# Conical Refraction in Generalized Biaxial Media

## A Geometric Algebra Approach

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*Abstract* — **It is well-know that conical refraction occurs for electric anisotropic biaxial crystals when the wave vector has the direction of the medium optic axes. In this paper, we show that conical refraction occurs – in an analogous away – for a more general type of biaxial media that have simultaneously electric and magnetic anisotropies. Furthermore, the new coordinate-free approach based on geometric algebra, developed by the authors in previous papers to address anisotropy, is shown to shed new light on this classic topic of optics that is conical refraction.**

***Electromagnetic Wave Propagation, Electric and Magnetic Anisotropies, Biaxial Media, Conical Refraction; Geometric Algebra***

## I. INTRODUCTION

The phenomenon of conical refraction is an ancient topic of optics, discovered theoretically in 1832 by Hamilton. In the same year, Lloyd experimentally demonstrated this effect using a rare specimen of arragonite. This was a remarkable experiment which was very difficult to reproduce due to the lack of suitable crystals. In fact, there have been few experimental confirmations of conical refraction (Lloyd 1833; Poggendorff 1839; Raman et al. 1941, Perkalsris & Mikhailichenko 1979; Mikhailichenko 2004). Only in 2004, the engineers of the company Vision Crystal Technology AG found an easy setup to demonstrate conical refraction using a monoclinic double tungstate crystal, $KY(WO_4)_2$. With the flexibility brought by artificial crystals, a new interest was regained in this old topic of optics, and new applications for conical refraction are now being under investigation [1],[2]. One the other hand, with the advent of metamaterials more general classes of anisotropic media are now being studied. Namely, media with simultaneous electric and magnetic responses at the same frequency range (either in the microwave or in the optical domains) [3],[4]. Metamaterials are nowadays a very active topic of research, with new exciting applications, such as the perfect lens [5], subwavelength imaging [6], and invisibility cloaking [7]. Media with electric and magnetic anisotropies represent an important subclass of metamaterials. The authors have presented a novel approach to address these media based on (Clifford) geometric algebra [8],[9]. Using this new coordinate-free approach, we defined a broader class of biaxial media. In this general case, the direct correspondence between the mathematical form (uniaxial or biaxial) of permittivity and permeability with the global behavior of the medium is lost. We show that a new constitutive function should be defined in order to characterize these materials. However, in this analysis an important question was left unsolved: does conical refraction occurs for these more general kind of biaxial media? Thus, the goal of this paper is to analyze conical refraction for biaxial materials stemming from electric and magnetic anisotropic properties. In this analysis, we will continue to use the same approach developed in [9] since geometric algebra provides a natural framework to address anisotropy. In Section II, we define a broader class of biaxial media using a new constitutive function. This function gives directly the direction of the optic axes and characterizes the type of anisotropy, i.e., if this function is uniaxial (biaxial) the medium is uniaxial (biaxial). In section III, it is considered that the wave vector is aligned with one of the optic axes. We show that the conical refraction can also occur for these generalized biaxial media, i.e. that the Poynting vector lies on the surface of a cone. We then obtain the general expression for the light cone that depends on the electric and magnetic properties of the material, and not only on the electric parameters, as is for the conventional biaxial case. This result confirms that our new definition of biaxial media capture the true physics of the biaxial media and it is not just an artificial mathematical classification. Finally, in Section IV we outline the main conclusions of this work.

## II. GEOMETRIC ALGEBRA APPROACH TO ANISOTROPY

Monochromatic plane waves of the form $\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]=\exp\left[i\,k_0\left(\mathbf{n}\cdot\mathbf{r}-c\,t\right)\right]$, with $\mathbf{k}=k_0\,\mathbf{n}$ and $k_0=\omega/c$, in a source-free region are considered. For this case, using Euclidean geometric algebra $\left(C\ell_3\right)$, the macroscopic Maxwell equations can be written in terms of complex-constant amplitude vectors as [9]

$$\begin{cases}\mathbf{n}\wedge\mathbf{E}=c\,\mathbf{B}\,\mathbf{e}_{123}\\ \mathbf{n}\wedge\mathbf{H}=-c\,\mathbf{D}\,\mathbf{e}_{123}\end{cases}\tag{1}$$

where $\wedge$ represents the Grassmann outer product and $\mathbf{e}_{123}=\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$ is the pseudoscalar of $C\ell_3$. On the other hand, the constitutive relations of anisotropic media are

$$\begin{cases}\mathbf{D}_0=\varepsilon_0\,\upepsilon\left(\mathbf{E}_0\right)\\ \mathbf{H}_0=\mu_0^{-1}\,\upmu^{-1}\left(\mathbf{B}_0\right)\end{cases}.\tag{2}$$

where $\varepsilon_0$ ($\mu_0$) is the permittivity (permeability) of vacuum, $\varepsilon$ is the electric function (permittivity) and $\mu$ is the magnetic function (permeability). Replacing Eq. (2) in Eq.(1), and after some algebra, we can obtain the characteristic equation as function of the electric field:

$$\hat{\mathbf{k}} \wedge \mathsf{w}\left(\mathbf{E}_\perp\right) = 0 \tag{3}$$

where $\mathsf{w}\left(\mathbf{E}_\perp\right) = \kappa\left(\hat{\mathbf{k}}\right) n^2 \eta\left(\mathbf{E}_\perp\right) - \mathbf{E}_\perp$ with $\kappa\left(\hat{\mathbf{k}}\right) = \dfrac{\hat{\mathbf{k}} \cdot \mu\left(\hat{\mathbf{k}}\right)}{\det\left(\mu\right)}$ and $\mathbf{E}_\perp = \mathbf{E} - \dfrac{\mathbf{E} \cdot \mu\left(\hat{\mathbf{k}}\right)}{\hat{\mathbf{k}} \cdot \mu\left(\hat{\mathbf{k}}\right)} \hat{\mathbf{k}}$. Throughout this paper we considered that the eigenvectors of the linear functions $\varepsilon$ and $\mu$ are the same, or equivalently the condition $\varepsilon\mu = \mu\varepsilon$ is verified. Under this restriction, we defined a broader class of biaxial media stemming from simulateously electric and magnetic properties [9]. We showed that these general media can be characterized by the function

$$\eta_a\left(\mathbf{a}\right) = \varepsilon_2\left(\varepsilon_1 - \varepsilon_3\right)\eta\left(\mathbf{a}\right) + \mu_2\left(\mu_1 - \mu_3\right)\eta^{-1}\left(\mathbf{a}\right) \tag{4}$$

where $\varepsilon_i$ ($\mu_i$) are the permittivity(permeability) eigenvalues and $\eta = \varepsilon^{-1}(\mu)$. The mathematical biaxial representation of this function, i.e. $\eta_a\left(\mathbf{a}\right) = \alpha\mathbf{a} + \beta\left[\left(\mathbf{c}_a \cdot \mathbf{a}\right)\mathbf{c}_b + \left(\mathbf{c}_b \cdot \mathbf{a}\right)\mathbf{c}_a\right]$ with

$$\begin{cases} \alpha = \mu_2\left(\varepsilon_1 - \varepsilon_3\right) + \varepsilon_2\left(\mu_1 - \mu_3\right) \\ \beta = \dfrac{1}{2}\left(\dfrac{\mu_3\varepsilon_1 - \mu_1\varepsilon_3}{\varepsilon_3\varepsilon_1}\right)\left[\left(\varepsilon_1 - \varepsilon_3\right)\varepsilon_2 + \varepsilon_1\varepsilon_3\dfrac{\mu_2\left(\mu_3 - \mu_1\right)}{\mu_1\mu_3}\right] \end{cases} \tag{5}$$

gives directly the medium optic axes,

$$\begin{cases} \mathbf{c}_a = \dfrac{\sqrt{\mu_3}\gamma_1\mathbf{e}_1 + \sqrt{\mu_1}\gamma_3\mathbf{e}_3}{\sqrt{\mu_3\gamma_1^2 + \mu_1\gamma_3^2}} \\ \mathbf{c}_b = \dfrac{-\sqrt{\mu_3}\gamma_1\mathbf{e}_1 + \sqrt{\mu_1}\gamma_3\mathbf{e}_3}{\sqrt{\mu_3\gamma_1^2 + \mu_1\gamma_3^2}} \end{cases} \tag{6}$$

where $\mathbf{e}_i$ are the eigenvectors of $\varepsilon$ and

$$\begin{cases} \gamma_1 = \sqrt{\left(\dfrac{\mu_2}{\varepsilon_2} - \dfrac{\mu_1}{\varepsilon_1}\right) \Big/ \left(\dfrac{\mu_3}{\varepsilon_3} - \dfrac{\mu_1}{\varepsilon_1}\right)} \\ \gamma_3 = \sqrt{\left(\dfrac{\mu_3}{\varepsilon_3} - \dfrac{\mu_2}{\varepsilon_2}\right) \Big/ \left(\dfrac{\mu_3}{\varepsilon_3} - \dfrac{\mu_1}{\varepsilon_1}\right)} \end{cases}. \tag{7}$$

This kind of media are biaxial if $\eta_a$ is biaxial, i.e. $\mathbf{c}_a \neq \mathbf{c}_b$, thus even for non biaxial permitivities and permeabilities it is possible to obtain biaxial media. In Figure 1, we plot a biaxial medium stemming from two uniaxial permittivity and permeability functions.

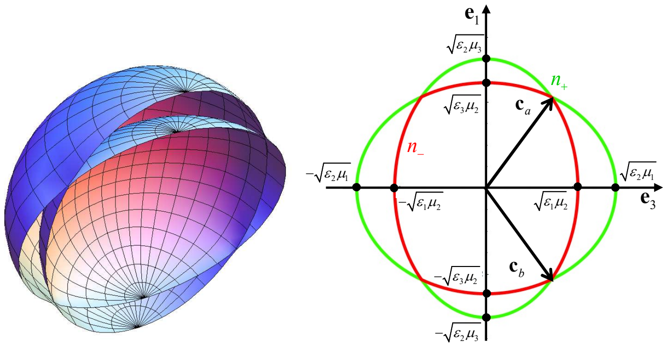


Figure 1. Biaxial medium: 3D and polar plots of the refractive index surfaces for $\varepsilon_1 = 1.5$, $\varepsilon_2 = \varepsilon_3 = 3$ and $\mu_1 = \mu_2 = 3$, $\mu_3 = 2$.

## III. CONICAL REFRACTION

When $\mathbf{k} \neq \mathbf{c}_i$ with $i = a, b$ we obtain from Eq. (3), the refractive indices and polarization of the characteristic waves, as we have shown in [9]. When $\mathbf{k} = \mathbf{c}_i$, Eq. (3) does not impose any restriction to the eigenfield polarizations. In fact, the field vectors are limited only by the conditions implied in Maxwell equations $\mathbf{k} \cdot \mathbf{D} = 0$ and $\mathbf{k} \cdot \mathbf{B} = 0$. Taking into account that $\varepsilon$ and $\mu$ are considered to be symmetric functions, we can obtain the two only constrains that the electric and magnetic fields must verify

$$\begin{cases} \mathbf{E} \cdot \varepsilon\left(\hat{\mathbf{k}}\right) = 0 \\ \mathbf{H} \cdot \mu\left(\hat{\mathbf{k}}\right) = 0 \end{cases}. \tag{8}$$

Considering $\mathbf{s}$ as the energy flow direction, i.e. the direction of the average Poynting vector, we know that $\mathbf{E} \cdot \mathbf{s} = 0, \mathbf{H} \cdot \mathbf{s} = 0$, hence

$$\begin{cases} \mathbf{E} = E_0\left[\mathbf{s} \wedge \varepsilon\left(\hat{\mathbf{k}}\right)\right]\mathbf{e}_{123} \\ \mathbf{H} = H_0\left[\mathbf{s} \wedge \mu\left(\hat{\mathbf{k}}\right)\right]\mathbf{e}_{123} \end{cases} \Rightarrow \begin{cases} \mathbf{D} = D_0\left[\varepsilon^{-1}\left(\mathbf{s}\right) \wedge \hat{\mathbf{k}}\right]\mathbf{e}_{123} \\ \mathbf{B} = B_0\left[\mu^{-1}\left(\mathbf{s}\right) \wedge \hat{\mathbf{k}}\right]\mathbf{e}_{123} \end{cases}. \tag{9}$$

Taking into account that $\mathbf{B} \cdot \mathbf{E} = 0$ and $\mathbf{D} \cdot \mathbf{H} = 0$, it follows from Eq. (9) after some algebra two equivalent expressions for the cone generated by the ray vector $\mathbf{s}$

$$\begin{cases} \left[\eta^{-1}\left(\mathbf{k}\right) \cdot \mathbf{s}\right]\left(\mathbf{s} \cdot \mathbf{k}\right) - \left[\varepsilon\left(\mathbf{k}\right) \cdot \mathbf{k}\right]\left[\mu^{-1}\left(\mathbf{s}\right) \cdot \mathbf{s}\right] = 0 \\ \left[\eta\left(\mathbf{k}\right) \cdot \mathbf{s}\right]\left(\mathbf{s} \cdot \mathbf{k}\right) - \left[\mu\left(\mathbf{k}\right) \cdot \mathbf{k}\right]\left[\varepsilon^{-1}\left(\mathbf{s}\right) \cdot \mathbf{s}\right] = 0 \end{cases} \tag{10}$$

The direction of energy does not have a fixed direction but forms an entire cone of directions, i.e. conical refraction. In Figure 2, we show the cones formed by $\mathbf{s}$ when $\mathbf{k} = \pm\mathbf{c}_a$ and $\mathbf{k} = \pm\mathbf{c}_b$, considering a general biaxial medium resulting from a biaxial permittivity and a uniaxial permeability. We should stress again, that the individual mathematical forms of the electric and magnetic constitutive functions do not have any direct physical meaning on the global behavior of the medium.

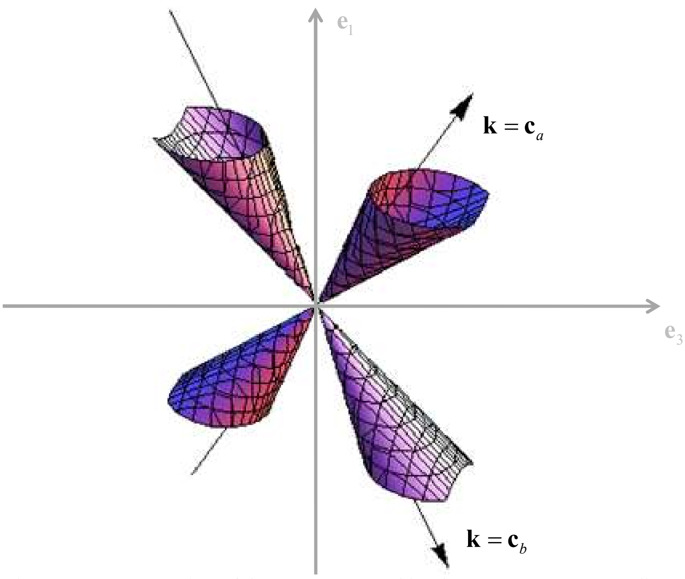


Figure 2. Representation of the cone generated by the ray vector $\mathbf{s}$, according to Eqs. (10) for $\varepsilon_1 = 2$ $\varepsilon_2 = 3/2$ $\varepsilon_3 = 1$ and $\mu_1 = \mu_2 = 2$, $\mu_3 = 3$

## IV. CONCLUSIONS

In this work we analyzed the conical refraction that occurs for biaxial media characterized by anisotropic electric and magnetic properties. The broader understanding on this kind of anisotropic media, gained with our new approach developed in previous papers, was fundamental to have an easy way to predict this effect. For example it is not intuitive that a media with uniaxial permittivity and permeability functions can be biaxial and have conical refraction, since a uniaxial electric (magnetic) crystal does not support conical refraction. We have shown that the interplay between the anisotropic electric and magnetic properties of the material is best captured by a new constitutive function. This function plays an analogous role that the inverse of permittivity does for simple electric biaxial crystals. The mathematical biaxial representation of this function has then a direct physical meaning, given the two medium optic axes in which conical refraction occurs. In this work we expect to bring a deeper insight on this ancient topic which, by the ongoing research on metamaterials, has regained a new interest for optical science.


## ACKNOWLEDGMENT

This work was partially funded by FCT - Foundation for Science and Technology, Portugal.


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