Forced vibration analysis of multi-degree-of-freedom nonlinear systems with an extended Galerkin method

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In this study, the dynamic response behaviour of a generalised nonlinear dynamic system is investigated using a newly proposed extended Galerkin method. The algebraic equations of vibration amplitudes are obtained through an integration of the weighted functions. The new method is equivalent to the harmonic balance method but with a much simpler calculation procedure and a higher efficiency. This is the first time to use the method for the analysis of nonlinear systems with high number of modes, manifesting that the method is applicable to forced vibrations of nonlinear behaviour. The method is further validated by the numerical Runge-Kutta method.

Keywords: Galerkin method; nonlinear vibration; multi-degree of freedom; Duffing oscillator; numerical continuation

1. Introduction

The Galerkin method based on the forced diminish of weighted residuals is a widely accepted technique to analyze the natural frequency, response behaviour, and structural vibration of complex dynamical systems[1-3]. The method has been widely used for free vibrations as a powerful and simple tool for approximate analysis of vibration frequency and mode shapes of continuous systems and structures. Furthermore, the method is usually demonstrated and applied to linear and nonlinear systems as can be found in textbooks and literatures[4-8]. This method mainly requires the integral of weighted residuals of basis functions of popular types such as trigonometric functions and polynomials over the physical domain to be zero. Further extension of Galerkin method is known as the foundation of the popular and powerful finite element method for numerical solutions of differential equations widely encountered in engineering and scientific problems these days. Undoubtedly, any further development of Galerkin method will also have impact on the future functions and algorithms of finite element software.

The use of the Galerkin method became popular since the early 1980s, it was found that the Galerkin method can be extended to nonlinear vibrations through the integration of harmonic terms in one period of the fundamental frequency. Lau et al.[9, 10] proposed the incremental harmonic balance method (IHBM) to investigate the nonlinear vibration of elastic systems including plates, shallow shells and columns. Kim and Perkins[11] used the Galerkin method for non-smooth dynamic systems, e.g., systems with friction, impact, clearances. Zhang et al.[12] investigated the vibration suppression performance of an elastic beam with inerter-based nonlinear energy sink. Recently, some studies[13-15] have been reported to investigate the vibration transmission characteristics in nonlinear smooth/non-smooth systems using HBM with alternating frequency time (AFT). All these methods can be used to obtain the approximate analytical results using different approximation orders and with different computational efficiency. In a recent study[16] searching for innovative techniques solving nonlinear vibration problems, it was found that the extension of Galerkin method has equally effective and accurate results compared with others approaches, but it was more efficient in the derivation and calculation. Actually, the equivalence of the extended Galerkin method (EGM) and other popular methods such as the HBM is known, and a new procedure is established and tested based on this fact. As a result, the new procedure can be adopted for vibration analyses of both linear and nonlinear systems as a unified approach to take the advantage of the popular method. However, previous studies are mainly focused on single or two-DOF nonlinear systems, and there is little research on the application of extended Galerkin method in dynamical systems with higher number of modes. Some work on multiple DOFs has been carried out on 3-DOF nonlinear vibration systems with modal interactions[17] and nonlinear energy sink[18].

In this study, the recently extended Galerkin method is developed to investigate the dynamic response of a generalised multi-degree of freedom (DOF) nonlinear dynamical system as a new solution technique. Such problems have wide engineering applications in machinery and structures with different solution techniques including propulsion shafting system[19, 20], spur gear system[21], robotics[22] and multi-storey building[23]. The nonlinearity of the system is characterized by a nonlinear spring with cubic stiffness coefficient. Three case studies are presented, including a single-DOF oscillator, a coupled 2-DOF system and a 3-DOF system, demonstrating the application of the method to forced vibration problems. The first-order, second-order, and third-order analytical results are obtained with the extended Galerkin method and compared with the numerical Runge-Kutta (RK) method with satisfaction.

2. Modelling and formulations

The validation of the extended Galerkin method starts with a typical nonlinear system at forced vibrations shown in Figure 1 with a schematic representation of a coupled Q-degree-of-freedom (Q-DOF) nonlinear system. The first subsystem has mass m_1 subjected to an external harmonic force $f\cos\omega t$, where f is the excitation amplitude and ω is the excitation frequency. A Duffing-type stiffness nonlinearity with nonlinear stiffness coefficient k_{nl} exists in the first subsystem so that the nonlinear restoring force has a cubic relationship with the displacement of m_1. Other stiffness and damping coefficients are all linear, e.g., the j-th oscillatory mass m_j links to linear springs with coefficients k_j and k_{j+1} and also connects linear viscous dampers with damping coefficients c_j and c_{j+1} ($1 \leq j \leq Q$). The displacements of each oscillator are denoted as $x_1, \ldots, x_j, \ldots, x_Q$. For the static equilibrium condition and the springs are un-deformed,

$x_1 = x_j = x_Q = 0$. The oscillators move horizontally without friction. It should also be noted that all parameters are dimensionless.

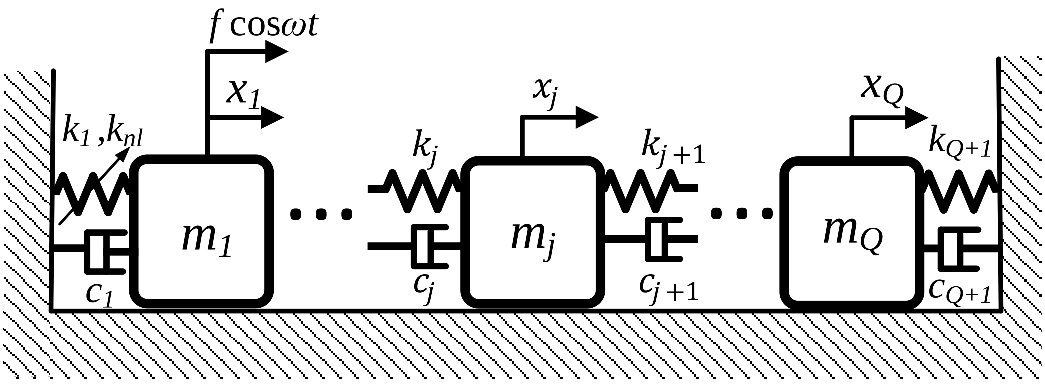

Figure 1. A schematic representation of a generalized Q-DOF vibration system

The governing equation of the j-th DOF mass in Figure 1 can be written as

$$m_j\ddot{x}_j + (x_j - x_{j-1})k_j + (\dot{x}_j - \dot{x}_{j-1})c_j - (x_{j+1} - x_j)k_{j+1} - (\dot{x}_{j+1} - \dot{x}_j)c_{j+1} + f_{j,\mathrm{nl}} = f_{j,\mathrm{ex}}, \tag{1}$$

where x_j, $\dot{x}_j$ and $\ddot{x}_j$ are the displacement, velocity and acceleration of the mass at the j-th coordinate, and $x_0 = \dot{x}_0 = x_{Q+1} = \dot{x}_{Q+1} = 0$; $f_{j,\mathrm{nl}}$ and $f_{j,\mathrm{ex}}$ are the nonlinear term and external force applying to the j-th DOF mass, and they are equal to zero if $j > 1$, i.e., only consider the nonlinearity and external force factors applying to the first coordinate. Such a problem has been studied before using many methods including the harmonic balance method for analytical approximations and direct numerical integrations by the Runge-Kutta method[24].

The total response of Eq. (1) is the sum of the transient solution and the steady-state solution. The former represents the natural response (complementary function) that approaches to zero as time goes to infinity. The latter is the forced response, also known as the particular integral. In engineering applications, for a dynamical system subjected to a harmonic excitation force, the periodic steady-state solution is normally of interest[4,

[25]. Therefore, the general solutions in the steady state of Eq. (1) can be expressed by N-th order truncated Fourier series:

$$x_j = \sum_{n=1}^{N} \left(x_{j,2n-1} \cos n\omega t + x_{j,2n} \sin n\omega t \right), \tag{2}$$

and consequently

$$\dot{x}_j = \omega \sum_{n=1}^{N} \left(-n x_{j,2n-1} \sin n\omega t + n x_{j,2n} \cos n\omega t \right), \tag{3}$$

$$\ddot{x}_j = -\omega^2 \sum_{n=1}^{N} \left(n^2 x_{j,2n-1} \cos n\omega t + n^2 x_{j,2n} \sin n\omega t \right). \tag{4}$$

where $n = 1, 2, \ldots, N$, $x_{j,2n-1}$ and $x_{j,2n}$ are the Fourier coefficients of the cosine and sine terms, respectively. Therefore, the nonlinear terms and external forces can be written as

$$f_{j,\mathrm{nl}} = \begin{cases} k_{\mathrm{nl}} x_1^3 = k_{\mathrm{nl}} \sum_{n=1}^{N} (q_{2n-1} \cos n\omega t + q_{2n} \sin n\omega t), \text{when } j = 1 \\ 0, \quad \text{when } j > 1 \end{cases}, \tag{5}$$

$$f_{j,\mathrm{ex}} = \begin{cases} f \cos \omega t, \text{when } j = 1 \\ 0, \text{when } j > 1 \end{cases}, \tag{6}$$

where the coefficients q_{2n-1} and q_{2n} are calculated by

$$q_{2n-1} = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x_1^3 \cos n\omega t \,\mathrm{d}t, \; q_{2n} = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x_1^3 \sin n\omega t \,\mathrm{d}t. \tag{7}$$

In this study, the first-, second- and third-order Fourier coefficients are obtained, which refer to the three frequency components $\omega, 3\omega$ and 5ω. Details of the calculations results are provided in the Appendix A. By substituting all related terms in Eq. (1) using Eqs (2)-(6), we obtain

$$m_j \omega^2 \sum_{n=1}^{N} \left(-n^2 x_{j,2n-1} \cos n\omega t - n^2 x_{j,2n} \sin n\omega t \right) + \Big(\sum_{n=1}^{N} \big(x_{j,2n-1} \cos n\omega t + x_{j,2n} \sin n\omega t \big) - \sum_{n=1}^{N} \big(x_{j-1,2n-1} \cos n\omega t + x_{j-1,2n} \sin n\omega t \big) \Big) k_j + \Big(\omega \sum_{n=1}^{N} \big(-n x_{j,2n-1} \sin n\omega t + n x_{j,2n} \cos n\omega t \big) - \omega \sum_{n=1}^{N} \big(-n x_{j-1,2n-1} \sin n\omega t + n x_{j-1,2n} \cos n\omega t \big) \Big) c_j - \Big(\sum_{n=1}^{N} \big(x_{j+1,2n-1} \cos n\omega t + x_{j+1,2n} \sin n\omega t \big) - \sum_{n=1}^{N} \big(x_{j,2n-1} \cos n\omega t + x_{j,2n} \sin n\omega t \big) \Big) k_{j+1} - \Big(\omega \sum_{n=1}^{N} \big(-n x_{j+1,2n-1} \sin n\omega t +$$

$$nx_{j+1,2n}\cos n\omega t\big) - \omega\sum_{n=1}^{N}\big(-nx_{j,2n-1}\sin n\omega t + nx_{j,2n}\cos n\omega t\big)\big)c_{j+1} + f_{j,\mathrm{nl}} - f_{j,\mathrm{ex}} = M_j, \tag{8}$$

where M_j is the solution error or residual at the j-th coordinate. Following the general procedure of the extended Galerkin method[16], it requires that the residual be orthogonal to each expansion function over one excitation period, i.e., $2\pi/\omega$. Therefore, the orthogonality condition of the residual function yields

$$\int_0^{2\pi/\omega} M_j \cos n\omega t\,\mathrm{d}t = 0\,, \tag{9}$$

$$\int_0^{2\pi/\omega} M_j \sin n\omega t\,\mathrm{d}t = 0, \tag{10}$$

The integration operations of Eqs. (9) and (10) lead to

$$-m_j\omega^2 n^2 x_{j,2n-1} + \big(x_{j,2n-1} - x_{j-1,2n-1}\big)k_j + \big(x_{j,2n} - x_{j-1,2n}\big)\omega n c_j - \big(x_{j+1,2n-1} - x_{j,2n-1}\big)k_{j+1} - \big(x_{j+1,2n} - x_{j,2n}\big)\omega n c_{j+1} + F_{j,\mathrm{nl\_c}} - F_{j,\mathrm{ex}} = 0\,, \tag{11}$$

$$-m_j\omega^2 n^2 x_{j,2n} + \big(x_{j,2n} - x_{j-1,2n}\big)k_j + \big(-x_{j,2n-1} + x_{j-1,2n-1}\big)\omega n c_j - \big(x_{j+1,2n} - x_{j,2n}\big)k_{j+1} - \big(-x_{j+1,2n-1} + x_{j,2n-1}\big)\omega n c_{j+1} + F_{j,\mathrm{nl\_s}} = 0\,, \tag{12}$$

where $F_{j,\mathrm{nl\_c}}$ and $F_{j,\mathrm{nl\_s}}$ are the nonlinear terms about the cosine and sine functions and $F_{j,\mathrm{ex}}$ is the external force term after the integration, and they can be expressed as

$$F_{j,\mathrm{nl\_c}} = \begin{cases} k_{\mathrm{nl}} q_{2n-1}, \text{when } j = 1 \\ 0, \text{when } j > 1 \end{cases}, F_{j,\mathrm{nl\_s}} = \begin{cases} k_{\mathrm{nl}} q_{2n}, \text{when } j = 1 \\ 0, \text{when } j > 1 \end{cases}, F_{j,\mathrm{ex}} = \begin{cases} f, \text{when } j = 1 \\ 0, \text{when } j > 1 \end{cases}. \tag{13}$$

Note that Eqs. (11) and (12) are coupled nonlinear algebraic equations for the mass at the j-th coordinate with the truncated order n, where $1 \le j \le Q$ and $1 \le n \le N$. For a generalized Q-DOF dynamical system with an N-th order approximation, the total equation numbers are $2QN$. These nonlinear algebraic equations can be solved by the Newton-Raphson method based on a numerical continuation scheme[26, 27].

3. Results and discussion

In this section, three case studies using the extended Galerkin method (EGM) are carried out, including the Duffing oscillator, a 2-DOF coupled system and a 3-DOF system. The first-, second- and third-order analytical results of each case are obtained and compared with the numerical Runge-Kutta method. It demonstrates the proposed method can be applied to multi-degree-of-freedom nonlinear vibrational systems with high accuracy and efficiency.

3.1 Case study 1-the Duffing oscillator ($Q = 1$)

A single-DOF oscillator is firstly considered with no constraints on the right. Therefore, the system is simplified to the Duffing oscillator[28], and the corresponding equation of motion is given in Eq. (B.1) in the Appendix B. The third-order (considering three frequency components $\omega, 3\omega,$ and 5ω) and the first-order approximations based on the EGM are used to obtain the steady-state dynamic response of the Duffing oscillator, see details in the Appendix B. For cross-verification and comparison, numerical results are also obtained from the fourth-order Runge-Kutta (RK) method. Non-dimensional system parameters are set as $m_1 = 1, c_1 = 0.02, k_1 = 1, k_{nl} = 0.1, f = 0.5$.

Figure 2 shows the frequency-response curve of the single-DOF oscillator obtained by three different approaches. The dashed and solid lines denote the first- and third-order approximations, respectively. The symbols are the numerical results based on the fourth-order RK method with variable time step. Due to the hardening stiffness nonlinearity, the response amplitude curve bends to the high-frequency range causing the multi-solution behaviour and jump phenomenon. In a wide range of the excitation frequency, response amplitude curves of each method merge in Fig. 2, indicating that the results obtained by the proposed analytical method have a good agreement with the numerical solution. In the range of $0.32 < \omega < 0.36$, the steady-state dynamic response of the third-order approximation is firstly lower and then higher than the first-order results, while the numerical results and the third-order approximation are well-matched. The reason for the discrepancies is that the super-harmonic occurs in this region, which introduces an additional frequency component 3ω in dynamic response. Therefore, the first-order approximation is insufficient to obtain the accurate response characteristics in this area and high-order approximation is required.

Figure 3 presents the frequency spectra and time history information in the super-harmonic region at a prescribed excitation frequency $\omega = 0.3414$. Fig. 3(a) shows that there are two frequency components ω and 3ω in the displacement motion, and the response is dominated by the fundamental excitation frequency $\omega_r = \omega$. In Fig. 3(b), the time histories of the three methods are presented. The samples are selected from time $t = 800T$ to $t = 804T$ with four periodic cycles, where T is one excitation period and $T = 2\pi/\omega$. Fig. 3(b) shows that the periodic time histories of the numerical and the third-order approximations are almost identical, which are slightly higher than the first-order results. It should be pointed out that the second-order approximations (considering two frequency components ω and 3ω) would also match well with the numerical results. The main reason is that there is no frequency component of $\omega_r = 5\omega$ and the corresponding Fourier coefficients $x_{1,9}$ and $x_{1,10}$ are omitted.

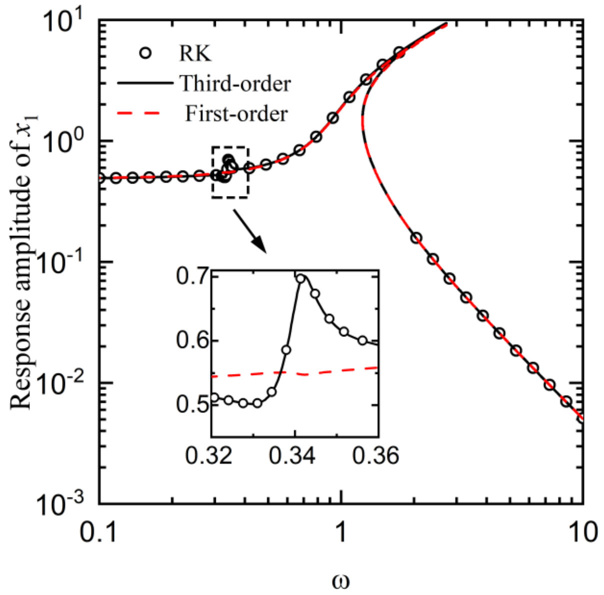

Figure 2. Frequency-response curve of a single-DOF nonlinear system with different methods: Runge-Kutta (symbols), third-order EGM (solid line) and first-order EGM (dashed line).

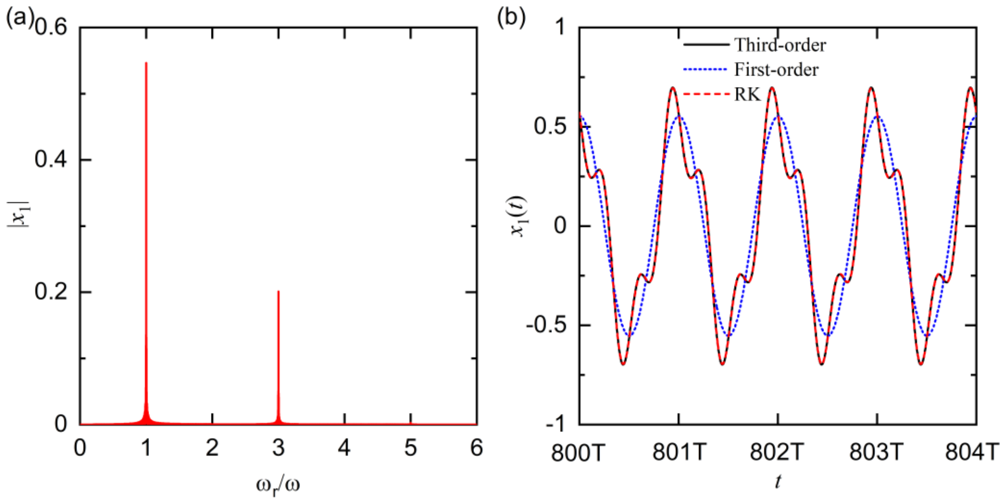

Figure 3. Dynamic response of the single-DOF system when super-harmonic occurs at the excitation frequency $\omega = 0.3414$ of (a) frequency spectra and (b) time history.

3.2 Case study 2-a two-DOF coupled system ($Q = 2$)

In this section, a 2-DOF (i.e., $Q = 2$) nonlinear vibration system is considered[29,30]. Thus, the generalised equation of motion can be transformed into Eqs. (C.1) and (C.2) in the Appendix C. The third-order approximations with frequency components ω, 3ω and 5ω are used. The first- and second-order approximations as well as the numerical integration results are also added for comparison, see more details in the Appendix C.

Dimensionless system parameters are set as: $m_1 = m_2 = 1, c_1 = c_2 = c_3 = 0.02, k_1 = k_2 = k_3 = 1, k_{nl} = 1, f = 1$.

Figure 4 shows the steady-state response amplitudes of masses one and two, respectively. The third-, second- and first-order results of EGM are denoted by solid, dashed and dotted lines, respectively. Due to the hardening stiffness, both the primary and the secondary resonant peaks twist to the high-frequency range. In the frequency range of 0.2 to 0.7, there are three super-harmonic resonant peaks, which are located at $\omega = 0.221$, 0.384 and 0.618, respectively, resulting in the differences between the methods. In the region around the second ($\omega = 0.384$) and the third ($\omega = 0.618$) super-harmonic peaks in Figure 4, the results obtained by the second-order approximations have a good agreement with the numerical results, but they are inconsistent near the first ($\omega = 0.221$) super-harmonic peak. In comparison, the third-order approximations are still compatible with the numerical RK results. It suggests that there may exists high-order frequency component, e.g., $\omega_r = 5\omega$. In the high-frequency range, lines of each approach merge, and the response amplitude decreases as the excitation frequency increases.

Figures 5(a) and (b) explores the frequency spectra and the time history behaviour of the displacement x_1 around the first super-harmonic peak at the excitation frequency $\omega = 0.221$. Figure 5(a) shows that there are two super-harmonic response components at $\omega_r = 3\omega$ and $\omega_r = 5\omega$ as well as a primary one at $\omega_r = \omega$. It is also noted that the response amplitude has a much lower level at the frequency components of $\omega_r = 3\omega$ compared with that of the fundamental frequency and $\omega_r = 5\omega$. Therefore, the effects of the frequency component $\omega_r = 3\omega$ on the dynamic response are negligible, and the dynamic response of x_1 is dominated by the fundamental frequency $\omega_r = \omega$ in this region. This is the reason for the sinusoidal movement of the second-order results in Figure 5(b), as denoted by the dotted line. Figure 5(b) also shows the steady-state displacement motions within four periodic cycles obtained by different methods. It demonstrates that the proposed method with the third-order approximation still has a good agreement with the numerical RK results in the 2-DOF coupled system.

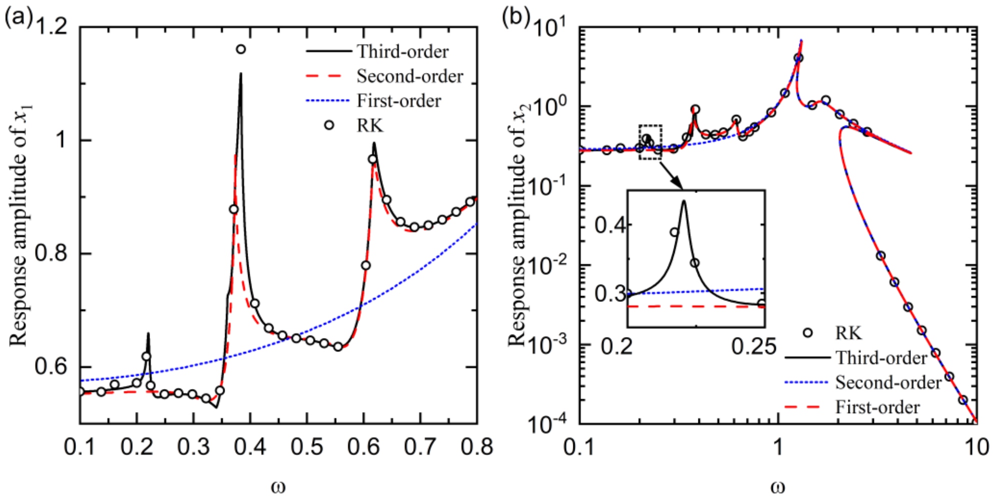

Figure 4. Frequency-response curve of (a) mass one and (b) mass two of the 2-DOF vibration system. Symbols: RK. Solid lines: third-order EGM. Dashed lines: second-order EGM. Dotted lines: first-order EGM.

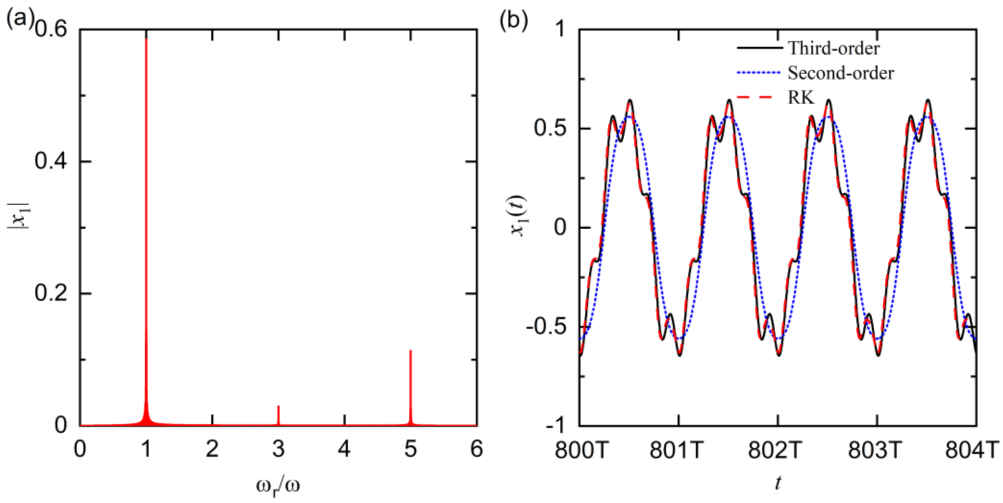

Figure 5. Dynamic response of the two-DOF system when super-harmonic occurs at the excitation frequency $\omega = 0.221$ of (a) frequency spectra and (b) time history.

3.3 Case study 3-a three-DOF coupled system ($Q = 3$)

In this section, a 3-DOF vibration system is considered[17], and the corresponding governing equations and the analytical expressions of steady-state solutions with different orders of approximations are presented in the Appendix D. Dimensionless system parameters are set as: $m_1 = m_2 = m_3 = 1, c_1 = c_2 = c_3 = c_4 = 0.02, k_1 = k_2 = k_3 = k_4 = 1, k_{nl} = 0.5, f = 1$.

Figure 6(a) and (b) shows the response amplitude of x_1 and x_3, respectively. Since it is a three-degree of freedom nonlinear dynamical system, there are at least three resonant peaks in the frequency-response curve, and all of which bend to the high-frequency range. In the excitation frequency range of 0.1 to 0.7, five different super-harmonic peaks are distinguished using the third-order approximation using EGM (i.e., solid lines) and numerical RK method (i.e., circle symbols), which are located at $\omega = 0.162$, 0.271, 0.298, 0.504 and 0.642, respectively. In comparison, the results obtained by the second-order approximation (i.e., dashed lines) only show the second ($\omega = 0.271$), the fourth ($\omega = 0.504$) and the fifth ($\omega = 0.642$) super-harmonic peaks. It indicates that the first ($\omega = 0.162$) and the third peaks ($\omega = 0.298$) can contain high-order frequency components, causing an underestimation of the second-order approximation compared with other results. In the high-frequency range, lines of each case merge. The reason is that the displacement amplitudes of the masses are small at higher frequencies, so that the effects of the stiffness nonlinearity are negligible.

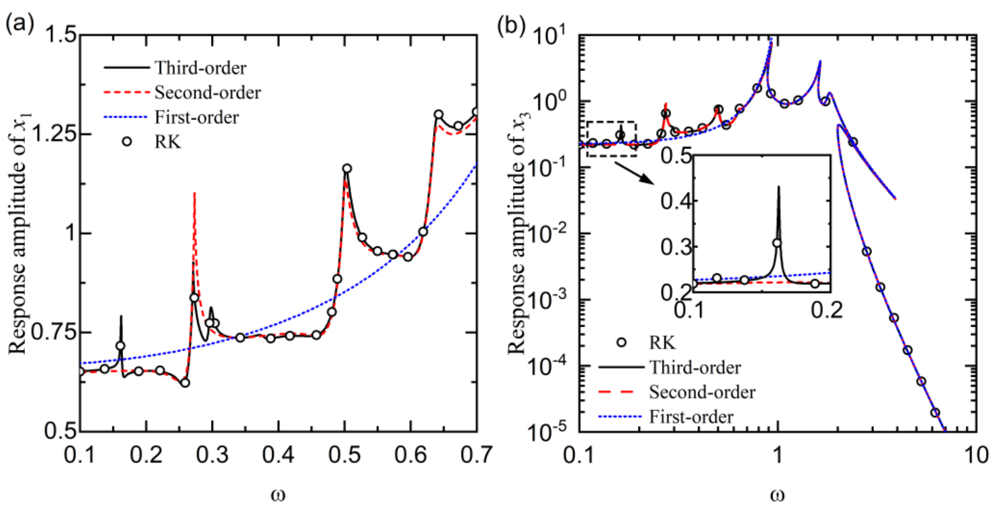

Figure 6. Frequency-response curve of (a) mass one and (b) mass three of the 3-DOF vibration system. Symbols: RK. Solid lines: third-order EGM. Dashed lines: second-order EGM. Dotted lines: first-order EGM

Figure 7(a) and (b) shows the frequency spectra and phase portrait of x_1 around the first super-harmonic peak at the excitation frequency $\omega = 0.162$, respectively. Figure 7(a) shows that there are three frequency components, namely $\omega_r = \omega, 3\omega$ and 5ω. It is also noted that the primary frequency component $\omega_r = \omega$ is dominant in this region, since its response amplitude value is much larger than the other two components. Figure 7(b) shows the phase portrait diagram of x_1 in the steady-state condition obtained by the third-order approximation and the RK method, where samples are extracted in time domain from $800T$ to $1200T$ and $T = 2\pi/\omega$. Figure 7(b) indicates that the third-order approximation based on the extended Galerkin method yields an accurate response solution for a multi-DOF nonlinear dynamical system.

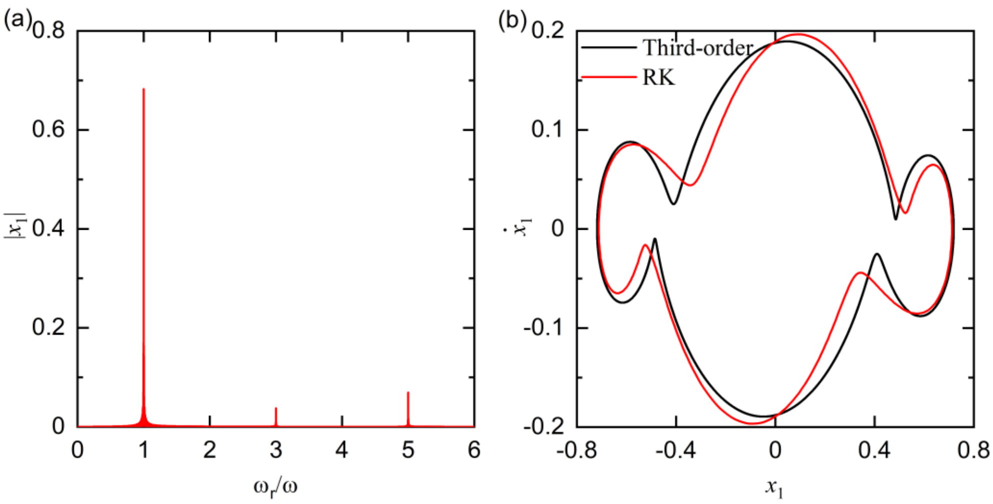

Figure 7. (a) Frequency spectra and (b) phase portrait of mass one at the excitation frequency $\omega = 0.162$

4. Conclusion

This study proposed an alternative procedure for the steady-state response solution based on the extended Galerkin method for a generalised multi-DOF nonlinear vibrational system. The proposed method transformed the ordinary governing equation into a set of algebraic equations of response amplitudes through an integration of the weighted equations of motion containing the coupled products of trigonometric terms. This approach was applied to three different case studies with a third-order approximation and have demonstrated its high accuracy and high efficiency in derivation and computation, compared with the conventional harmonic balance method. It was found that the results obtained by the third-order approximation match well with the numerical Runge-Kutta results in both the frequency and time domains. It should also be pointed out that the proposed method can be easily extended to a high-DOF linear or nonlinear system for forced vibration analysis as well as free vibration analysis. It is clear that the extended Galerkin method is a new and unique procedure, although equivalent to the harmonic balance method, for the approximate solutions of both linear and nonlinear problems of vibrations with a novel approach. The procedure will also change the current formulation of analysis of structural vibrations with the finite element method. It is also expected that the extended Galerkin and Rayleigh-Ritz methods will make these traditional techniques more capable for a unified analysis to vibration problems.

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Declaration of interest statement

The authors declare no conflict of interest.

Appendix

A- Formulations of Fourier coefficients of the nonlinear force

Based on Eq. (7), the first-order Fourier coefficients of the nonlinear force are

$$q_1 = \frac{3}{4}\left(x_{1,1}x_{1,2}^2 + x_{1,1}^3\right), \quad \text{(A.1)}$$

$$q_2 = \frac{3}{4}\left(x_{1,1}^2 x_{1,2} + x_{1,2}^3\right), \quad \text{(A.2)}$$

The second-order coefficients (ω and 3ω) of the nonlinear force are

$$q_1 = \frac{3}{4}\left(x_{1,1}x_{1,2}^2 + x_{1,1}^3 + \left(x_{1,1}^2 - x_{1,2}^2\right)x_{1,5} + 2x_{1,1}x_{1,2}x_{1,6} + 2\left(x_{1,5}^2 + x_{1,6}^2\right)x_{1,1}\right), \quad \text{(A.3)}$$

$$q_2 = \frac{3}{4}\left(x_{1,1}^2 x_{1,2} + x_{1,2}^3 + \left(x_{1,1}^2 - x_{1,2}^2\right)x_{1,6} - 2x_{1,1}x_{1,2}x_{1,5} + 2\left(x_{1,5}^2 + x_{1,6}^2\right)x_{1,2}\right), \quad \text{(A.4)}$$

$$q_5 = \frac{1}{4}\left(x_{1,1}^3 - 3x_{1,1}x_{1,2}^2 + 6\left(x_{1,1}^2 + x_{1,2}^2\right)x_{1,5} + 3x_{1,5}^3 + 3x_{1,5}x_{1,6}^2\right), \quad \text{(A.5)}$$

$$q_6 = \frac{1}{4}\left(-x_{1,2}^3 + 3x_{1,1}^2 x_{1,2} + 6\left(x_{1,1}^2 + x_{1,2}^2\right)x_{1,6} + 3x_{1,6}^3 + 3x_{1,5}^2 x_{1,6}\right), \quad \text{(A.6)}$$

The third-order coefficients (ω, 3ω and 5ω) of the nonlinear force are

$$q_1 = \frac{3}{4}\Big(x_{1,1}x_{1,2}^2 + x_{1,1}^3 + \left(x_{1,1}^2 - x_{1,2}^2\right)x_{1,5} + 2x_{1,1}x_{1,2}x_{1,6} + 2\left(x_{1,5}^2 + x_{1,6}^2\right)x_{1,1} + x_{1,9}\left(x_{1,5}^2 - x_{1,6}^2 + 2x_{1,1}x_{1,5} - 2x_{1,2}x_{1,6}\right) + 2x_{1,10}\left(x_{1,2}x_{1,5} + x_{1,1}x_{1,6} + x_{1,5}x_{1,6}\right) + 2x_{1,1}\left(x_{1,9}^2 + x_{1,10}^2\right)\Big), \quad \text{(A.7)}$$

$$q_2 = \frac{3}{4}\Big(x_{1,1}^2 x_{1,2} + x_{1,2}^3 + \left(x_{1,1}^2 - x_{1,2}^2\right)x_{1,6} - 2x_{1,1}x_{1,2}x_{1,5} + 2\left(x_{1,5}^2 + x_{1,6}^2\right)x_{1,2} + x_{1,10}\left(-x_{1,5}^2 + x_{1,6}^2 + 2x_{1,1}x_{1,5} - 2x_{1,2}x_{1,6}\right) + 2x_{1,9}\left(-x_{1,2}x_{1,5} - x_{1,1}x_{1,6} + x_{1,5}x_{1,6}\right) + 2x_{1,2}\left(x_{1,9}^2 + x_{1,10}^2\right)\Big), \quad \text{(A.8)}$$

$$q_5 = \frac{1}{4}\left(x_{1,1}^3 - 3x_{1,1}x_{1,2}^2 + 6\left(x_{1,1}^2 + x_{1,2}^2\right)x_{1,5} + 3x_{1,5}^3 + 3x_{1,5}x_{1,6}^2\right) + \frac{3}{4}x_{1,9}\left(x_{1,1}^2 - x_{1,2}^2 + 2x_{1,1}x_{1,5} + 2x_{1,2}x_{1,6}\right) + \frac{3}{2}x_{1,10}\left(x_{1,1}x_{1,2} - x_{1,2}x_{1,5} + x_{1,1}x_{1,6}\right) + \frac{3}{2}x_{1,5}\left(x_{1,9}^2 + x_{1,10}^2\right), \quad \text{(A.9)}$$

$$q_6 = \frac{1}{4}\left(-x_{1,2}^3 + 3x_{1,1}^2 x_{1,2} + 6\left(x_{1,1}^2 + x_{1,2}^2\right)x_{1,6} + 3x_{1,6}^3 + 3x_{1,5}^2 x_{1,6}\right) + \frac{3}{4}x_{1,10}\left(x_{1,1}^2 - x_{1,2}^2 + 2x_{1,1}x_{1,5} + 2x_{1,2}x_{1,6}\right) + \frac{3}{2}x_{1,9}\left(-x_{1,1}x_{1,2} + x_{1,2}x_{1,5} - x_{1,1}x_{1,6}\right) + \frac{3}{2}x_{1,6}\left(x_{1,9}^2 + x_{1,10}^2\right), \quad \text{(A.10)}$$

$$q_9 = \frac{3}{4}\left(x_{1,1}^2 x_{1,5} - x_{1,2}^2 x_{1,5} - 2x_{1,1}x_{1,2}x_{1,6} + x_{1,1}x_{1,5}^2 - x_{1,1}x_{1,6}^2 + 2x_{1,2}x_{1,5}x_{1,6} + x_{1,9}x_{1,10}^2\right) + \frac{3}{2}x_{1,9}\left(x_{1,1}^2 + x_{1,2}^2 + x_{1,5}^2 + x_{1,6}^2\right) + \frac{1}{2}x_{1,9}^3, \tag{A.11}$$

$$q_{10} = \frac{3}{4}\left(x_{1,1}^2 x_{1,6} - x_{1,2}^2 x_{1,6} + 2x_{1,1}x_{1,2}x_{1,5} - x_{1,2}x_{1,5}^2 + x_{1,2}x_{1,6}^2 + 2x_{1,1}x_{1,5}x_{1,6} + x_{1,9}^2 x_{1,10}\right) + \frac{3}{2}x_{1,10}\left(x_{1,1}^2 + x_{1,2}^2 + x_{1,5}^2 + x_{1,6}^2\right) + \frac{1}{2}x_{1,10}^3. \tag{A.12}$$

B- Formulations of the Duffing oscillator

Here, a single-DOF oscillator is considered with no constraints on the right, i.e., $k_{Q+1} = c_{Q+1} = 0$ and $Q = 1$. Therefore, the generalized vibration system of Eq. (1) is simplified to the Duffing oscillator as

$$\ddot{x}_1 + c_1\dot{x}_1 + k_1 x_1 + k_{nl}x_1^3 = f\cos\omega t. \tag{B.1}$$

The third-order approximations with frequency components ω, 3ω and 5ω are used, i.e., $N = 5$ and $x_{1,3} = x_{1,4} = x_{1,7} = x_{1,8} = q_3 = q_4 = q_7 = q_8 = 0$ (ignore the frequency components 2ω, 4ω and its coefficients). Eqs. (11) and (12) become

$$-m_1\omega^2 x_{1,1} + \omega c_1 x_{1,2} + k_1 x_{1,1} + k_{nl}q_1 = f, \tag{B.2}$$

$$-m_1\omega^2 x_{1,2} - \omega c_1 x_{1,1} + k_1 x_{1,2} + k_{nl}q_2 = 0, \tag{B.3}$$

$$-9m_1\omega^2 x_{1,5} + k_1 x_{1,5} + 3\omega c_1 x_{1,6} + k_{nl}q_5 = 0, \tag{B.4}$$

$$-9m_1\omega^2 x_{1,6} + k_1 x_{1,6} - 3\omega c_1 x_{1,5} + k_{nl}q_6 = 0, \tag{B.5}$$

$$-25m_1\omega^2 x_{1,9} + k_1 x_{1,9} + 5\omega c_1 x_{1,10} + k_{nl}q_9 = 0, \tag{B.6}$$

$$-25m_1\omega^2 x_{1,10} + k_1 x_{1,10} - 5\omega c_1 x_{1,9} + k_{nl}q_{10} = 0, \tag{B.7}$$

where q_1, q_2, q_5, q_6, q_9 and q_{10} can be calculated by Eqs. (A.7)-(A.12), respectively. Note that Eqs. (B.2)-(B.7) are six nonlinear algebraic equations with six unknowns $x_{1,1}$, $x_{1,2}$, $x_{1,5}$, $x_{1,6}$, $x_{1,9}$ and $x_{1,10}$, which can be solved by Newton-Raphson based numerical continuation scheme. It is also found that the third-order solutions can be easily degraded to second-order and first-order approximations. By solving Eqs. (B.2)-(B.5), we obtain the second-order approximate solutions where q_1, q_2, q_5 and q_6 can be calculated by Eqs. (A.3)-(A.6), respectively. Similarly, by solving Eqs. (B.2) and (B.3), the first-order solutions of the Duffing oscillator can be obtained, where q_1 and q_2 can be calculated by Eqs. (A.1) and (A.2), respectively.

C- Formulations of the two-DOF system

In this case, a 2-DOF (i.e., $Q = 2$) nonlinear vibration system is considered. Thus, the generalised governing Eq. (1) can be written as

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + k_{nl}x_1^3 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) = f\cos\omega t\,, \tag{C.1}$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + c_3\dot{x}_2 + k_3x_2 = 0. \tag{C.2}$$

The formulations of the third-order approximations with frequency components ω, 3ω and 5ω are presented here, i.e., $N = 5$ and $x_{j,3} = x_{j,4} = x_{j,7} = x_{j,8} = q_3 = q_4 = q_7 = q_8 = 0$ ($j = 1$ or 2, ignore the frequency components 2ω, 4ω and its coefficients). Eqs. (11) and (12) become

$$-m_1\omega^2x_{1,1} + (c_1 + c_2)\omega x_{1,2} - c_2\omega x_{2,2} + (k_1 + k_2)x_{1,1} - k_2x_{2,1} + k_{nl}q_1 - f = 0, \tag{C.3}$$

$$-m_1\omega^2x_{1,2} - (c_1 + c_2)\omega x_{1,1} + c_2\omega x_{2,1} + (k_1 + k_2)x_{1,2} - k_2x_{2,2} + k_{nl}q_2 = 0, \tag{C.4}$$

$$-m_2\omega^2x_{2,1} - c_2\omega x_{1,2} - k_2x_{1,1} + (c_3 + c_2)\omega x_{2,2} + (k_3 + k_2)x_{2,1} = 0, \tag{C.5}$$

$$-m_2\omega^2x_{2,2} + c_2\omega x_{1,1} - (c_2 + c_3)\omega x_{2,1} - k_2x_{1,2} + (k_2 + k_3)x_{2,2} = 0, \tag{C.6}$$

$$-9m_1\omega^2x_{1,5} + 3\omega(c_1 + c_2)x_{1,6} + (k_1 + k_2)x_{1,5} - 3c_2\omega x_{2,6} - k_2x_{2,5} + k_{nl}q_5 = 0, \tag{C.7}$$

$$-9m_1\omega^2x_{1,6} - 3\omega(c_1 + c_2)x_{1,5} + (k_1 + k_2)x_{1,6} - k_2x_{2,6} + 3\omega c_2x_{2,5} + k_{nl}q_6 = 0, \tag{C.8}$$

$$-9m_2\omega^2x_{2,5} - x_{1,5}k_2 - 3\omega c_2x_{1,6} + x_{2,5}(k_3 + k_2) + 3\omega(c_3 + c_2)x_{2,6} = 0\,, \tag{C.9}$$

$$-9m_2\omega^2x_{2,6} - x_{1,6}k_2 + 3\omega c_2x_{1,5} + x_{2,6}(k_3 + k_2) - 3\omega(c_3 + c_2)x_{2,5} = 0\,, \tag{C.10}$$

$$-25m_1\omega^2 x_{1,9} + 5\omega(c_1 + c_2)x_{1,10} + (k_1 + k_2)x_{1,9} - 5c_2\omega x_{2,10} - k_2 x_{2,9} + k_{nl}q_9 = 0, \tag{C.11}$$

$$-25m_1\omega^2 x_{1,10} - 5\omega(c_1 + c_2)x_{1,9} + (k_1 + k_2)x_{1,10} - k_2 x_{2,10} + 5\omega c_2 x_{2,9} + k_{nl}q_{10} = 0, \tag{C.12}$$

$$-25m_2\omega^2 x_{2,9} - x_{1,9}k_2 - 5\omega c_2 x_{1,10} + x_{2,9}(k_3 + k_2) + 5\omega(c_3 + c_2)x_{2,10} = 0, \tag{C.13}$$

$$-25m_2\omega^2 x_{2,10} - x_{1,10}k_2 + 5\omega c_2 x_{1,9} + x_{2,10}(k_3 + k_2) - 5\omega(c_3 + c_2)x_{2,9} = 0, \tag{C.14}$$

where q_1, q_2, q_5, q_6, q_9 and q_{10} can be obtained by Eqs. (A.7)-(A.12), respectively. It is found that Eqs. (C.3)-(C.14) are nonlinear algebraic equations with 12 unknowns, which can be solved by Newton-Raphson based numerical continuation scheme. It should also be pointed out that the second-order and the first-order approximations can be easily obtained from the third-order solutions. By solving Eqs. (C.3)-(C.10), we have the second-order approximate solutions where q_1, q_2, q_5 and q_6 can be calculated by Eqs. (A.3)-(A.6), respectively. The first-order solutions of the 2-DOF system can be obtained from Eqs. (C.3)-(C.6), where q_1 and q_2 are determined by Eqs. (A.1) and (A.2), respectively.

D- Formulations of the three-DOF system

In this section, a 3-DOF vibration system is considered (Q=3), and the corresponding governing equation can be written as

$$m_1\ddot{x_1} + c_1\dot{x_1} + k_1x_1 + f_{nl}x_1^3 - k_2(x_2 - x_1) - c_2(\dot{x_2} - \dot{x_1}) = f\cos\omega t, \tag{D.1}$$

$$m_2\ddot{x_2} + k_2(x_2 - x_1) + c_2(\dot{x_2} - \dot{x_1}) - k_3(x_3 - x_2) - c_3(\dot{x_3} - \dot{x_2}) = 0, \tag{D.2}$$

$$m_3\ddot{x_3} + k_3(x_3 - x_2) + c_3(\dot{x_3} - \dot{x_2}) + k_4x_3 + c_4\dot{x_3} = 0, \tag{D.3}$$

The third-order approximations with frequency components ω, 3ω and 5ω are considered here, i.e., $N = 5$ and $x_{j,3} = x_{j,4} = x_{j,7} = x_{j,8} = q_3 = q_4 = q_7 = q_8 = 0$ ($j =$

1, 2 or 3, ignore the frequency components 2ω, 4ω and its coefficients). Therefore, Eqs. (11) and (12) become

$$-m_1\omega^2 x_{1,1} + (c_1 + c_2)\omega x_{1,2} - c_2\omega x_{2,2} + (k_1 + k_2)x_{1,1} - k_2 x_{2,1} + k_{nl}q_1 - f = 0, \tag{D.4}$$

$$-m_1\omega^2 x_{1,2} - (c_1 + c_2)\omega x_{1,1} + c_2\omega x_{2,1} + (k_1 + k_2)x_{1,2} - k_2 x_{2,2} + k_{nl}q_2 = 0, \tag{D.5}$$

$$-m_2\omega^2 x_{2,1} - c_2\omega x_{1,2} + (c_2 + c_3)\omega x_{2,2} - c_3\omega x_{3,2} - k_2 x_{1,1} + (k_2 + k_3)x_{2,1} - k_3 x_{3,1} = 0, \tag{D.6}$$

$$-m_2\omega^2 x_{2,2} + c_2\omega x_{1,1} - (c_2 + c_3)\omega x_{2,1} + c_3\omega x_{3,1} - k_2 x_{1,2} + (k_2 + k_3)x_{2,2} - k_3 x_{3,2} = 0, \tag{D.7}$$

$$-m_3\omega^2 x_{3,1} - c_3\omega x_{2,2} + (c_3 + c_4)\omega x_{3,2} - k_3 x_{2,1} + (k_3 + k_4)x_{3,1} = 0, \tag{D.8}$$

$$-m_3\omega^2 x_{3,2} + c_3\omega x_{2,1} - (c_3 + c_4)\omega x_{3,1} - k_3 x_{2,2} + (k_3 + k_4)x_{3,2} = 0, \tag{D.9}$$

$$-9m_1\omega^2 x_{1,5} + 3(c_1 + c_2)\omega x_{1,6} - 3c_2\omega x_{2,6} + (k_1 + k_2)x_{1,5} - k_2 x_{2,5} + k_{nl}q_5 = 0, \tag{D.10}$$

$$-9m_1\omega^2 x_{1,6} - 3(c_1 + c_2)\omega x_{1,5} + 3c_2\omega x_{2,5} + (k_1 + k_2)x_{1,6} - k_2 x_{2,6} + k_{nl}q_6 = 0, \tag{D.11}$$

$$-9m_2\omega^2 x_{2,5} - 3c_2\omega x_{1,6} + 3(c_2 + c_3)\omega x_{2,6} - 3c_3\omega x_{3,6} - k_2 x_{1,5} + (k_2 + k_3)x_{2,5} - k_3 x_{3,5} = 0, \tag{D.12}$$

$$-9m_2\omega^2 x_{2,6} + 3c_2\omega x_{1,5} - 3(c_2 + c_3)\omega x_{2,5} + 3c_3\omega x_{3,5} - k_2 x_{1,6} + (k_2 + k_3)x_{2,6} - k_3 x_{3,6} = 0, \tag{D.13}$$

$$-9m_3\omega^2 x_{3,5} - 3c_3\omega x_{2,6} + 3(c_3 + c_4)\omega x_{3,6} - k_3 x_{2,5} + (k_3 + k_4)x_{3,5} = 0, \tag{D.14}$$

$$-9m_3\omega^2 x_{3,6} + 3c_3\omega x_{2,5} - 3(c_3 + c_4)\omega x_{3,5} - k_3 x_{2,6} + (k_3 + k_4)x_{3,6} = 0,$$

(D.15)

$$-25m_1\omega^2x_{1,9} + 5(c_1 + c_2)\omega x_{1,10} - 5c_2\omega x_{2,10} + (k_1 + k_2)x_{1,9} - k_2x_{2,9} + k_{nl}q_9 = 0,$$

(D.16)

$$-25m_1\omega^2x_{1,10} - 5(c_1 + c_2)\omega x_{1,9} + 5c_2\omega x_{2,9} + (k_1 + k_2)x_{1,10} - k_2x_{2,10} + k_{nl}q_{10} = 0,$$

(D.17)

$$-25m_2\omega^2x_{2,9} - 5c_2\omega x_{1,10} + 5(c_2 + c_3)\omega x_{2,10} - 5c_3\omega x_{3,10} - k_2x_{1,9} + (k_2 + k_3)x_{2,9} - k_3x_{3,9} = 0,$$

(D.18)

$$-25m_2\omega^2x_{2,10} + 5c_2\omega x_{1,9} - 5(c_2 + c_3)\omega x_{2,9} + 5c_3\omega x_{3,9} - k_2x_{1,10} + (k_2 + k_3)x_{2,10} - k_3x_{3,10} = 0,$$

(D.19)

$$-25m_3\omega^2x_{3,9} - x_{2,9}k_3 - 5\omega c_3x_{2,10} + x_{3,9}(k_3 + k_4) + 5\omega(c_3 + c_4)x_{3,10} = 0,$$

(D.20)

$$-25m_3\omega^2x_{3,10} - x_{2,10}k_3 + 5\omega c_3x_{2,9} + x_{3,10}(k_3 + k_4) - 5\omega(c_3 + c_4)x_{3,9} = 0,$$

(D.21)

where q_1, q_2, q_5, q_6, q_9 and q_{10} can be obtained from Eqs. (A.7)-(A.12), respectively. Eqs. (D.4)-(D.21) are the coupled nonlinear algebraic equations which yields the third-order steady-state solutions. As mentioned in the previous sections, the third-order solutions can be degraded to the second-order and the first-order approximations. By solving Eqs. (D.4)-(D.15), the second-order approximation solutions are obtained where q_1, q_2, q_5 and q_6 can be calculated by Eqs. (A.3)-(A.6), respectively. Similarly, From Eqs. (D.4)-(D.9), the first-order solutions of the 3-DOF system can be obtained, where q_1 and q_2 are calculated by Eqs. (A.1) and (A.2), respectively.

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