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On the Vertex Position Number of Graphs

Maya G.S. Thankachy[a], Ullas Chandran S. V.[a], James Tuite[b], Elias John Thomas[c], Gabriele Di Stefano[d], Grahame Erskine[b]

[a]*Department of Mathematics, Mahatma Gandhi College, University of Kerala, Thiruvananthapuram-695004, Kerala, India*
[b]*School of Mathematics and Statistics, Open University, Walton Hall, Milton Keynes, UK*
[c]*Department of Mathematics, Mar Ivanios College, University of Kerala, Thiruvananthapuram-695015, Kerala, India*
[d]*Department of Information Engineering, Computer Science and Mathematics, University of L'Aquila, Italy*

Abstract

In this paper we generalise the notion of visibility from a point in an integer lattice to the setting of graph theory. For a vertex x of a connected graph G, we say that a set $S \subseteq V(G)$ is an *x-position set* if for any $y \in S$ the shortest x, y-paths in G contain no point of $S \setminus \{y\}$. We investigate the largest and smallest orders of maximum x-position sets in graphs, determining these numbers for common classes of graphs and giving bounds in terms of the girth, vertex degrees, diameter and radius. Finally we discuss the complexity of finding maximum vertex position sets in graphs.

Keywords: geodesic, vertex position set, vertex position number, general position
2000 MSC: 05C12, 05C69

1. Introduction

All graphs considered in this paper are finite, undirected and simple. For a graph G we will denote the subgraph induced by a subset $S \subseteq V(G)$ by

$G[S]$. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u,v-path in G; any such path is called a *geodesic*. The distance is a metric on the vertex set $V(G)$. The diameter $\mathrm{diam}(G)$ of a connected graph G is the length of any longest geodesic. For any vertex u of G, the *eccentricity* of u is $\mathrm{e}(u) = \max\{d(u,v) : v \in V\}$. A vertex v of G such that $d(u,v) = \mathrm{e}(u)$ is called an *eccentric vertex* of u.

If a vertex u of G is adjacent to a vertex v, we write $u \sim v$. The *open neighbourhood* (or simply *neighbourhood*) of a vertex u in a graph G is the set $N_G(u) = \{v \in V(G) : u \sim v\}$ consisting of all vertices v which are adjacent to u, whilst the *closed neighbourhood* of u is defined by $N_G[u] = \{u\} \cup N(u)$. The *degree* of u is $\mathrm{degree}(u) = |N(u)|$. The *second neighbourhood* $N^2(u)$ is the set $\{v \in V(G) : d(u,v) = 2\}$ of vertices at distance two from u. If it is clear in which graph the neighbourhood is taken we will omit the subscript.

The *complement* $\overline{G}$ is the graph on the same vertex set $V(G)$ as G, but a pair of vertices u,v of $\overline{G}$ are adjacent in $\overline{G}$ if and only if they are nonadjacent in G. A vertex u is *simplicial* if the subgraph induced by its neighborhood $N(u)$ is complete; we will denote the number of simplicial vertices of a graph G by $\mathrm{s}(G)$ and the set of all simplicial vertices of G by $\mathrm{Ext}(G)$. A set of vertices in a graph is *independent* if no two vertices in the set are adjacent; the independence number $\alpha(G)$ is the number of vertices in a largest independent set of G. A graph is a *block graph* if every maximal 2-connected component is a clique. The graph G is *vertex-transitive* if the automorphism group $\mathrm{Aut}(G)$ acts transitively on $V(G)$. The *join* $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$, i.e. $G \vee H$ is obtained from the disjoint union $G \dot\cup H$ by adding all possible edges between G and H. For other basic graph theoretic terminology not defined here we refer to [4, 6].

Visibility and illumination problems are among the most attractive and interesting research topics in combinatorics, geometry and number theory [3]. Such problems have been studied intensively in the context of the integer lattice; a set Λ of points of the lattice is *visible* from a point x if for any $y \in \Lambda$ the straight line segment from x to y contains no other points of Λ. A well-known result from elementary analytic number theory first proved by Sylvester [20] states that the density of the set of points in the integer lattice that are visible from the origin is $\frac{6}{\pi^2}$ [2]. In particular, in Chapter III of [14], it is shown how to place a set of points with positive integer coordinates (i,j), $j \le i$, in such a way that each point is visible from the origin $(0,0)$, by also maximising the number of points with the same abscissa; this construction is

related to the Farey series and Euler's totient function ϕ. Other interesting visibility problems in integer lattices can be found in [9, 13].

In recent years the algorithmic component of visibility problems has attracted great attention under the name *art gallery* or *watchman* problems, which lie in the intersection of combinatorial and computational geometry [19]. Art gallery problems, theorems and algorithms are so named after the following celebrated question of V. Klee, which he posed 50 years ago in 1973: 'What is the minimum number of guards sufficient to cover the interior of an n-wall gallery?'. This problem was solved by Chvátal and subsequently by Fisk. By creating idealised situations such as obstacles, guards, etc., the theory succeeds in abstracting the algorithmic essence of many visibility problems.

Taking our inspiration from the result of Sylvester [20], in this paper we consider a generalisation of 'local visibility problems' to the context of the general position problem in graph theory. The general position problem originated in Dudeney's no-three-in-line problem [8] and the general position subset selection problem [10, 18] from discrete geometry. These problems were generalised to graphs independently in [5] and [15]. A set S of vertices of a graph G is in *general position* if for any $u, v \in S$ any u, v-geodesic does not intersect $S \setminus \{u, v\}$. The *general position number* $\mathrm{gp}(G)$ of G is the number of vertices in a largest general position set in G. We refer the reader to some recent papers [1, 16, 17, 21, 22, 23] for more information on the general position problem.

In a recent paper Di Stefano [7] introduced the concept of a *mutual visibility set* in a graph; a set S of points in a graph G are *mutually visible* if for any $u, v \in S$ there exists a shortest u, v-path in G that does not pass through $S \setminus \{u, v\}$; the *mutual visibility number* $\mu(G)$ of G is the number of vertices in a largest mutual visibility set in G. In [7] the mutual visibility number of some classes of graphs are determined and it is shown that the problem of finding a largest mutual visibility set is NP-complete for general graphs.

We now study a 'local' version of these problems using a parameter that we call the *vertex position number* of a graph. The plan of this paper is as follows. In Section 2 we provide some bounds on the vertex position numbers of a graph. In Section 3 the vertex position numbers of some common classes of graphs are determined. We characterise the graphs with very large or small vertex position numbers in Section 4. Finally in Section 5 we consider the computational complexity of finding the vertex position number of a graph.
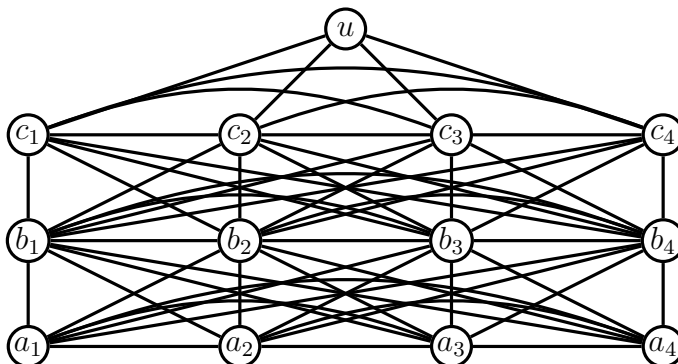
Figure 2.1

2. Vertex position sets in graphs

In this section we derive bounds for the vertex position numbers of a graph in terms of the minimum and maximum degrees, radius and diameter. First we formally define the vertex position numbers.

Definition 2.1. *For any graph G and a fixed vertex $x \in V(G)$, a set $S_x \subseteq V(G)$ is an x-position set if for any $y \in S_x$ no vertex of $S_x \setminus \{y\}$ lies on any x, y-geodesic in G. The x-position number of G is defined to be the maximum cardinality of an x-position set and is denoted by $p_x(G)$ or simply p_x. An x-position set of cardinality $p_x(G)$ is called a p_x-set. The maximum value of $p_x(G)$ among all vertices x of G is called the* vertex position number $\mathrm{vp}(G)$ *of G. Similarly, the minimum value of $p_x(G)$ among all vertices x of G is called the* minimum vertex position number $\mathrm{vp}^-(G)$ *of G.*

To illustrate these concepts, consider the graph G in Figure 2.1. We give the x-position numbers of G for representative vertices in Table 1, together with a (not necessarily unique) x-position set. We see from the table that $\mathrm{vp}^-(G) = 4$ and $\mathrm{vp}(G) = 11$.

Unless stated otherwise (for example in Theorem 2.13) we assume all graphs to be connected. However, Definition 2.1 also applies to disconnected graphs; if x belongs to a component C of a disconnected graph G, then any vertex y from another component D of G can be included in an x-position set, as there is no x, y-path in G. Hence in this case $p_x(G) = (n - |C|) + p_x(C)$.

For any vertex $x \in V(G)$ the set $\{x\}$ is an x-position set; however, by the convention in Definition 2.1, if G is connected, then x is not contained in any x-position set of order ≥ 2. Hence for any connected graph G with

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Vertex	p_x-set	$p_x(G)$
u	$\{c_1, c_2, c_3, c_4\}$	4
c_1	$\{u, c_2, c_3, c_4, b_1, b_2, b_3, b_4\}$	8
b_1	$\{c_1, c_2, c_3, c_4, b_2, b_3, b_4, a_1, a_2, a_3, a_4\}$	11
a_1	$\{a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$	7

Table 1

order $n \geq 2$ we have $1 \leq p_x(G) \leq n-1$ for any $x \in V(G)$ and more generally a (not necessarily connected) graph G has $\mathrm{vp}(G) = n$ if and only if G has an isolated vertex. These bounds are sharp: for any path P_n of length ≥ 1 we have $p_x(P_n) = 1$ for either terminal vertex, whilst for $n \geq 2$ we have $p_x(K_n) = n - 1$ for every vertex of a complete graph K_n. In this section we derive several bounds for the vertex position numbers in terms of various graph parameters. First we compare the vertex position number with the general position number.

Lemma 2.2. *The vertex position number and general position number of a graph are related by* $\mathrm{vp}(G) \geq \mathrm{gp}(G) - 1$.

Proof. Let S be a gp-set of G with $|S| = \mathrm{gp}(G)$. Choose a vertex $x \in S$. Then $S \setminus \{x\}$ is an x-position set, implying that $\mathrm{vp}(G) \geq p_x(G) \geq \mathrm{gp}(G) - 1$. ∎

The bound in Lemma 2.2 is met by the complete graph K_n. However, we now give an example to show that the numbers $\mathrm{vp}^-(G)$, $\mathrm{gp}(G)$ and $\mathrm{vp}(G)$ can be arbitrarily far apart. For $r \geq 2$ we define the vertex set of the graph $G(r)$ to be $\{u_{i,j} : 1 \leq i \leq 7, 1 \leq j \leq r\} \cup \{u\}$. Let $H(r)$ be the graph on the same vertex set as $G(r)$ with adjacencies defined as follows:

- $u \sim u_{1,j}$ for $1 \leq j \leq r$,

- $u_{i,j} \sim u_{i,j'}$, $1 \leq i \leq 7$, $1 \leq j, j' \leq r$ and $j \neq j'$, and

- $u_{i,j} \sim u_{i+1,j'}$ for $1 \leq i \leq 6$ and $1 \leq j, j' \leq r$.

Now define $G(r)$ to be the graph formed by deleting all edges to the vertices $u_{3,1}$, $u_{4,1}$ and $u_{5,1}$ except for the path $u_{2,1}, u_{3,1}, u_{4,1}, u_{5,1}, u_{6,1}$. See Figure 2.2 for an example. The minimum vertex position number of $G(r)$ is r (attained at the vertex u) and the vertex position number is $6r - 4$ (attained at the vertex $u_{4,1}$).
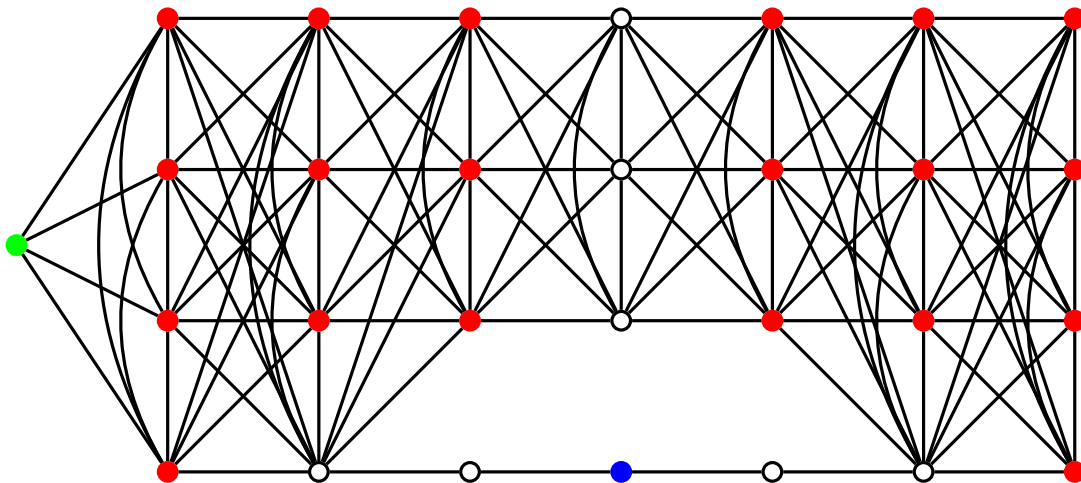
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Figure 2.2: A graph $G(r)$ with $\mathrm{vp}^- = r$ (green vertex) and $\mathrm{vp} = 6r - 4$ (blue vertex). A largest vp-set of the blue vertex is shown in red. Here $r = 4$.

Lemma 2.3. *For $r \geq 3$, we have $\mathrm{vp}^-(G(r)) = r, \mathrm{gp}(G(r)) = 2r, \mathrm{vp}(G(r)) = 6r - 4$.*

In Theorem 3.8 we will see that $\mathrm{vp}^-(G)$ can be larger than $\mathrm{gp}(G)$. Lemma 2.3 raises the question of how far apart the numbers $\mathrm{vp}^-(G)$ and $\mathrm{vp}(G)$ can be.

Problem 2.4. *Is the ratio $\frac{\mathrm{vp}(G)}{\mathrm{vp}^-(G)}$ bounded for connected graphs?*

We now bound the vertex position numbers in terms of the vertex degrees.

Lemma 2.5. *Let G have minimum degree δ and maximum degree Δ. Then $\mathrm{vp}^-(G) \geq \delta$ and $\mathrm{vp}(G) \geq \Delta$.*

Proof. It follows from Definition 2.1 that for any vertex x of G the neighbourhood $N(x)$ is an x-position set of G. Therefore for all vertices $x \in V(G)$ we have $p_x(G) \geq \mathrm{degree}(x)$. This implies that all p_x-sets have order at least δ and there exists a p_x-set with order at least Δ. ∎

We now generalise this result to sets of vertices at given distance from a fixed vertex; this leads to bounds on $p_x(G)$ in terms of the order n of G and the eccentricity $\mathrm{e}(x)$ of the vertex x.

Lemma 2.6. *For any vertex x of G with eccentricity $\mathrm{e}(x)$ the vertex position number satisfies $p_x(G) \geq \frac{n-1}{\mathrm{e}(x)}$. Thus $\mathrm{vp}^-(G) \geq \frac{n-1}{\mathrm{diam}(G)}$ and $\mathrm{vp}(G) \geq \frac{n-1}{\mathrm{rad}(G)}$.*

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Proof. For $1 \leq t \leq \mathrm{e}(x)$, let V_t be the set of vertices at distance exactly t from x in G. Each of the sets V_t is an x-position set and one of them must have order at least $\frac{n-1}{\mathrm{e}(x)}$, so that $p_x(G) \geq \frac{n-1}{\mathrm{e}(x)}$. Since $\mathrm{rad}(G) \leq \mathrm{e}(x) \leq \mathrm{diam}(G)$ for all $x \in V(G)$, the result follows. ∎

This bound immediately characterises the graphs with vertex position number one.

Corollary 2.7. *A graph G satisfies* $\mathrm{vp}^-(G) = 1$ *if and only if G is a path. The only connected graphs with* $\mathrm{vp}(G) = 1$ *are K_1 and K_2.*

Proof. If x is the terminal vertex of a path P_n, then $p_x(P_n) = 1$. Conversely, if $\mathrm{vp}^-(G) = 1$, then G is connected and by Lemma 2.6 we must have $\mathrm{diam}(G) = n - 1$, which implies that G is a path. By Lemma 2.5 any graph G with $\mathrm{vp}(G) = 1$ must have maximum degree $\Delta \leq 1$, which proves the latter statement. ∎

The argument of Lemma 2.6 also easily yields the vertex position number of the join of two graphs.

Corollary 2.8. *If G_i has order n_i and maximum degree Δ_i for $i = 1, 2$, then the vertex position number of the join $G_1 \vee G_2$ is*

$$\mathrm{vp}(G_1 \vee G_2) = \max\{n_1 + \Delta_2, n_2 + \Delta_1\}.$$

Proof. If both G_1 and G_2 are complete, then the result is trivial, so we can assume that the diameter of $G_1 \vee G_2$ is two. For any vertex x with degree Δ_1 in G_1 the set $V(G_2) \cup N_{G_1}(x)$ is an x-position set with order $n_2 + \Delta_1$ by Lemma 2.5, with a similar result for vertices y of maximum degree in G_2. Hence $\mathrm{vp}(G_1 \vee G_2) \geq \max\{n_1 + \Delta_2, n_2 + \Delta_1\}$.

Suppose without loss of generality that the largest vertex position set is attained at a vertex $x \in V(G_1)$. Then we can assume that a p_x-set contains a vertex of G_2, for otherwise $\mathrm{vp}(G_1 \vee G_2) = p_x(G_1 \vee G_2) \leq n_1 < p_y(G_1 \vee G_2)$ for any $y \in V(G_2)$, a contradiction. As a maximum p_x-set S_x contains a vertex of G_2, S_x cannot contain any vertex of $V(G_1) \setminus N_{G_1}[x]$, so that $p_x(G_1 \vee G_2) \leq n_2 + \Delta_1$. The result follows. ∎

Theorem 2.9. *For any connected graph G we have* $\mathrm{vp}^-(G) \geq \left\lceil \frac{\Delta+1}{3} \right\rceil$. *If G is bipartite, then* $\mathrm{vp}^-(G) \geq \left\lceil \frac{\Delta}{2} \right\rceil$.
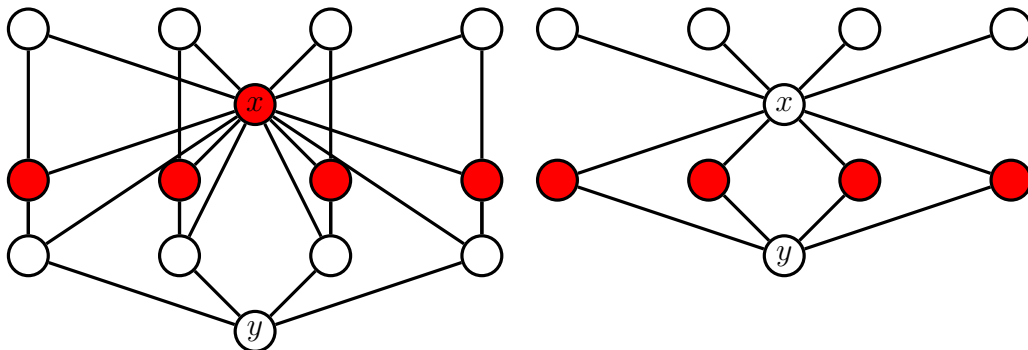
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Figure 2.3: Graphs with $\text{vp}^-(G) = \left\lceil \frac{\Delta+1}{3} \right\rceil$ (left) and $\text{vp}^-(G) = \frac{\Delta}{2}$ (right) with a p_y-set in red.

Proof. Let G be a connected graph with maximum degree Δ and let x be a vertex of G with this degree. Let y be any vertex of G. If $y = x$, we have $p_y(G) \geq \Delta$, so suppose that $y \neq x$ and let $r \geq 0$ be the length of the shortest path from y to $N(x)$. Then it can be seen that the distance from y to any vertex of $N[x]$ is one of r, $r+1$ or $r+2$. It follows that one of the level sets of G in the distance partition with respect to y must have order at least $\frac{\Delta+1}{3}$. Hence $p_y(G) \geq \frac{\Delta+1}{3}$ and our proof is complete. If G is bipartite, then the distance from y to any vertex of $N(x)$ is either r or $r+2$, so we can improve the bound to $\text{vp}^-(G) \geq \frac{\Delta}{2}$ in this case. The constructions in Figure 2.3 show that both of these bounds are tight (in both cases x is a vertex with maximum degree and $p_y(G) = \text{vp}^-(G)$). ∎

Now we give an upper bound for the vertex position number in terms of vertex eccentricity.

Lemma 2.10. *For any vertex x of G with eccentricity $\text{e}(x)$, the x-vertex position number of G is bounded above by $p_x(G) \leq n - \text{e}(x)$. Thus $\text{vp}(G) \leq n - \text{rad}(G)$ and $\text{vp}^-(G) \leq n - \text{diam}(G)$.*

Proof. Fix a vertex x of G with eccentricity $\text{e}(x)$. Let y be an eccentric vertex of x, i.e. $d(x,y) = \text{e}(x)$. Let $x = u_0, u_1, \ldots, u_{\text{e}(x)} = y$ be an x, y-geodesic in G and S_x be an x-position set of G with order $p_x(G)$. Suppose that $u_i, u_j \in S_x$ for some i, j with $0 \leq i < j \leq \text{e}(x)$; then u_i lies on an x, u_j-geodesic, contradicting the fact that S_x is an x-position set. Hence any x-position set contains at most one vertex from the set $\{x, u_1, u_2, \ldots, u_{\text{e}(x)}\}$. Thus $p_x(G) = |S_x| \leq n - 1 - \text{e}(x) + 1 = n - \text{e}(x)$. Thus $\text{vp}(G) \leq n - \text{rad}(G)$ and $\text{vp}^-(G) \leq n - \text{diam}(G)$. ∎

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The following theorem improves the upper bound for $\mathrm{vp}(G)$ from Lemma 2.10.

Theorem 2.11. *For any graph G with $\mathrm{rad}(G) \geq 3$ we have $\mathrm{vp}(G) \leq n - \mathrm{rad}(G) - 1$.*

Proof. Suppose that G has radius $\mathrm{rad}(G) \geq 3$ and meets the upper bound in Lemma 2.10. Then the largest value of the vertex position number is achieved by a central vertex, call it u. Let S_u be any u-position set of order $n - \mathrm{rad}(G)$. By the argument of Lemma 2.10 there is a path P from u to one of its eccentric vertices such that P contains just one vertex of S_u and all vertices of $V(G) \setminus V(P)$ belong to S_u. As the shortest path from u to any vertex $x \in V(G) \setminus V(P)$ cannot pass through another vertex of $V(G) \setminus V(P)$, the shortest path from u to x consists of a section of P followed by an edge from P to x. Hence each vertex of $V(G) \setminus V(P)$ has an edge to P; however, this contradicts our supposition that u is a central vertex. ∎

A vertex v in a connected graph G is a *boundary vertex* of a vertex u if $d(u, w) \leq d(u, v)$ for each neighbour w of v. The set of all boundary vertices of u is denoted by $\partial(u)$.

Proposition 2.12. *For any connected graph G and any vertex $x \in V(G)$, the boundary $\partial(x)$ is an x-position set of G.*

Proof. Assume to the contrary that $\partial(x)$ is not an x-position set; hence there must be a geodesic $x = u_0, u_1, \ldots, u_i = z, u_{i+1}, \ldots, u_k = y$ such that $z, y \in \partial(x)$ and $z \neq y$. This shows that $d(x, u_{i+1}) > d(x, z)$, contradicting the fact that z is a boundary vertex of x. Hence $\partial(x)$ is an x-position set. ∎

It follows from Proposition 2.12 that for any $x \in V(G)$ the set $\mathrm{Ext}(G) \setminus \{x\}$ is an x-position set. The bound in Proposition 2.12 is tight for the vertex x in the graph in Figure 2.4, but this is not true in general. In fact for any $s \geq t$ if we take a vertex x in the partite set of order $t + 1$ in the complete bipartite graph $K_{s,t+1}$ then $|\partial(x)| = t$ but $p_x(K_{s,t+1}) = s$.

Finally we present a Nordhaus-Gaddum relation for the vertex position number. In the following we do not assume that both G and $\overline{G}$ are connected.

Theorem 2.13. *For any graph G we have $n - 1 \leq \mathrm{vp}(G) + \mathrm{vp}(\overline{G}) \leq 2n - 1$. Both bounds are tight.*
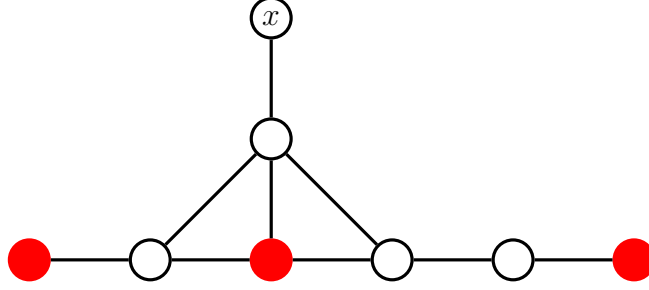
Figure 2.4: The boundary $\partial(x)$ (shown in red) is a p_x-set

Proof. Let x be any vertex of a graph G with degree $\mathrm{degree}(x)$. In the complement $\overline{G}$ the vertex x has degree $\mathrm{degree}'(x) = n - 1 - \mathrm{degree}(x)$. By Lemma 2.5 we thus have

$$\mathrm{vp}(G) + \mathrm{vp}(\overline{G}) \geq p_x(G) + p_x(\overline{G}) \geq \mathrm{degree}(x) + n - 1 - \mathrm{degree}(x) = n - 1. \quad (1)$$

To show that equality can hold, consider the cycle C_n for $n \geq 5$. If $n = 5$ the result is simple, as $\overline{C_5} \cong C_5$, so take $n \geq 6$. Label the vertices of the cycle $x_0, x_1, \ldots, x_{n-1}$, where $x_0 \sim x_{n-1}$ and $x_i \sim x_{i+1}$ for $0 \leq i \leq n - 2$. As will be shown in Corollary 4.2, we have $\mathrm{vp}(C_n) = 2$. Consider the vertex x_0 (as $\overline{C}_n$ is vertex-transitive the choice is arbitrary) and let S be a largest x_0-position set in $\overline{C}_n$. The degree of x_0 is $\mathrm{degree}(x_0) = n - 3$, so that by Lemma 2.5 we have $\mathrm{vp}(\overline{C}_n) \geq n - 3$. Suppose that a vertex x_i, $3 \leq i \leq n - 3$, belongs to S; then as x_0, x_i, x_1 and x_0, x_i, x_{n-1} are shortest paths we must have $x_1, x_{n-1} \notin S$, so that $|S| \leq n - 3$. Furthermore S cannot contain both vertices x_{n-2} and x_1 and likewise cannot contain both x_2 and x_{n-1}, so that if $S \cap \{x_3, \ldots, x_{n-3}\} = \emptyset$ we have $|S| \leq 2 < n - 3$. Thus $\mathrm{vp}(\overline{C}_n) = n - 3$ and we have $\mathrm{vp}(C_n) + \mathrm{vp}(\overline{C}_n) = n - 1$.

Applying this argument to a vertex of maximum degree Δ and a vertex with minimum degree δ gives the stronger bound $\mathrm{vp}(G) + \mathrm{vp}(\overline{G}) \geq n - 1 + \Delta - \delta$, so we see that we have equality in Equation 1 if and only if G is regular and both G and $\overline{G}$ have vertex position number equal to their maximum degree.

Trivially for $n \geq 2$ we have $\mathrm{vp}(G) \leq n$, with equality if and only if G has an isolated vertex. Not both of G and $\overline{G}$ can have an isolated vertex, for if G has an isolated vertex, then $\overline{G}$ contains a universal vertex. Therefore we do not have equality in both $\mathrm{vp}(G) \leq n$ and $\mathrm{vp}(\overline{G}) \leq n$, so it follows that $\mathrm{vp}(G) + \mathrm{vp}(\overline{G}) \leq 2n - 1$. Equality holds if and only if G contains an isolated vertex or a universal vertex. ∎

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3. Vertex position numbers of certain classes of graphs

In this section, we determine the x-position number of certain standard classes of graphs.

Lemma 3.1. *Let x be a vertex of a connected graph G and $S_x \subseteq V(G)$ be an x-position set of G. If $C_1, C_2, \ldots, C_k$ are the components of $G[S_x]$, then there exist $r_1, r_2, \ldots, r_k$ such that $d(x, y) = r_i$ for all $y \in C_i$.*

Proof. Suppose that there is a component C of $G[S_x]$ and $y, y' \in V(C)$ such that $d(x, y) \neq d(x, y')$. Then, considering a shortest path from y to y' in $G[C]$, we see that there is a pair $z, z' \in V(C)$ such that $z \sim z'$ and $d(x, z') = d(x, z) + 1$. However, this implies that an x, z-geodesic followed by the edge $z \sim z'$ is a shortest x, z'-path that passes through z, a contradiction. ∎

Theorem 3.2. *If G is a bipartite graph, then $\mathrm{vp}(G) \leq \alpha(G)$.*

Proof. Let S_x be an x-position set of G and suppose for a contradiction that S_x is not an independent set. Then there are $y, z \in S_x$ such that $y \sim z$ in G. By Lemma 3.1 we have $d(x, y) = d(x, z) = r$ for some $r \geq 1$. A shortest x, y-path, a shortest x, z-path and the edge $y \sim z$ together constitute an odd circuit, implying the existence of an odd cycle; since G is bipartite, this is impossible and it follows that $p_x(G) \leq \alpha(G)$. ∎

Theorem 3.3. *For $r \geq 2$, let $K_{n_1, n_2, \ldots, n_r}$ be the complete multipartite graph with partite sets $V_1, V_2, \ldots, V_r$, where $n_i = |V_i|$ and $n_1 \geq n_2 \geq \cdots \geq n_r$. Set $n = \sum_{i=1}^r n_i$. Then if the vertex x lies in V_i, the x-position number is given by*

$$p_x(K_{n_1, n_2, \ldots, n_r}) = \max\{n - n_i, n_i - 1\}.$$

Thus $\mathrm{vp}(K_{n_1, n_2, \ldots, n_r}) = n - n_r$.

Proof. Let $x \in V_i$ and let S be a maximum x-position set of the graph. Set $V = \bigcup_{i=1}^r V_i$. Suppose that S contains a vertex $y \in V_i \setminus \{x\}$; for any vertex $z \in V \setminus V_i$ the path x, z, y is a geodesic, so that in this case $S \cap (V \setminus V_i) = \emptyset$. Thus either $S \subseteq V \setminus V_i$ or $S \subseteq V_i \setminus \{x\}$. Conversely, both of these sets are x-position sets by the argument of Lemma 2.6, which yields the claimed bounds. ∎

Theorem 3.3 shows that equality holds in the bound of Theorem 3.2 for all complete bipartite graphs.

Lemma 3.4. *Let G be a connected graph of order n. Then for each $x \in V(G)$ there is a maximum x-position set without cut-vertices.*

Proof. Suppose that there is a maximum x-position set S containing a cut-vertex $v \neq x$ of G. Let $C_1, C_2, \ldots, C_k$ be the components of $G \setminus \{v\}$, where $k \geq 2$. Without loss of generality we may assume that $x \in V(C_1)$. Then it follows that $S \cap V(C_i) = \emptyset$ for all $i = 2, 3, \ldots, k$. Let u_i be any vertex in C_i for all $i = 2, 3, \ldots, k$. If $k \geq 3$, then the set $S' = (S \setminus \{v\}) \cup \{u_2, u_3, \ldots, u_k\}$ is an x-position set with order greater than S, a contradiction to the maximality of S. Hence $k = 2$. Let u be a farthest vertex from v in C_2. Then u is not a cut-vertex in G. Moreover, $S' = (S \setminus \{v\}) \cup \{u\}$ is a maximum x-position set containing fewer cut-vertices than S; this implies the existence of a maximum x-position set without cut-vertices in G. $\blacksquare$

Theorem 3.5. *For any connected block graph G,*

$$p_x(G) = \begin{cases} \text{s}(G) - 1, & \text{if } x \text{ is a simplicial vertex,} \\ \text{s}(G), & \text{otherwise.} \end{cases}$$

Proof. Let x be a vertex of G and S_x a maximum x-position set. By Lemma 3.4 we can assume that S_x contains no cut-vertices; every vertex of a block graph is either simplicial or a cut-vertex, so it follows that $p_x(G) \leq \text{s}(G)$ and, if x is itself simplicial, then $p_x(G) \leq \text{s}(G) - 1$. Conversely, by Proposition 2.12 the set $\text{Ext}(G)$ is an x-position set if $x \notin \text{Ext}(G)$ and $\text{Ext}(G) \setminus \{x\}$ is an x-position set if $x \in \text{Ext}(G)$, so that we have equality. $\blacksquare$

Corollary 3.6. *For any tree T with ℓ leaves we have*

$$p_x(T) = \begin{cases} \ell - 1, & \text{if } x \text{ is a leaf,} \\ \ell, & \text{otherwise.} \end{cases}$$

Corollary 3.6 implies the following bound for the vertex position numbers in terms of the girth of the graph.

Theorem 3.7. *If a graph G has girth g and minimum degree $\delta \geq 2$ and there are N vertices at distance at most $\left\lfloor \frac{g-1}{2} \right\rfloor - 1$ from a vertex u, then $p_u(G) \leq n - N$.*

Proof. Set $r = \left\lfloor \frac{g-1}{2} \right\rfloor$. Fix a vertex u of G and consider the subgraph G' induced by the vertices at distance at most r from u. G' is isomorphic to
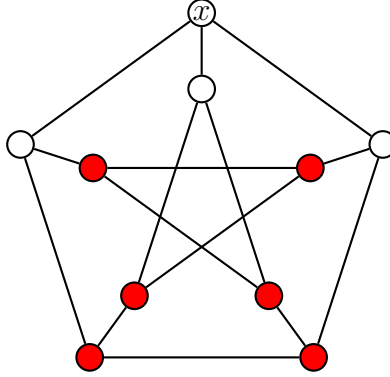
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Figure 3.1: A largest x-position set (red vertices) in the Petersen graph

a tree, possibly with some edges added between the vertices at distance r from u. It follows from Corollary 3.6 that the largest number of vertices from G' that can belong to a u-position set is the number of vertices at distance exactly r from u; hence there are at least N vertices missing from any u-position set and $p_u(G) \leq n - N$. ∎

Theorem 3.7 is tight for the Petersen and Hoffman-Singleton graphs by the argument in Lemma 2.6. Finally, we note that Lemma 2.5 gives the vertex position number of sufficiently large Kneser graphs. The Kneser graph $K(n,k)$ is the graph with vertex set equal to all k-subsets of $\{1, 2, \ldots, n\}$ with an edge between any two such subsets if and only if they are disjoint.

Theorem 3.8. *For sufficiently large n we have* $\mathrm{vp}(K(n,k)) = \binom{n-k}{k}$.

Proof. For $n \geq 3k$ the Kneser graph $K(n,k)$ has diameter two. Note that $K(n,k)$ is vertex-transitive, so $\mathrm{vp}(K(n,k)) = \mathrm{vp}^-(K(n,k))$ and we can without loss of generality consider the vertex $\{1, 2, \ldots, k\}$; let S be a largest $\{1, 2, \ldots, k\}$-position set. Lemma 2.5 gives $|S| = \mathrm{vp}(K(n,k)) \geq \binom{n-k}{k}$. Suppose that S contains a vertex at distance two from $\{1, 2, \ldots, k\}$, say $\{1, 2, \ldots, i, j_{i+1}, j_{i+2}, \ldots, j_k\}$, where

$$\{1, 2, \ldots, k\} \cap \{1, 2, \ldots, i, j_{i+1}, j_{i+2}, \ldots, j_k\} = \{1, 2, \ldots, i\}.$$

As $\{1, 2, \ldots, k\}$ and $\{1, 2, \ldots, i, j_{i+1}, j_{i+2}, \ldots, j_k\}$ have exactly $\binom{n-2k+i}{k}$ common neighbours, we would have $|S| \leq \binom{n}{k} - \binom{n-2k+i}{k}$, which is a polynomial in n of degree $k - 1$, whereas the vertex degree is a polynomial function of n with degree k. Thus for sufficiently large n compared to k the bound in Lemma 2.5 is best possible. ∎

Interestingly Theorem 3.8 implies that for large n the vertex position number of $K(n, k)$ is significantly larger than the general position number, as given in [11]; hence, since the Kneser graphs are vertex-transitive, $\text{vp}^-(K(n, k))$ is larger than $\text{gp}(K(n, k))$ for sufficiently large n. For small orders Lemma 2.5 is not optimal; for example, as previously noted for the Petersen graph P (isomorphic to $K(5, 2)$) the second neighbourhood of a vertex is a largest vertex position set and $\text{vp}(P) = \text{gp}(P) = 6$.

4. Characterisation results

We now make use of the bounds derived in Section 2 to characterise graphs with very large or very small vertex position numbers.

Corollary 4.1. *A connected graph G with order n satisfies $\text{vp}(G) = n - 1$ if and only if G contains a universal vertex, whilst $\text{vp}^-(G) = n - 1$ if and only if G is a complete graph.*

Proof. By Lemma 2.5 any universal vertex u has vertex position number $p_u(G) = n - 1$. Conversely, by Lemma 2.10 any vertex u with $p_u(G) = n - 1$ has eccentricity one and hence is universal. If $\text{vp}^-(G) = n - 1$, it follows that G is complete. ∎

Corollary 4.2. *A connected graph G satisfies $\text{vp}^-(G) = \text{vp}(G) = 2$ if and only if G is a cycle. Similarly $\text{vp}(G) = 2$ only for cycles and paths of length ≥ 2.*

Proof. Let C_n be a cycle for some $n \geq 3$ with vertex set identified with $\mathbb{Z}_n$. As C_n is 2-regular, by Lemma 2.5 we have $\text{vp}^-(C_n) \geq 2$. We now show that $\text{vp}(C_n) \leq 2$. If n is odd, then this follows immediately by Theorem 3.7, so suppose that n is even. The x-vertex position number of any graph G is bounded above by the number of geodesics with x as initial vertex needed to cover all vertices of $V(G)$, since each of these paths can contain at most one vertex of any x-position set. (essentially this is Theorem 3.3 in [15]). Taking $x = 0$ in C_n, the shortest paths $0, 1, \ldots, \frac{n}{2}$ and $0, n - 1, n - 2, \ldots, \frac{n}{2} + 1$ cover $V(C_n)$, so that $\text{vp}(C_n) = p_0(C_n) \leq 2$.

Conversely, suppose that $\text{vp}(G) = 2$; it follows from Lemma 2.5 that G has maximum degree $\Delta = 2$, so that G is either a path or a cycle. As $\text{vp}^-(P_n) = 1$ for paths by Corollary 2.7, it follows that if also $\text{vp}^-(G) = 2$, then G is a cycle. ∎

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Now we characterise some graphs with very large vertex position number.

Lemma 4.3. *A vertex u of a connected graph G with order n has $p_u(G) = n-2$ if and only if either i) u has degree $\text{degree}(u) = n-2$ or ii) $\text{degree}(u) \leq n-3$ and there is a $v \in N(u)$ such that for any vertex $w \notin N[u]$ the vertex v is the unique common neighbour of u and w.*

Proof. Let G be a graph with order n and let $u \in V(G)$ satisfy $p_u(G) = n-2$, with S_u a largest u-position set. By Lemma 2.5 the vertex u is not universal, so by Lemma 2.10 the eccentricity of u is two. Suppose that $\text{degree}(u) \leq n-3$ and $|N^2(u)| \geq 2$.

Let x, y be any vertices in $N^2(u)$. Suppose that one of these vertices, say x, has at least two common neighbours with u. If $x \in S_u$, then no vertex of $N(u) \cap N(x)$ can belong to S_u; as $u \notin S_u$, this accounts for at least three vertices missing from S_u. Thus $S_u = V(G) \setminus \{u, x\}$, so that $\{y\} \cup N(u) \subseteq S_u$; however, as the shortest u, y-paths pass through $N(u)$ this is impossible. Therefore every vertex in $N^2(u)$ has just one neighbour in $N(u)$. Similarly, suppose that $N(u) \cap N(x) = \{v\}$ and $N(u) \cap N(y) = \{v'\}$, where $v \neq v'$. Then $|S_u \cap \{v, x\}| \leq 1$ and $|S_u \cap \{v', y\}| \leq 1$, which, together with u, again accounts for at least three vertices missing from S_u. Thus we must have $v = v'$ and there is a vertex $v \in N(u)$ that is the unique neighbour in $N(u)$ of every vertex in $N^2(u)$.

Conversely, it is evident that any vertex u satisfying either condition i) or ii) has $p_u(G) = n-2$. In Case ii) our argument shows that $V(G) \setminus \{u, v\}$ is the unique maximum u-position set. In Case i) if the unique vertex $w \notin N[u]$ has degree ≥ 2, then $V(G) \setminus \{u, w\}$ is the unique maximum u-position set, whereas if w is a leaf with support vertex w', then $V(G) \setminus \{u, w\}$ and $V(G) \setminus \{u, w'\}$ are the two maximum u-position sets. ∎

Theorem 4.4. *For $n \geq 4$, a graph G with order n satisfies $\text{vp}^-(G) = \text{vp}(G) = n - 2$ if and only if G is isomorphic to an even clique with a perfect matching deleted, i.e. if and only if $G \cong K_{2,2,\dots,2}$.*

Proof. Assume that G is a graph such that $p_u(G) = n - 2$ for every $u \in V(G)$. Suppose that G contains a vertex u with degree $\leq n - 3$, so that by Lemma 4.3 the eccentricity of u is two and u has a neighbour v such that for any vertex $w \notin N[u]$ the vertex v is the unique common neighbour of u and w. As $\text{vp}(G) = n - 2$, G contains no universal vertex, so that there is a neighbour z of u such that $v \not\sim z$. Hence if $x \in N^2(u)$ we have

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$d(x, z) \geq 3$ and x has eccentricity at least three, so that by Lemma 2.10 we have $p_x(G) \leq n - 3$, a contradiction. Therefore every vertex of G has degree $n - 2$ and G is isomorphic to a K_{2r} with a perfect matching deleted. Conversely, by Lemma 4.3 in such a graph every vertex has vertex position number $n - 2$. ∎

Theorem 4.5. *For $n \geq 4$, a graph G has $\mathrm{vp}^-(G) = n - 2$ and $\mathrm{vp}(G) = n - 1$ if and only if i) G is isomorphic to a clique with a non-empty, non-perfect matching deleted or ii) G is the join of K_1 with a disjoint union of cliques.*

Proof. Let G be a graph with $\mathrm{vp}^-(G) = n - 2$ and $\mathrm{vp}(G) = n - 1$. We can assume that G contains $r \geq 1$ universal vertices as well as at least two vertices with degree $\leq n - 2$. If every vertex has degree either $n - 1$ or $n - 2$, then G is isomorphic to a clique with a matching deleted. To avoid the graph having $\mathrm{vp}^-(G) = \mathrm{vp}(G) = n - 1$ the matching must be non-empty and to avoid having $\mathrm{vp}^-(G) = \mathrm{vp}(G) = n - 2$ the matching is not perfect by Theorem 4.4.

Suppose that G contains a vertex u with $p_u(G) = n - 2$ and degree $\mathrm{degree}(u) \leq n - 3$; then u has a neighbour v satisfying condition ii) in Lemma 4.3. By Lemma 2.10 every vertex of G has eccentricity at most two, so, considering the vertices in $N^2(u)$, we see that v is a universal vertex; in fact, as v is a cut-vertex, it is the unique universal vertex of G, so that every other vertex w of G must have $p_w(G) = n - 2$. Consider the components $C_1, C_2, \ldots, C_t$ of $G \setminus v$, where $t \geq 2$. Suppose that there is a component C_i that is not a clique and choose vertices $z, z' \in C_i$ such that $z \nsim z'$. Then z has degree at most $n - 3$ and must also satisfy condition ii) of Lemma 4.3. Then as $d(z, z') \geq 2$ condition ii) of Lemma 4.3 implies that every z, z'-path passes through v; however, this contradicts the fact that z and z' lie in the same component of $G \setminus v$. Therefore G is the join of K_1 with a disjoint union of cliques. Conversely, any such join has the required parameters. ∎

5. Computational complexity

In this section we show that, for a given graph G, the vertex position number $p_x(G)$ can be computed in polynomial time for any vertex $x \in V(G)$. In particular, we will show that $p_x(G)$, for each $x \in V(G)$, can be computed as an independent set calculated on a graph obtained as a transformation of G. To this aim we need the following definitions.
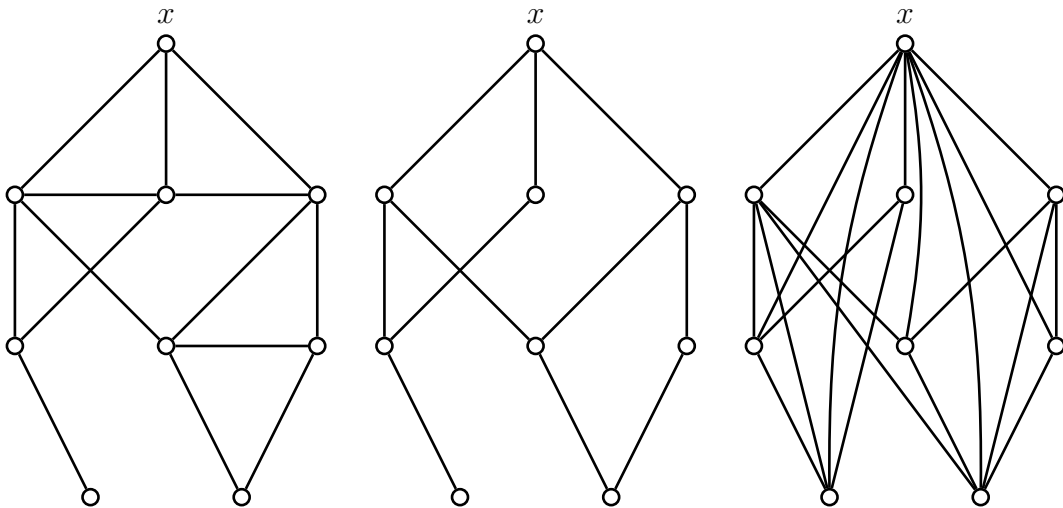
Figure 5.1: From left to right: a graph G with a vertex x, the graph $\widetilde{G}_x$, and the graph G_x^*. The graphs G and $\widetilde{G}_x$ are drawn by placing the vertices at the same distance from x on a common horizontal level.

Definition 5.1. *A graph is a* comparability graph *if the edges connect pairs of elements that are comparable to each other in a partial order.*

Definition 5.2. *Given a graph G and a vertex $x \in V(G)$, the* reduced *graph $\widetilde{G}_x$ is the graph on the same vertex set $V(G)$ obtained from G by removing all the edges connecting vertices at the same distance from x.*

Definition 5.3. *Given a graph G and a vertex $x \in V(G)$, the graph G_x^* is the graph on the same vertex set $V(G)$ obtained from the reduced graph $\widetilde{G}_x$ by adding an edge between any two vertices of any geodesic to x.*

See Figure 5.1 for a visualisation of $\widetilde{G}_x$ and G_x^*, starting from a graph G and a vertex x.

Lemma 5.4. *Given a graph G and a vertex $x \in V(G)$, G_x^* is a comparability graph.*

Proof. We define a partial order $<$ on $V(G_x^*)$ by setting $u < v$ for $u, v \in V(G_x^*)$ if and only if u and v are on the same geodesic to x and $d(u, x) < d(v, x)$. Since G_x^* has been built from $\widetilde{G}_x$ by adding an edge between every pair of vertices on each geodesic to x, it follows that any two comparable vertices are adjacent in G_x^* and hence G_x^* is a comparability graph. ∎

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Lemma 5.5. *Given a graph G and a vertex $x \in V(G)$, S_x is an x-position set of G if and only if S_x is an x-position set for $\widetilde{G}_x$. Hence $p_x(G) = p_x(\widetilde{G}_x)$.*

Proof. Note that any geodesic to x in G is also a geodesic to x in $\widetilde{G}_x$. The set S_x is a x-position set for G if and only if no two vertices of S_x lie on a common geodesic to x in G; as the geodesics to x in G coincide with the geodesics to x in $\widetilde{G}_x$, it follows that S_x is an x-position set for G if and only if it is an x-position set for $\widetilde{G}_x$. Consequently, any maximum vertex position set of G_x is a maximum vertex position set for $\widetilde{G}_x$ and $p_x(G) = p_x(\widetilde{G}_x)$. ■

Given a graph G, let us denote the graph induced by vertices in $V(G)\backslash\{x\}$ by $G - x$.

Lemma 5.6. *Given a graph G and a vertex $x \in V(G)$, S_x is an x-position set of G if and only if S_x is an independent set of $G_x^* - x$. Hence $p_x(G) = \alpha(G_x^* - x)$.*

Proof. Let S_x be an x-position set in G. By Lemma 5.5, S_x is also an x-position set in $\widetilde{G}_x$. Assume that S_x is not an independent set in $G_x^* - x$. Then there are two adjacent vertices $u, v \in V(G_x^* - x) \cap S_x$. By definition of G_x^* and Lemma 5.5, u and v would lie on the same geodesic to x in G, a contradiction.

For the converse, assume that S_x is an independent set of $G_x^* - x$. Let $u \in S_x$ and let P be any $u - x$ geodesic in G. Then by the construction of G_x^* the vertex u is adjacent to all the vertices of $V(P - u)$ in G_x^*. This immediately shows that $V(P) \cap S_x = \{u\}$. Consequently, S_x is an x-position set in G. Therefore $p_x(G) = \alpha(G_x^* - x)$. ■

Theorem 5.7. *Given a graph G and a vertex $x \in V(G)$, a maximum x-position set can be computed in $O(nm \log(n^2/m))$ time, where $n = |V(G_x^*)|$ and $m = |E(G_x^*)|$.*

Proof. Algorithm A in Figure 5.2 computes the distances of each vertex $v \in V(G)$ from x at Line 1. This requires $O(n + m)$ time. With the loop at Line 2, $\widetilde{G}_x$ is computed from G by removing edges between vertices at the same distance from x. This requires $O(m)$ time. The loop at Line 5 add edges to the graph in order to build G_x^*. This requires $O(n^2 + nm)$ time, since Lines from 6 to 14 codify for a breadth-first visit of the vertices on a geodesic to x passing through a vertex u. This visit, based on a queue

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Algorithm A:
Input: A connected graph G, a vertex $x \in V(G)$
Output: A maximum x-position set S_x, and $p_x(G)$

1 Let $D[u] := d(u, x)$, for each $u \in V(G)$;
2 for *each $uv \in E(G)$* **do**
3   **if** $D[u] = D[v]$ **then**
4   remove uv from $E(G)$;

5 for *each $u \in V(G - x)$* **do**
6   Let Q be a queue and $R := \{u\}$;
7 $Q.enqueue(u)$;
8   **while** Q *is not empty* **do**
9 $v := Q.dequeue()$;
10   **for** *each w in $N_G(v) \setminus R$ such that $D[w] > D[v]$* **do**
11 $Q.enqueue(w)$;
12 $R := R \cup \{w\}$;
13   **if** $D[w] > D[u] + 1$ **then**
14   add uv to $E(G)$;

15 Let S be a maximum independent set of $G - x$;
16 return S, $|S|$

Figure 5.2: Algorithm A to compute a maximum x-position set S_x of a graph G and $p_x(G)$, for a given $x \in V(G)$.

Q and a set R of the visited vertices, requires $O(n + m)$ time and since it is repeated for each vertex u in $V(G - x)$, the total time is $O(n^2 + nm)$. Finally, at Line 15 an independent set S of the resulting comparability graph $G^* - x$ is computed. According to [12], the computation of an independent set for a comparability graph requires $O(nm \log(n^2/m))$ time; the last step determines the computational time of the whole algorithm. By Lemma 5.6, the set S is also an x-position set of G, so that Algorithm A correctly returns S and its order. ∎

Corollary 5.8. *Given a connected graph G, $\mathrm{vp}^-(G)$ and $\mathrm{vp}(G)$ can be computed in $O(n^4 \log(n))$ time, where $n = |V(G)|$.*

Proof. Given a graph G, by calling Algorithm A for each vertex x of G,

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vp$^-(G)$ and vp(G) can be easily computed. By Theorem 5.7 each call requires $O(nm \log(n^2/m))$ steps, where $n = |V(G_x^*)|$ and $m = |E(G_x^*)|$. Considering that $n = |V(G_x^*)| = |V(G)|$ and $n - 1 \leq m \leq \binom{n}{2}$, each call requires at most $O(n^3 \log(n))$ time, yielding a total of $O(n^4 \log(n))$ time. ∎

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