



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

## Global dynamics for the two-dimensional stochastic nonlinear wave equations

**Citation for published version:**

Gubinelli, M, Koch, H, Oh, T & Tolomeo, L 2022, 'Global dynamics for the two-dimensional stochastic nonlinear wave equations', *International Mathematics Research Notices*, vol. 2022, no. 21, pp. 16954–16999. <https://doi.org/10.1093/imrn/rnab084>

**Digital Object Identifier (DOI):**

[10.1093/imrn/rnab084](https://doi.org/10.1093/imrn/rnab084)

**Link:**

[Link to publication record in Edinburgh Research Explorer](#)

**Document Version:**

Peer reviewed version

**Published In:**

International Mathematics Research Notices

**General rights**

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

**Take down policy**

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [openaccess@ed.ac.uk](mailto:openaccess@ed.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.



# GLOBAL DYNAMICS FOR THE TWO-DIMENSIONAL STOCHASTIC NONLINEAR WAVE EQUATIONS

MASSIMILIANO GUBINELLI, HERBERT KOCH, TADAHIRO OH, AND LEONARDO TOLOMEO

ABSTRACT. We study global-in-time dynamics of the stochastic nonlinear wave equations (SNLW) with an additive space-time white noise forcing, posed on the two-dimensional torus. Our goal in this paper is two-fold. (i) By introducing a hybrid argument, combining the  $I$ -method in the stochastic setting with a Gronwall-type argument, we first prove global well-posedness of the (renormalized) cubic SNLW in the defocusing case. Our argument yields a double exponential growth bound on the Sobolev norm of a solution. (ii) We then study the stochastic damped nonlinear wave equations (SdNLW) in the defocusing case. In particular, by applying Bourgain's invariant measure argument, we prove almost sure global well-posedness of the (renormalized) defocusing SdNLW with respect to the Gibbs measure and invariance of the Gibbs measure.

## CONTENTS

1. Introduction	2
1.1. Stochastic nonlinear wave equations	2
1.2. Global well-posedness of the cubic SNLW	5
1.3. Hyperbolic $\Phi_2$ -model and the Gibbs measure	9
1.4. Remarks and comments	13
2. Preliminary lemmas	14
2.1. Preliminary results from stochastic analysis	14
2.2. Product estimates	17
3. $I$ -method for the renormalized cubic SNLW	17
3.1. Commutator and other preliminary estimates	18
3.2. Proof of Theorem 1.2	22
4. Almost sure global well-posedness of the hyperbolic $\Phi_2$ -model	28
References	31

---

2010 *Mathematics Subject Classification.* 35L71, 60H15.

*Key words and phrases.* stochastic nonlinear wave equation; nonlinear wave equation; damped nonlinear wave equation; renormalization; white noise; Gibbs measure.

## 1. INTRODUCTION

**1.1. Stochastic nonlinear wave equations.** In [21], the first three authors studied the following stochastic nonlinear wave equations (SNLW) on the two-dimensional torus  $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$  with an additive space-time white noise forcing:

$$\begin{cases} \partial_t^2 u + (1 - \Delta)u + u^k = \xi \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1) \end{cases} \quad (x, t) \in \mathbb{T}^2 \times \mathbb{R}_+, \quad (1.1)$$

where  $k \geq 2$  is an integer and  $\xi(x, t)$  denotes a (Gaussian) space-time white noise on  $\mathbb{T}^2 \times \mathbb{R}_+$ . In the following, we restrict our attention to the real-valued setting. By introducing an appropriate time-dependent renormalization, they proved local well-posedness of (the renormalized version of) SNLW (1.1) with (almost) critical initial data. Our main goal in this paper is to construct global-in-time dynamics to SNLW in the following two settings:

- (i) When  $k = 3$ , we introduce a hybrid argument, combining the so-called *I*-method [8, 9] and a Gronwall-type argument [6], and prove global well-posedness of (1.1). See Subsection 1.2.
- (ii) For  $k \in 2\mathbb{N} + 1$ , we consider SNLW with a damping term. More precisely, we study the following stochastic damped nonlinear wave equation (SdNLW):

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^k = \sqrt{2}\xi. \quad (1.2)$$

This equation is known as the hyperbolic counterpart of the stochastic quantization equation studied in the parabolic setting [11]. By exploiting (formal) invariance of the Gibbs measure for the dynamics, we prove almost sure global well-posedness of SdNLW (1.2). See Subsection 1.3.

The main difficulty in studying these problems, even locally in time, comes from the roughness of the space-time white noise. The stochastic convolution  $\Psi$ , solving the linear stochastic wave equation:

$$\partial_t^2 \Psi + (1 - \Delta)\Psi = \xi, \quad (1.3)$$

is not a classical function but is merely a distribution for the spatial dimension  $d \geq 2$ . In particular, there is an issue in making sense of powers  $\Psi^k$  and, consequently, of the full nonlinearity  $u^k$  in (1.1). This requires us to modify the equation by introducing a proper renormalization. In fact, for the models (1.1) and (1.2) without renormalization, a phenomenon of triviality is known to hold [1, 34]; roughly speaking, extreme oscillations make solutions to (1.1) (or (1.2)) with regularized noises tend to that to the linear stochastic wave equation (1.3) (or the trivial solution) as the regularization is removed.

In the following, let us briefly go over the local well-posedness argument in [21] and introduce a renormalized equation. See also [45]. We first express the stochastic convolution (with the zero initial data) in terms of a stochastic integral. With  $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ , let  $S(t)$  denote the linear wave propagator:

$$S(t) = \frac{\sin(t\langle \nabla \rangle)}{\langle \nabla \rangle}, \quad (1.4)$$

defined as a Fourier multiplier operator. Namely, we set<sup>1</sup>

$$S(t)f = \sum_{n \in \mathbb{Z}^2} \frac{\sin(t\langle n \rangle)}{\langle n \rangle} \widehat{f}(n) e_n,$$

where  $\widehat{f}(n)$  is the Fourier coefficient of  $f$  and  $e_n(x) = e^{in \cdot x}$ . Then, the stochastic convolution  $\Psi$ , solving (1.3), is given by

$$\Psi(t) = \int_0^t S(t-t') dW(t'), \quad (1.5)$$

where  $W$  denotes a cylindrical Wiener process on  $L^2(\mathbb{T}^2)$ :

$$W(t) := \sum_{n \in \mathbb{Z}^2} B_n(t) e_n \quad (1.6)$$

and  $\{B_n\}_{n \in \mathbb{Z}^2}$  is defined by  $B_n(t) = \langle \xi, \mathbf{1}_{[0,t]} \cdot e_n \rangle_{x,t}$ . Here,  $\langle \cdot, \cdot \rangle_{x,t}$  denotes the duality pairing on  $\mathbb{T}^2 \times \mathbb{R}$ . As a result, we see that  $\{B_n\}_{n \in \mathbb{Z}^2}$  is a family of mutually independent complex-valued<sup>2</sup> Brownian motions conditioned so that  $B_{-n} = \overline{B_n}$ ,  $n \in \mathbb{Z}^2$ . By convention, we normalized  $B_n$  such that  $\text{Var}(B_n(t)) = t$ .

Given  $N \in \mathbb{N}$ , we define the truncated stochastic convolution  $\Psi_N = \mathbf{P}_N \Psi$ , solving the truncated linear stochastic wave equation:

$$\partial_t^2 \Psi_N + (1 - \Delta) \Psi_N = \mathbf{P}_N \xi$$

with the zero initial data. Here,  $\mathbf{P}_N$  denotes the frequency cutoff onto the spatial frequencies  $\{|n| \leq N\}$ . Then, for each fixed  $x \in \mathbb{T}^2$  and  $t \geq 0$ , we see that  $\Psi_N(x, t)$  is a mean-zero real-valued Gaussian random variable with variance

$$\begin{aligned} \sigma_N(t) &\stackrel{\text{def}}{=} \mathbb{E}[\Psi_N(x, t)^2] = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \int_0^t \left[ \frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle} \right]^2 dt' \\ &= \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \left\{ \frac{t}{2\langle n \rangle^2} - \frac{\sin(2t\langle n \rangle)}{4\langle n \rangle^3} \right\} \sim t \log N \end{aligned} \quad (1.7)$$

for  $N \gg 1$ . We point out that the variance  $\sigma_N(t)$  is time-dependent. For any  $t > 0$ , we see that  $\sigma_N(t) \rightarrow \infty$  as  $N \rightarrow \infty$ , which can be used to show that  $\{\Psi_N(t)\}_{N \in \mathbb{N}}$  is almost surely unbounded in  $W^{0,p}(\mathbb{T}^2)$  for any  $1 \leq p \leq \infty$ .

Let  $u_N$  denote the solution to SNLW (1.1) with the regularized noise  $\mathbf{P}_N \xi$ . Proceeding with the following decomposition of  $u_N$  ([26, 4, 11]):

$$u_N = \Psi_N + v_N. \quad (1.8)$$

Then, we see that the residual term  $v_N$  satisfies

$$\partial_t^2 v_N + (1 - \Delta) v_N + \sum_{\ell=0}^k \binom{k}{\ell} \Psi_N^\ell v_N^{k-\ell} = 0. \quad (1.9)$$

<sup>1</sup>Hereafter, we drop the harmless factor  $2\pi$ .

<sup>2</sup>In particular,  $B_0$  is a standard real-valued Brownian motion.

Note that, due to the deficiency of regularity, the power  $\Psi_N^\ell$  does not converge to any limit as  $N \rightarrow \infty$ . This is where we introduce the Wick renormalization. Namely, we replace  $\Psi_N^\ell$  by its Wick ordered counterpart:

$$:\Psi_N^\ell(x, t): \stackrel{\text{def}}{=} H_\ell(\Psi_N(x, t); \sigma_N(t)), \quad (1.10)$$

where  $H_\ell(x; \sigma)$  is the Hermite polynomial of degree  $\ell$  with variance parameter  $\sigma$ . See Section 2. Then, for each  $\ell \in \mathbb{N}$ , the Wick power  $:\Psi_N^\ell:$  converges to a limit, denoted by  $:\Psi^\ell:$ , in  $C([0, T]; W^{-\varepsilon, \infty}(\mathbb{T}^2))$  for any  $\varepsilon > 0$  and  $T > 0$ , almost surely (and also in  $L^p(\Omega)$  for any  $p < \infty$ ). See Lemma 2.3 below. This Wick renormalization gives rise to the renormalized version of (1.9):

$$\partial_t^2 v_N + (1 - \Delta)v_N + \sum_{\ell=0}^k \binom{k}{\ell} :\Psi_N^\ell: v_N^{k-\ell} = 0.$$

By taking a limit as  $N \rightarrow \infty$ , we then obtain the limiting equation:

$$\partial_t^2 v + (1 - \Delta)v + \sum_{\ell=0}^k \binom{k}{\ell} :\Psi^\ell: v^{k-\ell} = 0. \quad (1.11)$$

Given the almost sure space-time regularity of the Wick powers  $:\Psi^\ell:$ ,  $\ell = 1, \dots, k$ , standard deterministic analysis with the Strichartz estimates and the product estimates (Lemma 2.5) yields local well-posedness of (1.11) (for  $v$ ). Recalling the decomposition (1.8), this argument also shows that the solution  $u_N = \Psi_N + v_N$  to the renormalized SNLW with the regularized noise  $\mathbf{P}_N \xi$ :

$$\partial_t^2 u_N + (1 - \Delta)u_N + :u_N^k: = \mathbf{P}_N \xi,$$

where the renormalized nonlinearity  $:u_N^k:$  is interpreted as

$$:u_N^k: = :(\Psi_N + v_N)^k: = \sum_{\ell=0}^k \binom{k}{\ell} :\Psi_N^\ell: v_N^{k-\ell},$$

converges almost surely to a stochastic process  $u = \Psi + v$ , where  $v$  satisfies (1.11). It is in this sense that we say that the renormalized SNLW:

$$\partial_t^2 u + (1 - \Delta)u + :u^k: = \xi$$

is locally well-posed (for initial data of suitable regularity).

**Remark 1.1.** The equation (1.1) is also known as the stochastic nonlinear Klein-Gordon equation. In the following, however, we simply refer to (1.1) as the stochastic nonlinear wave equation.

In [21], we treated the equation (1.1) with the mass-less linear part  $\partial_t^2 u - \Delta u$ . Note that the same results in [21] with inessential modifications also hold for (1.1) with the massive linear part  $\partial_t^2 u + (1 - \Delta)u$ . Conversely, Theorem 1.2 below also holds for SNLW with the mass-less linear part  $\partial_t^2 u - \Delta u$ . We point out, however, that for our second main result (Theorem 1.7), we need to work with the massive linear part in order to avoid a problem at the zeroth frequency in the Gibbs measure construction; see [44]. For this reason, we work with the massive case in this paper.

**1.2. Global well-posedness of the cubic SNLW.** Our first goal is to construct global-in-time dynamics for the renormalized cubic SNLW. In the following, we study (1.11) with  $k = 3$ :

$$\partial_t^2 v + (1 - \Delta)v + v^3 + 3v^2\Psi + 3v : \Psi^2 : + : \Psi^3 : = 0. \quad (1.12)$$

In [21], it was shown that (1.12) is locally well-posed in  $\mathcal{H}^s(\mathbb{T}^2) \stackrel{\text{def}}{=} H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$  for  $s > \frac{1}{4}$ . Furthermore, the following blowup alternative holds almost surely; either the solution  $v$  exists globally in time or there exists some finite time  $T_* = T_*(\omega) > 0$  such that

$$\lim_{t \nearrow T_*} \|\vec{v}(t)\|_{\mathcal{H}^\sigma} = \infty, \quad (1.13)$$

where  $\vec{v} = (v, \partial_t v)$  and  $\sigma = \min(s, 1 - \varepsilon)$  for any small  $\varepsilon > 0$ . While the blowup alternative (1.13) is not explicitly proven in [21], it easily follows as a consequence of the (deterministic) contraction argument used to study (1.12) in [21].

In the parabolic setting, there are recent works [29, 30, 20, 27] on global well-posedness of the parabolic  $\Phi_d^4$ -model via deterministic approaches. The main ingredient in [29, 30] is a (non-trivial) adaptation of a standard globalization argument for a nonlinear heat equation by controlling the (weighted)  $L^p$ -norm of the smoother part of a solution (corresponding to  $v$  in (1.12)). Due to a weaker smoothing property, however, the situation is much more involved in the case of the wave equation.

Essentially speaking, the only known way to prove global well-posedness for the deterministic cubic nonlinear wave equation (NLW):

$$\partial_t^2 v + (1 - \Delta)v + v^3 = 0 \quad (1.14)$$

(except in the small data<sup>3</sup> regime) is to exploit the energy  $E(\vec{v})$  given by

$$E(\vec{v}) = \frac{1}{2} \int_{\mathbb{T}^2} (v^2 + |\nabla v|^2) dx + \frac{1}{2} \int_{\mathbb{T}^2} (\partial_t v)^2 dx + \frac{1}{4} \int_{\mathbb{T}^2} v^4 dx, \quad (1.15)$$

which is conserved for smooth solutions. There are two sources of difficulty in proving global well-posedness of the cubic SNLW (1.12).

- (i) The first problem comes from the lack of regularity of the solution  $\vec{v} = (v, \partial_t v)$  to (1.12). Due to the roughness of the stochastic convolution  $\Psi$ , we easily see that  $\vec{v} \in C([0, T]; \mathcal{H}^s(\mathbb{T}^2))$  only for  $s < 1$ . Namely, for a solution  $\vec{v}$  to (1.12), the energy  $E(\vec{v})$  is infinite. In order to overcome this difficulty, we propose to use the  $I$ -method introduced by Colliander-Keel-Staffilani-Takaoka-Tao [8, 9]. See for example [47] on an application of the  $I$ -method to the cubic NLW (1.14) on  $\mathbb{T}^2$  in the deterministic setting.
- (ii) The second problem comes from the fact that  $v$  satisfies the cubic NLW with perturbations, which results in the non-conservation of the energy  $E(\vec{v})$  even if  $\vec{v}(t)$  were in  $\mathcal{H}^1(\mathbb{T}^2)$ .

The second problem could be easily remedied if  $\Psi$  were slightly smoother. Given  $\Psi \in C(\mathbb{R}_+; L^\infty(\mathbb{T}^2))$ , consider

$$\partial_t^2 v + (1 - \Delta)v + v^3 + 3v^2\Psi + 3v\Psi^2 + \Psi^3 = 0. \quad (1.16)$$

---

<sup>3</sup>This includes the construction of solutions near a particular solution.

In this case, we can apply the globalization argument by Burq-Tzvetkov [6], originally introduced in the context of the cubic NLW on  $\mathbb{T}^3$  with random initial data. Namely, by Cauchy-Schwarz inequality along with (1.16) and Young's inequality, we have

$$\begin{aligned} |\partial_t E(\vec{v})| &= \left| \int_{\mathbb{T}^2} (\partial_t v) \{ \partial_t^2 v + (1 - \Delta)v + v^3 \} dx \right| \\ &\leq (E(\vec{v}))^{\frac{1}{2}} \left( \|\Psi\|_{C_T L_x^\infty} \int_{\mathbb{T}^2} v^4 dx + \|\Psi\|_{C_T L_x^6} \right)^{\frac{1}{2}} \\ &\leq C(T, \Psi)(1 + E(\vec{v})) \end{aligned}$$

for any given  $T > 0$ , where  $C_T L_x^p = C([0, T]; L^p(\mathbb{T}^2))$ . Then, global well-posedness of (1.16) in  $\mathcal{H}^1(\mathbb{T}^2)$  follows from Gronwall's inequality.

As described above, we can handle each of the difficulties (i) and (ii) by a standard approach if it occurs one at a time. The main difficulty in proving global well-posedness of the cubic SNLW (1.12) lies in the fact that we need to handle the difficulties (i) and (ii) *at the same time*. This combination of the problems (i) and (ii) makes the problem significantly harder.

We now state our first main result.

**Theorem 1.2.** *Let  $s > \frac{4}{5}$ . Then, the renormalized cubic SNLW (1.12) on  $\mathbb{T}^2$  is globally well-posed in  $\mathcal{H}^s(\mathbb{T}^2)$ . More precisely, given  $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ , the solution  $v$  to (1.12) exists globally in time, almost surely, such that  $(v, \partial_t v) \in C(\mathbb{R}_+; \mathcal{H}^\sigma(\mathbb{T}^2))$ , where  $\sigma = \min(s, 1 - \varepsilon)$  for any small  $\varepsilon > 0$ .*

For simplicity, we only consider the case  $\frac{4}{5} < s < 1$  such that  $\sigma = s$ . The main approach is to combine the  $I$ -method with a Gronwall-type argument. Let us first recall the main idea of the  $I$ -method. Fix  $0 < s < 1$ . Given  $N \geq 1$ , we define a smooth, radially symmetric, non-increasing (in  $|\xi|$ ) multiplier  $m_N \in C^\infty(\mathbb{R}^2; [0, 1])$ , satisfying

$$m_N(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq N, \\ \left(\frac{N}{|\xi|}\right)^{1-s}, & \text{if } |\xi| \geq 2N. \end{cases} \quad (1.17)$$

We then define the  $I$ -operator  $I = I_N$  to be the Fourier multiplier operator with the multiplier  $m_N$ :

$$\widehat{I_N f}(n) = m_N(n) \widehat{f}(n). \quad (1.18)$$

Then, we see that  $I_N$  acts as the identity operator on low frequencies  $\{|n| \leq N\}$ , while it acts as a fractional integration operator of order  $1 - s$  on high frequencies  $\{|n| \geq 2N\}$ . From the definition, it is easy to see that  $I f \in H^1(\mathbb{T}^2)$  if and only if  $f \in H^s(\mathbb{T}^2)$  with the bound:

$$\|f\|_{H^s} \lesssim \|I f\|_{H^1} \lesssim N^{1-s} \|f\|_{H^s}. \quad (1.19)$$

Moreover, by the Littlewood-Paley theory, we have<sup>4</sup>

$$\|If\|_{W^{s_0+s_1,p}} \lesssim N^{s_1} \|f\|_{W^{s_0,p}} \quad (1.20)$$

for any  $s_0 \in \mathbb{R}$ ,  $0 \leq s_1 \leq 1 - s$ , and  $1 < p < \infty$ .

Let  $\frac{4}{5} < s < 1$ . Given initial data  $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ , we consider the  $I$ -SNLW:

$$\partial_t^2 I v + (1 - \Delta) I v + I(v^3) + 3I(v^2 \Psi) + 3I(v : \Psi^2 :) + I(: \Psi^3 :) = 0. \quad (1.21)$$

The local well-posedness of (1.12) implies local well-posedness of the  $I$ -SNLW (1.21). In view of the blowup alternative (1.13) and (1.19), our main task is to control the growth of the modified energy  $E(I\vec{v})$ . Note that there are two sources for the non-conservation of the modified energy  $E(I\vec{v})$ , reflecting the problems (i) and (ii) discussed above: (i) The main part of the nonlinearity is  $I(v^3)$ , not the cubic power  $(Iv)^3$ , and (ii) there are perturbative terms:  $3I(v^2 \Psi) + 3I(v : \Psi^2 :) + I(: \Psi^3 :)$ . Indeed, a direct computation with (1.15) and (1.21) gives

$$\begin{aligned} E(I\vec{v})(t) - E(I\vec{v})(0) &= \int_0^t \int_{\mathbb{T}^2} (\partial_t I v) \{ -I(v^3) + (Iv)^3 \} dx dt \\ &\quad - 3 \int_0^t \int_{\mathbb{T}^2} (\partial_t I v) I(v^2 \Psi) dx dt \\ &\quad - 3 \int_0^t \int_{\mathbb{T}^2} (\partial_t I v) I(v : \Psi^2 :) dx dt \\ &\quad - \int_0^t \int_{\mathbb{T}^2} (\partial_t I v) I(: \Psi^3 :) dx dt \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned} \quad (1.22)$$

The first term  $A_1$  represents the main commutator part, resulting from the application of the  $I$ -operator, and we estimate this part by establishing a certain commutator estimate (as in the deterministic setting). On the other hand, the second, third, and fourth terms  $A_2$ ,  $A_3$ , and  $A_4$  represent the contributions from the perturbative terms in (1.21), which are to be controlled by a Gronwall-type argument as above.<sup>5</sup> The worst contribution comes from  $A_2$ . In order to control this term, the standard estimate (1.20) with the fact that  $\Psi(t) \in W^{-\varepsilon, \infty}(\mathbb{T}^2)$  is too crude since it loses a positive power of  $N$ . We instead need to use a finer regularity property of  $\Psi(t)$ , namely, it is logarithmically divergent from  $L^p(\mathbb{T}^2)$ . See Lemma 2.4 below. At the end of the day, we end up with a Gronwall-type estimate, where the right-hand side has a logarithmically superlinear growth. Roughly speaking, we obtain an estimate of the form:

$$\partial_t E(I\vec{v})(t) \lesssim E(I\vec{v})(t) \log(E(I\vec{v})(t)). \quad (1.23)$$

See (3.22) and (3.39) below for precise bounds. We then implement an iterative argument, proceeding over time intervals of fixed size, by *choosing an increasing sequence of the parameters  $N_k$  for the  $I$ -operator*. See Subsection 3.2 for details.

<sup>4</sup>Here,  $W^{s,r}(\mathbb{T}^2)$  denotes the usual  $L^r$ -based Sobolev space (Bessel potential space) defined by the norm:

$$\|u\|_{W^{s,r}} = \|\langle \nabla \rangle^s u\|_{L^r} = \|\mathcal{F}^{-1}(\langle n \rangle^s \widehat{u}(n))\|_{L^r}.$$

When  $r = 2$ , we have  $H^s(\mathbb{T}^2) = W^{s,2}(\mathbb{T}^2)$ .

<sup>5</sup>As we see in Section 3, these terms also contain the commutator parts as well. For simplicity, we ignore this issue in this part of discussion.



**Remark 1.3.** (i) In a standard application of the  $I$ -method, one first fixes the large target time  $T \gg 1$  and then chooses a parameter  $N = N(T) \gg 1$ . For our problem, this is not sufficient. We instead need to choose an increasing sequence of the parameters  $N_k$  for the  $I$ -operator over different local-in-time intervals. It would be of interest to investigate a possible application of this new type of the  $I$ -method argument in the deterministic or random data setting (other than that mentioned in the following remark).

(ii) A standard application of the  $I$ -method yields a polynomial (in time) growth bound on the Sobolev norm of a solution. See, for example, Section 6 in [9]. A close examination of our hybrid argument yields a double exponential growth bound on the  $\mathcal{H}^s$ -norm of a solution. See Remark 3.7. We point out that such a double exponential bound would follow as a direct consequence of the estimate<sup>6</sup> (1.23). While it may be possible to improve this double exponential bound, we do not know how to do so at this point. Such an argument would require a new globalization approach. Lastly, we note that, while one may expect a subpolynomial growth in the deterministic setting, we expect at best a polynomial growth bound for the (undamped) SNLW due to the polynomial growth (in time) of the stochastic convolution (which is essentially a Brownian motion in time). Compare this with the damped case, where the invariant measure argument yields a logarithmic growth bound; see Remark 1.8.

**Remark 1.4.** In [45], Thomann and the third author proved almost sure global well-posedness of the renormalized defocusing cubic NLW on  $\mathbb{T}^2$  with the random data distributed by the massive Gaussian free field. The proof in [45] was based on (formal) invariance of the Gibbs measure and Bourgain's invariant measure argument. We point out that a slight modification of the proof of Theorem 1.2 provides another proof of this almost sure global well-posedness result via a pathwise argument (without using the invariant measure argument).

**Remark 1.5.** (i) Theorem 1.2 establishes global well-posedness of the renormalized cubic SNLW (1.12) on  $\mathbb{T}^2$  in  $\mathcal{H}^s(\mathbb{T}^2)$  for  $s > \frac{4}{5}$ , which leaves a gap to the local well-posedness threshold  $s > \frac{1}{4}$  from [21]. It may be possible to refine the  $I$ -method part (for example, by using analysis from [47]) to lower regularities (to some extent). We, however, decided not to pursue this issue since our globalization argument presented in Section 3 is already quite involved, and our main goal in this part is to present this hybrid argument of the  $I$ -method with a Gronwall-type argument in its simplest form.

(ii) In a recent work [53], the fourth author extended Theorem 1.2 to the renormalized cubic SNLW on  $\mathbb{R}^2$ . For this problem, one needs to handle not only the roughness of the noise but also its unboundedness.

(iii) In [17], Forlano recently adapted our globalization argument in studying the BBM equation with random initial data outside  $L^2(\mathbb{T})$ .

(iv) At this point, we do not know how to prove global well-posedness of the renormalized (undamped) SNLW with (super-)quintic nonlinearity. Even with a smoother noise, one would need to use a trick introduced in [38] to handle the high homogeneity. See for example [28] for global well-posedness of the stochastic nonlinear beam equations on  $\mathbb{T}^3$ .

<sup>6</sup>Note that we do not quite obtain the estimate (1.23) for the modified energy  $E(I\vec{v})$ . See (3.22) and (3.39) below for the actual bounds.

**Remark 1.6.** In order to prove global well-posedness of a stochastic PDE, we employ the *I*-method to study the equation (1.12) for  $v = u - \Psi$ . As such, our argument is essentially pathwise and thus entirely deterministic, once we have a control on the relevant stochastic terms.

In a recent work [7], the third author with Cheung and Li implemented the *I*-method to prove global well-posedness of stochastic nonlinear Schrödinger equations (SNLS) below the energy space. In estimating the growth of the modified energy, the authors used Ito's lemma, which lead to a careful stopping time argument (rather than a usual application of the *I*-method, where one iterates a local-in-time argument with a control on the modified energy). The argument introduced in [7] is a natural<sup>7</sup> extension of the *I*-method to the stochastic setting, which can be applied to a wide class of stochastic dispersive equations.

**1.3. Hyperbolic  $\Phi_2$ -model and the Gibbs measure.** In this subsection, we consider the following stochastic damped nonlinear wave equation (SdNLW):

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^k = \sqrt{2}\xi \quad (1.24)$$

for  $k \in 2\mathbb{N} + 1$ . This model is known as the so-called canonical stochastic quantization equation for the  $\Phi_2^{k+1}$ -model [48]; see also a discussion below. The local well-posedness argument from [21] for the undamped (renormalized) SNLW is readily applicable to yield local well-posedness of (the renormalized version of) SdNLW (1.24) for any  $k \in 2\mathbb{N} + 1$ . Moreover, when  $k = 3$ , a slight modification of the proof of Theorem 1.2 provides a deterministic (i.e. pathwise) argument, establishing global well-posedness in the damped case. As pointed out in Remark 1.5, such a deterministic argument is limited to  $k = 3$  at this point. In the damped case, however, we can rely on a probabilistic argument in order to construct global-in-times dynamics for (1.24) with general  $k \in 2\mathbb{N} + 1$ . More precisely, we construct global-in-time dynamics for (1.24), by exploiting (formal) invariance of the Gibbs measure with the density:

$$“d\bar{\rho}(u, \partial_t u) = Z^{-1} e^{-E(u, \partial_t u)} dud(\partial_t u)”, \quad (1.25)$$

where  $E(u, \partial_t u)$  denotes the energy (= Hamiltonian):

$$E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{T}^2} (u^2 + |\nabla u|^2) dx + \frac{1}{2} \int_{\mathbb{T}^2} (\partial_t u)^2 dx + \frac{1}{k+1} \int_{\mathbb{T}^2} u^{k+1} dx \quad (1.26)$$

for the (deterministic undamped) NLW:

$$\partial_t^2 u + (1 - \Delta)u + u^k = 0. \quad (1.27)$$

By drawing an analogy to finite-dimensional Hamiltonian systems, the Gibbs measure  $\bar{\rho}$  was expected to be invariant under the NLW dynamics (1.27). In [45], the third author and Thomann showed that this is indeed the case. As for SdNLW (1.24), we can view it as the superposition of the NLW dynamics (1.27) and the Ornstein-Uhlenbeck dynamics (for the component  $\partial_t u$ ):

$$\partial_t(\partial_t u) = -\partial_t u + \sqrt{2}dW,$$

<sup>7</sup>In particular, the authors in [7] studied the growth of the modified energy of a solution  $u$  (rather than the residual part  $v = u - \Psi$ ) via Ito's lemma, which is a natural extension of the  $H^1$ -global well-posedness result on SNLS by de Bouard and Debussche [14] to the low regularity setting.

each of which preserves the Gibbs measure  $\vec{\rho}$  in (1.25). Hence, we expect the Gibbs measure  $\vec{\rho}$  to be invariant under the dynamics of SdNLW (1.24).

By substituting (1.26) in the exponent of (1.25), we see that the Gibbs measure  $\vec{\rho}$  decouples into the  $\Phi_2^{k+1}$ -measure on  $u$  and the white noise measure on  $\partial_t u$ . The dynamical model (1.24) then corresponds to the canonical stochastic quantization equation<sup>8</sup> of the  $\Phi_2^{k+1}$ -model; see [48]. For this reason, we also refer to (1.24) as the hyperbolic  $\Phi_2^{k+1}$ -model.

In order to make our discussion rigorous, let us introduce some notations. Given  $s \in \mathbb{R}$ , let  $\mu_s$  denote a Gaussian measure on periodic distributions, formally defined by

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^2} e^{-\frac{1}{2}\langle n \rangle^{2s} |\hat{u}(n)|^2} d\hat{u}(n). \quad (1.28)$$

Note that  $\mu_1$  corresponds to the massive Gaussian free field, while  $\mu_0$  corresponds to the white noise. We set

$$\vec{\mu}_s = \mu_s \otimes \mu_{s-1}. \quad (1.29)$$

In particular, when  $s = 1$ , the measure  $\vec{\mu}_1$  is defined as the induced probability measure under the map:

$$\omega \in \Omega \mapsto (u^1(\omega), u^2(\omega)),$$

where  $u^1(\omega)$  and  $u^2(\omega)$  are given by

$$u^1(\omega) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e_n \quad \text{and} \quad u^2(\omega) = \sum_{n \in \mathbb{Z}^2} h_n(\omega) e_n. \quad (1.30)$$

Here,  $\{g_n, h_n\}_{n \in \mathbb{Z}^2}$  denotes a family of independent standard complex-valued Gaussian random variables conditioned so that  $\overline{g_n} = g_{-n}$  and  $\overline{h_n} = h_{-n}$ ,  $n \in \mathbb{Z}^2$ . It is easy to see that  $\vec{\mu}_1 = \mu_1 \otimes \mu_0$  is supported on  $\mathcal{H}^s(\mathbb{T}^2)$  for  $s < 0$  but not for  $s \geq 0$ .

With (1.26), (1.28), and (1.29), we can formally write the Gibbs measure  $\vec{\rho}$  in (1.25) as

$$d\vec{\rho}(u, \partial_t u) \sim e^{-\frac{1}{k+1} \int_{\mathbb{T}^2} u^{k+1} dx} d\vec{\mu}_1(u, \partial_t u). \quad (1.31)$$

In view of the roughness of the support of  $\vec{\mu}_1$ , the nonlinear term  $\int_{\mathbb{T}^2} u^{k+1} dx$  in (1.31) is not well defined and thus a proper renormalization is required to give a meaning to (1.31).

Given a random variable  $X$ , let  $\mathcal{L}(X)$  denote the law of  $X$ . Suppose that  $\mathcal{L}(u) = \mu_1$ . Then, given  $N \in \mathbb{N}$ , we have

$$\alpha_N \stackrel{\text{def}}{=} \mathbb{E}[(\mathbf{P}_N u(x))^2] = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \frac{1}{\langle n \rangle^2} \sim \log N \quad (1.32)$$

for  $N \gg 1$ , independent of  $x \in \mathbb{T}^2$ . Given  $N \in \mathbb{N}$ , define the truncated renormalized density:

$$R_N(u) = \exp\left(-\frac{1}{k+1} \int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{k+1}(x): dx\right), \quad (1.33)$$

where the Wick power  $:(\mathbf{P}_N u)^{k+1}(x):$  is defined by

$$:(\mathbf{P}_N u)^{k+1}(x): \stackrel{\text{def}}{=} H_{k+1}(\mathbf{P}_N u(x); \alpha_N).$$

<sup>8</sup>Namely, the Langevin equation with the momentum  $v = \partial_t u$ .

Then, it is known that  $\{R_N\}_{N \in \mathbb{N}}$  forms a Cauchy sequence in  $L^p(\mu_1)$  for any finite  $p \geq 1$ . Thus, there exists a random variable  $R(u)$  such that

$$\lim_{N \rightarrow \infty} R_N(u) = R(u) \quad \text{in } L^p(\mu_1). \quad (1.34)$$

See [50, 19, 13, 44] for details. In view of (1.33) and (1.34), we can write the limit as

$$R(u) = \exp \left( -\frac{1}{k+1} \int_{\mathbb{T}^2} :u^{k+1}(x): dx \right).$$

By defining the renormalized truncated Gibbs measure:

$$d\vec{\rho}_N(u, \partial_t u) = Z_N^{-1} R_N(u) d\vec{\mu}_1(u, \partial_t u), \quad (1.35)$$

we then conclude that the renormalized truncated Gibbs measure  $\vec{\rho}_N$  converges, in the sense of (1.34), to the renormalized Gibbs measure  $\vec{\rho}$  given by

$$\begin{aligned} d\vec{\rho}(u, \partial_t u) &= Z^{-1} e^{R(u)} d\vec{\mu}_1(u, \partial_t u) \\ &= Z^{-1} \exp \left( -\frac{1}{k+1} \int_{\mathbb{T}^2} :u^{k+1}(x): dx \right) d\vec{\mu}_1(u, \partial_t u). \end{aligned} \quad (1.36)$$

Furthermore, the resulting Gibbs measure  $\vec{\rho}$  is equivalent<sup>9</sup> to the Gaussian measure  $\vec{\mu}_1$ .

Next, we move onto the well-posedness theory of the hyperbolic  $\Phi_2^{k+1}$ -model (1.24). Let us first introduce the following renormalized truncated SdNLW:

$$\partial_t^2 u_N + \partial_t u_N + (1 - \Delta)u_N + \mathbf{P}_N(:(\mathbf{P}_N u)^k:) = \sqrt{2}\xi \quad (1.37)$$

and its formal limit:

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + :u^k: = \sqrt{2}\xi. \quad (1.38)$$

It is easy to check that the renormalized truncated Gibbs measure  $\vec{\rho}_N$  is invariant under the truncated dynamics (1.37). See Section 4.

We now state our second result.

**Theorem 1.7.** *The renormalized SdNLW (1.38) is almost surely globally well-posed with respect to the renormalized Gibbs measure  $\vec{\rho}$  in (1.36). Furthermore, the renormalized Gibbs measure  $\vec{\rho}$  is invariant under the dynamics.*

*More precisely, there exists a non-trivial stochastic process  $(u, \partial_t u) \in C(\mathbb{R}_+; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$  for any  $\varepsilon > 0$  such that, given any  $T > 0$ , the solution  $(u_N, \partial_t u_N)$  to the renormalized truncated SdNLW (1.37) with the random initial data  $(u_N, \partial_t u_N)|_{t=0}$  distributed according to the renormalized truncated Gibbs measure  $\vec{\rho}_N$  in (1.35), converges in probability to some stochastic process  $(u, \partial_t u)$  in  $C([0, T]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$ . Moreover, the law of  $(u(t), \partial_t u(t))$  is given by the renormalized Gibbs measure  $\vec{\rho}$  in (1.36) for any  $t \geq 0$ .*

In the context of the renormalized (deterministic) NLW:

$$\partial_t^2 u + (1 - \Delta)u + :u^k: = 0,$$

the third author with Thomann proved an analogous result; see [45].

In view of the convergence of  $\vec{\rho}_N$  to  $\vec{\rho}$ , the invariance of  $\vec{\rho}_N$  under the truncated SdNLW dynamics (1.37), and Bourgain's invariant measure argument [3, 4], Theorem 1.7 follows once we construct the limiting process  $(u, \partial_t u)$  locally in time with a good approximation

<sup>9</sup>Namely,  $\vec{\rho}$  and  $\vec{\mu}_1$  are mutually absolutely continuous.

property by the solution  $u_N$  to (1.37). Furthermore, in view of the equivalence of  $\vec{\rho}$ ,  $\vec{\rho}_N$ , and  $\vec{\mu}_1$ , it suffices to study the renormalized SdNLW (1.37) and (1.38) with the Gaussian random initial data  $(\phi_0, \phi_1)$  with  $\mathcal{L}(\phi_0, \phi_1) = \vec{\mu}_1$ .

As in the previous sections, we proceed with the first order expansion. For our damped model, we let  $\Phi$  be the solution to the linear stochastic damped wave equation:

$$\begin{cases} \partial_t^2 \Phi + \partial_t \Phi + (1 - \Delta)\Phi = \sqrt{2}\xi \\ (\Phi, \partial_t \Phi)|_{t=0} = (\phi_0, \phi_1), \end{cases} \quad (1.39)$$

where  $\mathcal{L}(\phi_0, \phi_1) = \vec{\mu}_1$ . Define the linear damped wave propagator  $\mathcal{D}(t)$  by

$$\mathcal{D}(t) = e^{-\frac{t}{2}} \frac{\sin\left(t\sqrt{\frac{3}{4} - \Delta}\right)}{\sqrt{\frac{3}{4} - \Delta}} \quad (1.40)$$

as a Fourier multiplier operator. Then, the stochastic convolution  $\Phi$  can be expressed as

$$\Phi(t) = \partial_t \mathcal{D}(t)\phi_0 + \mathcal{D}(t)(\phi_0 + \phi_1) + \sqrt{2} \int_0^t \mathcal{D}(t-t') dW(t'), \quad (1.41)$$

where  $W$  is as in (1.6). A direct computation shows that  $\Phi_N(x, t) = \mathbf{P}_N \Phi(x, t)$  is a mean-zero real-valued Gaussian random variable with variance

$$\mathbb{E}[\Phi_N(x, t)^2] = \mathbb{E}[(\mathbf{P}_N \Phi(x, t))^2] = \alpha_N$$

for any  $t \geq 0$ ,  $x \in \mathbb{T}^2$ , and  $N \geq 1$ , where  $\alpha_N$  is as in (1.32). We point out that unlike  $\sigma_N(t)$  in (1.7), the variance  $\alpha_N$  is time independent. This is due to the fact that the massive Gaussian free field  $\mu_1$  is invariant under the dynamics of the linear stochastic damped wave equation (1.39).

Let  $u_N$  be the solution to (1.37) with  $\mathcal{L}((u_N, \partial_t u_N)|_{t=0}) = \vec{\mu}_1$ . Then, by writing  $u_N$  as

$$u_N = v_N + \Phi = (v_N + \Phi_N) + \mathbf{P}_N^\perp \Phi_N, \quad (1.42)$$

where  $\mathbf{P}_N^\perp = \text{Id} - \mathbf{P}_N$ , we see that the dynamics of the renormalized truncated SdNLW (1.37) decouples into the linear dynamics for the high frequency part given by  $\mathbf{P}_N^\perp \Phi_N$  and the nonlinear dynamics for the low frequency part  $\mathbf{P}_N u_N$ :

$$\partial_t^2 \mathbf{P}_N u_N + \partial_t \mathbf{P}_N u_N + (1 - \Delta)\mathbf{P}_N u_N + \mathbf{P}_N (:\mathbf{P}_N u^k:) = \sqrt{2}\mathbf{P}_N \xi \quad (1.43)$$

Then, the residual part  $v_N = \mathbf{P}_N u_N - \Phi_N$  satisfies the following equation:

$$\begin{cases} \partial_t^2 v_N + \partial_t v_N + (1 - \Delta)v_N + \sum_{\ell=0}^k \binom{k}{\ell} \mathbf{P}_N (:\Phi_N^\ell: v_N^{k-\ell}) = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (0, 0), \end{cases} \quad (1.44)$$

where the Wick power is defined by

$$:\Phi_N^\ell(x, t): \stackrel{\text{def}}{=} H_\ell(\Phi_N(x, t); \alpha_N). \quad (1.45)$$

As in the undamped case discussed earlier, for each  $\ell \in \mathbb{N}$ , the Wick power  $:\Phi_N^\ell:$  converges to a limit, denoted by  $:\Phi^\ell:$ , in  $C([0, T]; W^{-\varepsilon, \infty}(\mathbb{T}^2))$  for any  $\varepsilon > 0$  and  $T > 0$ , almost surely (and also in  $L^p(\Omega)$  for any  $p < \infty$ ). See Lemma 2.3 below. This allows us to formally obtain the limiting equation:

$$\begin{cases} \partial_t^2 v + \partial_t v + (1 - \Delta)v + \sum_{\ell=0}^k \binom{k}{\ell} :\Phi^\ell: v^{k-\ell} = 0 \\ (v, \partial_t v)|_{t=0} = (0, 0). \end{cases} \quad (1.46)$$

Note that the damped wave propagator  $\mathcal{D}(t)$  in (1.40) satisfies the same Strichartz estimates as the standard wave propagator  $S(t)$  in (1.4). Hence, by following the argument in [21], we can prove local well-posedness of (1.46), using the Strichartz estimates. In Section 4, we instead present a simple argument for local well-posedness of (1.46) based on Sobolev's inequality. See Proposition 4.1. This local well-posedness can also be applied to the truncated equation (1.44), uniformly in  $N \in \mathbb{N}$ . Once we prove (uniform in  $N$ ) local well-posedness of (1.44) and (1.46) and check invariance of the truncated Gibbs measure  $\bar{\rho}_N$  under the truncated SdNLW dynamics (1.37), the rest of the proof of Theorem 1.7 follows from a standard application of Bourgain's invariant measure argument, whose details we omit. See, for example, [40] for further details, where Robert, Tzvetkov, and the third author extended Theorem 1.7 to the case of two-dimensional compact Riemannian manifolds without boundary.

**Remark 1.8.** (i) In Section 4, we present a proof of local well-posedness of (1.46) based on Sobolev's inequality and construct a solution  $v$  to (1.46) in  $C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$  for any  $\varepsilon > 0$ , where  $T = T(\omega)$  is an almost surely positive local existence time. In this argument, we assume a priori that a solution  $v$  belongs only to  $C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$  (without intersecting with any auxiliary function space). As a consequence, we obtain *unconditional uniqueness* for the solution  $v$  to (1.46). Unconditional uniqueness is a concept of uniqueness which does not depend on how solutions are constructed; see [23]. As a result, we obtain the uniqueness of the limiting process  $u = \Phi + v$  in the entire class:

$$\Phi + C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2)).$$

Compare this with the solutions constructed in [21], where we assume a priori that they also belong to some Strichartz space such that the uniqueness statement in [21] is only conditional (namely in  $C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$  intersected with the Strichartz space).

(ii) Let  $(u, \partial_t u)$  the limiting process be constructed in Theorem 1.7. Then, as a consequence of Bourgain's invariant measure argument, we obtain the following logarithmic growth bound:

$$\|(u(t), \partial_t u(t))\|_{\mathcal{H}^{-\varepsilon}} \leq C(\omega) (\log(1+t))^{\frac{k}{2}}$$

for any  $t \geq 0$ . See [40] for details.

**1.4. Remarks and comments.** (i) The stochastic nonlinear wave equations have been studied extensively in various settings; see [12, Chapter 13] for the references therein. In recent years, we have witnessed a rapid progress on the theoretical understanding of SNLW with singular stochastic forcing. Since the work [21] on local well-posedness of the renormalized SNLW on  $\mathbb{T}^2$ , there have been a number of works on the subject: SNLW with a power-type nonlinearity on  $\mathbb{T}^2$  and  $\mathbb{T}^3$  [22, 40, 34, 33, 35, 5, 36] and SNLW with trigonometric and exponential nonlinearities on  $\mathbb{T}^2$  [41, 43, 42]. See also [39, 37] for a related study on the deterministic NLW with random initial data. We also mention the work [15, 16] by Deya on SNLW with more singular (both in space and time) noises on bounded domains in  $\mathbb{R}^d$  and the work [53] by the fourth author on global well-posedness of the renormalized cubic SNLW on  $\mathbb{R}^2$ .

(ii) In [52], the fourth author introduced a new approach to establish unique ergodicity of Gibbs measures for stochastic dispersive/hyperbolic equations. In particular, ergodicity of

the Gibbs measures was shown in [52] for the cubic SdNLW on  $\mathbb{T}$  and the cubic stochastic damped nonlinear beam equation on  $\mathbb{T}^3$ . More recently, the fourth author further developed the methodology and managed to prove ergodicity of the hyperbolic  $\Phi_2^{k+1}$ -model (1.38) for any odd integer  $k \geq 3$ ; see [54].

(iii) For simplicity of the presentation, we only consider the regularization by the sharp frequency cutoff  $\mathbf{P}_N$  in this paper. A straightforward modification allows us to treat regularization by a smooth mollifier. Furthermore, by a standard argument, we can show that the limiting processes obtained through regularization by a smooth mollifier agree with the limiting processes constructed in Theorems 1.2 and 1.7 via the sharp frequency cutoff  $\mathbf{P}_N$ . See [37] for such an argument in the context of the deterministic NLW with random initial data.

## 2. PRELIMINARY LEMMAS

In this section, we introduce some notations and go over basic lemmas.

**2.1. Preliminary results from stochastic analysis.** In this subsection, by recalling some basic tools from probability theory and Euclidean quantum field theory ([25, 32, 49, 50]), we establish some preliminary estimates on the stochastic convolutions and their Wick powers. First, recall the Hermite polynomials  $H_k(x; \sigma)$  defined through the generating function:

$$F(t, x; \sigma) \stackrel{\text{def}}{=} e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma).$$

For readers' convenience, we write out the first few Hermite polynomials:

$$H_0(x; \sigma) = 1, \quad H_1(x; \sigma) = x, \quad H_2(x; \sigma) = x^2 - \sigma, \quad H_3(x; \sigma) = x^3 - 3\sigma x.$$

Next, we recall the Wiener chaos estimate. Let  $(H, B, \mu)$  be an abstract Wiener space. Namely,  $\mu$  is a Gaussian measure on a separable Banach space  $B$  with  $H \subset B$  as its Cameron-Martin space. Given a complete orthonormal system  $\{e_j\}_{j \in \mathbb{N}} \subset B^*$  of  $H^* = H$ , we define a polynomial chaos of order  $k$  to be an element of the form  $\prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j \rangle)$ , where  $x \in B$ ,  $k_j \neq 0$  for only finitely many  $j$ 's,  $k = \sum_{j=1}^{\infty} k_j$ ,  $H_{k_j}$  is the Hermite polynomial of degree  $k_j$ , and  $\langle \cdot, \cdot \rangle = {}_B \langle \cdot, \cdot \rangle_{B^*}$  denotes the  $B$ - $B^*$  duality pairing. We then denote the closure of the span of polynomial chaoses of order  $k$  under  $L^2(B, \mu)$  by  $\mathcal{H}_k$ . The elements in  $\mathcal{H}_k$  are called homogeneous Wiener chaoses of order  $k$ . We also set

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j$$

for  $k \in \mathbb{N}$ .

Let  $L = \Delta - x \cdot \nabla$  be the Ornstein-Uhlenbeck operator.<sup>10</sup> Then, it is known that any element in  $\mathcal{H}_k$  is an eigenfunction of  $L$  with eigenvalue  $-k$ . Then, as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup  $U(t) = e^{tL}$  due to Nelson [31], we have the following Wiener chaos estimate [50, Theorem I.22]. See also [51, Proposition 2.4].

<sup>10</sup>For simplicity, we write the definition of the Ornstein-Uhlenbeck operator  $L$  when  $B = \mathbb{R}^d$ .

**Lemma 2.1.** *Let  $k \in \mathbb{N}$ . Then, we have*

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}$$

for any  $p \geq 2$  and any  $X \in \mathcal{H}_{\leq k}$ .

Before proceeding further, we recall the following corollary to the Garsia-Rodemich-Rumsey inequality ([18, Theorem A.1]).

**Lemma 2.2.** *Let  $(E, d)$  be a metric space. Given  $u \in C([0, T]; E)$ , suppose that there exist  $c_0 > 0$ ,  $\theta \in (0, 1)$ , and  $\alpha > 0$  such that*

$$\int_{t_1}^{t_2} \int_{t_1}^{t_2} \exp \left\{ c_0 \left( \frac{d(u(t), u(s))}{|t-s|^\theta} \right)^\alpha \right\} dt ds =: F_{t_1, t_2} < \infty \quad (2.1)$$

for any  $0 \leq t_1 \leq t_2 \leq T$  with  $t_2 - t_1 \leq 1$ . Then, we have

$$\exp \left\{ \frac{c_0}{C} \left( \sup_{t_1 \leq s < t \leq t_2} \frac{d(u(t), u(s))}{\zeta(t-s)} \right)^\alpha \right\} \leq \max(F_{t_1, t_2}, e) \quad (2.2)$$

for any  $0 \leq t_1 \leq t_2 \leq T$  with  $t_2 - t_1 \leq 1$ , where  $\zeta(t)$  is defined by

$$\zeta(t) = \int_0^t \tau^{\theta-1} \left\{ \log \left( 1 + \frac{4}{\tau^2} \right) \right\}^{\frac{1}{\alpha}} d\tau. \quad (2.3)$$

When  $\alpha = 2$ , Lemma 2.2 reduces to Corollary A.5 in [18]. While Lemma 2.2 for general  $\alpha > 0$  follows in an analogous manner, we present a proof for readers' convenience.

*Proof.* Let  $\Psi(t) = e^{c_0 t^\alpha} - 1$  and  $p(t) = t^\theta$ . Then, from the Garsia-Rodemich-Rumsey inequality ([18, Theorem A.1]) with (2.1), we obtain

$$d(u(t_1), u(t_2)) \leq 8\theta c_0^{-\frac{1}{\alpha}} \int_0^{t_2-t_1} t^{\theta-1} \left\{ \log \left( 1 + \frac{4F_{t_1, t_2}}{t^2} \right) \right\}^{\frac{1}{\alpha}} dt. \quad (2.4)$$

Note that we have

$$\log(1 + AB) \leq \log(1 + A) + \log B \leq 2 \log(1 + A) \cdot \log B \quad (2.5)$$

for  $A \geq e - 1$  and  $B \geq e$ . Then, it follows from (2.4) and (2.5) with (2.3) that

$$d(u(t_1), u(t_2)) \leq C\theta c_0^{-\frac{1}{\alpha}} \zeta(t_2 - t_1) (\log(\max(F_{t_1, t_2}, e)))^{\frac{1}{\alpha}}, \quad (2.6)$$

provided that  $\frac{4}{(t_2-t_1)^2} \geq e - 1$ , which is certainly satisfied for  $0 < t_2 - t_1 \leq 1$ . The desired estimate (2.2) follows directly from (2.6).  $\square$

Let  $\Psi$  and  $\Phi$  be the stochastic convolutions defined in (1.5) and (1.41), respectively. Then, using standard stochastic analysis with the Wiener chaos estimate (Lemma 2.1), we have the following regularity and convergence result.

**Lemma 2.3.** *Let  $Z = \Psi$  or  $\Phi$ . Given  $k \in \mathbb{N}$  and  $N \in \mathbb{N}$ , let  $:Z_N^k := :(\mathbf{P}_N Z)^k$ : denote the truncated Wick power defined in (1.10) or (1.45), respectively. Then, given any  $T, \varepsilon > 0$  and finite  $p \geq 1$ ,  $\{ :Z_N^k : \}_{N \in \mathbb{N}}$  is a Cauchy sequence in  $L^p(\Omega; C([0, T]; W^{-\varepsilon, \infty}(\mathbb{T}^2)))$ , converging to some limit  $:Z^k :$  in  $L^p(\Omega; C([0, T]; W^{-\varepsilon, \infty}(\mathbb{T}^2)))$ . Moreover,  $:Z_N^k :$  converges almost surely*



to the same limit in  $C([0, T]; W^{-\varepsilon, \infty}(\mathbb{T}^2))$ . Given any finite  $q \geq 1$ , we have the following tail estimate:

$$P\left(\| :Z^k : \|_{L_T^q W_x^{-\varepsilon, \infty}} > \lambda\right) \leq C \exp\left(-c \frac{\lambda^{\frac{2}{k}}}{T^{1+\frac{2}{qk}}}\right) \quad (2.7)$$

for any  $T \geq 1$  and  $\lambda > 0$ . When  $q = \infty$ , we also have the following tail estimate:

$$P\left(\| :Z^k : \|_{L^\infty([j, j+1]; W_x^{-\varepsilon, \infty})} > \lambda\right) \leq C \exp\left(-c \frac{\lambda^{\frac{2}{k}}}{j+1}\right) \quad (2.8)$$

for any  $j \in \mathbb{Z}_{\geq 0}$  and  $\lambda > 0$ .

*Proof.* In the following, we briefly discuss the case of the stochastic convolution  $\Psi$  associated with the linear wave operator. A straightforward modification yields the corresponding result for  $\Phi$ . As for the convergence part of the statement, see [21, Proposition 2.1] and [22, Lemma 3.1] for the details. As for the exponential tail estimate (2.7), by repeating the argument in the proof of [21, Proposition 2.1], we have

$$\mathbb{E}[|\langle \nabla \rangle^{-\varepsilon} : \Psi^k(x, t) :|^2] \lesssim \sum_{n_1, \dots, n_k \in \mathbb{Z}^2} \frac{t^k}{\langle n_1 \rangle^2 \cdots \langle n_k \rangle^2 \langle n_1 + \cdots + n_k \rangle^{2\varepsilon}} \leq C_\varepsilon t^k \quad (2.9)$$

for any  $\varepsilon > 0$ , uniformly in  $x \in \mathbb{T}^2$  and  $t \geq 0$ . Then, Minkowski's integral inequality and the Wiener chaos estimate (Lemma 2.1), we obtain

$$\left\| \| : \Psi^k : \|_{L_T^q W_x^{-\varepsilon, \infty}} \right\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} T^{\frac{k}{2} + \frac{1}{q}} \quad (2.10)$$

for any sufficiently large  $p \gg 1$  (depending  $q \geq 1$ ). The exponential tail estimate (2.7) follows from (2.10) and Chebyshev's inequality (see also Lemma 4.5 in [55]).

Fix  $j \in \mathbb{Z}_{\geq 0}$  and  $\lambda > 0$ . Then, we have

$$\begin{aligned} P\left(\| : \Psi^k : \|_{L^\infty([j, j+1]; W_x^{-\varepsilon, \infty})} > \lambda\right) &\leq P\left(\| : \Psi^k(j) : \|_{W_x^{-\varepsilon, \infty}} > \frac{\lambda}{2}\right) \\ &\quad + P\left(\sup_{t \in [j, j+1]} \| : \Psi^k(t) : - : \Psi^k(j) : \|_{W_x^{-\varepsilon, \infty}} > \frac{\lambda}{2}\right). \end{aligned} \quad (2.11)$$

In view of (2.9), we see that the first term on the right-hand side of (2.11) is controlled by the right-hand side of (2.8). As for the second term on the right-hand side of (2.11), we first recall from the proof of [21, Proposition 2.1] that

$$\left\| |h|^{-\rho} \|\delta_h(: \Psi^k(t) :)\|_{W_x^{-\varepsilon, \infty}} \right\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} (j+1)^{\frac{k}{2}}$$

for any sufficiently large  $p \gg 1$ ,  $t \in [j, j+1]$ , and  $|h| \leq 1$ , where  $\delta_h f(t) = f(t+h) - f(t)$  and  $0 < \rho < \varepsilon$ . Then, by applying Lemma 4.5 in [55], we obtain the following exponential bound:

$$\mathbb{E} \left[ \exp \left\{ (j+1)^{-1} \left( \frac{\| : \Psi^k(\tau_2) - : \Psi^k(\tau_1) : \|_{W_x^{-\varepsilon, \infty}}}{|\tau_2 - \tau_1|^\rho} \right)^\alpha \right\} \right] \leq C < \infty, \quad (2.12)$$

uniformly in  $j \leq \tau_1 < \tau_2 \leq j+1$  (and  $j \in \mathbb{Z}_{\geq 0}$ ). By integrating (2.12) in  $\tau_1$  and  $\tau_2$ , this verifies the hypothesis (2.1) of Lemma 2.2 (under an expectation). Finally, applying

Lemma 2.2 and then Chebyshev's inequality, we conclude that

$$P\left(\sup_{t \in [j, j+1]} \|\Psi^k(t) - \Psi^k(j)\|_{W_x^{-\varepsilon, \infty}} > \frac{\lambda}{2}\right) \leq C \exp\left(-c \frac{\lambda^{\frac{2}{k}}}{j+1}\right).$$

This proves (2.8).  $\square$

In order to prove Theorem 1.2, Lemma 2.3 is not sufficient. The following lemma shows a finer regularity property of  $\Psi$ , namely, it is only logarithmically divergent from being a function. We recall that the  $I$ -operator depends on the underlying  $0 < s < 1$  and  $N \in \mathbb{N}$ .

**Lemma 2.4.** *Let  $\Psi$  be as in (1.5) and fix  $0 < s < 1$ . Then, given any  $x \in \mathbb{T}^2$  and  $t \in \mathbb{R}_+$ ,  $I\Psi(x, t)$  is a mean-zero Gaussian random variable with variance bounded by  $C_0 t \log N$ , where the constant  $C_0$  is independent of  $x \in \mathbb{T}^2$  and  $t \in \mathbb{R}_+$ ,*

*Proof.* Given any  $x \in \mathbb{T}^2$  and  $t \in \mathbb{R}_+$ ,  $I\Psi(x, t)$  is obviously a mean-zero Gaussian random variable (if the variance is finite). By writing  $\Psi = \mathbf{P}_N \Psi + \mathbf{P}_N^\perp \Psi$ , we separately estimate the contributions from  $\mathbf{P}_N \Psi$  and  $\mathbf{P}_N^\perp \Psi$ . For the low frequency part, we have  $I\mathbf{P}_N \Psi = \mathbf{P}_N \Psi$  and thus from (1.7), we have

$$\mathbb{E}[(I\mathbf{P}_N \Psi(x, t))^2] = \mathbb{E}[(\mathbf{P}_N \Psi(x, t))^2] \sim t \log N$$

uniformly in  $x \in \mathbb{T}^2$ . For the high frequency part, it follows from (1.5), and (1.17) that

$$\begin{aligned} \mathbb{E}[(I\mathbf{P}_N^\perp \Psi(x, t))^2] &= \int_0^t \sum_{|n| > N} \mathbb{E}[|\widehat{\Psi}(n, t')|^2] m_N^2(n) dt' \\ &\lesssim t \sum_{|n| > N} \frac{N^{2-2s}}{|n|^{4-2s}} \\ &\sim t, \end{aligned}$$

uniformly in  $x \in \mathbb{T}^2$ . This proves Lemma 2.4.  $\square$

**2.2. Product estimates.** We recall the following product estimates. See [21] for the proof.

**Lemma 2.5.** *Let  $0 \leq s \leq 1$ .*

(i) *Suppose that  $1 < p_j, q_j, r < \infty$ ,  $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$ ,  $j = 1, 2$ . Then, we have*

$$\|\langle \nabla \rangle^s (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \left( \|f\|_{L^{p_1}(\mathbb{T}^d)} \|\langle \nabla \rangle^s g\|_{L^{q_1}(\mathbb{T}^d)} + \|\langle \nabla \rangle^s f\|_{L^{p_2}(\mathbb{T}^d)} \|g\|_{L^{q_2}(\mathbb{T}^d)} \right).$$

(ii) *Suppose that  $1 < p, q, r < \infty$  satisfy the scaling condition:  $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$ . Then, we have*

$$\|\langle \nabla \rangle^{-s} (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|\langle \nabla \rangle^{-s} f\|_{L^p(\mathbb{T}^d)} \|\langle \nabla \rangle^s g\|_{L^q(\mathbb{T}^d)}.$$

Note that while Lemma 2.5 (ii) was shown only for  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{d}$  in [21], the general case  $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$  follows from the inclusion  $L^{r_1}(\mathbb{T}^d) \subset L^{r_2}(\mathbb{T}^d)$  for  $r_1 \geq r_2$ .

### 3. $I$ -METHOD FOR THE RENORMALIZED CUBIC SNLW

In this section, we prove global well-posedness of the renormalized cubic SNLW (1.12) on  $\mathbb{T}^2$  (Theorem 1.2). In Subsection 3.1, we go over preliminary estimates. Then, we present a proof of Theorem 1.2 in Subsection 3.2.

**3.1. Commutator and other preliminary estimates.** In the following, we fix  $N \in \mathbb{N}$  and  $0 < s < 1$  and set<sup>11</sup>  $I = I_N$ . Moreover, we use the following notations:

$$f_{\lesssim N} = \mathbf{P}_{\frac{N}{3}} f \quad \text{and} \quad f_{\gtrsim N} = \mathbf{P}_{\frac{1}{3}} f = f - f_{\lesssim N}. \quad (3.1)$$

We first go over basic commutator estimates in Lemmas 3.1, 3.2, and 3.3.

**Lemma 3.1.** *Let  $\frac{2}{3} \leq s < 1$ . Then, we have*

$$\|(If)^k - I(f^k)\|_{L^2} \lesssim N^{-1+k(1-s)} \|If\|_{H^1}^k \quad (3.2)$$

for  $k = 1, 2, 3$ .

*Proof.* By the definition of the  $I$ -operator and (3.1), we have  $I(f_{\lesssim N}^k) = f_{\lesssim N}^k$  for  $k = 1, 2, 3$ . Thus, we have

$$\begin{aligned} (If)^k - I(f^k) &= (I(f_{\lesssim N} + f_{\gtrsim N}))^k - I((f_{\lesssim N} + f_{\gtrsim N})^k) \\ &= (f_{\lesssim N} + I(f_{\gtrsim N}))^k - I((f_{\lesssim N} + f_{\gtrsim N})^k) \\ &= \underbrace{f_{\lesssim N}^k - I(f_{\lesssim N}^k)}_{=0} + \sum_{j=0}^{k-1} \binom{k}{j} \left( f_{\lesssim N}^j (If_{\gtrsim N})^{k-j} - I(f_{\lesssim N}^j f_{\gtrsim N}^{k-j}) \right). \end{aligned} \quad (3.3)$$

In the following, we use Hölder's inequality with  $\frac{1}{2} = \frac{j}{q} + \frac{1}{2+\delta}$  for (i) some large but finite  $q \gg 1$  and small  $\delta > 0$  when  $j \geq 1$  and (ii)  $q = \infty$  and  $\delta = 0$  when  $j = 0$ . Then, by Hölder's and Sobolev's inequalities, we have

$$\begin{aligned} \|f_{\lesssim N}^j (If_{\gtrsim N})^{k-j}\|_{L^2} &\leq \|f_{\lesssim N}\|_{L^q}^j \|If_{\gtrsim N}\|_{L^{(2+\delta)(k-j)}}^{k-j} \\ &\lesssim \|f_{\lesssim N}\|_{H^1}^j \|If_{\gtrsim N}\|_{H^{1-\frac{2}{(2+\delta)(k-j)}}}^{k-j} \\ &\lesssim N^{-1+\varepsilon} \|If\|_{H^1}^k \end{aligned} \quad (3.4)$$

for some small  $\varepsilon > 0$ . Proceeding similarly with the boundedness of the multiplier  $m_N$  and (1.19), we have

$$\begin{aligned} \|I(f_{\lesssim N}^j f_{\gtrsim N}^{k-j})\|_{L^2} &\lesssim \|f_{\lesssim N}^j f_{\gtrsim N}^{k-j}\|_{L^2} \\ &\leq \|f_{\lesssim N}\|_{L^q}^j \|f_{\gtrsim N}\|_{L^{(2+\delta)(k-j)}}^{k-j} \\ &\lesssim \|If_{\lesssim N}\|_{H^1}^j \|f_{\gtrsim N}\|_{H^{1-\frac{2}{(2+\delta)(k-j)}}}^{k-j} \\ &\lesssim N^{-1+k(1-s)} \|If_{\lesssim N}\|_{H^1}^j \|f_{\gtrsim N}\|_{H^s}^{k-j} \\ &\lesssim N^{-1+k(1-s)} \|If\|_{H^1}^k \end{aligned} \quad (3.5)$$

since  $\frac{2}{3} \leq s < 1$ . Therefore, the desired estimate (3.2) follows from (3.3), (3.4), and (3.5).  $\square$

**Lemma 3.2.** *Let  $0 < \sigma < 1$ . Given  $\delta > 0$ , there exist small  $\sigma_0 = \sigma_0(\delta) > 0$  and large  $p = p(\delta) \gg 1$  such that*

$$\|(If)(Ig) - I(fg)\|_{L^2} \lesssim N^{-\frac{1-\sigma}{2}+\delta} \|f\|_{H^{1-\sigma}} \|g\|_{W^{-\sigma_0,p}} \quad (3.6)$$

for any sufficiently large  $N \gg 1$ .

<sup>11</sup>Recall that the  $I$ -operator also depends on  $0 < s < 1$ .

*Proof.* By writing  $f = f_{\lesssim N^{\frac{1}{2}}} + f_{\gtrsim N^{\frac{1}{2}}}$  and  $g = g_{\lesssim N} + g_{\gtrsim N}$ , we have

$$\begin{aligned}
(I f)(I g) - I(f g) &= \left\{ (I f_{\lesssim N^{\frac{1}{2}}})(I g_{\lesssim N}) - I(f_{\lesssim N^{\frac{1}{2}}} g_{\lesssim N}) \right\} \\
&\quad + \left\{ (I f_{\lesssim N^{\frac{1}{2}}})(I g_{\gtrsim N}) - I(f_{\lesssim N^{\frac{1}{2}}} g_{\gtrsim N}) \right\} \\
&\quad + (I f_{\gtrsim N^{\frac{1}{2}}})(I g) \\
&\quad - I(f_{\gtrsim N^{\frac{1}{2}}} g) \\
&=: B_1 + B_2 + B_3 + B_4.
\end{aligned} \tag{3.7}$$

From the definition of the  $I$ -operator with (3.1), we see that

$$B_1 = 0 \tag{3.8}$$

for any sufficiently large  $N \gg 1$  since  $\text{supp} \{ \mathcal{F}(f_{\lesssim N^{\frac{1}{2}}} g_{\lesssim N}) \} \subset \{ n \in \mathbb{Z}^2 : |n| \leq \frac{5}{6} N \}$  for  $N \gg 1$ .

For  $|n_1| \lesssim N^{\frac{1}{2}}$  and  $|n_2| \gtrsim N$ , from the mean value theorem with (1.17), we have

$$|m(n_1 + n_2) - m(n_2)| \lesssim N^{1-s} |n_2|^{-2+s} |n_1|. \tag{3.9}$$

Let  $\star_n = \{ n_1, n_2 \in \mathbb{Z}^2 : n = n_1 + n_2, |n_1| \leq \frac{N^{\frac{1}{2}}}{3}, |n_2| > \frac{N}{3} \}$ . By (3.9), the fact that  $m(n_1) \equiv 1$  on  $\star_n$ , and Young's inequality followed by Cauchy-Schwarz inequality (in  $n_1$ ), we have

$$\begin{aligned}
\|B_2\|_{L^2} &= \left\| \sum_{\star_n} (m(n_2) - m(n_1 + n_2)) \widehat{f}(n_1) \widehat{g}(n_2) \right\|_{\ell_n^2} \\
&\lesssim N^{1-s} \left\| \sum_{\star_n} \frac{\langle n_1 \rangle^{1+\sigma+\delta}}{|n_2|^{2-s-\delta}} \left| \frac{\widehat{f}(n_1)}{\langle n_1 \rangle^{\sigma+\delta}} \right| \left| \frac{\widehat{g}(n_2)}{|n_2|^\delta} \right| \right\|_{\ell_n^2} \\
&\lesssim N^{-\frac{1-\sigma}{2} + \frac{3}{2}\delta} \|\langle n_1 \rangle^{-\sigma-\delta} \widehat{f}(n_1)\|_{\ell_{n_1}^1} \|g\|_{H^{-\delta}} \\
&\lesssim N^{-\frac{1-\sigma}{2} + \frac{3}{2}\delta} \|f\|_{H^{1-\sigma}} \|g\|_{H^{-\delta}}.
\end{aligned} \tag{3.10}$$

As for  $B_3$ , by Hölder's inequality, Sobolev's embedding theorem, and applying (1.20) twice, we have

$$\begin{aligned}
\|B_3\|_{L^2} &\leq \|I f_{\gtrsim N^{\frac{1}{2}}}\|_{L^2} \|I g\|_{L^\infty} \\
&\lesssim N^{-\frac{1-\sigma}{2}} \|f\|_{H^{1-\sigma}} \|I g\|_{W^{3\delta, \delta-1}} \\
&\lesssim N^{-\frac{1-\sigma}{2} + 4\delta} \|f\|_{H^{1-\sigma}} \|g\|_{W^{-\delta, \delta-1}}
\end{aligned} \tag{3.11}$$

for  $\delta > 0$  sufficiently small.

Lastly, from (1.20) and Lemma 2.5 (ii), we have

$$\begin{aligned}
\|B_4\|_{L^2} &\lesssim N^{2\delta} \|f_{\gtrsim N^{\frac{1}{2}}} g\|_{H^{-2\delta}} \\
&\lesssim N^{2\delta} \|f_{\gtrsim N^{\frac{1}{2}}}\|_{H^{2\delta}} \|g\|_{W^{-2\delta, \delta-1}} \\
&\lesssim N^{-\frac{1-\sigma}{2} + 3\delta} \|f_{\gtrsim N^{\frac{1}{2}}}\|_{H^{1-\sigma}} \|g\|_{W^{-2\delta, \delta-1}}
\end{aligned} \tag{3.12}$$

for  $\delta > 0$  sufficiently small.

Putting (3.7), (3.8), (3.10), and (3.11), and (3.12) together, we obtain (3.6).  $\square$

From Lemmas 3.1 and 3.2, we obtain the following commutator estimate. For our application, we will use this lemma with  $g = : \Psi^{3-k} :$ .

**Lemma 3.3.** *Let  $\frac{2}{3} \leq s < 1$  and  $k = 1, 2$ . Given  $\delta > 0$ , there exist small  $\sigma_0 = \sigma_0(\delta) > 0$  and  $p = p(\delta) \gg 1$  such that*

$$\|I(f^k g) - (If)^k Ig\|_{L^2} \lesssim N^{-\frac{1-k(1-s)}{2} + \delta} \|If\|_{H^1}^k \|g\|_{W^{-\sigma_0, p}} \quad (3.13)$$

for any sufficiently large  $N \gg 1$ .

*Proof.* By the triangle inequality, we have

$$\begin{aligned} \|I(f^k g) - (If)^k Ig\|_{L^2} &\leq \|I(f^k g) - I(f^k)Ig\|_{L^2} + \|(I(f^k) - (If)^k)Ig\|_{L^2} \\ &=: D_1 + D_2. \end{aligned} \quad (3.14)$$

By Sobolev's inequality (with  $s > \frac{1}{2}$ ) and the fractional Leibniz rule (Lemma 2.5 (i)), we have

$$\|f^k\|_{H^{1-k(1-s)}} \lesssim \|f^k\|_{W^{s, \frac{2}{1+(k-1)(1-s)}}} \lesssim \|f\|_{H^s} \|f\|_{L^{\frac{2}{1-s}}}^{k-1} \lesssim \|f\|_{H^s}^k. \quad (3.15)$$

Thus, by Lemma 3.2 with  $\sigma = k(1-s)$ , (3.15), and (1.19), given  $\delta > 0$ , we have

$$\begin{aligned} \|D_1\|_{L^2} &\lesssim N^{-\frac{1-k(1-s)}{2} + \delta} \|f\|_{H^s}^k \|g\|_{W^{-\sigma_0, p}} \\ &\lesssim N^{-\frac{1-k(1-s)}{2} + \delta} \|If\|_{H^1}^k \|g\|_{W^{-\sigma_0, p}} \end{aligned} \quad (3.16)$$

for some small  $\sigma_0 = \sigma_0(\delta) > 0$  and large  $p = p(\delta) \gg 1$ . On the other hand, by Hölder's inequality, Lemma 3.1, Sobolev's embedding theorem, and (1.20), we have

$$\begin{aligned} \|D_2\|_{L^2} &\leq \|I(f^k) - (If)^k\|_{L^2} \|Ig\|_{L^\infty} \\ &\lesssim N^{-1+k(1-s)} \|If\|_{H^1}^k \|Ig\|_{W^{3\delta, \delta-1}} \\ &\lesssim N^{-1+k(1-s)+4\delta} \|If\|_{H^1}^k \|g\|_{W^{-\delta, \delta-1}}. \end{aligned} \quad (3.17)$$

Putting (3.14), (3.16), and (3.17) together, we obtain (3.13).  $\square$

We conclude this subsection by presenting useful estimates for controlling the Gronwall part of our hybrid  $I$ -method argument.

**Lemma 3.4.** (i) *Let  $k = 0, 1$ . Then, for any  $0 \leq \gamma \leq 1-s$ , we have*

$$\left| \int_{\mathbb{T}^2} (\partial_t Iv(t))(Iv(t))^k Iw(t) dx \right| \lesssim N^\gamma (1 + E^{\frac{3}{4}}(I\vec{v})(t)) \|w(t)\|_{W_x^{-\gamma, 4}}$$

for any  $t \geq 0$ , where  $E$  is the energy defined in (1.15).

(ii) *There exists  $c > 0$  such that*

$$\begin{aligned} &\left| \int_{t_1}^{t_2} \int_{\mathbb{T}^2} (\partial_t Iv)(Iv)^2 Iw dx dt \right| \\ &\lesssim \left\{ \int_{t_1}^{t_2} \left( E^{1+c\eta}(I\vec{v})(t) + \frac{\eta}{(t-t_1)^{\frac{1}{2}}} \right) dt \right\} \|Iw\|_{L_{[t_1, t_2], x}^{\eta-1}}, \end{aligned} \quad (3.18)$$

uniformly in  $0 < \eta < \frac{1}{8}$  and  $t_2 \geq t_1 \geq 0$ , where  $L_{I, x}^p = L^p(I; L^p(\mathbb{T}^2))$  for a given time interval  $I \subset \mathbb{R}_+$ .

For our application, we will use Part (i) with  $w = : \Psi^{3-k} :$ ,  $k = 0, 1$ , and Part (ii) with  $w = \Psi$ .

*Proof.* (i) Let  $k = 0, 1$ . Then, by Hölder's inequality, (1.15), and (1.20), we have

$$\begin{aligned} \left| \int_{\mathbb{T}^2} (\partial_t I v(t)) (I v(t))^k I w(t) dx \right| &\leq \|\partial_t I v(t)\|_{L^2} \|I v(t)\|_{L^4}^k \|I w(t)\|_{L^{\frac{4}{2-k}}} \\ &\lesssim \|\partial_t I v(t)\|_{L^2} \|I v(t)\|_{L^4}^k \|I w(t)\|_{L^4} \\ &\lesssim N^\gamma E(I \bar{v})^{\frac{1}{2} + \frac{k}{4}}(t) \|w(t)\|_{W^{-\gamma, 4}}. \end{aligned}$$

(ii) By interpolation with (1.15), we have

$$\|I v\|_{W^{\theta, \frac{4}{1+\theta}}} \lesssim \|I v\|_{H^1}^\theta \|I v\|_{L^4}^{1-\theta} \lesssim E^{\frac{\theta}{2}}(I \bar{v}) E^{\frac{1-\theta}{4}}(I \bar{v}) = E^{\frac{1+\theta}{4}}(I \bar{v})$$

for  $0 \leq \theta \leq 1$ . Then, by Sobolev's inequality, we have

$$\|I v\|_{L^{\frac{4}{1-\theta}}} \lesssim E^{\frac{1+\theta}{4}}(I \bar{v}), \quad (3.19)$$

where the implicit constant is uniform in  $\theta$  as long as  $0 \leq \theta \leq \theta_{\max} < 1$ . Set  $\theta = 4\eta$ . Then, by Hölder's inequality (in  $x$ ), (1.15), (3.19), and Hölder's inequality (in  $t$ ), we obtain

$$\begin{aligned} \left| \int_{t_1}^{t_2} \int_{\mathbb{T}^2} (\partial_t I v) (I v)^2 I w dx dt \right| &\leq \int_{t_1}^{t_2} \|\partial_t I v\|_{L_x^2} \|I v\|_{L_x^4} \|I v\|_{L_x^{\frac{4}{1-4\eta}}} \|I w\|_{L_x^{\eta-1}} dt \\ &\lesssim \int_{t_1}^{t_2} E^{1+\eta}(I \bar{v}) \|I w\|_{L_x^{\eta-1}} dt \\ &\lesssim \left( \int_{t_1}^{t_2} E^{\frac{1+\eta}{1-\eta}}(I \bar{v}) dt \right)^{1-\eta} \|I w\|_{L_{[t_1, t_2], x}^{\eta-1}}, \end{aligned} \quad (3.20)$$

uniformly in  $0 < \eta < \frac{1}{8}$ .

Next, we estimate the first factor on the right-hand side in (3.20). Let

$$p = p(\eta) = \frac{1-\eta}{1-2\eta} \quad \text{and} \quad q = q(\eta) = \frac{1}{1-2\eta}.$$

This implies that

$$p' = \frac{1-\eta}{\eta} \quad \text{and} \quad q' = \frac{1}{2\eta}.$$

Then, by Hölder's and Young's inequalities, we have

$$\begin{aligned} \left( \int_{t_1}^{t_2} f(t) dt \right)^{1-\eta} &\leq \left( \int_{t_1}^{t_2} |f(t)|^p dt \right)^{\frac{1-\eta}{p}} (t_2 - t_1)^{\frac{1-\eta}{p'}} \\ &\leq \frac{1}{q} \left( \int_{t_1}^{t_2} |f(t)|^p dt \right)^{q \cdot \frac{1-\eta}{p}} + \frac{1}{q'} (t_2 - t_1)^{q' \cdot \frac{1-\eta}{p'}} \\ &= (1-2\eta) \int_{t_1}^{t_2} |f(t)|^{\frac{1-\eta}{1-2\eta}} dt + 2\eta (t_2 - t_1)^{\frac{1}{2}}. \end{aligned}$$

Applying this to the first factor on the right-hand side in (3.20), we obtain

$$\left( \int_{t_1}^{t_2} E^{\frac{1+\eta}{1-\eta}}(I \bar{v})(t) dt \right)^{1-\eta} \lesssim \int_{t_1}^{t_2} \left( E^{\frac{1+\eta}{1-2\eta}}(I \bar{v})(t) + \frac{\eta}{(t-t_1)^{\frac{1}{2}}} \right) dt. \quad (3.21)$$

Putting (3.20) and (3.21) together, we obtain (3.18).  $\square$

**3.2. Proof of Theorem 1.2.** In this subsection, we use the estimates in the previous subsection and implement an iterative argument to construct a solution to (1.12) on a time interval  $[0, T]$  for any given  $T \gg 1$ . Unlike the usual application of the  $I$ -method (where the parameter  $N$  depends only on the target time  $T \gg 1$ ), we will need to construct an increasing sequence  $\{N_k\}_{k \in \mathbb{Z}_{\geq 0}}$  of parameters over local-in-time intervals, which allows us to proceed over a time interval of fixed length at each iteration step.

Fix  $\frac{4}{5} < s < 1$  and a target time  $T \gg 1$ . Our main goal is to control growth of the modified energy  $E(I\vec{v})(t)$  on the time interval  $[0, T]$ . We use the following short-hand notation for the modified energy:

$$E(t) = E(I\vec{v})(t).$$

Then, from (1.22), Lemmas 3.1, 3.3, and 3.4, we have

$$\begin{aligned} & E(t_2) - E(t_1) \\ & \lesssim \int_{t_1}^{t_2} N^{-1+3(1-s)} E^2(t) dt \\ & \quad + \sum_{k=1}^2 \int_{t_1}^{t_2} N^{-\frac{1-k(1-s)}{2} + \delta} E^{\frac{k+1}{2}}(t) \|\Psi^{3-k}(t)\|_{W_x^{-\sigma_0, p}} dt \\ & \quad + \sum_{k=0}^1 \int_{t_1}^{t_2} N^\gamma (1 + E^{\frac{3}{4}}(t)) \|\Psi^{3-k}(t)\|_{W_x^{-\gamma, 4}} dt \\ & \quad + \left\{ \int_{t_1}^{t_2} \left( E^{1+c\eta}(t) + \frac{\eta}{(t-t_1)^{\frac{1}{2}}} \right) dt \right\} \|I\Psi\|_{L_{[t_1, t_2], x}^{\eta^{-1}}} \end{aligned} \tag{3.22}$$

for any  $t_2 \geq t_1 \geq 0$ .

Before proceeding to the following crucial proposition, let us introduce some notations.

Given  $j \in \mathbb{Z}_{\geq 0}$ , set  $V_j = V_j(\omega)$

$$V_j = \max_{k=1,2} \|\Psi^{3-k}\|_{L_{[j, j+1]}^\infty W_x^{-\sigma_0, p}} + \max_{k=0,1} \|\Psi^{3-k}\|_{L_{[j, j+1]}^\infty W_x^{-\gamma, 4}}$$

and define  $V = V(\omega)$  by

$$e^{V^{\frac{1}{3}}} = \sum_{j=0}^{\infty} e^{-\theta j} e^{V_j^{\frac{1}{3}}} \tag{3.23}$$

for some  $\theta > 0$ . Note that  $V$  is almost surely finite, since, by applying (2.8) in Lemma 2.3 and choosing  $\theta$  sufficiently large, we have

$$\mathbb{E} \left[ e^{V^{\frac{1}{3}}} \right] = \sum_{j=0}^{\infty} e^{-\theta j} \mathbb{E} \left[ e^{V_j^{\frac{1}{3}}} \right] \leq \sum_{j=0}^{\infty} e^{-\theta j} e^{c(j+1)} < \infty.$$

Now, given  $T \gg 1$ , set

$$M_T = \max_{k=1,2} \|\Psi^{3-k}\|_{L_T^\infty W_x^{-\sigma_0, p}} + \max_{k=0,1} \|\Psi^{3-k}\|_{L_T^\infty W_x^{-\gamma, 4}}. \tag{3.24}$$

Then, noting from (3.23) that

$$V_j^{\frac{1}{3}} \leq V^{\frac{1}{3}} + \theta j,$$

we have

$$M_T = \max_{j \leq T} V_j \leq V^{\frac{1}{3}} + \theta T \lesssim V + T^3. \quad (3.25)$$

We also define  $R = R(\omega)$  by

$$R = \sum_{N=1}^{\infty} \sum_{j=1}^{\infty} e^{-\theta j \log N} \int_0^j \int_{\mathbb{T}^2} e^{|I_N \Psi(x,t)|} dx dt, \quad (3.26)$$

where  $I = I_N$  is the  $I$ -operator defined in (1.18). Then, by applying Lemma 2.4 and choosing  $\theta$  sufficiently large, we have

$$\begin{aligned} \mathbb{E}[R] &= \sum_{N=1}^{\infty} \sum_{j=1}^{\infty} e^{-\theta j \log N} \int_0^j \int_{\mathbb{T}^2} \mathbb{E} \left[ e^{|I_N \Psi(x,t)|} \right] dx dt \\ &\lesssim \sum_{N=1}^{\infty} \sum_{j=1}^{\infty} e^{-\theta j \log N} j e^{c_j \log N} < \infty. \end{aligned}$$

Thus,  $R$  is finite almost surely. In the following, we assume that  $R = R(\omega) \geq 1$ .

In the following, we fix  $\omega \in \Omega$  such that  $V = V(\omega) < \infty$  and  $R = R(\omega) < \infty$  and prove global well-posedness by pathwise analysis. The following proposition plays a fundamental role in our iterative argument to prove Theorem 1.2.

**Proposition 3.5.** *Let  $\frac{2}{3} < s < 1$ ,  $T \geq T_0 \gg 1$ , and  $N \in \mathbb{N}$ . Moreover, let  $V = V(\omega) < \infty$  and  $R = R(\omega) < \infty$  be as in (3.23) and (3.25). Then, there exist  $\alpha = \alpha(s)$ ,  $\beta = \beta(s) > 0$  with  $\alpha > \beta$  such that if*

$$E(t_0) \leq N^\beta \quad (3.27)$$

for some  $0 \leq t_0 < T$ , then there exists  $\tau = \tau(s, N, T, V, R) = \tau(s, N, T, \omega) > 0$  with  $\tau \leq t_*(R) \leq 1$  such that

$$E(t) \leq N^\alpha \quad (3.28)$$

for any  $t$  such that  $t_0 \leq t \leq \min(T, t_0 + \tau)$ .

*Proof.* Without loss of generality, we assume that  $E(t) \geq 1$ . (This can be guaranteed by replacing  $E(t)$  by  $E(t) + 1$ .) Then, from (3.22) with (3.24), we have

$$\begin{aligned} &E(t) - E(t_0) \\ &\lesssim \int_{t_0}^t N^{-1+3(1-s)} E^2(t') dt' \\ &\quad + M_T \sum_{k=1}^2 \int_{t_0}^t N^{-\frac{1-k(1-s)}{2} + \delta} E^{\frac{k+1}{2}}(t') dt' \\ &\quad + M_T \int_{t_0}^t N^\gamma E^{\frac{3}{4}}(t') dt' \\ &\quad + \left\{ \int_{t_0}^t \left( E^{1+c\eta}(t') + \frac{\eta}{(t' - t_1)^{\frac{1}{2}}} \right) dt' \right\} \|I\Psi\|_{L_{[t_0, t], x}^{\eta-1}} \end{aligned} \quad (3.29)$$

for any  $t \geq t_0$ .



In the following, we assume

$$\max_{t_0 \leq \tau \leq t} E(\tau) \leq 100N^\alpha \quad (3.30)$$

for some  $t \geq t_0$ , where  $\alpha > \beta$  is to be determined later. Then, we show that (3.28) holds for this  $t$ . It follows from the continuity in time of  $E(t)$  and (3.27) with  $\alpha > \beta$  that there exists  $t_1 > t_0$  sufficiently close to  $t_0$  such that (3.30) holds true for  $t_0 \leq t \leq t_1$ .

Letting  $\eta^{-1} = n \in \mathbb{N}$ , it follows from (3.26) that

$$\begin{aligned} \|I_N \Psi\|_{L^n_{[t_1, t_2], x}}^n &= \int_{t_1}^{t_2} \int_{\mathbb{T}^2} |I_N \Psi(x, t)|^n dx dt \leq n! \int_0^T \int_{\mathbb{T}^2} e^{|I_N \Psi(x, t)|} dx dt \\ &\leq n! e^{\theta T \log N} R. \end{aligned}$$

With  $n! \leq n^n$ , this implies

$$\|I_N \Psi\|_{L^n_{[t_1, t_2], x}} \leq n e^{\frac{1}{n} \theta T \log N} R^{\frac{1}{n}}.$$

We now choose

$$n \sim \theta T \log N + c\alpha \log(100N) \sim T \log N \gg 1.$$

Then, under the assumption (3.30) and  $\eta = n^{-1}$ , we can estimate the last term on the right-hand side of (3.29) as

$$\begin{aligned} &\left\{ \int_{t_0}^t \left( E^{1+c\eta}(t') + \frac{\eta}{(t' - t_1)^{\frac{1}{2}}} \right) dt' \right\} \|I \Psi\|_{L^{\eta^{-1}}_{[t_0, t], x}} \\ &\leq \int_{t_0}^t \left( E(t') n e^{\frac{1}{n} (\theta T \log N + c\alpha \log(100N))} + \frac{e^{\frac{1}{n} \theta T \log N} R^{\frac{1}{n}}}{(t' - t_1)^{\frac{1}{2}}} \right) dt' \\ &\leq \int_{t_0}^t \left( T E(t') \log N + \frac{R}{(t' - t_1)^{\frac{1}{2}}} \right) dt', \end{aligned} \quad (3.31)$$

where we used the assumption that  $n \geq 1$  and  $R = R(\omega) \geq 1$  in the last step.

Next, we define  $F$  by

$$F(t) = \max_{t_0 \leq \tau \leq t} E(\tau) - E(t_0) + \max(E(t_0), N^\beta). \quad (3.32)$$

Then, under the assumption (3.30), we have we have

$$N^\beta \leq F(t) \leq 200N^\alpha \quad (3.33)$$

for  $t_0 \leq t \leq t_1$ . In particular, we have

$$\log F(t) \sim \log N. \quad (3.34)$$

Moreover, from (3.33), we have

$$\begin{cases} N^{-1+3(1-s)} F^2(t) \lesssim N^{-\alpha} F^2(t) \leq F(t), \\ N^{-\frac{1-2(1-s)}{2} + \delta} F^{\frac{3}{2}}(t) \lesssim N^{-\frac{\alpha}{2}} F^{\frac{3}{2}}(t) \leq F(t), \\ N^{-\frac{1-(1-s)}{2} + \delta} F(t) \leq F(t), \\ N^\gamma F^{\frac{3}{4}}(t) \lesssim N^\gamma F^{-\frac{1}{4}}(t) F(t) \leq F(t), \end{cases} \quad (3.35)$$

provided that

$$\alpha \leq 1 - 3(1 - s) = -2 + 3s, \quad \delta \leq \min\left(\frac{2s-1-\alpha}{2}, \frac{s}{2}\right), \quad \text{and} \quad \gamma \leq \frac{\beta}{4}. \quad (3.36)$$

Here,  $\gamma = \gamma(s) > 0$  is a small constant, appearing in Lemma 3.4. The first condition in (3.36) with  $\alpha > 0$  requires  $s > \frac{2}{3}$ . Hence, from (3.29) with (3.31), (3.32), (3.34), and (3.35) followed by (3.25), we obtain

$$\begin{aligned}
& F(t) - F(t_0) \\
& \lesssim (1 + M_T) \int_{t_0}^t F(t') dt' + \int_{t_0}^t \left( TF(t') \log F(t') + \frac{R}{(t' - t_1)^{\frac{1}{2}}} \right) dt' \\
& \lesssim (1 + V + T^3) \int_{t_0}^t F(t') dt' + \int_{t_0}^t \left( TF(t') \log F(t') + \frac{R}{(t' - t_1)^{\frac{1}{2}}} \right) dt' \quad (3.37) \\
& \lesssim (1 + V + R + T) \int_{t_0}^t \left( F(t')(\log F(t') + T^2) + \frac{R}{(t' - t_1)^{\frac{1}{2}}} \right) dt'
\end{aligned}$$

for any  $t_0 \leq t \leq t_1$  such that (3.33) holds. Denoting by  $C_0$  the implicit constant in (3.37), we define  $G$  by

$$G(t) = F(t) - 2C_0 R(t - t_0)^{\frac{1}{2}}. \quad (3.38)$$

Then, it follows from (3.37) that

$$G(t) - G(t_0) \lesssim (1 + V + R + T) \int_{t_0}^t G(t')(\log G(t') + T^2) dt' \quad (3.39)$$

for any  $t_0 \leq t \leq \min(t_1, t_0 + t_*(C_0, R))$  such that

$$2C_0 R(t - t_0)^{\frac{1}{2}} \sim 1 \quad (3.40)$$

(which guarantees  $G(t) \sim F(t)$  in view of (3.38)).

Now, note that the equation

$$\partial_t H(t) = \kappa H(t)(\log H(t) + T^2)$$

has an explicit solution

$$H(t) = \exp \left( e^{\kappa t} (\log H(0) + T^2) - T^2 \right).$$

Then, by comparison, we deduce from (3.39) that

$$G(t) \leq \exp \left( e^{C(1+V+R+T)(t-t_0)} (\log G(t_0) + T^2) - T^2 \right). \quad (3.41)$$

Recall from (3.38) and (3.32) that  $G(t_0) = N^\beta$ . Then, under the condition

$$e^{C(1+V+R+T)(t-t_0)} (\beta \log N + T^2) \leq \alpha \log N + T^2 - \log 2, \quad (3.42)$$

the bound (3.41) implies

$$G(t) \leq \frac{1}{2} N^\alpha \quad (3.43)$$

for any  $t_0 \leq t \leq \min(t_1, t_0 + t_*(C_0, R))$ . Then, we conclude from (3.32), (3.38), and (3.40) that

$$E(t) \leq F(t) \leq N^\alpha \quad (3.44)$$

for any  $t_0 \leq t \leq \min(t_1, t_0 + t_*(C_0, R))$ . This in turn guarantees the conditions (3.30) and (3.33). Therefore, by a standard continuity argument, we conclude that the bounds (3.43) and (3.44) hold for any  $t_0 \leq t \leq t_0 + t_*(C_0, R)$  sufficiently close to  $t_0$  such that the condition (3.42) holds.

Finally, let us rewrite the condition (3.42). Let  $\alpha = \alpha(s) > \beta = \beta(s)$  satisfy the conditions (3.36). Then, there exists small  $0 < \tau \leq t_*(C_0, R)$  such that

$$\alpha - e^{C(1+V+R+T)\tau} \beta \geq c_0 > 0. \quad (3.45)$$

Then, by choosing  $\tau = \tau(s, N, T, V, R) > 0$  sufficiently small such that

$$e^{C(1+V+R+T)\tau} - 1 \leq \frac{c_0 \log N - \log 2}{T^2}, \quad (3.46)$$

we can guarantee the condition (3.42) and hence the desired bound (3.44) for  $0 \leq t - t_0 \leq \tau$ . This concludes the proof of Proposition 3.5.  $\square$

**Remark 3.6.** By choosing  $\tau \sim_{V,R} T^{-1}$  sufficiently small, we can guarantee the condition (3.45).

We now present a proof of Theorem 1.2. Fix  $\frac{4}{5} < s < 1$  and  $T \gg 1$ . Moreover, we fix  $\omega \in \Omega$  such that  $V = V(\omega) < \infty$  and  $R = R(\omega) < \infty$ . Then, let the parameters  $\alpha, \beta, \tau$  be as in Proposition 3.5.

Fix  $N_0 \gg 1$  (to be determined later). Then, for  $k \in \mathbb{Z}_{\geq 0}$ , define an increasing sequence  $\{N_k\}_{k \in \mathbb{Z}_{\geq 0}}$  by setting

$$N_k = N_0^{\sigma^k} \quad (3.47)$$

for some  $\sigma > 1$  such that

$$N_{k+1}^{2(1-s)} N_k^\alpha + N_k^{2\alpha} \ll N_{k+1}^\beta, \quad (3.48)$$

which requires  $\beta > 2(1-s)$ . Recalling that  $\alpha > \beta$  and (3.36), we have the following constraints:

$$2(1-s) < \beta < \alpha \leq 1 - 3(1-s),$$

which imposes the condition  $s > \frac{4}{5}$ . Suppose that

$$E(I_{N_k} \vec{v})(t) \leq N_k^\alpha \quad (3.49)$$

for some  $k$  and  $t \geq 0$ . Then, by (1.19), Sobolev's inequality, (3.49), and (3.48), we have

$$\begin{aligned} E(I_{N_{k+1}} \vec{v})(t) &\lesssim \|I_{N_{k+1}} \vec{v}\|_{\mathcal{H}^1}^2 + \|I_{N_{k+1}} v\|_{L^4}^4 \\ &\lesssim N_{k+1}^{2(1-s)} \|\vec{v}\|_{\mathcal{H}^s}^2 + \|v\|_{H^{\frac{1}{2}}}^4 \\ &\lesssim N_{k+1}^{2(1-s)} \|I_{N_k} \vec{v}\|_{\mathcal{H}^1}^2 + \|I_{N_k} v\|_{H^1}^4 \\ &\lesssim N_{k+1}^{2(1-s)} E(I_{N_k} \vec{v}) + E(I_{N_k} \vec{v})^2 \\ &\lesssim N_{k+1}^{2(1-s)} N_k^\alpha + N_k^{2\alpha} \\ &\ll N_{k+1}^\beta. \end{aligned} \quad (3.50)$$

We are now ready to implement an iterative argument. Given  $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ , choose  $N_0 = N_0(\phi_0, \phi_1, s) \gg 1$  such that

$$E(I_{N_0} \vec{v})(0) \leq N_0^\beta. \quad (3.51)$$

By applying Proposition 3.5, we have

$$E(I_{N_0} \vec{v})(t) \leq N_0^\alpha$$

for any  $0 \leq t \leq \tau$ . By (3.49) and (3.50), this then implies

$$E(I_{N_1} \vec{v})(\tau) \leq N_1^\beta.$$

Applying Proposition 3.5 once again, we in turn obtain

$$E(I_{N_1} \vec{v})(t) \leq N_1^\alpha$$

for  $0 \leq t \leq 2\tau$ . By (3.49) and (3.50), this then implies

$$E(I_{N_2} \vec{v})(2\tau) \leq N_2^\beta.$$

After iterating this argument  $\lceil \frac{T}{\tau} \rceil + 1$  times, we obtain a solution  $v$  to the renormalized cubic SNLW (1.12) on the time interval  $[0, T]$ . Since the choice of  $T \gg 1$  was arbitrary, this proves global well-posedness of (1.12).

**Remark 3.7.** Fix  $T \gg 1$  and let the other parameters be as above. Then, it follows from the argument above and (1.19) that

$$\|\vec{v}(t)\|_{\mathcal{H}^s} \lesssim (E(I_{N_k} \vec{v})(t))^{\frac{1}{2}} \leq N_k^{\frac{\alpha}{2}}$$

for any  $0 \leq t \leq T$  such that  $k\tau \leq t \leq (k+1)\tau$ ,  $k \in \mathbb{Z}_{\geq 0}$ . Then, using (3.47), we have

$$\|\vec{v}(t)\|_{\mathcal{H}^s} \lesssim \exp\left(\frac{\alpha}{2} \sigma^k \log N_0\right) \leq \exp\left(\frac{\alpha}{2} \log N_0 \cdot \exp\left(\frac{(\log \sigma)t}{\tau}\right)\right) \quad (3.52)$$

for  $0 \leq t \leq T$ . Moreover, in view of (3.51), we choose  $N_0 \in \mathbb{N}$  such that  $1 + E(I_{N_0} \vec{v})(0) \sim N_0^\beta$  and thus we have

$$\log N_0 \sim \log(2 + \|\vec{v}(0)\|_{\mathcal{H}^s}). \quad (3.53)$$

In order to reach the target time  $T$ , we iteratively apply Proposition 3.5  $K \sim \frac{T}{\tau}$ -many times. For this purpose, we need to guarantee the condition (3.46). In view of Remark 3.6 and (3.47) with  $k = K \sim \frac{T}{\tau}$ , the condition (3.46) now reads as

$$e^{C(1+V+R+T)T^{-1}} - 1 \leq \frac{c_0 \sigma^{T^2} \log N_0 - \log 2}{T^2},$$

which holds true for any sufficiently large  $T \gg 1$ .

Finally, from (3.52), (3.53), and Remark 3.6, we conclude the following double exponential bound:

$$\|\vec{v}(t)\|_{\mathcal{H}^s} \leq C \exp\left(c \log(2 + \|\vec{v}(0)\|_{\mathcal{H}^s}) \cdot e^{C(\omega)t^2}\right) \quad (3.54)$$

for any  $t \geq 0$ .

We conclude this section by pointing out that by implementing a more involved version of Proposition 3.5 (see for example the paper [53] by the fourth author, studying SNLW on  $\mathbb{R}^2$ ), it is possible to improve  $t^2$  in (3.54) to  $t^\alpha$  for some  $\alpha < 2$ . For readers' convenience, however, we decided to include the current slightly simpler and more intuitive approach, with a Gronwall-type argument with  $G \log G$  as in (3.39). We point out that we do not know how to improve  $t^2$  in (3.54) to  $t$  at this point.

4. ALMOST SURE GLOBAL WELL-POSEDNESS OF THE HYPERBOLIC  $\Phi_2$ -MODEL

We present a simple local well-posedness argument for (1.46) based on Sobolev's inequality. We first consider the following deterministic NLW:

$$\begin{cases} \partial_t^2 v + \partial_t v + (1 - \Delta)v + \sum_{\ell=0}^k \binom{k}{\ell} \Xi_\ell v^{k-\ell} = 0 \\ (v, \partial_t v)|_{t=0} = (v_0, v_1) \end{cases} \quad (4.1)$$

for given initial data  $(v_0, v_1)$  and a source  $(\Xi, \dots, \Xi_k)$  with the understanding that  $\Xi_0 \equiv 1$ .

Given  $s \in \mathbb{R}$ , define  $\mathcal{X}^s(\mathbb{T}^2)$  by

$$\mathcal{X}^s(\mathbb{T}^2) \stackrel{\text{def}}{=} \mathcal{H}^s(\mathbb{T}^2) \times (L^2([0, 1]; W^{s-1, \infty}(\mathbb{T}^2)))^{\otimes k}$$

and set

$$\|\Xi\|_{\mathcal{X}^s} = \|(v_0, v_1)\|_{\mathcal{H}^s} + \sum_{j=1}^k \|\Xi_j\|_{L^2([0, 1]; W^{s-1, \infty})}$$

for  $\Xi = (v_0, v_1, \Xi_1, \Xi_2, \dots, \Xi_k) \in \mathcal{X}^s(\mathbb{T}^2)$ . Then, we have the following local well-posedness result for (4.1).

**Proposition 4.1.** *Given an integer  $k \geq 2$ , there exists  $\varepsilon_k > 0$  such that, for  $0 \leq \varepsilon < \varepsilon_k$ , (4.1) is unconditionally local well-posed in  $\mathcal{X}^{1-\varepsilon}(\mathbb{T}^2)$ . More precisely, given an enhanced data set:*

$$\Xi = (v_0, v_1, \Xi_1, \Xi_2, \dots, \Xi_k) \in \mathcal{X}^{1-\varepsilon}(\mathbb{T}^2), \quad (4.2)$$

there exist  $T = T(\|\Xi\|_{\mathcal{X}^{1-\varepsilon}}) \in (0, 1]$  and a unique solution  $v$  to (4.1) in the class:

$$C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2)). \quad (4.3)$$

In particular, the uniqueness of  $v$  holds in the entire class (4.3). Furthermore, the solution map:  $\Xi \in \mathcal{X}^{1-\varepsilon}(\mathbb{T}^2) \mapsto v \in C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$  is locally Lipschitz continuous.

We point out that Proposition 4.1 is completely deterministic. Once we prove Proposition 4.1, the claimed local well-posedness of the renormalized SdNLW (1.46) follows from Proposition 4.1 and Lemma 2.3, stating that the (random) enhanced data set  $\Xi = (v_0, v_1, \Phi, :\Phi^2:, \dots, :\Phi^k:)$  almost surely belongs to  $\mathcal{X}^{1-\varepsilon}(\mathbb{T}^2)$ ,  $\varepsilon > 0$ .

*Proof.* By writing (4.1) in the Duhamel formulation, we have

$$\begin{aligned} v(t) &= \Gamma(v) \stackrel{\text{def}}{=} \partial_t \mathcal{D}(t)v_0 + \mathcal{D}(t)(v_0 + v_1) \\ &\quad + \sum_{\ell=0}^k \binom{k}{\ell} \int_0^t \mathcal{D}(t-t') (\Xi_\ell v^{k-\ell})(t') dt', \end{aligned} \quad (4.4)$$

where the map  $\Gamma = \Gamma_\Xi$  depends on the enhanced data set  $\Xi$  in (4.2). Fix  $0 < T < 1$ .

We first treat the case  $\ell = 0$ . From (1.40) and applying Sobolev's inequality twice, we obtain

$$\begin{aligned} \left\| \int_0^t \mathcal{D}(t-t') v^k(t') dt' \right\|_{C_T H_x^{1-\varepsilon}} &\lesssim T \|v^k\|_{C_T H_x^{-\varepsilon}} \lesssim T \|v^k\|_{C_T L_x^{\frac{2}{1+\varepsilon}}} \lesssim T \|v\|_{C_T L_x^{\frac{2k}{1+\varepsilon}}}^k \\ &\lesssim T \|v\|_{C_T H_x^s}^k, \end{aligned} \quad (4.5)$$

provided that

$$0 \leq \varepsilon \leq \frac{1}{k-1}.$$

For  $1 \leq \ell \leq k-1$ , it follows from Lemma 2.5(ii) and then (i) followed by Sobolev's inequality that

$$\begin{aligned} \left\| \int_0^t \mathcal{D}(t-t') (\Xi_\ell v^{k-\ell})(t') dt' \right\|_{C_T H_x^{1-\varepsilon}} &\lesssim T^{\frac{1}{2}} \|\Xi_\ell v^{k-\ell}\|_{L_T^2 H_x^{-\varepsilon}} \\ &\lesssim T^{\frac{1}{2}} \|\langle \nabla \rangle^{-\varepsilon} \Xi_\ell\|_{L_T^2 L_x^{\frac{2}{\varepsilon}}} \|\langle \nabla \rangle^\varepsilon v^{k-\ell}\|_{C_T L_x^2} \\ &\lesssim T^{\frac{1}{2}} \|\Xi\|_{\mathcal{X}^{1-\varepsilon}} \|\langle \nabla \rangle^\varepsilon v\|_{C_T L_x^{2(k-\ell)}}^{k-\ell} \\ &\lesssim T^{\frac{1}{2}} \|\Xi\|_{\mathcal{X}^{1-\varepsilon}} \|v\|_{C_T H_x^{1-\varepsilon}}^{k-\ell}, \end{aligned} \quad (4.6)$$

provided that

$$0 \leq \varepsilon \leq \frac{1}{2(k-1)}. \quad (4.7)$$

Lastly, from (1.40), we have

$$\left\| \int_0^t \mathcal{D}(t-t') \Xi_k(t') dt' \right\|_{C_T H_x^{1-\varepsilon}} \lesssim T^{\frac{1}{2}} \|\Xi_k\|_{L_T^2 H_x^{-\varepsilon}} \leq T^{\frac{1}{2}} \|\Xi\|_{\mathcal{X}^{1-\varepsilon}}. \quad (4.8)$$

Putting (4.4), (4.5), (4.6), and (4.8) together, we have

$$\|\Gamma(v)\|_{C_T H_x^{1-\varepsilon}} \leq C_1 \|(v_0, v_1)\|_{\mathcal{H}^{1-\varepsilon}} + C_2 T^{\frac{1}{2}} (1 + \|\Xi\|_{\mathcal{X}^{1-\varepsilon}}) (1 + \|v\|_{C_T H_x^{1-\varepsilon}})^k,$$

as long as (4.7) is satisfied. An analogous difference estimate also holds. Therefore, by choosing  $T = T(\|\Xi\|_{\mathcal{X}^{1-\varepsilon}}) > 0$  sufficiently small, we conclude that  $\Gamma$  is a contraction in the ball  $B_R \subset C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$  of radius  $R = 2C_1 \|(v_0, v_1)\|_{\mathcal{H}^{1-\varepsilon}} + 1$ . At this point, the uniqueness holds only in the ball  $B_R$  but by a standard continuity argument, we can extend the uniqueness to hold in the entire  $C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$ . We omit details.  $\square$

Next, we provide a brief discussion on invariance of the truncated Gibbs measure  $\vec{\rho}_N$  in (1.35) under the dynamics of the renormalized truncated SdNLW (1.37) for  $u_N$ .

Given  $N \in \mathbb{N}$ , define the marginal probabilities measures  $\vec{\mu}_{1,N}$  and  $\vec{\mu}_{1,N}^\perp$  on  $\mathbf{P}_N \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)$  and  $\mathbf{P}_N^\perp \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)$ , respectively, as the induced probability measures under the following maps:

$$\omega \in \Omega \longmapsto (\mathbf{P}_N u^1(\omega), \mathbf{P}_N u^2(\omega))$$

for  $\vec{\mu}_{1,N}$  and

$$\omega \in \Omega \longmapsto (\mathbf{P}_N^\perp u^1(\omega), \mathbf{P}_N^\perp u^2(\omega))$$

for  $\vec{\mu}_{1,N}^\perp$ , where  $u^1$  and  $u^2$  are as in (1.30). Then, we have

$$\vec{\mu}_1 = \vec{\mu}_{1,N} \otimes \vec{\mu}_{1,N}^\perp. \quad (4.9)$$

From (1.35) and (4.9), we then have

$$\vec{\rho}_N = \vec{\nu}_N \otimes \vec{\mu}_{1,N}^\perp. \quad (4.10)$$

where  $\vec{\nu}_N$  is given by

$$d\vec{\nu}_N = Z_N^{-1} R_N(u) d\vec{\mu}_{1,N}$$

with the density  $R_N$  as in (1.33).

Recalling the decomposition (1.42), we see that the dynamics for the high frequency part  $\mathbf{P}_N^\perp u_N = \mathbf{P}_N^\perp \Phi$  is given by

$$\partial_t^2 \mathbf{P}_N^\perp \Phi + \partial_t \mathbf{P}_N^\perp \Phi + (1 - \Delta) \mathbf{P}_N^\perp \Phi = \sqrt{2} \mathbf{P}_N^\perp \xi. \quad (4.11)$$

This is a linear dynamics and thus we can readily verify that the Gaussian measure  $\bar{\mu}_{1,N}^\perp$  is invariant under the dynamics of (4.11) (for example, by studying (4.11) for each frequency  $|n| > N$  on the Fourier side).

On the other hand, the low frequency part  $\mathbf{P}_N u_N$  satisfies (1.43). With  $(u_N^1, u_N^2) = (\mathbf{P}_N u_N, \partial_t \mathbf{P}_N u_N)$ , we can write (1.43) in the following Ito formulation:

$$\begin{aligned} d \begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} + \left\{ \begin{pmatrix} 0 & -1 \\ 1 - \Delta & 0 \end{pmatrix} \begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{P}_N ( : (u_N^1)^k : ) \end{pmatrix} \right\} dt \\ = \begin{pmatrix} 0 \\ -u_N^2 dt + \sqrt{2} \mathbf{P}_N dW \end{pmatrix}. \end{aligned} \quad (4.12)$$

This shows that the generator  $\mathcal{L}^N$  for (4.12) can be written as  $\mathcal{L}^N = \mathcal{L}_1^N + \mathcal{L}_2^N$ , where  $\mathcal{L}_1^N$  denotes the generator for the deterministic NLW with the truncated nonlinearity:

$$d \begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} + \left\{ \begin{pmatrix} 0 & -1 \\ 1 - \Delta & 0 \end{pmatrix} \begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{P}_N ( : (u_N^1)^k : ) \end{pmatrix} \right\} dt = 0 \quad (4.13)$$

and  $\mathcal{L}_2^N$  denotes the generator for the Ornstein-Uhlenbeck process (for the second component  $u_N^2$ ):

$$d \begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -u_N^2 dt + \sqrt{2} \mathbf{P}_N dW \end{pmatrix}. \quad (4.14)$$

Note that (4.13) is a Hamiltonian equation with the Hamiltonian:

$$E(u_N^1, u_N^2) = \frac{1}{2} \int_{\mathbb{T}^2} ((u_N^1)^2 + |\nabla u_N^1|^2) dx + \frac{1}{2} \int_{\mathbb{T}^2} (u_N^2)^2 dx + R_N(u_N^1),$$

where  $R_N$  is as in (1.33). Then, from the conservation of the Hamiltonian  $E(u_N^1, u_N^2)$  and Liouville's theorem (on a finite-dimensional phase space), we conclude that  $\bar{\nu}_N$  is invariant under the dynamics of (4.13). In particular, we have  $(\mathcal{L}_1^N)^* \bar{\nu}_N = 0$ . On the other hand, by recalling that the Ornstein-Uhlenbeck process preserves the standard Gaussian measure, we conclude that  $\bar{\nu}_N$  is also invariant under the dynamics of (4.14) since the measure  $\bar{\nu}_N$  is nothing but the white noise (projected onto the low frequencies  $\{|n| \leq N\}$ ) on the second component  $u_N^2$ . Thus, we have  $(\mathcal{L}_2^N)^* \bar{\nu}_N = 0$ . Hence, we obtain

$$(\mathcal{L}^N)^* \bar{\nu}_N = (\mathcal{L}_1^N)^* \bar{\nu}_N + (\mathcal{L}_2^N)^* \bar{\nu}_N = 0.$$

This shows invariance of  $\bar{\nu}_N$  under (4.12) and hence under (1.43).

Therefore, from (4.10) and invariance of  $\bar{\nu}_N$  and  $\bar{\mu}_{1,N}^\perp$  under (4.12) and (4.11), respectively, we conclude that the truncated Gibbs measure  $\bar{\rho}_N$  in (1.35) is invariant under the dynamics of the renormalized truncated SdNLW (1.37).

The rest of the proof of Theorem 1.7 follows from a standard application of Bourgain's invariant measure argument and thus we omit details. See, for example, [40] for details.

**Acknowledgements.** T.O. was supported by the European Research Council (grant no. 637995 “ProbDynDispEq” and grant no. 864138 “SingStochDispDyn”). L.T. was supported by the European Research Council (grant no. 637995 “ProbDynDispEq”). M.G., H.K., and L.T. were supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the Hausdorff Center for Mathematics under Germany’s Excellence Strategy - EXC-2047/1 - 390685813 and through CRC 1060 - project number 211504053. The authors would like to thank the anonymous referees for helpful comments.

## REFERENCES

- [1] S. Albeverio, Z. Haba, F. Russo, *Trivial solutions for a non-linear two-space-dimensional wave equation perturbed by space-time white noise*, Stochastics Stochastics Rep. 56 (1996), no. 1-2, 127–160.
- [2] Á. Bényi, T. Oh, O. Pocovnicu, *On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on  $\mathbb{R}^d$ ,  $d \geq 3$* , Trans. Amer. Math. Soc. Ser. B 2 (2015), 1–50.
- [3] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. 166 (1994), no. 1, 1–26.
- [4] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. 176 (1996), no. 2, 421–445.
- [5] B. Bringmann, *Invariant Gibbs measures for the three-dimensional wave equation with a Hartree non-linearity II: dynamics*, arXiv:2009.04616 [math.AP].
- [6] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. II. A global existence result*, Invent. Math. 173 (2008), no. 3, 477–496.
- [7] K. Cheung, G. Li, T. Oh, *Almost conservation laws for stochastic nonlinear Schrödinger equations*, J. Evol. Equ. <https://doi.org/10.1007/s00028-020-00659-x>
- [8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation*, Math. Res. Lett. 9 (2002), no. 5-6, 659–682.
- [9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp global well-posedness for KdV and modified KdV on  $\mathbb{R}$  and  $\mathbb{T}$* , J. Amer. Math. Soc. 16 (2003), no. 3, 705–749.
- [10] J. Colliander, T. Oh, *Almost sure well-posedness of the cubic nonlinear Schrödinger equation below  $L^2(\mathbb{T})$* , Duke Math. J. 161 (2012), no. 3, 367–414.
- [11] G. Da Prato, A. Debussche, *Strong solutions to the stochastic quantization equations*, Ann. Probab. 31 (2003), no. 4, 1900–1916.
- [12] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Second edition. Encyclopedia of Mathematics and its Applications, 152. Cambridge University Press, Cambridge, 2014. xviii+493 pp.
- [13] G. Da Prato, L. Tubaro, *Wick powers in stochastic PDEs: an introduction*, Technical Report UTM, 2006, 39 pp.
- [14] A. de Bouard, A. Debussche, *The stochastic nonlinear Schrödinger equation in  $H^1$* , Stochastic Anal. Appl. 21 (2003), no. 1, 97–126.
- [15] A. Deya, *A nonlinear wave equation with fractional perturbation*, Ann. Probab. 47 (2019), no. 3, 1775–1810.
- [16] A. Deya, *On a non-linear 2D fractional wave equation*, Ann. Inst. Henri Poincaré Probab. Stat. 56 (2020), no. 1, 477–501.
- [17] J. Forlano, *Almost sure global well posedness for the BBM equation with infinite  $L^2$  initial data*, Discrete Contin. Dyn. Syst. 40 (2020), no. 1, 267–318.
- [18] P. Friz, N. Victoir, *Multidimensional stochastic processes as rough paths. Theory and applications*, Cambridge Studies in Advanced Mathematics, 120. Cambridge University Press, Cambridge, 2010. xiv+656 pp.
- [19] J. Glimm, A. Jaffe, *Quantum physics. A functional integral point of view*, Second edition. Springer-Verlag, New York, 1987. xxii+535 pp.
- [20] M. Gubinelli, M. Hofmanová, *Global solutions to elliptic and parabolic  $\Phi^4$  models in Euclidean space*, Comm. Math. Phys. 368 (2019), no. 3, 1201–1266.
- [21] M. Gubinelli, H. Koch, T. Oh, *Renormalization of the two-dimensional stochastic nonlinear wave equations*, Trans. Amer. Math. Soc. 370 (2018), no 10, 7335–7359.
- [22] M. Gubinelli, H. Koch, T. Oh, *Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity*, to appear in J. Eur. Math. Soc.



- [23] T. Kato, *On nonlinear Schrödinger equations. II.  $H^s$ -solutions and unconditional well-posedness*, J. Anal. Math. 67 (1995), 281–306
- [24] M. Keel, T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), no. 5, 955–980.
- [25] H. Kuo, *Introduction to stochastic integration*, Universitext. Springer, New York, 2006. xiv+278 pp.
- [26] H.P. McKean, *Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger*, Comm. Math. Phys. 168 (1995), no. 3, 479–491. *Erratum: Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger*, Comm. Math. Phys. 173 (1995), no. 3, 675.
- [27] A. Moinat, H. Weber *Space-time localisation for the dynamic  $\Phi_3^4$  model*, Comm. Pure Appl. Math. 73 (2020), no. 12, 2519–2555.
- [28] R. Mosincat, O. Pocovnicu, L. Tolomeo, Y. Wang, *Global well-posedness of three-dimensional periodic stochastic nonlinear beam equations*, preprint.
- [29] J.-C. Mourrat, H. Weber, *Global well-posedness of the dynamic  $\Phi^4$  model in the plane*, Ann. Probab. 45 (2017), no. 4, 2398–2476.
- [30] J.-C. Mourrat, H. Weber, *The dynamic  $\Phi_3^4$  model comes down from infinity*, Comm. Math. Phys. 356 (2017), no. 3, 673–753.
- [31] E. Nelson, *A quartic interaction in two dimensions*, 1966 Mathematical Theory of Elementary Particles (Proc. Conf., Dedham, Mass., 1965) pp. 69–73 M.I.T. Press, Cambridge, Mass.
- [32] D. Nualart, *The Malliavin calculus and related topics*, Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006. xiv+382 pp.
- [33] T. Oh, M. Okamoto, *Comparing the stochastic nonlinear wave and heat equations: a case study*, Electron. J. Probab. 26 (2021), paper no. 9, 44 pp.
- [34] T. Oh, M. Okamoto, T. Robert, *A remark on triviality for the two-dimensional stochastic nonlinear wave equation*, Stochastic Process. Appl. 130 (2020), no. 9, 5838–5864.
- [35] T. Oh, M. Okamoto, L. Tolomeo, *Focusing  $\Phi_3^4$ -model with a Hartree-type nonlinearity*, arXiv:2009.03251 [math.PR].
- [36] T. Oh, M. Okamoto, L. Tolomeo, *Stochastic quantization of the  $\Phi_3^3$ -model*, preprint.
- [37] T. Oh, M. Okamoto, N. Tzvetkov, *Uniqueness and non-uniqueness of the Gaussian free field evolution under the two-dimensional Wick ordered cubic wave equation*, preprint.
- [38] T. Oh, O. Pocovnicu, *Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on  $\mathbb{R}^3$* , J. Math. Pures Appl. 105 (2016), 342–366.
- [39] T. Oh, O. Pocovnicu, N. Tzvetkov, *Probabilistic local Cauchy theory of the cubic nonlinear wave equation in negative Sobolev spaces*, to appear in Ann. Inst. Fourier (Grenoble).
- [40] T. Oh, T. Robert, N. Tzvetkov, *Stochastic nonlinear wave dynamics on compact surfaces*, arXiv:1904.05277 [math.AP].
- [41] T. Oh, T. Robert, P. Sosoe, Y. Wang, *On the two-dimensional hyperbolic stochastic sine-Gordon equation*, Stoch. Partial Differ. Equ. Anal. Comput. 9 (2021), 1–32.
- [42] T. Oh, T. Robert, P. Sosoe, Y. Wang, *Invariant Gibbs dynamics for the dynamical sine-Gordon model*, Proc. Roy. Soc. Edinburgh Sect. A (2020), 17 pages. doi: <https://doi.org/10.1017/prm.2020.68>
- [43] T. Oh, T. Robert, Y. Wang, *On the parabolic and hyperbolic Liouville equations*, arXiv:1908.03944 [math.AP].
- [44] T. Oh, L. Thomann, *A pedestrian approach to the invariant Gibbs measure for the 2-d defocusing nonlinear Schrödinger equations*, 6 (2018), 397–445.
- [45] T. Oh, L. Thomann, *Invariant Gibbs measure for the 2-d defocusing nonlinear wave equations*, Ann. Fac. Sci. Toulouse Math. 29 (2020), no. 1, 1–26.
- [46] O. Pocovnicu, *Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on  $\mathbb{R}^d$ ,  $d = 4$  and 5*, J. Eur. Math. Soc. 19 (2017), 2321–2375.
- [47] T. Roy, *On the interpolation with the potential bound for global solutions of the defocusing cubic wave equation on  $\mathbb{T}^2$* , J. Funct. Anal. 270 (2016), no. 9, 3280–3306.
- [48] S. Ryang, T. Saito, K. Shigemoto, *Canonical stochastic quantization*, Progr. Theoret. Phys. 73 (1985), no. 5, 1295–1298.
- [49] I. Shigekawa, *Stochastic analysis*, Translated from the 1998 Japanese original by the author. Translations of Mathematical Monographs, 224. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2004. xii+182 pp.
- [50] B. Simon, *The  $P(\varphi)_2$  Euclidean (quantum) field theory*, Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1974. xx+392 pp.

- [51] L. Thomann, N. Tzvetkov, *Gibbs measure for the periodic derivative nonlinear Schrödinger equation*, Nonlinearity 23 (2010), no. 11, 2771–2791.
- [52] L. Tolomeo, *Unique ergodicity for a class of stochastic hyperbolic equations with additive space-time white noise*, Comm. Math. Phys. 377 (2020), no. 2, 1311–1347.
- [53] L. Tolomeo, *Global well-posedness of the two-dimensional stochastic nonlinear wave equation on an unbounded domain*, to appear in Ann. Probab.
- [54] L. Tolomeo, *Ergodicity for the hyperbolic  $P(\Phi)_2$ -model*, in preparation.
- [55] N. Tzvetkov, *Construction of a Gibbs measure associated to the periodic Benjamin-Ono equation*, Probab. Theory Related Fields 146 (2010), no. 3-4, 481–514.

MASSIMILIANO GUBINELLI, HAUSDORFF CENTER FOR MATHEMATICS & INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT BONN, ENDENICHER ALLEE 60, D-53115 BONN, GERMANY

*Email address:* `gubinelli@iam.uni-bonn.de`

HERBERT KOCH, MATHEMATISCHES INSTITUT, UNIVERSITÄT BONN, ENDENICHER ALLEE 60, D-53115 BONN, GERMANY

*Email address:* `koch@math.uni-bonn.de`

TADAHIRO OH, SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, AND THE MAXWELL INSTITUTE FOR THE MATHEMATICAL SCIENCES, JAMES CLERK MAXWELL BUILDING, THE KING'S BUILDINGS, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, UNITED KINGDOM

*Email address:* `hiro.oh@ed.ac.uk`

LEONARDO TOLOMELO, THE UNIVERSITY OF EDINBURGH, AND THE MAXWELL INSTITUTE FOR THE MATHEMATICAL SCIENCES, JAMES CLERK MAXWELL BUILDING, THE KING'S BUILDINGS, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, UNITED KINGDOM AND MATHEMATICAL INSTITUTE, HAUSDORFF CENTER FOR MATHEMATICS, UNIVERSITÄT BONN, BONN, GERMANY

*Email address:* `tolomeo@math.uni-bonn.de`