivatives $\mu^{(k)}(0) \neq 0$. In our case it turns out that $\dot{\mu}(0) = 0$, however one can show that $\ddot{\mu}(0) \neq 0$.

This is being proved by the use of the Lyapunov-Schmidt reduction. To discuss the Lyapunov-Schmidt reduction and the bifurcation formulas in the next two sections we first make some definitions suitable for our purposes. Let $\mu^* := \mu_1$ be the bifurcation point from Proposition 3.2 and let

$$\varphi^*(x) := \cos(x).$$

Let furthermore

$$M := \left\{ \sum_{k \neq 1} a_k \cos(kx) \in C^\alpha(\mathbb{S}) \right\},$$

and

$$N := \ker\left(D_\varphi F_s[0, \mu^*]\right) = \operatorname{span}(\varphi^*).$$

Then $C^\alpha_{\text{even}}(\mathbb{S}) = M \oplus N$ and we can use the canonical embedding $C^\alpha(\mathbb{S}) \hookrightarrow L^2(\mathbb{S})$ to define a continuous projection $\coprod : C^\alpha(\mathbb{S}) \to \mathbb{C}$ by

$$\coprod_\varphi := \langle \varphi, \varphi^* \rangle_{L^2(\mathbb{S})} \varphi^*,$$

with

$$\langle u, v \rangle_{L^2(\mathbb{S})} := \frac{1}{\pi} \int_{\mathbb{S}} uv \, dx.$$

Lyapunov-Schmidt reduction. The Lyapunov-Schmidt procedure is a method for reducing the question of existence of solutions to an infinite-dimensional equation, locally in a neighbourhood of a known solution, to an equivalent one involving an equation in finite dimensions, quite commonly (though not always) in just two dimensions [28].

Theorem 4.5 (Lyapunov-Schmidt Reduction [29]). *There exist a neighborhood* $\mathcal{O} \times Y \subset U$ *around* $(0, \mu^*)$ *in which the problem*

$$F_s(\varphi, \mu) = 0 \tag{4.3}$$

is equivalent to

$$\Phi(\varepsilon\varphi^*, \mu) := \coprod F_s(\varepsilon\varphi^* + (\varepsilon\varphi^*, \mu), \mu) = 0 \tag{4.4}$$

for functions $\psi \in C^\infty(\mathcal{O}_N \times Y, M)$, $\Phi \in C^\infty(\mathcal{O}_N \times Y, N)$, *and* $\mathcal{O}_N \subset N$ *an open neighborhood of the zero function in N. One has*

$$\Phi(0, \mu^*) = 0,$$
$$\psi(0, \mu^*) = 0,$$
$$D_\varphi \psi(0, \mu^*) = 0,$$

and solving the finite-dimensional problem (4.4) *provides a solution*

$$\varphi = \varepsilon\varphi^* + \psi(\varepsilon\varphi^*, \mu)$$

of the infinite-dimensional problem (4.3).

We next discuss the concept of bifurcation formulas in relation to the solution curve (bifurcation curve) of the Whitham equation.

Bifurcation formulas. The shape of the bifurcation curve follows from the bifurcation formulas. If $D^2_{\varphi\varphi}F_s[0,\mu^*](\varphi^*,\varphi^*) \notin R(D_\varphi F_s[0,\mu^*])$, the number $\dot{\mu}(0)$ is nonzero, and the bifurcation is called *transcritical* (see Figure 1).

However, if $D^2_{\varphi\varphi}F_s[0,\mu^*](\varphi^*,\varphi^*) \in R(D_\varphi F_s[0,\mu^*])$ then $\dot{\mu}(0)$ and the local shape of the curve is determined by $\ddot{\mu}(0)$. Now, if $\dot{\mu}(0) < 0$, the bifurcation is *subcritical*, and if $\dot{\mu}(0) > 0$, it is *supercritical*. In both cases the diagram is referred to as a *pitchfork bifurcation* (see Figure 1).

The bifurcation formulas in [8, Theorem 4.6] are modified with general $s \in (0,1)$. The proof is again an adaption of the one in [8].

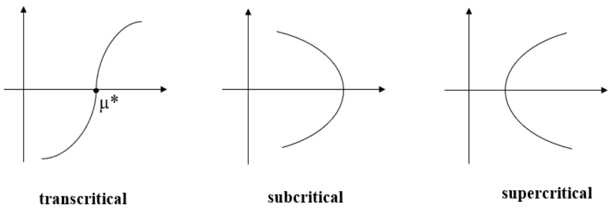

Figure 1. An illustration of the pitchfork bifurcation.

Theorem 4.6 (Bifurcation Formulas). *Let*

$$\mu^* = (\tanh(1))^s,$$

$$C_1 = \frac{1}{\mu^* - 1},$$

and

$$C_2 = \frac{1}{2\left(\mu^* - \left(\frac{\tanh(2)}{2}\right)^s\right)}.$$

The main bifurcation curve $(k = 1)$ *for the Whitham equation found in Proposition 3.2 satisfies*

$$\varphi(\varepsilon) = \varepsilon\cos(x) + \varepsilon^2\left(\tfrac{1}{2}C_1 + C_2\cos(2x)\right) + \mathcal{O}(\varepsilon^3) \tag{4.5}$$

and

$$\mu(\varepsilon) = \mu^* + \varepsilon^2(C_1 + C_2) + \mathcal{O}(\varepsilon^3) \tag{4.6}$$

in the limit as $\varepsilon \to 0$ *for* $(\varphi(\varepsilon), \mu(\varepsilon)) \in C^\alpha_{\text{even}}(\mathbb{S}) \times (0,1)$. *In particular,* $\ddot{\mu}(0) < 0$ *and Proposition 3.2 describes a subcritical pitchfork bifurcation*.

Proof of Theorem 4.6. The analysis for μ is performed first, followed by that of φ. It is known that $\varepsilon \mapsto \mu(\varepsilon)$ is analytic at $\varepsilon = 0$ and that $\mu(0) = \mu^*$, however it remains to show that $\dot{\mu}(0) = 0$ and also to determine $\ddot{\mu}(0)$. We refer to [29, Section I.6] for the bifurcation formulas used in this proof. We have that

$$D^2_{\varphi\varphi} F_s[0,\mu^*](\varphi^*,\varphi^*) = -2\varphi^{*2},$$
$$D^2_{\varphi\mu} F_s[0,\mu^*]\varphi^* = \varphi^*,$$

and the value of $\dot{\mu}(0)$ may be explicitly calculated as

$$\dot{\mu}(0) = -\frac{1}{2}\frac{\langle D^2_{\varphi\varphi} F_s[0,\mu^*](\varphi^*,\varphi^*),\varphi^*\rangle_{L^2(\mathbb{S})}}{\langle D^2_{\varphi\mu} F_s[0,\mu^*]\varphi^*\rangle_{L^2(\mathbb{S})}} = 0,$$

since

$$\int_{\mathbb{S}} \cos^3(x)\ dx = 0.$$

Moreover, when $\dot{\mu}(0) = 0$ one has that

$$\ddot{\mu}(0) = -\frac{1}{3}\frac{\langle D^3_{\varphi\varphi\varphi}\, \Phi[0,\mu^*](\varphi^*,\varphi^*,\varphi^*),\varphi^*\rangle_{L^2(\mathbb{S})}}{\langle D^2_{\varphi\mu} F_s[0,\mu^*]\varphi^*\rangle_{L^2(\mathbb{S})}}.$$

Since $D^2_{\varphi\mu} F_s[0,\mu^*] = \mathrm{id}$ we find that the denominator is of unit size. One calculates

$$D_\varphi \Phi[\varphi,\mu]\varphi^* = \coprod D_\varphi F_s[\varphi+\psi(\varphi,\mu),\mu\,]\big(\varphi^* + D_\varphi\psi(\varphi,\mu)\varphi^*\big),$$

$$\begin{aligned} D^2_{\varphi\varphi}\ \Phi[\varphi,\mu](\varphi^*,\varphi^*) &\\ &= \coprod D^2_{\varphi\varphi} F_s[\varphi+\psi(\varphi,\mu),\mu\,]\big(\varphi^* + D_\varphi\psi(\varphi,\mu)\varphi^*, \varphi^* + D_\varphi\psi(\varphi,\mu)\varphi^*\big) \\ &\quad + \coprod D_\varphi F_s[\varphi+\psi(\varphi,\mu),\mu\,] D^2_{\varphi\varphi}\psi[\varphi,\mu](\varphi^*,\varphi^*), \end{aligned}$$

and, in view of that F_s is quadratic in φ,

$$\begin{aligned} D^3_{\varphi\varphi\varphi}\ \Phi[\varphi,\mu](\varphi^*,\varphi^*,\varphi^*) &\\ &= 3\coprod D^2_{\varphi\varphi} F_s[\varphi+\psi(\varphi,\mu),\mu\,]\Big(\varphi^* + D_\varphi\psi(\varphi,\mu)\varphi^*, D^2_{\varphi\varphi}\psi[\varphi,\mu](\varphi^*,\varphi^*)\Big) \\ &\quad + \coprod D_\varphi F_s[\varphi+\psi(\varphi,\mu),\mu\,] D^3_{\varphi\varphi\varphi}\psi[\varphi,\mu](\varphi^*,\varphi^*,\varphi^*). \end{aligned}$$

Applying the form of $D_\varphi F_s$ together with

$$\psi(0,\mu^*) = D_\varphi\psi[0,\mu^*]\varphi^* = 0,$$

one finds that

$$\begin{aligned} D^3_{\varphi\varphi\varphi}\ \Phi[\varphi,\mu](\varphi^*,\varphi^*,\varphi^*) &\\ &= \coprod(\mu^*\mathrm{id} - L_s) D^3_{\varphi\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*,\varphi^*) - 6\coprod \varphi^* D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*). \end{aligned}$$

We have $\mathrm{ran}(\mu^*\mathrm{id} - L_s) = M$, so that $\coprod(\mu^*\mathrm{id} - L_s) = 0$. We thus need to determine $\varphi^* D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*)$. Since $D_\varphi F_s[0,\mu^*] = \mu^*\mathrm{id} - L_s$ is an isomorphism on M, it is possible (see again [29, Section I.6]) to rewrite $D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*)$ as

$$\begin{aligned} D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*) &= -\big(D_\varphi F_s[0,\mu^*]\big)^{-1}(\mathrm{id} - \coprod) D^2_{\varphi\varphi} F_s[0,\mu^*](\varphi^*,\varphi^*) \\ &= -\big(D_\varphi F_s[0,\mu^*]\big)^{-1}(\mathrm{id} - \coprod)(-2\varphi^{*2}) \\ &= \big(D_\varphi F_s[0,\mu^*]\big)^{-1}(2\cos^2(x)) \\ &= \big(D_\varphi F_s[0,\mu^*]\big)^{-1}(1+\cos(2x)) \qquad (4.9) \\ &= \frac{1}{\mu^*-1} + \frac{\cos(2x)}{\mu^* - \left(\frac{\tanh(2)}{2}\right)^s}. \end{aligned}$$

After multiplication with $\cos(x)$ this equals

$$\frac{\cos(x)}{\mu^*-1}+\frac{\cos(x)}{2\left(\mu^*-\left(\frac{\tanh(2)}{2}\right)^s\right)}+\frac{\cos(3x)}{2\left(\mu^*-\left(\frac{\tanh(2)}{2}\right)^s\right)}.$$

In view of (4.7) and (4.8) the coefficient in front of $\cos(x)$ equals $\frac{1}{2}\ddot{\mu}(0)$. All taken into consideration, we obtain (4.6) via a Maclaurin series, and one easily checks that $\ddot{\mu}(0)<0$.

To prove (4.5), we make use of the formula

$$\varphi(\varepsilon)=\varepsilon\varphi^*+\psi(\varepsilon\varphi^*,\mu(\varepsilon)) \tag{4.10}$$

from the Lyapunov-Schmidt reduction (cf. Theorem 4.5). We already know that $\varphi(0)=0$ and $\dot{\varphi}(0)=\cos(x)$, so it remains to calculate $\ddot{\varphi}(0)$. It follows from (4.10) that $\ddot{\varphi}(\varepsilon)=D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*)+2D^2_{\varphi\mu}\psi[0,\mu^*](\varphi^*,\dot{\mu}(0))+D^2_{\mu\mu}\psi[0,\mu^*](\dot{\mu}(0),\dot{\mu}(0))+D_\mu\psi[0,\mu^*]\ddot{\mu}(0)$. Since $\psi(0,\mu)\equiv 0$ where ψ exists, we have $D_\mu\psi(0,\mu^*)=0$. Combining this with $\dot{\mu}(0)=0$ one finds that

$$\ddot{\varphi}(0)=D^2_{\varphi\varphi}\psi[0,\mu^*](\cos(x),\cos(x)),$$

so that the proposition now follows from (4.9).

□

Remark 4.7. *We note that*

$$(\mu^*-L_s)^{-1}\sum_{k=0} a_k\cos(kx)=\sum_{k=0}\frac{a_k}{\mu^*-\left(\frac{\tanh(k)}{k}\right)^s}\cos(kx).$$

We next discuss some properties along the bifurcation branch of the Whitham equation.

Properties along the bifurcation branch. In considering a sequence of Whitham solutions $(\varphi_n,\mu_n)\in S$ where $\mu_n\in(0,1)$, then Lemma 4.1 implies that φ_n is uniformly bounded in $C(\mathbb{S})$.That is

$$\|\varphi\|^2_\infty\le\|\mu\varphi\|_\infty-\|L_s\|_{L^\infty(\mathbb{R})}\|\varphi\|_\infty=(|\mu|+1)\|\varphi\|_\infty, \tag{4.11}$$

so that $(\varphi_n)_n$ is bounded whenever $(\mu_n)_n$ is bounded. We know that the kernel K_s of the Whitham equation is integrable and continuous almost everywhere, hence it follows by dominated convergence that $(L_s\varphi_n)_n$ is equicontinuous. The Arzela-Ascoli Lemma can be applied to conclude that a subsequence of φ_n converges uniformly in $C(\mathbb{S})$, when dealing with periodic solutions.

We restate Lemma 4.3 in [2], suitable for our purposes as we consider the generalized Whitham equation (3.1).

Lemma 4.8. *Let φ be an even, non-constant,* 2π*-periodic solution of* (3.1) *such that φ is non-decreasing on $(-\pi,0)$ with $\varphi\le\frac{\mu}{2}$ on $(-\pi,\pi)$. Then φ smooth and strictly increasing on* $(-\pi,0)$*, and as $x\to 0$ we have*

$$\frac{\mu}{2}-\varphi(x)\gtrsim|x|^s, \tag{4.12}$$

for s in the symbol m_s, and $|x|\ll\delta$.

See [1] and [2] for the proof of Lemma 4.8. The proofs in [1] and [2] are for the cases $s=\frac{1}{2}$ and $s=1$, and that these proofs may be generalized.

Theorem 4.9 (Uniform Convergence). *Any sequence of Whitham solutions* $(\varphi_n, \mu_n) \in S$ *has a subsequence which converges uniformly to a solution* φ *in* $C(\mathbb{S})$. *If* $\varphi < \frac{\mu}{2}$ *uniformly on* $\mathbb{R}$, *then the solution is smooth. Assuming* φ *is even and strictly increasing on* $(-\pi, 0)$, *if* $\varphi \leq \frac{\mu}{2}$ *attains the maximal value* $\varphi(0) = \frac{\mu}{2}$, *then the solution is of regularity* $C^r(\mathbb{S})$ *for all* $r < s$, *with*

$$\left|\frac{\mu}{2} - \varphi(x)\right| \lesssim |x|^s$$

for $|x| \ll 1$.

The global regularity is not stated as $C^s(\mathbb{S})$ in the theorem, even though the pointwise estimate is. And to establish the global C^s-Hölder regularity ($\varphi \in C^s(\mathbb{S})$), one needs to use a second double symmetrisation formula and also combine the lower and upper estimates in Lemma 4.8 and Theorem 4.9 respectively.

Proof of Theorem 4.9. We begin the proof by considering φ as a limit in $C^0(\mathbb{S})$. The first part of Theorem 4.9 has already been established in connection to (4.11). It remains to prove the global regularity results in the third part of the theorem. To establish this, we first prove the global regularity with no particular restriction on r and that $r + s < 1$ with $s \leq \frac{1}{2}$. We know from the previous section that L_s maps $C^r(\mathbb{S})$ into $C^{r+s}(\mathbb{S})$ for $r + s \notin \mathbb{Z}$ and $r, s \in (0,1)$. In the case when $2\varphi < \mu$ everywhere, we have from Lemma 4.3 that $\varphi \in C^\infty(\mathbb{S})$, which in particular implies that $\varphi \in C^r(\mathbb{S})$, for any $r \in (0,1)$. Assume now that $\varphi \in C^0(\mathbb{S})$ with $\varphi \leq \frac{\mu}{2}$ and $\varphi(0) = \frac{\mu}{2}$ is even and strictly increasing on the interval $(-\pi, 0)$, then we have

$$\begin{aligned} |\varphi(0) - \varphi(y)| &= \frac{\mu}{2} - \varphi(y) \\ &= \left(\frac{\mu^2}{4} - L_s\varphi(y)\right)^{\frac{1}{2}} \\ &= (L_s\varphi(0) - L_s\varphi(y))^{\frac{1}{2}} \\ &\lesssim |y|^{\frac{r+s}{2}}, \end{aligned}$$

for any $r \in (0,1)$ (i.e there is no particular restriction on r). This means that if $\varphi \in C^r(\mathbb{S})$ then $L_s\varphi \in C^{r+s}(\mathbb{S})$ and φ has Hölder regularity $\frac{1}{2}(r+s)$ at 0. However, this does not prove the regularity at all points, hence we give an argument for $r + s < 2$ for all $r \in (0,1)$ and $\varphi(0) = \frac{\mu}{2}$, for a regularity at the highest point. To establish the regularity at all points, we first consider the situation where $r + s < 1$. Assuming that $0 \leq x < y < \pi$ and from (3.1) we have that

$$L_s\varphi(x) = N(\varphi(x)) = \mu\varphi - \varphi^2, \tag{4.13}$$

with $N'(\varphi(x)) = \mu - 2\varphi$ and $N''(\varphi(x)) = -2$. By expansion of N, we have

$$N(\varphi(y)) = N(\varphi(x)) + N'(\varphi(x))(\varphi(y) - \varphi(x)) + \frac{1}{2}N''\varphi(\xi)(\varphi(y) - \varphi(x))^2$$

$$\begin{aligned} N(\varphi(x)) - N(\varphi(y)) &= (\varphi(y) - \varphi(x))(\mu - 2\varphi(x)) + (\varphi(y) - \varphi(x))^2 \\ &= 2(\varphi(y) - \varphi(x))\left(\frac{\mu}{2} - \varphi(x)\right) + (\varphi(y) - \varphi(x))^2. \end{aligned}$$

It follows from (4.13) that the above estimate yields

$$L_s\varphi(x) - L_s\varphi(y) \geq \big(\varphi(y) - \varphi(x)\big)\left(\frac{\mu}{2} - \varphi(x)\right) \tag{4.14}$$

and

$$L_s\varphi(x) - L_s\varphi(y) \geq \big(\varphi(y) - \varphi(x)\big)^2. \tag{4.15}$$

Now for $r + s < 1$, we have that $\frac{1}{2}(r+s) > r$ for $r < s$. We know from before that if $\varphi \in C^r(\mathbb{S})$ then $L_s\varphi \in C^{r+s}(\mathbb{S})$ and by this $\varphi \in C(\mathbb{S})$ implies $L_s\varphi \in C^s(\mathbb{S})$, hence $\varphi \in C^s(\mathbb{S})$ whenever $2\varphi \neq \mu$. Now, considering the map $L_s\colon \varphi \in C(\mathbb{S}) \to \varphi \in C^s(\mathbb{S})$, we have from (4.15) that

$$|\varphi(y) - \varphi(x)| \leq |x-y|^{\frac{s}{2}}.$$

Hence $\varphi \in C^{\frac{s}{2}}(\mathbb{S})$. This argument can be repeated for $\varphi \in C^{\frac{s}{2}}(\mathbb{S})$ and so forth. Thus

$$\varphi \in C^r(\mathbb{S}),\ \ 0 < r < s.$$

In the case when $s \in \left(\frac{1}{2}, 1\right)$, the value of $r + s$ will at some point exceed 1 but not 2 (i.e $r + s < 2$). We now consider $0 < r < s < 1$ for $r + s > 1$ with $s \in \left(\frac{1}{2}, 1\right)$ and by the mean value theorem if $f \in C^{1+\gamma}(\mathbb{S})$ with $f'(0) = 0$, then

$$\begin{aligned} |f(x) - f(y)| &= |x-y||f'(0) - f'(\xi)| \\ &\leq |x-y||\xi|^\gamma \\ &\leq |x-y||y|^\gamma. \end{aligned}$$

We know from before that if $\varphi \in C^r(\mathbb{S})$ then $L_s\varphi \in C^{r+s}(\mathbb{S})$, however, applying the estimate of the mean value theorem to the function $L_s\varphi$ and that $(L_s\varphi)'(0) = 0$, it follows that for $0 \leq x < y < \pi$ we have that

$$|L_s\varphi(x) - L_s\varphi(y)| \lesssim |x-y||y|^{r+s-1}. \tag{4.16}$$

If we assume that $x < |x-y|$ and whenever $\varphi \in C^r(\mathbb{S})$, $r \in (0,1)$, the estimate (4.15), (4.16) and the triangle inequality altogether yield

$$|\varphi(x) - \varphi(y)| \lesssim |x-y|^{\frac{r+s}{2}}, \tag{4.17}$$

which is valid uniformly for all $0 \leq x < y < \pi$ and all solutions φ. On the other hand, when $|x-y| \leq x$ we have from (4.14), (4.16) and the triangle inequality that

$$\big(\varphi(x) - \varphi(y)\big)\left(\frac{\mu}{2} - \varphi(x)\right) \lesssim |x-y||x|^{r+s-1}. \tag{4.18}$$

Now, the estimates (4.12) and (4.18) gives that

$$\varphi(x) - \varphi(y) \lesssim \frac{|x-y||x|^{r+s-1}}{|x|^s} = \frac{|x-y|}{x^{1-r}}, \tag{4.19}$$

whenever $\varphi \in C^r(\mathbb{S})$ for some $r \in (0,1)$.

We observe that if $|f(x) - f(y)| < 1$, then for a given $\beta \in (0,1)$ we have that

$$|f(x) - f(y)| \leq (f(x) - f(y))^\beta$$

Now for a given $\beta \in (0,1)$, we interpolate between (4.17) (at the point $x = 0$) and (4.19) with the estimate (4.20). Applying the reverse triangle inequality to $|x-y| \leq x$ we have that $\big||x| - |y|\big| \leq |x-y| \leq x$, which implies that $y < 2x$. Now, using $y < 2x$ we have

$$\frac{1}{|x-y|^{\beta}} \cdot \frac{\big(\varphi(x) - \varphi(y)\big)}{\left(\frac{\mu}{2} - \varphi(y)\right)} \leq \frac{\big(\varphi(x) - \varphi(y)\big)^{\beta}}{\left(\frac{\mu}{2} - \varphi(y)\right)^{\beta}} \cdot \frac{1}{|x-y|^{\beta}}$$

$$\frac{\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{\beta}} \leq \frac{\big(\varphi(x) - \varphi(y)\big)^{\beta}}{\left(\frac{\mu}{2} - \varphi(y)\right)^{\beta}} \left(\frac{\mu}{2} - \varphi(y)\right)^{1-\beta}$$

$$\lesssim \left(\frac{x}{x^{1-r}}\right)^{\beta} x^{-\beta} \left(x^{\frac{r+s}{2}}\right)^{1-\beta}$$

$$\lesssim x^{(r-1)\beta + \left(\frac{r+s}{2}\right)(1-\beta)}$$

which is bounded for all $0 \leq x < y < \pi$, provided that $\beta \leq \frac{r+s}{2-r+s}$ and $r + s < 2$. In particular,when we consider $\beta = \frac{r+s}{2-r+s}$ then the estimate above becomes

$$\varphi(x) - \varphi(y) \lesssim |x-y|^{\frac{r+s}{2-r+s}}, \tag{4.21}$$

valid for all solutions φ when $|x-y| \leq x$ and $r + s < 2$ and for all $0 \leq x < y < \pi$ whenever $\varphi \in C^{r}(\mathbb{S})$ with $r \in (0,1)$, $r + s < 2$, and $s > r$. We will from now establish the uniformity for all solutions $\varphi \in C^{r}(\mathbb{S})$. It follows that if $\varphi \in C^{r}(\mathbb{S})$ is a solution of the generalized Whitham equation for some $r \in (0,1)$ with $s > r$ and $r + s < 2$, then $\varphi \in C^{\frac{r+s}{2-r+s}}(\mathbb{S})$. If we fix $r_0 \in \left(0, \frac{s}{2}\right)$ and define the recurrence relation

$$\lambda_n = r_0, \quad \lambda_{n+1} = \frac{\lambda_n + s}{2 - \lambda_n + s}, \qquad n \geq 0,$$

yielding that $\varphi \in C^{\lambda_n}(\mathbb{S})$ for all $n \in \mathbb{N}$ and $\frac{\lambda_n + s}{2 - \lambda_n + s} > \lambda_n$ which implies $\lambda_n < s$. We observe that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is clearly strictly increasing with $\lambda_n \mapsto s$, that is φ belongs to $C^{s}(\mathbb{S})$, and the estimates of $C^{r}(\mathbb{S})$ are uniform for all $r \in (0, s)$.

We note that if $A \lesssim B$ then there exist a constant c such that $A \leq cB$ and with this estimate we now show that there is a constant c_r (depending on r) such that at the point $x = 0$ the estimate (4.17) becomes

$$\left|\frac{\mu}{2} - \varphi(x)\right| \leq c_r |y|^{\frac{r+s}{2}} \tag{4.22}$$

for $r + s < 2$ and $r < s$. In considering $|y| \ll 1$, $s > r$ and $\frac{r_n + s}{2} > r_n$ we have from (4.22) that

$$\left|\frac{\mu}{2} - \varphi(x)\right| \leq c_r |y|^{\frac{r+s}{2}} \leq c_r |y|^{r_n}$$

for $r_n \in (0, s)$. In other to control the constant c_r and also by observing that $\varphi \in C^{r}(\mathbb{S})$ for all $r \in (0, s)$, we define the constant by

$$c_r \coloneqq \sup_{y \in \mathbb{S}} \frac{\frac{\mu}{2} - \varphi(y)}{|y|^{r}} < \infty$$

for all $r \in (0, s)$ and the goal is to establish that one may let $r \nearrow s$ in other to obtain the desired bound

$$\frac{\mu}{2} - \varphi(x) \lesssim |x|^{s}$$

for all x sufficiently small. To derive that, we first let

$$v(y) = \frac{\mu}{2} - \varphi(y) = \varphi(0) - \varphi(y)$$

and by letting $0 < \delta \ll 1$ and also noting that for all $y \in (0, \delta)$ we have from (4.15) that

$$\left(\frac{\mu}{2} - \varphi(y)\right)^2 \leq L_s\varphi(0) - L_s\varphi(y).$$

Now using the fact that K_s and φ are even and 2π-periodic, and also noting that

$$\varphi(0) = \varphi\widehat{K}_s(0) = \int_{\mathbb{R}} K_s(x)\varphi(0),$$

we have that

$$\begin{aligned}
\left(v(x)\right)^2 &\leq L_s\varphi(0) - L_s\varphi(y) \\
&= \int_{-\pi}^{\pi} \left(K_s(y) - K_s(x-y)\right)\varphi(y)\,dy \\
&= \int_{-\pi}^{\pi} \left(K_s(y) - K_s(x-y)\right)\varphi(y)\,dy - \int_{-\pi}^{\pi} K_s(y)\varphi(0)\,dy + \int_{-\pi}^{\pi} K_s(x-y)\varphi(0)\,dy \\
&= \int_{-\pi}^{\pi} \left(K_s(x-y) - K_s(y)\right)\left(\varphi(0) - \varphi(y)\right)dy \\
&= \int_{-\pi}^{\pi} \left(K_s(x-y) - K_s(y)\right)v(y)\,dy.
\end{aligned}$$

Taking $y \mapsto -y$ and the fact that K_s is even, we have

$$K_s(x-y) = K_s\left(-(-x+y)\right) = K_s\left(-(-x-y)\right) = K_s(x+y).$$

Putting all estimates together, we have the representation

$$\left(v(x)\right)^2 = \frac{1}{2}\int_{-\pi}^{\pi} |K_s(-x+y) + K_s(x+y) - 2K_s(y)|\,v(y)\,dy. \tag{4.24}$$

We claim that there is a constant c_r' such that

$$\frac{1}{2}\int_{-\pi}^{\pi} |K_s(-x+y) + K_s(x+y) - 2K_s(y)|\,\psi(y)^{r+s}\,dy \leq c_r'\left(\psi(x)\right)^{2r}, \quad 0 \leq r < s \tag{4.25}$$

where

$$\psi(x) = \min\{|x|, 1\}.$$

For $|x| \geq 1$, this follows directly from the integrability of K_s and the fact that $\|\psi\|_\infty \leq 1$. For $|x| \leq 1$, we use the splitting

$$K_s(x) = \mathcal{F}^{-1}\left\{\frac{1}{|\xi|^s} + \frac{(\tanh|\xi|)^s - 1}{|\xi|^s}\right\}, \tag{4.26}$$

where the first term has inverse Fourier transform $\frac{1}{|x|^s}$, while the second term is integrable and exponentially decaying and hence has a real-analytic transform. Since we already considered the case when $s \in \left(\frac{1}{2}, 1\right)$, we have that $r + s < r + 2s - 1$ for $r < s$, and for $|x| \leq 1$ implies that $|x|^{r+s} \leq |x|^{r+2s-1}$. Now, for the first part in (4.26), we use the identity

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left|\frac{1}{|-x+y|^s} + \frac{1}{|x+y|^s} - \frac{2}{|y|^s}\right| |y|^{r+2s-1}\,dy \\
&\qquad = x^{r+s}\int_{\mathbb{R}} \left|\frac{1}{|m-1|^s} + \frac{1}{|m+1|^s} - \frac{2}{|m|^s}\right| |m|^{r+2s-1}dm
\end{aligned}$$

Where $y = xm$ and the integral converges since

$$\left|\frac{1}{|m-1|^s} + \frac{1}{|m+1|^s} - \frac{2}{|m|^s}\right| \lesssim |m|^{-s-1}, \qquad |m| \gg 1.$$

The estimate above could be $|m|^{-s-2}$ for $s \in \left(0, \frac{1}{2}\right)$ but in the case as $s \to 1$, $|m|^{-s-2}$ is the appropriate estimate for $|m| \gg 1$. We have the estimate (4.25) by using the fact that $|x|^{r+s} \leq |x|^{2r}$ for $|x| \leq 1$ and $r < s$. Now, (4.24) and (4.25) gives the estimate

$$\big(v(x)\big)^2 \leq c_r'\big(\psi(x)\big)^{2r} \tag{4.27}$$

which is uniform for all $(r, x) \in (0, s) \times (0, \delta)$ with $0 < \delta \ll 1$. Rearranging (4.27) yields the estimate

$$\left(\frac{\frac{\mu}{2} - \varphi(x)}{|x|^r}\right)^2 \leq c_r'$$

valid for all $x \in (0,\delta]$ and $r \in (0, s)$. For $x \in (\delta,1]$ we note that the left-hand side in the above inequality is uniformly bounded for all $r \in [0,s)$, hence we find

$$\left(\frac{\frac{\mu}{2} - \varphi(x)}{|x|^r}\right)^2 \leq \max(c_r', 1)$$

valid for all $x \in (0,1)$ and $r \in (0, s)$. Taking the supremum over $r < s$ now yields $(c_r')^2 \leq \max(c_r', 1)$, hence $c_r' \leq 1$ uniformly in $r \in (0, s)$. With this uniform bound and now taking $r \mapsto s$ in (4.23), we now have

$$\frac{\mu}{2} - \varphi(x) \leq c_r|x|^s \tag{4.28}$$

valid for all $|x| \ll 1$ and $s \in (0,1)$.

□

Proposition 4.10. *In Theorem 4.4, alternative (ii) occurs and that alternative (i) implies (ii) for $\alpha > s$. Given any sequence of positive numbers $\varepsilon_n \nearrow \infty$, there exist a limiting wave φ obtained as the uniform limit of sequence $\{\varphi(\varepsilon_{nk})\}_k$. The limiting wave is a solution of (3.1) with*

$$\mu = \lim_{k\to\infty} \mu(\varepsilon_{nk})$$

and that it is even and satisfies $\varphi(0) = \frac{\mu}{2}$. Further, it is smooth, strictly increasing on $(-\pi, 0)$, and satisfies

$$\frac{\mu}{2} - \varphi(x) \lesssim |x|^s$$

for all $|x| \ll 1$ sufficiently small.

Lemma 5.5 in [2] gives a detailed account as to why alternative (iii) in Theorem 4.4 cannot occur. We now establish the argument that alternative (i) in Theorem 4.4 implies alternative (ii). The proof of Theorem 4.11 is an adaption of the one in [8], but with general $s \in (0,1)$.

Theorem 4.11 (Characterization of Blow-up). *Alternative (i) in Theorem 4.4 can happen only if*

$$\lim_{\varepsilon\to\infty} \inf \inf_{x\in\mathbb{R}} \left(\frac{\mu(\varepsilon)}{2} - \varphi(x;\varepsilon)\right) = 0. \tag{4.29}$$

In particular, alternative (i) implies alternative (ii).

Proof of Theorem 4.11. Assume that

$$\lim_{\varepsilon\to\infty} \inf \inf_{x\in\mathbb{R}} \left(\frac{\mu(\varepsilon)}{2} - \varphi(x;\varepsilon)\right) \geq \delta, \tag{4.30}$$

for some $\delta > 0$. From (4.14) we have the estimate

$$(\varphi(x) - \varphi(y))\left(\frac{\mu}{2} - \varphi(x)\right) \lesssim L_s\varphi(x) - L_s\varphi(y)$$

$$|\varphi(x) - \varphi(y)| \lesssim \frac{|L_s\varphi(x) - L_s\varphi(y)|}{\delta}.$$

Any solution in (4.30) of the generalized Whitham equation satisfies the estimate above. Now since $L_s: C(\mathbb{S}) \to C^s(\mathbb{S})$ is continuous and the family $\{\varphi(\varepsilon)\}_\varepsilon$ is uniformly bounded in $C(\mathbb{S})$ (see Lemma 4.1), it follows that $\{\varphi(\varepsilon)\}_\varepsilon$ is uniformly bounded in $C^s(\mathbb{S})$ too. Repeating the argument for L_s as a continuous operator $C^s(\mathbb{S}) \to C^\alpha(\mathbb{S})$ where $\alpha < 1$ and $\alpha > s$ yields that

$$\|\varphi(\varepsilon)\|_{C^\alpha(\mathbb{S})} \leq C_s\delta^{-1}, \quad \alpha \in (0,1),$$

for some constant C_s depending only on L_s. By assumption μ is bounded (see Lemma 4.1) and $\|\varphi(\varepsilon)\|_{C^\alpha(\mathbb{S})} \to \infty$ is possible only if (4.29) holds.

□

We finally conclude that $\mu(\varepsilon)$ is bounded and hence, according to Theorem 4.9, there is a subsequence $(\varphi_{nk})_k$ which converges uniformly to a solution φ_o as $k \to \infty$. If we consider μ_o as the wave speed associated to φ_o, then by the uniform convergence properties of φ_{nk}, it follows that $\varphi_o(0) = \frac{\mu_o}{2}$ and $\varphi \in C^\alpha(\mathbb{S})$. When the alternative (i) in Theorem 4.4 happens, alternative (iii) does not and that alternative (i) implies alternative (ii) with the condition that $0 < s < \alpha < 1$. The solution of the normalised equation (3.1) from the generalized Whitham equation (1.4) is even, strictly increasing on $(-\pi, 0)$, smooth on $\mathbb{S}$, and has the regularity $C^s(\mathbb{S})$ for $\alpha < s$ at the point when the maximal value is $\varphi(0) = \frac{\mu}{2}$.

Acknowledgements. The author would like to thank Professor Mats Ehrnström for useful conversations and also for providing valuable feedback on previous versions of this paper. The author is also thankful to Professor Olav Kristian Gunnarson Dovland for helpful comments on an earlier draft of this paper.

References

[1] Ehrnström, Mats, and Erik Wahlén. "On Whitham's conjecture of a highest cusped wave for a nonlocal dispersive equation." *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*. Vol. 36. No. 6. Elsevier Masson, 2019.

[2] M. Ehrnström, M. A. Johnson, and K. M. Claassen, "Existence of a highest wave in a fully dispersive two-way shallow water model," *Archive for Rational Mechanics and Analysis*, vol. 231, no. 3, pp. 1635–1673, 2019.

[3] T. Bridges, M. Groves, P. Milewski, and D. Nicholls, "Theory of water waves." https://www.newton.ac.uk/files/reports/scientific/tww.pdf, 2014. [Online; accessed 8-January-2020].

[4] A. Constantin and J. Escher, "Wave breaking for nonlinear nonlocal shallow water equa-tions," *Acta Mathematica*, vol. 181, no. 2, pp. 229–243, 1998.

[5] G. B. Whitham, *Linear and nonlinear waves*, vol. 42. John Wiley & Sons, 2011.

[6] H. McKean, "Integrable systems and algebraic curves," in *Global Analysis*, pp. 83–200, Springer, 1979.

[7] G. B. Whitham, "Variational methods and applications to water waves," in *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 299, pp. 6–25, The Royal Society, 1967.

[8] M. Ehrnström and H. Kalisch, "Global bifurcation for the whitham equation," *Mathematical Modelling of Natural Phenomena*, vol. 8, no. 5, pp. 13–30, 2013.

[9] M. Ehrnström and H. Kalisch, "Traveling waves for the Whitham equation," *Differential and Integral Equations*, vol. 22, no. 11/12, pp. 1193–1210, 2009.

[10] V. M. Hur and L. Tao, "Wave breaking for the Whitham equation with fractional dispersion," *Nonlinearity*, vol. 27, no. 12, p. 2937, 2014.

[11] P. I. Naumkin and I. A. Shishmarev, "Nonlinear nonlocal equations in the theory of waves," *American Mathematical Society*, vol. 133, 1994.

[12] V. M. Hur, "Wave breaking in the Whitham equation," *Advances in Mathematics*, vol. 317, pp. 410–437, 2017.

[13] V. M. Hur and M. A. Johnson, "Modulational instability in the Whitham equation for water waves," *Studies in Applied Mathematics*, vol. 134, no. 1, pp. 120–143, 2015.

[14] H. Borluk, H. Kalisch, and D. P. Nicholls, "A numerical study of the Whitham equation as a model for steady surface water waves," *Journal of Computational and Applied Mathematics*, vol. 296, pp. 293–302, 2016.

[15] T. Truong, E. Wahlén, and M. H. Wheeler, "Global bifurcation of solitary waves for the Whitham equation," *arXiv preprint arXiv:2009.04713*, 2020.

[16] D. Moldabayev, H. Kalisch, and D. Dutykh, "The Whitham equation as a model for surface water waves," *Physica D: Nonlinear Phenomena*, vol. 309, pp. 99–107, 2015.

[17] N. Sanford, K. Kodama, J. D. Carter, and H. Kalisch, "Stability of traveling wave solutions to the Whitham equation," *Physics Letters A*, vol. 378, no. 30, pp. 2100–2107, 2014.

[18] N. Bottman and B. Deconinck, "KdV cnoidal waves are spectrally stable," *Discrete and Continuous Dynamical Systems-Series A (DCDS-A)*, vol. 25, no. 4, p. 1163, 2009.

[19] J. C. Bronski and M. A. Johnson, "The modulational instability for a generalized Korteweg–de Vries equation," *Archive for Rational Mechanics and Analysis*, vol. 197, no. 2, pp. 357–400, 2010.

[20] R. L. Schilling, R. Song, and Z. Vondracek, *Bernstein functions: theory and applications*, vol. 37. Walter de Gruyter, 2012.

[21] S. A. Gabov, “On Whitham’s equation,” *Dokl. Akad. Nauk SSSR: English transl. in Soviet Math. Dokl*. 19, vol. 242, no. 5, pp. 993–996, 1978.

[22] A. Zaitsev, “Stationary Whitham waves and their dispersion relation,” in *Doklady Akademii Nauk*, vol. 286, pp. 1364–1369, Russian Academy of Sciences, 1986.

[23] O. G. Jørsboe and L. Mejlbro, *The Carleson-Hunt theorem on Fourier series*, vol. 911. Springer, 2006.

[24] H. Amann, “Operator-valued Fourier multipliers, vector-valued Besov spaces, and applica-tions,” *Mathematische Nachrichten*, vol. 186, no. 1, pp. 5–56, 1997.

[25] R. S. Strichartz, *A guide to distribution theory and Fourier transforms*. World Scientific Publishing Co. Inc., 2003.

[26] V. I. Arnold, V. S. Afrajmovich, Y. S. Il’yashenko, and L. P. Shil’nikov, *Dynamical systems V: bifurcation theory and catastrophe theory*, vol. 5. Springer Science & Business Media, 2013.

[27] Y. Katznelson, *An introduction to harmonic analysis*. Cambridge University Press, 2004.

[28] B. Buffoni and J. Toland, *Analytic theory of global bifurcation: An introduction*. Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2003.

[29] H. Kielhöfer, *Bifurcation theory: An introduction with applications to PDEs*, vol. 156. Springer Science & Business Media, 2006

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITETET I AGDER, 4604 KRISTIANSAND, NORWAY
E-mail: **obed.afram@uia.no**

Notes to contributors

Transactions of The Royal Norwegian Society of Sciences and Letters is a multidisciplinary journal, covering subjects from sciences in broad sense. Publication is irregular, and the type of publication varies, from single articles to larger monographies and anthologies. Acceptance of large manuscripts and those containing color prints may be conditional on financial contribution from the author. Transactions is at the scientific Level 1 in the NSD register of authorised publishing channels in Norway, and is indexed in national and international databases for scientific publications.

Peer review and anonymity: All submissions will be subject to peer review. Manuscripts must therefore be anonymized by the author(s), and information about the author(s) must be given on a title page. In the paper, please omit name(s) of author(s), except when citing previous work. Self cites should always be done in the third person and in such a way that reviewers cannot identify author(s).

Title page: The title page must contain the title and the author(s) name, institution, mailing address, phone number, and email address, as well as any acknowledgements. Co-authors must be indicated. This page must also contain a statement that the manuscript has not been published or submitted for publication elsewhere, and co-authors must give their consent to publication.

Manuscript standard: Manuscripts are preferably written in a Scandinavian language or English. All manuscripts are required to have an English summary and title. Submitted manuscripts should contain the following:
- Title
- Abstract, approx. 200 words in the paper's language
- The manuscript
- Notes as footnotes. Notes should be kept to a minimum and used only for comment
- Reference list in alphabetical order
- Numbered figures and table in separate files

Tables, figures and illustrations: All figures and tables in the manuscript must be allowed published. If someone other than the author(s) has made a figure or table, the name of the source should be stated in the text. The author(s) is responsible for obtaining the permission to use earlier published illustrations. Each table/figure should be numbered, and each placing in the text should be marked.

Reference system: Transactions uses APA style for citation and referencing. Within the text, identify sources using the author-date system, e.g. Smith & Jones (2002, pp. 32–45) or (Smith & Jones, 2002, pp. 32–45). If a cited work has more than two authors, use 'et al.'. In the reference list, however, the names of all authors should be given. When citing more than one reference at a time, list them in alphabetical order, divided by semicolon. Citations in the text must agree exactly with the list of references. References must include DOI for sources that have one. Place the DOI-URL at the end of the reference. URL is used for electronic sources without DOI. Include the accessed date.

The list of references is written in alphabetical order. Examples:
- *Article in journal:* Frøland, H. O. & Hatlehol, G. (2000). Organisation Todt and Forced Labour in Norway during the Nazi Occupation: Preliminary Remarks from an ongoing Research Project. *Transactions of The Royal Norwegian Society of Sciences and Letters, 2015*(4), pp. 45–61.
- *Chapter in book:* Searchy, W. A. & Nowicki, S. (2000). Male-male competition and female choice in the evolution of vocal signalling. In Y. Espmark, T. Amundsen & G. Rosenqvist (Eds.), *Animal Signals. Signalling and Signal Design in Animal Communication* (pp. 301–315). Trondheim: Tapir Academic Press.
- *Book:* Borgen, P. (1996). *Early Christianity and Hellenistic Judaism.* Edinburgh: T. & T. Clark.
- *Thesis:* Jensen H. (2002). *Causes and consequences of individual variation in fitness-related traits in house sparrows.* Unpublished Ph.D. thesis, Norwegian University of Science and Technology, Trondheim.

Offprints: 5 offprints are supplied free of charge for each author. Additional offprints must be ordered beforehand and paid by authors.

Copyright: Authors retain the copyright of their article, but allow The Royal Norwegian Society of Sciences and Letters to copy and redistribute the material in any medium or format.

Submit your manuscript: Manuscripts are submitted as Word- or TeX-files by email to post@dknvs.no.

The Royal Norwegian Society of Sciences and Letters website: www.dknvs.no

Transactions of The Royal Norwegian Society of Sciences and Letters

The Royal Norwegian Society of Sciences and Letters was founded in 1760, and the Transactions (Skrifter) first appeared in 1761. The Transactions series is among the oldest scientific publications in the world.

Det Kongelige Norske Videnskabers Selskab ble stiftet i 1760, og Skrifter utkom første gang i 1761. Det er en av verdens eldste vitenskapelige skriftserier som fremdeles utgis.

The Royal Norwegian Society of Sciences and Letters

www.dknvs.no