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ARTICLE

Hermann von Helmholtz and the Quantification Problem of Psychophysics

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Abstract

Hermann von Helmholtz has been widely acknowledged as one of the forerunners of contemporary theories of measurement. However, his conception of measurement differs from later, representational conceptions in two main respects. Firstly, Helmholtz advocated an empiricist philosophy of arithmetic as grounded in some psychological facts concerning quantification. Secondly, his theory implies that mathematical structures are common to both subjective experiences and objective ones. My suggestion is that both of these differences depend on a classical approach to measurement, according to which the arithmetic laws of addition define what is measurable as a particular domain for their application, and, at the same time, the extensibility of these laws to all known physical processes works as a heuristic principle for empirical research. Such an approach is worth reconsidering, not only because it lends plausibility to some of the controversial aspects of Helmholtz's theory, but also because it offers a philosophical perspective on quantification problems that originated in the nineteenth-century.

This paper draws insights on Helmholtz's philosophical views from his engagement with the measurability of sensations via Fechner's psychophysical law. This seems to be in contrast with the fact that the reception of Helmholtz's theory culminated with the formulation of the theory of extensive measurement. I will contend that Helmholtz reached a no less important standpoint in the nineteenth-century debate on whether sensations are different (i.e., intensive) kinds of magnitudes and on how, if at all, they can be measured.

Keywords Measurement · Psychophysics · Nineteenth-century psychology · Sign theory · Hermann von Helmholtz · Gustav Theodor Fechner

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1 Introduction

Hermann von Helmholtz's paper *Counting and Measuring from an Epistemological Point of View* (1887) has been acknowledged as a milestone in the development of the representational theory of extensive measurement (Krantz et al. 1971, 71). However, more recent literature has challenged this picture and called for a more nuanced assessment of Helmholtz's contribution to modern measurement theory. Not only is Helmholtz's formalism significantly different from later axiomatizations, but some of the philosophical assumptions underlying his theory are different if not opposed to a representational view of measurement. This is particularly the case with Helmholtz's sign theory of numbers, which is deeply interwoven with his theory of perception. Michell (1993) has drawn attention to this theory to classify Helmholtz as one of the last proponents of a classical view of measurement, according to which numbers are 'discovered' or abstracted from ratios of physical quantities. By contrast, the representational view holds that numbers are 'assigned' to empirical entities, and therefore contributed to the measurement situation.

From a more historical perspective, Heidelberger (1993) and Darrigol (2003) have given a contextual account of how quantification problems encountered by Helmholtz in his scientific research motivated his philosophical account of measurement. These studies have emphasized the need to move beyond the classical/representational dichotomy to better assess Helmholtz's contribution to the development of measurement theory.

This paper elaborates on this suggestion by investigating Helmholtz's engagement with a central quantification problem of his time, namely, the measurability of sensations via Gustav Theodor Fechner's psychophysical law. Fechner addressed the problem in two steps.[1] The first step is to establish the general presuppositions for the possibility of measurement: (1) The magnitudes of a quantity increase and diminish in a continuous way; (2) There is a degree of difference, such that one can discern whether or not it is equal to the degree of difference given between any two other magnitudes; (3) There are discernible circumstances under which a magnitude equals zero; (4) A definition for a standard unit can be obtained by relating that unit to other magnitudes, with which it is linked functionally; (5) In order to obtain standards for measuring magnitudes, we must rely on the mental impressions that are produced in us by material quantities.

The second step is to find a principle that applies to psychic processes and fulfills 1–5. Fechner addressed this problem by introducing the psychophysical law that took his name:

$$E = k \cdot logR$$

Mathematically, this law is the integral of Weber's law for calculating discernible increments of stimuli (R). Fechner's law states that there is a functional relation between a constant increase in sensation (E) and the increase in stimulus necessary for it.

Fechner's argument initiated a debate on the measurability of sensations that involved Jules Tannery, Joseph Delboeuf, Johannes von Kries, Hermann Cohen, August Stadler, and Adolf Elsas, among other scientists and philosophers. This debate concerned both the

[1] I rely in the following on the presentation given by Heidelberger (2004, 193–200), who has offered a thorough account of how Fechner developed his formulation of the preconditions for the possibility of measurement from 1858 to 1887. For a formal presentation of Fechner's argument, see also Falmagne (1985).

empirical adequacy of the psychophysical law and the more fundamental issues concerning the measurability of extensive magnitudes and intensive ones.

It is beyond the scope of this paper to show how this debate paved the way for a systematic treatment of the preconditions for the possibility of measurement. My aim here is to reconsider Helmholtz (1887) as a turning point in this prehistory of measurement theory. I will argue, firstly, that, while Helmholtz did not discuss Fechner's law in Helmholtz (1887), there is evidence that the quantification problem of psychophysics was one of the main objectives of Helmholtz's paper. In particular, I will draw attention to the fact that Helmholtz himself in the second volume of his *Handbook of Physiological Optics* (1867) had discussed several applications of the psychophysical law that are found again in Helmholtz (1887) as examples of quantification of different kinds. Secondly, I will argue that the way in which Helmholtz engaged with this problem in 1887 sheds light on his philosophical assumptions. I have argued elsewhere that Helmholtz in his account of measurement reached a new synthesis between his empiricist theory of perception and some of the tenets of neo-Kantian philosophy (Biagioli 2016, Ch. 4; 2018). My suggestion here is that, in Helmholtz's view, such an account promised to offer conceptual resources to address a variety of quantification problems, including those concerning intensive magnitudes. This seems to be in contrast with the fact that the reception of Helmholtz's paper culminated with the formulation of the theory of extensive measurement. I will contend that Helmholtz's paper reached a no less important standpoint in the nineteenth-century debate on whether sensations are different (i.e., intensive) kinds of magnitudes and on how, if at all, they can be measured.

2 Helmholtz on Psychophysics in the *Handbook of Physiological Optics*

Helmholtz's *Handbook of Physiological Optics*, appeared in three volumes in 1856, 1860, and 1867[2], laid down new standards for conducting empirical research in the physiology of vision. This famously includes the empirical explanation of how spatial notions are acquired by association of various sense impressions in the psychological part of Helmholtz's work (Helmholtz 1867, 427–820).[3] This work marked a no less important standpoint in the experimental physiology of his time. Not only does the book contain a comprehensive presentation of Helmholtz's experimental results on retina adjustments, but it offers a likewise rigorous presentation of others' achievements in the physiology of vision. Notably, this includes Fechner's study of brightness contrast and the optical applications of the psychophysical law.

Fechner investigated various contrast phenomena in 1838–1840. Afterwards, he was forced to discontinue his experimental work due to eye damage.[4] Nevertheless, he continued to work on a comprehensive account of brightness contrast and simultaneously on the psychophysical program. The phenomena under consideration include colored shadows, the fading of brightness and color differences on long fixation, as well as afterimage, that is, an optical effect in which an image continues to appear after exposure to an original image.

[2] In the following, all references are given according to the 1867 edition, which includes all of the volumes.

[3] On how Helmholtz positioned himself within the nativism/empiricism debate on the origins of spatial notions, see Hatfield (1990, Ch. 5).

[4] On Fechner's work from experimentation to the development of the psychophysical program, see Brožek and Gundlach (1988), Heidelberger (2004).

A positive afterimage, in which the colors of the original image are maintained, can occur after a brief exposure to a bright stimulus, especially when the surrounding conditions are darker than the stimulus. Negative afterimages occur after prolonged exposure to a colored stimulus, which results in a perception of the complementary color. A negative afterimage is obtained, for example, by staring at a colored object against a dark field. The same kind of image appears in the closed eye after exposure to an initial stimulus independently of the light then available to the eye or in the absence of external light. Whereas Joseph Plateau and others explained these phenomena by assuming that exposure to light of a particular kind produces a physiological response in the retina, Fechner maintained that afterimages depend on retinal fatigue (see Plateau 1834; Fechner 1840). In particular, he pointed out that black surfaces reflect considerable amounts of light. Regarding afterimages in the closed eye, Fechner appealed to the constant effect of intrinsic retinal light. Fechner took this to reveal the patterns of exhaustion of afterimages.[5] The details of Fechner's explanation need not concern us further. What matters for our considerations about Fechner's influence on Helmholtz is the fact that afterimages showed the subjective nature of color contrast. In general, visual contrast offered the example of a "sensation of a difference between sensations," which involves a higher cognitive capacity or "psychical connection" (Fechner 1860, 72).

Fechner urged a closer examination of contrast phenomena in order to explain deviations from the psychophysical law. However, he left it as an open question whether this explanation should be psychological or physiological. In the former case, deviations from the law would depend on an unconscious error of comparison of the two impressions. In the latter case, the contrast situation would produce a real change of sensitivity, that is, a physiological distortion occurring between the physical stimulus and the higher neural activity that constitutes the psychophysical event.

As Turner (1988) has pointed out, Fechner's twofold conjecture influenced two opposing approaches. Fechner's former student Ewald Hering adopted the supposition of specific physiological processes intervening between the stimulus and the psychophysical event. Helmholtz developed a psychological explanation of contrast phenomena in line with his own empirical approach.

Helmholtz made use of the psychophysical law as a "first approximation to the truth" (Helmholtz 1867, 314): within certain limits (to be determined empirically for each color), the linear variation of the light corresponds to constant increments (or decrements) of the sensation of brightness contrast. This function accounts for contrast phenomena[6], as well as for more familiar experiences, such as the fact that the perception of the contrast between dark and light objects is increased by a softer illumination. At the other end of the spectrum, the contrast of homogenously illuminated objects, such as black letters on a white paper, is perceived most clearly in a fully illuminated environment. The contrast ceases to appear at the point when the light is perceived as blinding. The psychophysical law is an approximation to the true correspondence between light and contrast phenomena, because, as Helmholtz explained, some of the circumstances that determine the limits of the perception of these phenomena intervene also in the intermediate degrees of the quality measured. It is easy to notice that, for example, the contrast becomes sharper when the light approximates the limit of complete darkness. To mention an intuitive example also used by Helmholtz,

[5] On the different interpretations of afterimages in nineteenth-century physiology of vision, see Turner (1988).

[6] In particular, Helmholtz relies here on Fechner (1838).

night paintings reproduce the effect of dim light by sharpening the contrast between the moon's brightness and the darkness of the sky. Helmholtz went on to analyze Fechner's experiences with color shadows and afterimages (Helmholtz 1867, 316–27).

As Turner (1994) has shown in detail, Helmholtz's discussion of these phenomena was largely based on Fechner, although Helmholtz departed from Fechner in two important respects. The first is that Helmholtz relied on the three-receptor theory of Thomas Young to account for the physiological process of retinal fatigue. The second point of disagreement with Fechner relates to Helmholtz's defense of an empirical approach: while Fechner hypothesized a physiological mechanism to account for contrast phenomena that deviate from the psychophysical law, Helmholtz rejected such a hypothesis as an unjustified metaphysical assumption; his own explanation of how the deviations occurred rather involved unconscious inferences from sense impressions to the comparative judgments implicit in the experience of contrast. Fechner, in turn, emphasized various empirical objections against Helmholtz's account of contrast in a section added to his (1860) paper.

For our present purpose, it is important to emphasize the philosophical level of this discussion. While Helmholtz and Fechner, in different ways, defended the role of experiment in physiology, Helmholtz departed from Fechner in the discussion of a philosophical problem that had its roots in the classical theory of knowledge and had become known as 'the psychophysical parallelism.' (see Heidelberger 2004, Ch. 5). The problem was to account for an asymmetric relation between body and mind: for every inner change, there is a corresponding outer change, but the opposite is not necessarily true. A single type of stimuli (e.g., pressure) can produce different types of sensation depending on where it is applied (e.g., to the eye or the skin). This shows that there is no causal determination of mental properties. In this regard, as Heidelberger suggests, Fechner foreshadowed the view that mental properties supervene on a physical basis, in the sense that an object cannot alter in some mental respect without altering in some physical respect. This allowed Fechner to advocate what Heidelberger (2004, 99) calls a nonreductive form of materialism. Helmholtz called into question the materialist view altogether by taking the principle that no effect is without cause to be a contribution of the mind, which cannot be derived from experience because it is a necessary presupposition of experience in the Kantian sense. According to Helmholtz, the determination of the cause of a given sensation is an inferential process that, once learned, takes place unconsciously. This explains the possibility of error, when it is overlooked that different outer changes can produce the same sensation. To use an example from optics (Helmholtz 1867, 428), an electric current and a blow in the eye can produce the same visual effect as external light. Naturally, if confronted with that particular sensation, our unconscious inference would identify external light as the cause. However, it would be wrong to assume that such a conclusion is necessarily true.

In summary, both Fechner and Helmholtz found it problematic to assume that there is a causal dependence of mental properties from physical phenomena in the traditional (deterministic) sense. Fechner deemed the psychophysical parallelism a functional relation, which found a precise mathematical expression in the psychophysical law. Starting from a similar consideration, Helmholtz addressed the psychophysical parallelism by identifying sensations as 'signs', whose meaning has to be learned by an intellectual process (Helmholtz 1867, 797). On the one hand, he held that such a process happens via unconscious inferences (from sensations to their unknown causes) and is subject to error, which is consistent with an empiricist view of knowledge. On the other hand, Helmholtz denied that the senses have immediate access to external reality. He argued that learning to interpret the meaning of sensations is a complex process, which is mediated by the structures of the mind. These are interpreted by Helmholtz in various ways as naturalized Kantian a prioris, including space, time, causality, and the lawfulness of nature. Hatfield (1990) has introduced the useful term 'empirism' to characterize how Helmholtz's theory of spatial perception differs from empiricism in incorporating a priori elements. This strategy offered a starting point also for Helmholtz's philosophy of geometry.[7] Following Hatfield's suggestion, I call such an approach 'empirist'.

3 Helmholtz's Sign Theory of Numbers

Helmholtz himself emphasized the connection between his inquiries into the foundations of mathematics and his theory of perception in the introduction to *Counting and Measuring*. He began by noticing that, while he had invoked an empirist account of geometrical axioms against his Kantian contemporaries, he still subscribed to Kant's transcendental philosophy for the view of space as a "form of intuition" (Helmholtz 1977, 72).[8] He went on to describe his plan to extend the same approach to arithmetic in the following way:

> If the empirist theory—which I besides others advocate—regards the axioms of geometry no longer as propositions unprovable and without need of proof, it must also justify itself regarding the origin of the axioms of arithmetic, which are correspondingly related to the form of intuition of time. (Helmholtz 1977, 72)

The first part of *Counting and Measuring* aims to spell out such a relation between arithmetical axioms and the form of intuition of time.

Helmholtz assumed the axioms of arithmetic to be the following laws of addition:

AI. If two magnitudes are both equal with a third, they are equal amongst themselves.

AII. The associative law of addition: $(a + b) + c = a + (b + c)$.

AIII. The commutative law of addition: $a + b = b + a$.

AIV. If equals are added to equals, their sums are equal.

AV. If equals are added to unequals, their sums are unequal.

AVI. If two numbers are different, one of them must be higher than the other.

AVII. If a number c is higher than another one a, then I can portray c as the sum of a and a positive whole number b to be found.

Helmholtz then identified the form of intuition of time as the linear order of the time sequence. He introduced the concept of ordinal numbers as 'signs' that stand for places in such a sequence: different signs ought to stand for different places, so that the sequence can be symbolized without omissions or repetitions. Helmholtz went on to show that the axioms of arithmetic apply to ordinal numbers, insofar as the axioms themselves define the ordinal relations of equality (AI) and of being higher and lower (AVI). The remaining axioms define

[7] See Hatfield (1990), Friedman (1999), Biagioli (2014; 2016), Patton (2018).

[8] As Helmholtz had put it in *The Facts in Perception* from 1878: "Kant's doctrine of the a priori given forms of intuition is a very fortunate and clear expression of the state of affairs; but these forms must be devoid of content and free to an extent sufficient for absorbing any content whatsoever that can enter the relevant form of perception" (Helmholtz 1977, 162).

the concept of addition. Helmholtz proved, firstly, that they apply to the whole sequence of numbers based on a procedure introduced by Hermann Grassmann.[9]

Secondly, Helmholtz proved that the same axioms apply to number sequences regardless of the order of the elements. Given two numbers n and $(n + 1)$, on the one side, and two signs ε and ζ, on the other, there are two possible manners of correlation:

a) $n \rightarrow \varepsilon$, $(n + 1) \rightarrow \zeta$; or

b) $n \rightarrow \zeta$, $(n + 1) \rightarrow \varepsilon$.

If a) is substituted for b), the series α, β, γ, etc. can be put into one-to-one correspondence with the series $(n + 1)$, $(n + 2)$, etc. By continued exchanging of neighboring elements, one can bring about any possible sequence without gaps or repetitions. This allowed Helmholtz to prove a theorem stating that:

> *Attributes of a series of elements which do not alter when arbitrarily neighboring elements are exchanged in order with each other, are not altered by any possible alteration of the order of the elements*. (Helmholtz 1977, 85)

This theorem enabled him to define a cardinal number n as the sign correlated to a group of objects, if the complete number series from 1 to n is needed in order to correlate a number to each element of the group. The theorem thus provides an equivalent formulation of AVII for cardinal numbers, as well as a means to generalizing the remaining axioms to the concept of a sum between arbitrarily many numbers.

It appears clearly now that Helmholtz's formalism is still quite far from the later axiomatizations of measurement theory. Whereas Helmholtz's starting points were his empirist account of numbers and the above axioms of addition, measurement theory as we know it begins with a set of axioms of quantity and requires a representation theorem to show that attributes that can be characterized by those axioms can also be given a numerical quantification. The first to adopt this procedure was Hölder (1901). This axiomatization departed from Helmholtz's also for the fact that Hölder's theory included an axiom of continuity. By contrast, Helmholtz formulated a theory of discrete magnitudes. We will turn back to this important point in connection with Helmholtz's discussion of intensive magnitudes. For now, it is noteworthy that the different formalism did not prevent Hölder from referring to Helmholtz's paper as an important step towards his own theory of measurement (Hölder 1901, 2, note). A reconstruction of the development from Helmholtz to modern representational theory has been given by Diez (1997).

I believe that the main contrast with these later developments depends not so much on the fact that the means of modern axiomatizations were not available to Helmholtz, but on the philosophical view underlying Helmholtz's approach. This is less evident, because Helmholtz's very definition of cardinal numbers as arbitrarily chosen signs might suggest a representational view of measurement. To a closer look, however, it becomes clear that the concept of number in Helmholtz's account is deeply interwoven with perception, and so is arithmetic, which Helmholtz deemed "a method based on psychological facts" (Helmholtz 1977, 103). This is exactly the opposite of what representationalists presuppose. As Michell (1993, 189) has pointed out: "The representational view of measurement is made necessary by any view about the ontological status of numbers that removes them from the empirical domain." Therefore, Michell has argued that Helmholtz sided with a classical view of

[9] For a thorough explanation of Grassmann's procedure, see Cantù (2013, Ch. 5).

numbers as ratios of quantities rather than with the modern view, which clearly separates numbers from quantities.

My discussion of Helmholtz's philosophical background suggests otherwise. It is, namely, a characterizing point of Helmholtz's empirism that outer experience is necessarily mediated by subjective forms. Helmholtz separated numbers from quantities, insofar as he 'removed' numbers from (outer) experience, so to speak. As he put it:

> The concept of addition described above coincides with the concept of it which proceeds from determining the total cardinal number of several groups of numerable objects, but has the advantage of being obtainable without reference to external experience. One has thereby proved, for the concepts of number and of a sum—taken only from inner intuition—from which we started out, the series of axioms of addition which are necessary for the foundation of arithmetic; and also proved, at the same time, that the outcome of this kind of addition coincides with the kind which can be derived from the numbering of external numerable objects. (Helmholtz 1977, 87)

This quotation spells out the objective of Helmholtz's introductory remarks, and also outlines the plan for the rest of the inquiry. What has been proved so far is that the fundamental concepts of arithmetic can be derived by abstracting away from any perceptual content and starting only from the form of the time sequence. These concepts offer the first and characteristic model for the axioms of addition. With reference to his theorem, Helmholtz added that there is a way to know a priori that the same axioms apply to the composition of external numerable objects. However, this way of proceeding required him to start with ordinal numbers in order to arrive at a definition of cardinal numbers as derivative terms. By a similar reasoning, Helmholtz proceeded in the second part of his paper to characterize magnitudes by a further extension of his additive principles to different kinds of empirical attributes.

4 From Numbers to Measurement

The second part of Helmholtz's paper deals with the question: "What is the objective sense of our expressing relationships between real objects as magnitudes by using denominate numbers; and under what conditions can we do this?" (Helmholtz 1977, 75). This can be considered to be the first formulation of the key question of measurement theory (Diez 1997; Darrigol 2003). However, besides the important differences between the modern representational theory and Helmholtz's already discussed in the previous section, Helmholtz proceeded in a different way also when it comes to the interpretation of his theory. Modern measurement theory in its formal development provides an indispensable tool for adequately addressing quantification problems; however, the theory as such does not decide which of its interpretations are admissible. Which interpretation is adopted in contemporary debates about measurability largely depends on the author's metaphysical and epistemic commitments (see Tal 2017).

What is characteristic of Helmholtz's approach is that the determination of the objective sense of quantitative judgments takes place in the following two stages. In a first stage, Helmholtz gave a provisional definition of magnitudes as "objects, or attributes of objects, which allow a distinction into greater, equal or smaller when compared with similar ones" (Helmholtz 1977, 85). All of the terms that appear in this definition, however, require a more precise characterization based on the initial set of axioms of addition. Helmholtz began by claiming that AI provides a definition of 'equality.' In addition, the actual determination that the attributes of two similar objects are equal presupposes that, under suitable circumstances, one can observe an interaction between these objects which does not occur, as a rule, between other pairs of similar objects. Helmholtz called such a procedure a "method of comparison."[10] To mention a well-known example from Helmholtz's geometrical papers, methods of comparison include the superposition of a measuring rod on solid bodies that can be brought into congruent coincidence. Further examples are examined in the next section. For now it is important to notice that having a method of comparison does not by itself account for the objective sense of quantitative judgments. In the example of superposition, Helmholtz had emphasized elsewhere that the observability of congruent coincidences depends on our more general assumptions about the structure of space as a whole (Helmholtz 1868). Notably, this includes the free mobility of rigid bodies understood as the requirement that each point of a system in motion can be brought to the place of each other, provided that all points of the system remain fixedly interlinked. Therefore, Helmholtz in (1878) identified the structure of a space of constant curvature as the a priori form of outer intuition in a (generalized) Kantian sense.[11]

This example can serve well to illustrate the second stage of Helmholtz's argument. Helmholtz himself emphasized again in (1887) that further assumptions are required in order for attributes such as lengths to be comparable (i.e., to be capable of being equal, bigger or smaller). These are assumptions about the structure of space. The condition for assigning a numerical value to the number of times that the extremities of a measuring rod can be superposed to pairs of point at the same distance on a solid body, is that, as Helmholtz put it, "the points must be fixedly linked" (Helmholtz 1977, 92). This is tantamount to the assumption of the free mobility of rigid bodies, along with the other conditions that, according to Helmholtz, determine the homogeneity of space.[12]

More generally, Helmholtz pointed out that the composition of physical magnitudes rests on the following interpretation of the principle of homogeneity of the sum and the summands:

> *The issue of whether the result of connection remains the same, when parts are exchanged, must be decided by the same method of comparison with which we ascertained the equality of the parts to be exchanged*. (Helmholtz 1977, 95)

[10] I will use the expression 'method of comparison' throughout the paper to translate 'Methode der Vergleichung'. Quotes from the existing English translation of Helmholtz (1887) will be modified accordingly.

[11] See note 8. The possibility of generalizing the Kantian notion of space has been subjected to compelling criticisms by Moritz Schlick in his comments to the centenary edition of Helmholtz's *Epistemological Writings*. Consequently, Schlick's reading of Helmholtz attached more importance to the points of agreement with his own logical empiricism. More recent literature has shown that it is nonetheless possible to give a consistent reading of Helmholtz's account of space as inherently Kantian, and that such a reading illuminates no less important aspects of Helmholtz's epistemology (see Friedman 1997; Ryckman, 2005, Ch. 3; Patton 2009; Biagioli 2016).

[12] Helmholtz's original axiomatization (Helmholtz 1868) included the free mobility of rigid bodies and the monodromy of space. Subsequently, in 1893, he recognized the possibility of giving equivalent axiomatizations without monodromy in the way indicated by Sophus Lie and Felix Klein (see Königsberger 1903, 81).

Notice that this requirement has a parallel in Helmholtz's previous theorem about the additivity of cardinal numbers: as the above theorem allowed Helmholtz to show that the laws of addition can be extended from ordinal to cardinal numbers, the claim here is that the requirement of the conformity of measurement results with measuring procedures accounts for the extensibility of AII-AVII to empirical magnitudes. In order to emphasize this point, Helmholtz called a composition that satisfies the said axioms 'addition' and generalized his requirement as follows:

> *A physical method of connecting magnitudes of the same kind can be regarded as addition, if the result of the connection—when compared as a magnitude of the same kind—is not altered either by exchanging individual elements with each other, or by exchanging terms of the connection with equal magnitudes of the same kind*. (Helmholtz 1977, 96)

This concludes Helmholtz's explanation of the above definition of magnitudes. What he initially described in more intuitive terms as 'similar' magnitudes, is specified in the above requirement in terms of similarity in kind, that is, homogeneity. However, this also makes it clear that the definition applies only to those magnitudes that can be added according to the arithmetical laws. In other words, these are the magnitudes that can be expressed as a sum of units of magnitudes of the same kind.

This seems in contrast with some of the examples mentioned by Helmholtz to illustrate the notion of physical comparison. Besides the distance between a pair of points, magnitudes for which there is a method of composition according to Helmholtz include physical magnitudes that can be summed such as weights, magnitudes that can be represented by a numerical scale, such as duration and temperature, as well as psychophysical events, such as brightness and tones. The latter two examples, which are taken from Helmholtz's physiological work, deserve a closer examination also for a better understanding of the scope of his theory of measurement. Helmholtz himself left the door open for a treatment of nonadditive magnitudes after listing the laws of addition, by saying: "It seems to me an unnecessary restriction of the domain of validity of the propositions discovered, that one should from the outset treat physical magnitudes only as ones composed of units" (Helmholtz 1977, 73). However, the question arises whether his theory provides the required conceptual resources for such a treatment. Furthermore, we know from the previous sections that in order to deal with contrast phenomena, Helmholtz in (1867) made use of the psychophysical law. However, the interpretation of those empirical results had been lively debated in the period that preceded the publication of *Counting and Measuring*. So, a further question arises as to how, if at all, he reconsidered his position towards psychophysics in 1887.

I will address these questions after giving a brief account of the debate that is at the background of Helmholtz's (1887) discussion of intensive magnitudes.

5 The Problem with Intensive Magnitudes

In the time between Helmholtz's *Handbook of Physiological Optics* (1867) and *Counting and Measuring* (1887), several objections had been raised against the possibility of quantifying psychical phenomena.[13] This debate in Germany involved physiologists such as Georg Elias Müller, Julius Bernstein, Johannes von Kries, neo-Kantian philosophers such as Hermann Cohen, August Stadler, Ferdinand August Müller, the physicist close to Hermann Cohen, Adolf Elsas, as well as Eduard Zeller, to whom the collection where *Counting and Measuring* appeared was dedicated. However, it was the French mathematician Jules Tannery who initiated the debate with two contributions to the journal *Revue Scientifique* (1875a; 1875b). Both of these contributions appeared anonymously in response to Théodule Ribot's articles on contemporary psychology in Germany, which had been reprinted in the same journal in 1874. Subsequently, Tannery's objections to Fechner were discussed by the Belgian philosopher and psychophysicist Joseph Delboeuf in his (1878) review of Fechner's *The Case for Psychophysics* (1877). Delboeuf's review contributed to disseminate Tannery's arguments in the German debate, although Delboeuf regretted that Fechner had not responded to Tannery's objections. Delboeuf himself abandoned Fechner's law after considering Tannery's arguments.

Tannery's objections concerned the psychophysical law as an expression of the relation between stimuli and sensations, as well as the measurability of sensations, more generally. Tannery identified the preconditions for measurability as: (1) Additivity, stating that "the only dimensions that can be measured directly are those for which we can define equality and summation" (English translation from Heidelberger 2004, 208); (2) Homogeneity, stating that one can only add things of the same kind. In stating 2, Tannery made it clear that homogeneity is a necessary precondition for direct measurement: "Directly measurable dimensions necessarily have this quality, because measurement itself requires that dimensions of the same kind be comparable" (Heidelberger 2004, 209). Sensations, according to Tannery, fail to satisfy 2. Therefore, 1 in this case relies on arbitrary stipulations concerning the equality and summation of sensations. It follows that we have no empirical grounds for an inference from the nature of the stimuli to that of sensations, least of all for expressing such an inference in the highly abstract mathematical terms of Fechner's logarithm. A logarithmic equation can be stated here only as a conventional stipulation rather than as an empirical law. It is always possible to define differentials of sensation and to equate them with increments of the corresponding stimuli. However, such a procedure remains arbitrary and without any grounds in the phenomena.

Helmholtz himself did not explicitly address Tannery's objections in (1887). However, he engaged with Adolf Elsas' pamphlet *On Psychophysics: Physical and Epistemological Considerations* (from 1886). After offering a reconstruction of the debate on the measurability of sensations thus far, Elsas argued for the negative conclusion that sensations are not and cannot be made measurable based on three premises. The first is that all physical processes can be reduced to causal relations between forces and movements. Secondly, Elsas assumed that sensations belong to a different, psychical kind of experience. Thirdly, Elsas

[13] For a thorough discussion of the objections raised against psychophysics, see Heidelberger (2004, Ch. 3). While some of these objections specifically concern Fechner's psychophysical program, the following paragraphs focus on those objections that are more directly relevant to Helmholtz's account of intensive magnitudes.

restricted the applicability of mathematics to physical phenomena, which for Elsas coincide with causal connections. Fechner himself had identified the psychophysical parallelism as a weaker kind of functional relation between increments of sensation and stimulus. Elsas concluded that this relation cannot be expressed mathematically at all. Sensations differ in this respect even from magnitudes that can be made extensive, such as temperature. While temperature is not measured directly, it is possible to perform such a measurement on various extensive representatives of it, for example a column of mercury. The quantification in this case is justified in Elsas' eyes by the fact that there is a causal dependency between temperature and the extension of its representative in the thermometer. By contrast, sense qualities present themselves as something that is essentially intensive and lie outside the scope of natural science. With this, Elsas believed to have excluded not only psychophysics, but also all mathematical and physiological psychologies as "absurd designations" (Elsas 1886, 70). As he put it: "Mathematics cannot be applied any further than in fields where the concepts of movement and of force find application; physics ends where causality no longer rules, and physiology has to do nothing more than measuring the organism" (Elsas 1886, 70).

Notice that this argument differs from Tannery's for its implications on the nature of sensations as well as for Elsas' conclusion. Tannery, unlike Elsas, had limited himself to pointing out there are no empirical grounds to assume the measurability of sensations, which does not imply that there are grounds for excluding it once and for all. By contrast, Elsas believed to have derived such grounds from a mechanistic image of nature, as implied in the first of the above premises combined with a neo-Kantian epistemology. In particular, Elsas referred to Cohen's account of the principle of infinitesimal magnitudes for the view that intensive magnitudes cannot be reduced to extensive ones.[14]

In Kant's terminology, intensive magnitudes are distinguished from extensive ones by the way in which they are apprehended: whereas an extensive magnitude is represented as the sum of its parts, Kant called intensive "that magnitude which can only be apprehended as a unity, and in which multiplicity can only be represented through approximation to negation = 0" (Kant 1998, 210, B edition). Kant went on to identify the property of intensive magnitudes on account of which no part of them is the smallest as their continuity. Kant stated that every reality in appearance and, consequently, every sensation has intensive magnitude, i.e., a degree (Kant 1998, 210, B edition).

In addressing the distinction between extensive and intensive magnitudes, Helmholtz took a position that is opposed to Elsas'. Helmholtz made it clear that the theory based on the above axioms of addition applies to extensive magnitudes only. At the same time, he outlined how to extend the consideration to other kinds of magnitudes in the concluding sections of *Counting and Measuring*. Helmholtz distinguished, firstly, between those magnitudes that can be divided into equal parts and those that cannot be thus divided without remainder. It follows from Helmholtz's theory that only the former magnitudes can find an exact expression in terms of denominate numbers. Considering that Helmholtz's axiomatization does not include continuity, an exact numerical representation is excluded in the case of irrational proportions. He pointed out that, nevertheless, it is always possible

[14] On Cohen's own account cf. Giovanelli (2017). A comparison with Helmholtz's theory of measurement is beyond the scope of this paper. It is noteworthy, however, that Cohen himself talked about different forms of quantification rather than imposing absolute limits on quantification in physics. In this respect, and despite their diverging interpretations of Kant, Cohen agreed in important ways with Helmholtz's approach (see Biagioli 2018).

to assign an approximate value to such magnitudes by enclosing their measure between arbitrarily narrow limits. Helmholtz believed that such a procedure would suffice to provide an adequate representation of any given magnitudes. More complicated cases, such as Weierstrass' everywhere continuous but nowhere differentiable functions, had been studied in mathematics. However, these cases have not yet found applications in physics or geometry (Helmholtz 1977, 92).

In order to illustrate the distinction between extensive and intensive magnitudes, Helmholtz went on to consider the case of physical magnitudes for which a unit of measure and a method of comparison are not available. Examples of this include the refraction of light relative to the refractive index of the medium, specific gravity, thermal conductivity, electrical conductivity. These are examples of values for which the physicists have found indirect modes of calculation by combining the fundamental units of space, time and mass. In this sense, Helmholtz suggested that the distinction between extensive and intensive magnitudes can be rephrased in terms of a distinction between magnitudes and coefficients, or directly and indirectly measurable magnitudes. This formulation, however, also makes it clear that the distinction is provisional. Helmholtz emphasized this by saying that: "Occasionally, new discoveries can lead to ways of additively conjoining coefficients, whereby they would move into the class of directly determinable magnitudes" (Helmholtz 1977, 99).

Finally, Helmholtz considered the case of multidimensional magnitudes, such as the displacements of a point in space, its velocity and acceleration, the force propelling it, electricity, heat, and magnetic moment. These magnitudes are nonhomogeneous according to Helmholtz, in the sense that their determination involves two or more methods of comparison. He maintained that also in this case it is possible to consider the comparison an additive composition in terms of the calculus of segments. Helmholtz emphasized that thereby the possibility of the geometrical representation as additive rests on the observed facts about how the object under consideration are composed. Helmholtz considered the Newtonian theory of color mixture. This requires that any quantum of colored light can be represented as the composition of three light quanta of suitably chosen fundamental colors. This is because the mixture of more colors would produce in the eye the same impression as the corresponding mixture of three fundamental colors (Helmholtz 1977, 100). Color mixture offered the example of how the axioms of addition can be applied to nonhomogeneous magnitudes, when their comparison is found to satisfy the requirement that the result remain the same by exchanging of individual elements with each other or with equivalent ones.

6 Concluding Remarks

Although Helmholtz did not go into the details of the debate on psychophysics in (1887), the above argumentation offers a possible strategy to counter some of the main objections. Helmholtz himself emphasized in the introductory remarks of his paper (1977, 74) that his way to rephrase the received metaphysical distinction between intensive and extensive magnitudes contradicted especially Elsas' 'strict Kantian view.' In particular, Helmholtz rejected the view that at least some intensive magnitudes are essentially different from extensive ones and cannot be made extensive by any scientific development. At the same time, we saw that Helmholtz presented his own argumentation as a sort of nonorthodox Kantian explanation of the possibility of counting and measuring, by drawing the concepts of number and of sum from the inner intuition of the time sequence. This allowed Helmholtz to account for the a priori validity of the laws of addition for any sequence of numerable objects. What distinguishes Helmholtz's empirism from strict Kantianism is that the further extension of the laws of addition to empirical objects can be defined only relative to the procedures of the best current scientific theories rather than once and for all from the standpoint of the transcendental architecture of the knowing subject. Therefore, Helmholtz emphasized that the definitions of equality and homogeneity provided by the axioms also require an interpretation in terms of methods of comparison and of composition, if they are to be referred to magnitudes in an objective way.

Helmholtz's reliance on empirical procedures sheds light also on the fact that he did not propose again his previous reading of the psychophysical law as an approximation in 1887. Arguably, by the time he wrote *Counting and Measuring*, he might have shared with many working physiologists the view that had been first expressed by Tannery, namely, that there are no empirical grounds for reading Fechner's logarithmic equation as anything more than a conventional stipulation (cf. Heidelberger 1993).

My suggestion is that, nevertheless, Helmholtz's approach to measurement naturally led him to address this kind of quantification problem. In introducing it, he still made ample use of examples from his and others' physiological works related to the psychophysical program. More notably, he gave a principled argument for the meaningfulness of psychophysical quantification, by extending his considerations to the measurability of continuous, intensive, and nonhomogeneous magnitudes. Without determining which of these paths should lead to a solution of this intricate problem, Helmholtz's theory offered a new and nonreductive perspective on it, which in my view is a no less significant contribution to the epistemology of measurement than his theory of extensive magnitudes.

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